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Spectral Properties of the Complex Airy Operator on the Half-Line*

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Abstract. We prove a theorem on the completeness of the system of root functions of the Schrödinger operator $L = -d^2/dx^2 + p(x)$ on the half-line $\mathbb{R}_+$ with a potential p for which L appears to be maximal sectorial. An application of this theorem to the complex Airy operator $\mathscr{L}_c = -d^2/dx^2 + cx$, $c = \text{const}$, implies the completeness of the system of eigenfunctions of $\mathscr{L}_c$ for the case in which $|\arg c| < 2\pi/3$. We use subtler methods to prove a theorem stating that the system of eigenfunctions of this special operator remains complete under the condition that $|\arg c| < 5\pi/6$.

Key words: Schrödinger operator, complex Airy operator, nonself-adjoint operator, completeness of the eigenfunctions of a differential operator.

Introduction

This article mainly studies the operator

$$\mathscr{L}_c = -\frac{d^2}{dx^2} + cx \tag{1}$$

on the half-line $x \in [0, +\infty)$ with the Dirichlet boundary condition at zero. We are interested in the case where the constant c is not real. The main result is given by Theorem 1, which states that *the eigenfunctions of this operator correspond to simple eigenvalues and form a complete system in the space* $L_2(\mathbb{R}_+)$ *provided that* $|\arg c| < 5\pi/6$.

We also consider the operator

$$\mathscr{L}_{c,\alpha} = -\frac{d^2}{dx^2} + cx^\alpha, \qquad x \in [0, \infty), \tag{2}$$

where $\alpha > 0$ and $c \in \mathbb{C} \setminus (-\infty, 0]$. It is convenient to study this operator as a special case of operators of the more general form

$$Ly = -y'' + p(x)y, \quad p(x) = q(x) \pm ir(x), \qquad x \in [0, \infty), \tag{3}$$

where

$$r(x) \geqslant M_0, \quad q(x) \geqslant c_0 r(x) + M_1, \quad \varliminf_{x\to+\infty} x^{-\alpha} r(x) \geqslant a > 0, \quad \alpha > 0, \tag{4}$$

and M_0, M_1, and c_0 are real (possibly, negative) constants. It suffices to assume that the functions q and r are locally integrable. We obtain Theorem 2, which states that *if conditions* (4) *are satisfied and the operator* L_D *is generated by the differential expression* (3) *and the Dirichlet boundary condition at zero, then the system of its root functions is complete in* $L_2(\mathbb{R}_+)$ *provided that* $|\gamma| < 2\alpha\pi/(2+\alpha)$, *where* $\gamma = \arg(\pm i + c_0) \in (0, \pi)$. Moreover, this system is a basis for the Abel–Lidskii summation method.

Note that the potential cx^α can be represented in the form $|c|(\cos\gamma + i\sin\gamma)x^\alpha$; hence the completeness theorem for the operator (2) holds for $|\arg c| < 2\alpha\pi/(2+\alpha)$. In particular, if $c = i$, then the completeness theorem holds for $\alpha > 2/3$. Although Theorem 2 seems to be more general, it only gives the completeness result for the operator (1) with $|\arg c| < 2\pi/3$. We obtain a proof of Theorem 2 from the general theory stemming from Keldysh's paper [1]. This theory was further developed by numerous authors (see [2, Sec. 4] for details). The theorem given here generalizes a

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theorem due to Lidskii [3], who proved it under the assumption that the constant c_0 in (4) is zero. The claim in Theorem 1 for $|\arg c| \in [2\pi/3, 5\pi/6)$ (which does not follow from Theorem 2) is a much subtler result, and it is the most important part of the paper.

We obtained the results presented here in 1999, soon after we had discussed these problems with Davies (see the paper [4]). However, we postponed the publication, because we hoped to solve the completeness problem for the operator $\mathscr{L}_c$ completely (no pun intended). Recently, Mityagin has drawn our attention to the following problem stated by Almog [5]: *Is the eigenfunction system of the operator* $\mathscr{L}_{i,\alpha} = -d^2/dx^2 + ix^\alpha$ *complete for* $\alpha \in (0, 2/3]$? We do not know the answer, but the relation between this problem and the result stated in Theorem 1 is obvious. In particular, we have no doubt that the following conjecture is true: *there exists a number* $\alpha_0 < 2/3$ *such that the eigenfunctions of the operator* $\mathscr{L}_{i,\alpha}$ *form a complete system in* $L_2(\mathbb{R}_+)$ *for* $\alpha \in (\alpha_0, 2/3]$.

Here we consider operators on the half-line $\mathbb{R}_+$, but the results remain valid for the entire line if one extends the potentials to the entire line as even functions (see the remark at the end of the paper). The complex Airy operator is usually understood to be the operator $\mathscr{L}_c$ with $c = i$ (e.g., see [6]), but we preserve the name for arbitrary $c \in \mathbb{C} \setminus \{0\}$. The Airy operator was studied in connection with the Orr–Sommerfeld problem, well known in fluid mechanics (see the papers [7]–[10] and references therein). This operator is also related to other problems of mechanics (e.g., see Almog [6]). For nonself-adjoint operators, it is important not only to know where the spectrum is located but also to have some information on the ε-pseudospectrum $\sigma_\varepsilon := \{\lambda \in \mathbb{C} : \|(L - \lambda I)^{-1}\| \geqslant \varepsilon^{-1}\}$. On this topic, note, e.g., the papers by Trefethen–Embree [11], Krejčiřík–Siegl–Tater–Viola [12], and Henry–Krejčiřík [13].

One can single out yet another three directions of research in the study of the Schrödinger operator with complex potential $p = q + ir$. The first direction deals with the case in which the function r is in some sense subordinate to q, so that the corresponding operator is a perturbation of a self-adjoint operator. Here we note the papers [2] and [14]—[20] by Adduci, Djakov, Mityagin, Shkalikov, Siegl, and Viola. In the second direction, the potential is pure imaginary, $q(x) \equiv 0$. Here we note the papers [4] and [21]–[26] by Davies, Henry, Krejčiřík, Kuijlaars, Shkalikov, and Tumanov. The third direction deals with the Schrödinger operator with PT-symmetric potential, $p(x) = \overline{-p(-x)}$. Here we note the papers [27]–[29] by Bender, Boettcher, Eremenko, Gabrielov, Krejčiřík, Shapiro, and Siegl. In the context of the present paper, it is important to note that the complex Airy operator with $c = i$ was studied on the half-line and the entire line in the paper by Grebenkov–Helffer–Henry [30]. In particular, it was proved there that the eigenfunctions of this operator on the half-line form a complete system but not a basis and that the spectrum of this operator on the entire line is empty.

To make the exposition simple and specific, we restrict ourselves to the Dirichlet boundary condition. The theorems and their proofs remain valid if one replaces the Dirichlet condition with the condition $y'(0) + hy(0) = 0,\ h \in \mathbb{R}$.

1. Definition of the Operator $\mathscr{L}_c$ and Its Main Properties

Let us give a more precise definition of the operator $\mathscr{L}_c$ with the Dirichlet boundary condition. Namely, $\mathscr{L}_c$ is the operator defined on the space $L_2(\mathbb{R}_+)$ by the differential expression

$$l(y) = -y'' + cxy, \qquad x \in [0, +\infty),$$

with the domain

$$\mathfrak{D}(\mathscr{L}_c) = \{y \in L_2(\mathbb{R}_+) : y \in W^2_{2,\mathrm{loc}},\ l(y) \in L_2(\mathbb{R}_+),\ y(0) = 0\}.$$

The constant c is assumed to be complex and satisfy $c \in \mathbb{C} \setminus (-\infty, 0]$. In other words, we are interested in the case of $\gamma := \arg c \in (-\pi, \pi)$. Obviously, the operator $\mathscr{L}_c$ is densely defined, because its domain contains infinitely differentiable functions compactly supported on $(0, +\infty)$; the set of these functions is well known to be dense in $L_2(\mathbb{R}_+)$.

In what follows, we deal with special functions satisfying the Airy equation $y'' = zy$. It is well known (e.g., see [31, Sec. IV.1] and [32, Sec. 10.4]) that this equation has two linearly independent solutions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ with the initial conditions

$$\mathrm{Ai}(0) = \frac{1}{3^{2/3}\Gamma(2/3)}, \quad \mathrm{Ai}'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)},$$
$$\mathrm{Bi}(0) = \frac{1}{3^{1/6}\Gamma(2/3)}, \quad \mathrm{Bi}'(0) = \frac{3^{1/6}}{\Gamma(1/3)};$$

the Wronskian $W(\mathrm{Ai}, \mathrm{Bi})$ of these functions satisfies

$$W(\mathrm{Ai}, \mathrm{Bi}) = \mathrm{Ai}(z)\,\mathrm{Bi}'(z) - \mathrm{Ai}'(z)\,\mathrm{Bi}(z) = 1/\pi.$$

Both functions are entire functions of order $3/2$ and type $2/3$. In any domain $|\arg z| < \pi - \varepsilon$, where $\varepsilon > 0$ is arbitrarily small, the function $\mathrm{Ai}(z)$ admits the following asymptotic representations as $|z| \to \infty$:*

$$\begin{aligned} \mathrm{Ai}(z) &= \frac{1}{2\sqrt{\pi}}\, z^{-1/4} e^{-2z^{3/2}/3}(1 + O(z^{-3/2})), \\ \mathrm{Ai}'(z) &= -\frac{1}{2\sqrt{\pi}}\, z^{1/4} e^{-2z^{3/2}/3}(1 + O(z^{-3/2})). \end{aligned} \tag{5}$$

The representations (5) can be differentiated arbitrarily many times. Clearly, the functions $\mathrm{Ai}(z)$ and $\mathrm{Ai}'(z)$ exponentially decay as $|z| \to \infty$ along any ray in the sector $|\arg z| < \pi/3$. Finally, the following representation holds on the ray $\arg z = \pi$ for the function $\mathrm{Ai}(-x)$, $x > 0$:

$$\mathrm{Ai}(-x) = \frac{1}{\sqrt{\pi}}|x|^{-1/4}\left(\sin\left(\frac{2}{3}\,|x|^{3/2} + \frac{\pi}{4}\right) + O(|x|^{-3/2})\right), \qquad |x| \to \infty.$$

All zeros z_k of $\mathrm{Ai}(z)$ are simple and lie on the ray $\arg z = -\pi$, and one has

$$z_k = -\left[\frac{3}{2}\,\pi\left(k - \frac{1}{4}\right)\right]^{2/3} + O(k^{-4/3}), \qquad k = 1, 2, \dots\,. \tag{6}$$

Instead of $\mathrm{Bi}(z)$, it will be convenient to use the function $U(z) := \mathrm{Bi}(z) - \sqrt{3}\,\mathrm{Ai}(z)$, which satisfies the initial conditions

$$U(0) = 0, \quad U'(0) = \frac{2 \cdot 3^{1/6}}{\Gamma(1/3)}; \quad \text{furthermore, } W(\mathrm{Ai}, U) = \frac{1}{\pi}\,.$$

In the sector $|\arg z| < \pi/3$, this function has the asymptotic representation

$$U(z) = \frac{1}{\sqrt{\pi}}\, z^{-1/4} e^{2z^{3/2}/3}(1 + O(z^{-3/2})), \qquad U'(z) = \frac{1}{\sqrt{\pi}}\, z^{1/4} e^{2z^{3/2}/3}(1 + O(z^{-3/2})). \tag{7}$$

Thus, the functions $U(z)$ and $U'(z)$ grow exponentially as $|z| \to \infty$ along any ray in this sector. The asymptotic representations of $U(z)$ and $U'(z)$ are also well known in other sectors of the complex plane, but here we do not need them.

Proposition 1. *The domain $\mathfrak{D}(\mathscr{L}_c)$ coincides with the set of functions of the form*

$$y(x) = \pi c^{-1/3}\left[\mathrm{Ai}(c^{1/3}x)\int_0^x U(c^{1/3}t)f(t)\,dt + U(c^{1/3}x)\int_x^\infty \mathrm{Ai}(c^{1/3}t)f(t)\,dt\right], \tag{8}$$

where $f(x)$ ranges over the entire space $L_2(\mathbb{R}_+)$. For each function $y \in \mathfrak{D}(\mathscr{L}_c)$, one has

$$x^{1/2}y(x) \to 0, \quad y'(x) \to 0 \quad \text{as } x \to +\infty, \qquad x^{1/2}y(x), y'(x) \in L_2(\mathbb{R}_+). \tag{9}$$

Equation (8) defines a bounded operator on $L_2(\mathbb{R}_+)$, which is the inverse of $\mathscr{L}_c$.

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*From now on, the branch of z^α is fixed by the condition $\arg z \in [-\pi, \pi)$.

Proof. First, note that the ray $c^{1/3}x$, $x > 0$, lies in the sector $|\arg z| < \pi/3$ of the complex plane. This means that the function $\mathrm{Ai}(c^{1/3}t)$ decays exponentially, and hence the improper integral in (8) converges. Since $f \in L_2(\mathbb{R}_+)$, it follows that the function y defined in (8) lies in $W_2^1[0,b]$ for every finite b. By differentiating, we obtain

$$y'(x) = \pi\biggl[\mathrm{Ai}'(c^{1/3}x)\int_0^x U(c^{1/3}t)f(t)\,dt + U'(c^{1/3}x)\int_x^\infty \mathrm{Ai}(c^{1/3}t)f(t)\,dt\biggr], \tag{10}$$

whence it follows that $y' \in W_2^1[0,b]$ for every finite b. By differentiating once more, we obtain

$$\begin{aligned} y'' &= \pi[\mathrm{Ai}'(c^{1/3}x)U(c^{1/3}x) - U'(c^{1/3}x)\,\mathrm{Ai}(c^{1/3}x)]f(x) \\ &\quad + \pi c^{1/3}\biggl[\mathrm{Ai}''(c^{1/3}x)\int_0^x U(c^{1/3}t)f(t)\,dt + U''(c^{1/3}x)\int_x^\infty \mathrm{Ai}(c^{1/3}t)f(t)\,dt\biggr] \\ &= -f(x) + cxy(x); \end{aligned}$$

i.e., $y \in W_2^2[0,b]$ for each $b > 0$ and $l(y) = f \in L_2(\mathbb{R}_+)$. Since $y(0) = 0$, it follows that $y \in \mathfrak{D}(\mathscr{L}_c)$. The converse is true as well. Namely, let $y \in \mathfrak{D}(\mathscr{L}_c)$ and $l(y) = f \in L_2(\mathbb{R}_+)$. By a classical theorem on the general form of a solution of a differential equation,

$$\begin{aligned} y &= C_1\,\mathrm{Ai}(c^{1/3}x) + C_2U(c^{1/3}x) \\ &\quad + \pi c^{-1/3}\biggl[\mathrm{Ai}(c^{1/3}x)\int_0^x U(c^{1/3}t)f(t)\,dt + U(c^{1/3}x)\int_x^\infty \mathrm{Ai}(c^{1/3}t)f(t)\,dt\biggr], \end{aligned}$$

where C_1 and C_2 are constants. It follows from the relation $y(0) = 0$ that $C_1 = 0$, while the condition $y \in L_2(\mathbb{R}_+)$ and the estimate (12), which will be proved below, imply that $C_2 = 0$; i.e., y admits the representation (8). Thus, formula (8) defines the inverse operator $\mathscr{L}_c^{-1}$. Its boundedness follows from the estimate $|y(x)| \leqslant M\|f\|$ on every finite interval $[0,b]$ and the estimate (12). Here and in what follows, the letter M (or M_1, or M_2) stands for various positive constants, and $\|\cdot\| = \|\cdot\|_{L_2(\mathbb{R}_+)}$.

Let us prove relations (9). First, note that, by virtue of (7), there exists a constant M such that the estimate

$$|U(c^{1/3}t)| \leqslant Mt^{-1/4}\exp(at^{3/2}), \qquad a := (2/3)|c|^{1/2}\cos(\gamma/2),$$

holds for $t > 0$. (Recall that $\gamma = \arg c$.) It is easily seen that the function $g(t) = t^{-1/4}\exp(at^{3/2})$ increases for sufficiently large $t \geqslant b = b(a)$. Hence

$$\begin{aligned} \biggl|\int_0^x U(c^{1/3}t)f(t)\,dt\biggr| &\leqslant \biggl(\int_0^b + \int_b^{x-1} + \int_{x-1}^x\biggr)|U(c^{1/3}t)|\,|f(t)|\,dt \\ &\leqslant M_1\|f\| + M\|f\|x^{1/4}\exp[a(x^{3/2} - x^{1/2})] + Mx^{-1/4}\exp(ax^{3/2})\biggl(\int_{x-1}^x |f(t)|^2\,dt\biggr)^{1/2} \end{aligned}$$

for $x > b+1$. Here the constant M_1 depends on b alone. When passing to the second inequality, we have taken into account the fact that the length of the integration interval in the second integral is less than x and used the inequality $(x-1)^{3/2} \leqslant x^{3/2} - x^{1/2}$, which holds for sufficiently large x. This estimate, together with the representation (5), implies that

$$\begin{aligned} &\biggl|\mathrm{Ai}(c^{1/3}x)\int_0^x U(c^{1/3}t)f(t)\,dt\biggr| \\ &\qquad \leqslant Mx^{-1/4}\exp(-ax^{3/2})\|f\| + M\exp(-ax^{1/2})\|f\| + Mx^{-1/2}\biggl(\int_{x-1}^x |f(t)|^2\,dt\biggr)^{1/2} \end{aligned}$$

for sufficiently large x. We have estimated the first summand in (8). In a similar way, we estimate the second summand. The function $g(t) = t^{1/2}\exp(-at^{3/2})$ is decreasing for sufficiently large t, and hence

$$\left|\int_x^\infty \mathrm{Ai}(c^{1/3}t)f(t)\,dt\right| \leqslant M\biggl(\int_{x+1}^\infty g^2(t)t^{-3/2}\,dt\biggr)^{1/2}\|f\| + M\int_x^{x+1} g(t)t^{-3/4}\,|f(t|\,dt$$
$$\leqslant 2^{1/2}Mx^{-1/4}g(x+1)\|f\| + Mg(x)x^{-3/4}\biggl(\int_x^{x+1}|f(t)|^2dt\biggr)^{1/2}$$

for large x. Since $(x+1)^{3/2} \geqslant x^{3/2} + x^{1/2}$, we have

$$|U(c^{1/3}x)| \leqslant Mx^{1/4}g^{-1}(x), \qquad g(x+1)g^{-1}(x) \leqslant M\exp(-ax^{1/2}),$$

and hence the absolute value of the second term on the right-hand side in (8) can be estimated by

$$M\exp(-ax^{1/2})\|f\| + Mx^{-1/2}\biggl(\int_x^{x+1}|f(t)|^2dt\biggr)^{1/2}.$$

By adding the resulting estimates, we arrive at the inequality

$$|y(x)| \leqslant M_2\exp(-ax^{1/2})\|f\| + M_2x^{-1/2}\biggl(\int_{x-1}^{x+1}|f(t)|^2dt\biggr)^{1/2}, \qquad x > b+1, \tag{11}$$

which proves the first relation in (9). The second relation in (9) can be obtained in the same way except that (10) is used instead of (8) and we take into account the fact that the estimates for the derivatives Ai' and U' differ from the estimates for the functions themselves by the factor $x^{1/2}$.

Let us prove that $x^{1/2}y(x) \in L_2(\mathbb{R}_+)$. It follows from the estimate (11) that

$$\int_b^\infty |x^{1/2}y(x)|^2\,dx \leqslant M\|f\|^2\int_b^\infty x\exp(-2ax^{1/2})\,dx + M\int_b^\infty\int_{x-1}^{x+1}|f(t)|^2dt\,dx$$
$$\leqslant M\|f\|^2 + M\int_{b-1}^\infty |f(t)|^2\int_{t-1}^{t+1}dx\,dt \leqslant M\|f\|^2. \tag{12}$$

The inclusion $y' \in L_2(\mathbb{R}_+)$ can be obtained in a similar way with regard to the fact that $|y'(x)|$ is bounded by the right-hand side of (11) multiplied by $x^{1/2}$. This completes the proof of the proposition. □

Proposition 2. *The numerical range of the operator* $\mathscr{L}_c$ *lies in the closed sector* S_γ *of the complex plane bounded by the rays* $\arg\lambda = 0$ *and* $\arg\lambda = \gamma$, *where* $\gamma = \arg c$.

Proof. The quadratic form of our operator is given by

$$(\mathscr{L}_c y, y) = \int_0^\infty (-y''\overline{y} + cxy\overline{y})\,dx = -\overline{y}(x)y'(x)\Big|_0^\infty + \int_0^\infty |y'|^2\,dx + c\int_0^\infty x|y|^2\,dx.$$

It remains to note that $y(0) = 0$ and $\overline{y}(x)y'(x) \to 0$ as $x \to \infty$ by (9). □

Proposition 3. *The operator* $\mathscr{L}_c$ *is closed and has zero deficiency numbers* (*the dimension of the kernel and the codimension of the range*). *The adjoint operator* $\mathscr{L}_c^*$ *coincides with the operator* $\mathscr{L}_{\overline{c}}$.

Proof. The first claim follows from Proposition 1, because we have presented the inverse operator $\mathscr{L}_c^{-1}$, which is bounded and hence closed. Consequently, the inverse of the inverse is closed as well. To prove the second claim, let us verify the Lagrange identity. Let $y(x) \in \mathfrak{D}(\mathscr{L}_c)$, and let $u(x) \in \mathfrak{D}(\mathscr{L}_{\overline{c}})$. Then

$$(\mathscr{L}_c y, u) = \int_0^\infty (-y''(x)\overline{u}(x) + cy(x)\overline{u}(x))\,dx$$
$$= \int_0^\infty (-y(x)\overline{u}''(x) + y(x)\overline{\overline{c}u}(x))\,dx + y(x)\overline{u}'(x)\Big|_0^\infty - y'(x)\overline{u}(x)\Big|_0^\infty$$
$$= \int_0^\infty (-y(x)\overline{u}''(x) + y(c)\overline{\overline{c}u}(x))\,dx = (y, \mathscr{L}_{\overline{c}}u).$$

Set $\mathscr{L}_c y = z$ and $\mathscr{L}_{\overline{c}} u = v$. It follows from the Lagrange identity that

$$(z, \mathscr{L}_{\overline{c}}^{-1} v) = (\mathscr{L}_c y, u) = (y, \mathscr{L}_{\overline{c}} u) = (y, v) = (\mathscr{L}_c^{-1} z, v) \quad \forall z, v \in L_2(\mathbb{R}_+).$$

Consequently, $\mathscr{L}_{\overline{c}}^{-1} = (\mathscr{L}_c^{-1})^*$. But then $\mathscr{L}_{\overline{c}} = \mathscr{L}_c^*$. □

Proposition 4. *The resolvent $\mathscr{R}_c(\lambda) = (\mathscr{L}_c - \lambda I)^{-1}$ is well defined and is a bounded operator for each $\lambda \in \mathbb{C} \setminus S_\gamma$, where the sector S_γ is defined in Proposition 2. Further,*

$$\|\mathscr{R}_c(\lambda)\|_{L_2(\mathbb{R}_+)} \leqslant \frac{1}{\operatorname{dist}(\lambda, S_\gamma)}\,. \tag{13}$$

Proof. For $\gamma = 0$, the operator $\mathscr{L}_c$ is self-adjoint and positive, $S_\gamma = \mathbb{R}_+$, and the claim to be proved is well known. Let $\gamma \neq 0$. By Proposition 2, both operators $T_1 = e^{i(\pi/2-\gamma)} \operatorname{sign}\gamma \cdot \mathscr{L}_c$ and $T_2 = -i \operatorname{sign}\gamma \cdot \mathscr{L}_c$ are accretive (i.e., $\operatorname{Re}(T_j y, y) \geqslant 0$ for each $y \in \mathfrak{D}(\mathscr{L}_c) = \mathfrak{D}(T_j)$, $j = 1, 2$). By Proposition 3, both operators are closed and have zero deficiency numbers; i.e., they are closed maximal accretive operators. Hence the operators $T_1 - zI$ and $T_2 - zI$ are invertible for each z in the open left half-plane (e.g., see [33, Chap. III.10]), and

$$\|(T_1 - zI)^{-1}\| \leqslant |\operatorname{Re} z|^{-1} \quad \text{and} \quad \|(T_2 - zI)^{-1}\| \leqslant |\operatorname{Re} z|^{-1}.$$

These estimates are obviously equivalent to the estimate (13). □

It follows from Proposition 2 that the numerical range of the operator $\mathscr{L}_c$ lies in the sector S_γ. Hence the operator $T = e^{-i\gamma/2}\mathscr{L}_c$ is sectorial, and it follows from Proposition 3 that this operator is m-sectorial. For each m-sectorial operator, there exists a unique sectorial sesquilinear form associated with it. Let us find an explicit expression for this form.

Proposition 5. *The closed sectorial sesquilinear form $\mathfrak{t}$ of the operator $T = e^{-i\gamma/2}\mathscr{L}_c$ is given by*

$$\mathfrak{t}[u, v] = e^{-i\gamma/2} \int_0^\infty u'(x)\overline{v}'(x)\,dx + |c| e^{i\gamma/2} \int_0^\infty x u(x)\overline{v}(x)\,dx,$$
$$u, v \in \mathfrak{D}(\mathfrak{t}) = \{y \in L_2(\mathbb{R}_+) : y', x^{1/2} y \in L_2(\mathbb{R}_+)\}. \tag{14}$$

Proof. Define the form $\mathfrak{t}_0[u, v] = (Tu, v)$ on the domain $\mathfrak{D}(T) = \mathfrak{D}(\mathscr{L}_c)$. By integrating by parts in the same way as in the proof of Proposition 2, we obtain

$$\mathfrak{t}_0[u, v] = e^{-i\gamma/2} \int_0^\infty u'(x)\overline{v}'(x)\,dx + |c| e^{i\gamma/2} \int_0^\infty x u(x)\overline{v}(x)\,dx.$$

Obviously, the linear manifold $\mathfrak{D}(\mathfrak{t})$ defined above is a closed set in the norm corresponding to the inner product

$$[u, v] = \int_0^\infty (x u(x)\overline{v}(x) + u'(x)\overline{v}'(x))\,dx;$$

i.e., $\mathfrak{D}(\mathfrak{t})$ equipped with this inner product is a Hilbert space. Note that $\mathfrak{D}(\mathscr{L}_c)$ is a core of the form $\mathfrak{t}$ (see [33, Chap. VI, Theorem 2.1]). Hence the domain of the closure of $\mathfrak{t}_0$ coincides with $\mathfrak{D}(\mathfrak{t})$, and the form $\mathfrak{t}$ corresponding to the operator T is given by (14). □

Proposition 6. *For each λ in the resolvent set, the operator $\mathscr{R}_c(\lambda) = (\mathscr{L}_c - \lambda I)^{-1}$ is compact. The spectrum of the operator $\mathscr{L}_c$ is discrete and consists of a sequence of simple eigenvalues*

$$\lambda_n = t_n c^{2/3}, \quad n \in \mathbb{N}, \quad \text{where } t_n > 0 \text{ and } t_n = [(3\pi/2)(n - 1/4)]^{2/3} + O(n^{-4/3}). \tag{15}$$

The corresponding eigenfunctions are

$$y_n(x) = \operatorname{Ai}(-t_n + x c^{1/3}), \qquad n = 1, 2, \dots . \tag{16}$$

Proof. The compactness of the resolvent of the operator $T = e^{-i\gamma/2}\mathscr{L}_c$ is equivalent to that of the resolvent of the operator $H = \operatorname{Re} T$ (see [33, Chap. VI, Theorem 3.3]). The operator H is generated by the quadratic form $\mathfrak{h} = \operatorname{Re}\mathfrak{t}$; i.e.,

$$\mathfrak{h}[u,v] = \cos\frac{\gamma}{2}\int_0^\infty u'(x)\overline{v}'(x)\,dx + |c|\cos\frac{\gamma}{2}\int_0^\infty xu(x)\overline{v}(x)\,dx, \qquad \mathfrak{D}(\mathfrak{h}) = \mathfrak{D}(\mathfrak{t}).$$

The operator H can be uniquely reconstructed from the quadratic form; hence from Proposition 5 we obtain

$$\begin{gathered} Hy = \cos(\gamma/2)(-y'' + |c|xy), \\ \mathfrak{D}(H) = \{y \in L_2(\mathbb{R}_+) : y \in W^2_{2,\mathrm{loc}},\ -y'' + |c|xy \in L_2(\mathbb{R}_+),\ y(0) = 0\}. \end{gathered} \tag{17}$$

The compactness of the resolvent $(H-\lambda)^{-1}$ follows from Molchanov's criterion (e.g., see [34, Sec. VII.24]). Thus, the resolvent $\mathscr{R}_c(\lambda)$ is compact, and the spectrum $\sigma(\mathscr{L}_c)$ is discrete. Let us write out the eigenvalue equation

$$-y'' + cxy = \lambda y, \qquad y(0) = 0,\ y \in L_2(\mathbb{R}_+).$$

We make the change of variables $t = -\lambda c^{-2/3} + xc^{1/3}$. Then the equation acquires the form $-y''_{tt} + ty = 0$; i.e.,

$$y(x) = C_1 \operatorname{Ai}(c^{1/3}x - \lambda c^{-2/3}) + C_2 U(c^{1/3}x - \lambda c^{-2/3}). \tag{18}$$

In view of (5) and (7), we find that $C_2 = 0$, and since $y(0) = 0$, we arrive at the eigenvalue equation $\operatorname{Ai}(-\lambda c^{-2/3}) = 0$. It remains to note that all zeros of the function $\operatorname{Ai}(z)$ are simple, and relations (15) now follow from (6). By substituting $\lambda = \lambda_n$ into (18), we obtain relations (16). □

2. Completeness Theorem for the Operator $\mathscr{L}_c$

Theorem 1. *The eigenfunction system of the operator $\mathscr{L}_c$ is complete and minimal in $L_2(\mathbb{R}_+)$ under the condition $|\arg c| < (5\pi)/6$.*

Proof. The minimality of the eigenfunction system $\{y_n\}$ of the operator $\mathscr{L}_c$ follows from the well-known relations

$$(y_n, z_k) = c_n\delta_{nk}, \qquad c_n \neq 0,$$

where $\{z_k\}$ is the eigenfunction system of the adjoint operator $\mathscr{L}_{\overline{c}}$. To prove the completeness, we use the Levinson method (see [35, Supplement 4]), taking into account the presence of the function $\operatorname{Ai}(\mathrm{w} + c^{1/3}x)$, which generates the eigenfunctions y_n for $\mathrm{w} = -t_n$. To be definite, we shall consider the case of $\operatorname{Im} c \geqslant 0$; i.e., $\gamma \in [0,\pi)$, where $\gamma = \arg c$. The case of $\gamma \in (-\pi, 0]$ can be treated in a similar way. (The estimates given below should be carried out in the same way but in sectors symmetric with respect to the real line.)

We split the proof into several steps. Let a function $f \in L_2(\mathbb{R}_+)$ be orthogonal to the eigenfunctions of $\mathscr{L}_c$. Consider the function

$$F(\mathrm{w}) = \frac{F_0(\mathrm{w})}{\operatorname{Ai}(\mathrm{w})}, \qquad F_0(\mathrm{w}) = \int_0^\infty \operatorname{Ai}(\mathrm{w} + xc^{1/3})f(x)\,dx. \tag{19}$$

At the first step, we show that F is an entire function of order $\rho \leqslant 3/2$ and finite type for $\rho = 3/2$. At the second step, we prove that F admits the estimate

$$|F(\mathrm{w})| \leqslant M\|f\|R^{1/2}, \qquad R = |\mathrm{w}| \geqslant 1, \tag{20}$$

in the sector

$$S = \{\mathrm{w} \in \mathbb{C} : -\pi + \gamma/3 \leqslant \arg \mathrm{w} \leqslant \pi - 2\gamma/3\}. \tag{21}$$

At the third step, we show that there exists a number $\alpha_0 \in (0, \gamma/3)$ such that the estimate (20) remains valid in the sector $S' = S \cup S_0$, where

$$S_0 = \{\mathrm{w} \in \mathbb{C} : -\pi + \alpha_0 \leqslant \arg \mathrm{w} \leqslant -\pi + \gamma/3\}. \tag{22}$$

The number α_0 cannot be computed in closed form (it is a root of a transcendental equation), but one can show that the opening angle of the complementary sector $\mathbb{C}\setminus S'$ is less than $2\pi/3$ provided that $\gamma < 5\pi/6$. Then it follows that the estimate (20) holds in the entire complex plane. At the fourth step, we show that $F(\mathrm{w}) \equiv 0$, and the fifth step gives $f(x) \equiv 0$. Let us proceed to the implementation of our plan.

Step 1. Let us show that the function F_0 in (19) is well defined and holomorphic in the parameter $\mathrm{w} \in \mathbb{C}$. Take an arbitrary number $\delta \in (0, \pi/4]$ such that $\gamma/2+\delta < \pi/2$. Let w run over the compact set $|\mathrm{w}| \leqslant R$. We represent the function F_0 in the form

$$F_0(\mathrm{w}) = \left(\int_0^{x_0} + \int_{x_0}^{\infty} \right) \mathrm{Ai}(\mathrm{w} + xc^{1/3}) f(x)\, dx := F_1(\mathrm{w}) + F_2(\mathrm{w}),$$
$$x_0 = x_0(R) = \frac{R}{|c|^{1/3} \sin(2\delta/3)}.$$

The first integral is proper, and hence it is holomorphic in the parameter w in the disk $|\mathrm{w}| \leqslant R$. To prove that the second integral is holomorphic, note that

$$\left| \arg\left(1 + \frac{\mathrm{w}}{c^{1/3}x}\right)\right| \leqslant \frac{2\delta}{3} \implies \arg(\mathrm{w} + c^{1/3}x)^{3/2} \in \left[\frac{\gamma}{2} - \delta, \frac{\gamma}{2} + \delta\right]$$

for $x > x_0(R)$. Then

$$\mathrm{Re}(\mathrm{w} + c^{1/3}x)^{3/2} \geqslant |\mathrm{w} + c^{1/3}x|^{3/2} \cos(\gamma/2 + \delta) \geqslant (3/2)a_1(|c|^{1/3}x - R)^{3/2} \tag{23}$$

for $x > x_0(R)$ and $|\mathrm{w}| \leqslant R$, where $a_1 := (2/3)\cos(\gamma/2 + \delta)$. Consequently,

$$|\,\mathrm{Ai}(\mathrm{w} + c^{1/3}x)| \leqslant M \exp(-(2/3)\,\mathrm{Re}(\mathrm{w} + c^{1/3}x)^{3/2}) \leqslant M \exp(-a_1(|c|^{1/3}x - R)^{3/2}) \tag{24}$$

for $x > x_0(R)$ by (5) and (23). A similar estimate with right-hand side multiplied by $|\mathrm{w} + c^{1/3}x|^{1/4}$ holds for $\mathrm{Ai}'(\mathrm{w} + c^{1/3}x)^{3/2}$. It follows that the integral $F_2(\mathrm{w})$ converges uniformly with respect to the parameter w in the disk $|\mathrm{w}| \leqslant R$ and remains uniformly convergent after the differentiation with respect to w; i.e., the function F_2 is holomorphic in the disk $|\mathrm{w}| \leqslant R$. Since R is arbitrary, we see that $F_0 = F_1 + F_2$ is an entire function. Further, so is $F(\mathrm{w})$, because all the points $\mathrm{w} = -t_k$ are its removable singularities. (By our assumption, the function $F_0(\mathrm{w})$ vanishes at these points, while the function $\mathrm{Ai}(\mathrm{w})$ has simple zeros there.)

Now let us estimate the growth of $F_0(\mathrm{w})$ as $|\mathrm{w}| \to \infty$. The length of the integration interval in the integral F_1 is proportional to R, and hence

$$|F_1(\mathrm{w})| \leqslant MR^{1/2}\|f\| \exp((2/3)(R + |c|^{1/3}x_0)^{3/2}) \leqslant MR^{1/2}\|f\| \exp(M_1 R^{3/2}). \tag{25}$$

Set $v(x) = \exp(-a_1(|c|^{1/3}x - R)^{3/2} + x)$, where the number a_1 is defined in (23). This function is decreasing for $x > x_0$ provided that the number $x_0 = x_0(R)$ is sufficiently large. Hence it follows from (24) that

$$\begin{aligned} |F_2(\mathrm{w})| &\leqslant M \int_{x_0}^{\infty} \exp(-a_1(|c|^{1/3}x - R)^{3/2} + x)|f(x)|e^{-x}\, dx \\ &\leqslant 2^{-1/2}M\|f\| \exp\{-a_1(|c|^{1/3}x_0(R) - R)^{3/2} + x_0)\}, \quad x_0 = \frac{R}{|c|^{1/3}\sin(2\delta/3)}\,. \end{aligned} \tag{26}$$

Consequently, $|F_2(\mathrm{w})| \to 0$ as $\mathrm{w} \to \infty$. It follows from (25) that $F_0 = F_1 + F_2$ is an entire function of order $\leqslant 3/2$ and of finite type if the order is $3/2$. Then the entire function $F(\mathrm{w})$ defined as the ratio of the entire functions $F_0(\mathrm{w})$ and $\mathrm{Ai}(\mathrm{w})$ has the growth characteristic that does not exceed the maximum of the growth characteristics of the numerator and denominator (e.g., see [35, Sec. I.9]); i.e., F has order $\rho \leqslant 3/2$ and finite type if $\rho = 3/2$.

Step 2. Consider the sectors

$$\begin{aligned} S_1 &= \{\mathrm{w} \in \mathbb{C} : -\pi + \gamma/3 \leqslant \arg \mathrm{w} \leqslant -\pi/3 + \varepsilon\}, \\ S_2 &= \{\mathrm{w} \in \mathbb{C} : \pi/3 - \varepsilon \leqslant \arg \mathrm{w} \leqslant \pi - 2\gamma/3\}, \end{aligned} \tag{27}$$

where the number $\varepsilon \in (0, \pi/3)$ will be chosen below. According to (26) and (5),

$$\frac{|F_2(\mathrm{w})|}{|\operatorname{Ai}(\mathrm{w})|} \leqslant M\|f\| \exp\left(\frac{2}{3}\left(\cos\frac{3\varphi}{2} - \frac{\cos(\frac{\gamma}{2}+\delta)(1-\sin(\frac{2\delta}{3})^{3/2}}{\sin^{3/2}\frac{2\delta}{3}}\right)R^{3/2} + \frac{R}{|c|^{1/3}\sin\frac{2\delta}{3}}\right), \tag{28}$$

where $\varphi = \arg \mathrm{w}$. The estimate $\cos(3\varphi/2) \leqslant \sin(3\varepsilon/2)$ holds in the union $S_1 \cup S_2$ of sectors defined in (27). Take a number $\varepsilon > 0$ such that

$$\sin\frac{3\varepsilon}{2} - \frac{\cos(\frac{\gamma}{2}+\delta)(1-\sin\frac{2\delta}{3})^{3/2}}{\sin^{3/2}\frac{2\delta}{3}} < 0.$$

Then as $R \to \infty$ we obtain the estimate

$$|F_2(\mathrm{w})|/|\operatorname{Ai}(\mathrm{w})| = o(1)\|f\|. \tag{29}$$

Further, once more applying (5), we obtain

$$|F_1(\mathrm{w})|/|\operatorname{Ai}(\mathrm{w})| \leqslant MR^{1/2}\|f\| \max_{x>0} \exp(\xi(x)), \tag{30}$$

where

$$\xi(x) = \frac{2}{3}\left(R^{3/2}\cos\frac{3\varphi}{2} - \operatorname{Re}(w + c^{1/3}x)^{3/2}\right), \qquad x \in [0, +\infty). \tag{31}$$

Note that $\xi'(x) = -\operatorname{Re}(c^{1/3}(\mathrm{w} + c^{1/3}x)^{1/2})$, whence it follows that $\xi'(x) \leqslant 0$ provided that

$$\arg(\mathrm{w} + c^{1/3}x) \in [-\pi, \pi - 2\gamma/3]. \tag{32}$$

In this case, $\xi(x)$ is not increasing, and since $\xi(0) = 0$, we have $\xi(x) \leqslant 0$. Hence, under condition (32), from inequality (30) we obtain

$$|F_1(\mathrm{w})|/|\operatorname{Ai}(\mathrm{w})| \leqslant M\|f\|R^{1/2} \quad \text{as } |\mathrm{w}| = R \to \infty. \tag{33}$$

Let us verify that condition (32) holds in the union $S_1 \cup S_2$ for every $x \geqslant 0$. Let us analyze two cases. If $\varphi \in [-\pi + \gamma/3, -\pi/3 + \varepsilon]$, then both vectors w and $c^{1/3}x$ lie in the half-plane $-\pi + \gamma/3 \leqslant \arg z \leqslant \gamma/3$, and hence so does their sum. If $\varphi \in [\pi/3 - \varepsilon, \pi - 2\gamma/3]$, then both vectors w and $c^{1/3}x$ lie in the sector $0 \leqslant \arg z \leqslant \pi - 2\gamma/3$ of opening angle $\leqslant \pi$, and hence so does their sum. In both cases, condition (32) is satisfied, and we have proved that the estimate (33) holds in the union $S_1 \cup S_2$ of sectors given by (27). In view of the estimate (29), we find that the asymptotic estimate

$$|F(\mathrm{w})| \leqslant M\|f\|R^{1/2} \tag{34}$$

holds in the union $S_1 \cup S_2$. The sector S is defined by formula (21), and the opening angle of the sector $S \setminus (S_1 \cup S_2)$ is less than $2\pi/3$. By applying the Phragmén–Lindelöf theorem to the function $\widetilde{F}(\mathrm{w}) = F(\mathrm{w})(\mathrm{w}+1)^{-1/2}$, we find that the estimate (34) holds in the entire sector S.

Step 3. It remains to estimate the function F in the remaining sector $S'' = \mathbb{C} \setminus S$. The opening angle of the sector S'' is γ; hence an estimate for F in the remaining sector can also be obtained from the Phragmén–Lindelöf theorem if $\gamma < 2\pi/3$. Now consider the case of $\gamma \in [2\pi/3, 5\pi/6)$; here additional estimates are needed.

Let $-\pi < \varphi < -\pi + \gamma/3$. The estimate of the fraction $|F_2(\mathrm{w})|/|\operatorname{Ai}(\mathrm{w})|$ does not change (this fraction is still bounded by some constant), because $\cos(3\varphi/2) < 0$ and we can use the estimate (28). Let us prove the estimate (33) in the sector S_0 defined by formula (22), where $\alpha_0 < \gamma/3$. Let us study the function $\xi(x)$ defined in (31) for its maximum. Recall that the branch of the function $z^{3/2}$ has been fixed by the choice of the argument $\arg z \in [-\pi, \pi)$. It is easily seen that if $\varphi \in (-\pi, -\pi + \gamma/3)$, then the ray $\mathrm{w} + c^{1/3}x$, $x > 0$, meets the ray $(-\infty, 0)$, and hence the curve $(\mathrm{w} + c^{1/3}x)^{3/2}$, $x \in [0, +\infty)$, has a jump discontinuity at some point, which we denote by x_1. Note, however, that the value of the jump at x_1 is pure imaginary, so that the function $\xi(x)$ is continuous. Let us study the function ξ separately on the intervals $x \in (0, x_1)$ and $x \in (x_1, +\infty)$. On the first interval, $\xi'(x) < 0$, because the argument of $\mathrm{w} + c^{1/3}x$ varies from φ (at $x = 0$) to $-\pi$ (at $x = x_1$); i.e., condition (32) is satisfied. On the second interval, the argument of $\mathrm{w} + c^{1/3}x$ is monotone decreasing from π (at $x = x_1 + 0$) to $\gamma/3$ (at $x \to +\infty$), and hence there exists a unique point, which we denote by x_2, at which $\arg(\mathrm{w} + c^{1/3}x) = \pi - 2\gamma/3$. The derivative ξ' changes the sign from plus to minus at x_2; i.e., x_2 is a point of local maximum of ξ. Thus, the estimate (33), which is equivalent to the condition $\xi(x) \leqslant 0$ as $x \in [0, +\infty)$ by virtue of (30), holds if and only if $\xi(x_2) \leqslant 0$. To compute the value $\xi(x_2)$, consider the triangle with vertices 0, w, and $\mathrm{w} + c^{1/3}x_2$ on the complex plane. The respective angles of this triangle are $\alpha + 2\gamma/3$, $\gamma/3 - \alpha$, and $\pi - \gamma$. (Here we write $\alpha = \pi + \varphi \in (0, \gamma/3)$, where $\varphi = \arg \mathrm{w}$.) By the sine theorem,

$$|\mathrm{w} + c^{1/3}x_2| = R\,\frac{\sin(\gamma/3 - \alpha)}{\sin\gamma}$$

$$\Longrightarrow\ \operatorname{Re}(\mathrm{w} + c^{1/3}x_2)^{3/2} = |\mathrm{w} + c^{1/3}x_2|^{3/2}\cos\left(\frac{3\pi}{2} - \gamma\right) = -R^{3/2}\,\frac{\sin^{3/2}(\gamma/3 - \alpha)}{\sin^{1/2}\gamma},$$

$$\xi(x_2) = \frac{2}{3}R^{3/2}\biggl(\cos\frac{3\varphi}{2} + \frac{\sin^{3/2}(\gamma/3 - \alpha)}{\sin^{1/2}\gamma}\biggr) = \frac{2}{3}R^{3/2}\biggl(\frac{\sin^{3/2}(\gamma/3 - \alpha)}{\sin^{1/2}\gamma} - \sin\frac{3\alpha}{2}\biggr).$$

Thus, inequality $\xi(x_2) \leqslant 0$ is equivalent to the condition

$$\sin^{3/2}(\gamma/3 - \alpha)\sin^{-1/2}\gamma - \sin(3\alpha/2) \leqslant 0. \tag{35}$$

Denote the left-hand side of this inequality, which is a function of the variable $\alpha \in [0, \gamma/3]$ with a parameter $\gamma \in [2\pi/3, \pi)$, by $\eta(\alpha)$. It is easily seen that $\eta(\alpha)$ is monotone decreasing, $\eta(0) > 0$, and $\eta(\gamma/3) < 0$. Hence inequality (35) holds on the interval $\alpha \in [\alpha_0, \gamma/3]$ for some $\alpha_0 \in (0, \gamma/3)$. Thus, we have proved the estimate (33), and hence also (34), for all rays $\arg \mathrm{w} = \varphi \in [-\pi + \alpha_0, \pi - 2\gamma/3]$. The Phragmen-Lindelöf principle permits us to extend this estimate to the entire complex plane provided that the opening angle $\alpha_0 + 2\gamma/3$ of the remaining sector is strictly less than $2\pi/3$. The number $\alpha_0 = \alpha_0(\gamma)$ is a root of the transcendental equation $\eta(\alpha) = 0$. We do not seek it in any form. Instead, we note that, since the function η is monotone, it follows that

$$\begin{aligned}\alpha_0 + 2\gamma/3 < 2\pi/3 \iff \alpha_0 < 2(\pi - \gamma)/3 \iff \eta(2(\pi - \gamma)/3) < 0\\ \iff \sin^{3/2}(\gamma - 2\pi/3)\sin^{-1/2}\gamma - \sin\gamma < 0\\ \iff \sin^{3/2}(\gamma - 2\pi/3) < \sin^{3/2}\gamma \iff \gamma < 5\pi/6\,.\end{aligned}$$

Step 4. Thus, for $\gamma < 5\pi/6$ the entire function $F(\mathrm{w})$ admits the asymptotic estimate (34) in the entire complex plane and hence is constant. Let us prove that this constant is zero. To this end, it suffices to verify that $F(\mathrm{w}) \to 0$ as $\mathrm{w} \to \infty$ along at least one ray in the complex plane. Let us return to the beginning of step 2, take the ray $\mathrm{w} = Re^{-i\pi/2}$, and note that it lies in the sector S_1, so that the estimate (29) is satisfied. It remains to strengthen the estimate (33). To this end, we split the integration path $x \in [0, x_0]$ into two parts by a point $x_3(R) = R^{-\theta}$, where $\theta \in (0, 1/2)$ is arbitrary. Since

$$\frac{|F_1(\mathrm{w})|}{|\operatorname{Ai}(\mathrm{w})|} \leqslant M\int_0^{x_0(R)} |f(x)|e^{\xi(x)}\,dx,$$

where the function $\xi(x)$ is defined in (31) and is monotone decreasing from $\xi(0) = 0$ to $-\infty$ on $[0, +\infty)$, we have

$$\biggl(\int_0^{x_3} + \int_{x_3}^{x_0}\biggr)|f(x)|e^{\xi(x)}\,dx \leqslant \|f\|(\sqrt{x_3} + \sqrt{x_0}\,e^{\xi(x_3)}) \leqslant \|f\|(R^{-\theta/2} + CR^{1/2}e^{\xi(x_3)}).$$

It remains to note that

$$\xi(x_3(R)) \sim -\operatorname{Re}(c^{1/3}x_3\mathrm{w}^{1/2}) = -|c|^{1/3}R^{-\theta+1/2}\cos(\gamma/3 - \pi/4)$$

as $R \to \infty$, and hence $R^{1/2}e^{\xi(x_3(R))} = o(1)$. Consequently, $F(\mathrm{w}) \equiv 0$.

Step 5. We have proved that $F_0(\mathrm{w}) = \int_0^\infty \mathrm{Ai}(\mathrm{w} + c^{1/3}x)\, d\mu \equiv 0$. (Here and in what follows, we write $d\mu = f(x)\, dx$ for brevity.) Recall that $\mathrm{Ai}''(t) = t\,\mathrm{Ai}(t)$, whence, by induction,

$$\mathrm{Ai}^{(n)}(t) = P_n(t)\,\mathrm{Ai}(t) + Q_n(t)\,\mathrm{Ai}'(t),$$

where P_n and Q_n are polynomials. Further,

$$\begin{gathered}P_0(t) = 1, \quad P_1(t) = 0, \quad P_2(t) = t, \quad P_3(t) = 1, \quad P_4(t) = t^2, \quad P_5(t) = 4t,\\ Q_0(t) = 0, \quad Q_1(t) = 1, \quad Q_2(t) = 0, \quad Q_3(t) = t, \quad Q_4(t) = 2, \quad Q_5(t) = t^2,\\ P_n(t) = P'_{n-1}(t) + tQ_{n-1}(t), \qquad Q_n(t) = P_{n-1}(t) + Q'_{n-1}(t).\end{gathered}$$

Set $\deg P_n(t) = p_n$ and $\deg Q_n(t) = q_n$. Then

$$\begin{cases} p_n = q_{n-1} + 1, \\ q_n = p_{n-1} \end{cases} \implies \begin{cases} p_n = p_{n-2} + 1, \\ q_n = q_{n-2} + 1 \end{cases}$$

for all $n \geqslant 5$, whence we readily find that

$$\begin{aligned} p_{2n} &= n, & p_{2n+1} &= n - 1, \\ q_{2n} &= n - 2, & q_{2n+1} &= n \end{aligned} \tag{36}$$

for all $n \geqslant 2$. By differentiating with respect to w, we obtain

$$F_0^{(j)}(\mathrm{w}) = \int_0^\infty (P_j(\mathrm{w} + c^{1/3}x)\,\mathrm{Ai}(\mathrm{w} + c^{1/3}x) + Q_j(\mathrm{w} + c^{1/3}x)\,\mathrm{Ai}'(\mathrm{w} + c^{1/3}x))\, d\mu \equiv 0, \qquad j \in \mathbb{N} \cup \{0\}. \tag{37}$$

In particular,

$$\begin{aligned} F_0(\mathrm{w}) &= \int_0^\infty \mathrm{Ai}(\mathrm{w} + c^{1/3}x)\, d\mu \equiv 0, \quad F_0'(\mathrm{w}) = \int_0^\infty \mathrm{Ai}'(\mathrm{w} + c^{1/3}x)\, d\mu \equiv 0, \\ F_0''(\mathrm{w}) &= \int_0^\infty (\mathrm{w} + c^{1/3}x)\,\mathrm{Ai}(\mathrm{w} + c^{1/3}x)\, d\mu \equiv 0 \implies \int_0^\infty x\,\mathrm{Ai}(\mathrm{w} + c^{1/3}x)\, d\mu \equiv 0, \\ F_0'''(\mathrm{w}) &= \int_0^\infty (\mathrm{Ai}(\mathrm{w} + c^{1/3}x) + (\mathrm{w} + c^{1/3}x)\,\mathrm{Ai}'(\mathrm{w} + c^{1/3}x))\, d\mu \equiv 0 \\ &\qquad\qquad \implies \int_0^\infty x\,\mathrm{Ai}'(\mathrm{w} + c^{1/3}x)\, d\mu \equiv 0. \end{aligned}$$

Let us prove by induction that

$$\int_0^\infty x^n\,\mathrm{Ai}(\mathrm{w} + c^{1/3}x)\, d\mu \equiv \int_0^\infty x^n\,\mathrm{Ai}'(\mathrm{w} + c^{1/3}x)\, d\mu \equiv 0, \qquad n = 0, 1, 2, \dots . \tag{38}$$

The base of induction is the case of $n = 0$ and $n = 1$. We have already proved these relations in this case. Now if Eqs. (38) have been proved for all $n < N$, where $N \geqslant 2$, then, by writing out (37) for $j = 2N$ and by taking into account (36), we obtain

$$\int_0^\infty \left(\left((\mathrm{w} + c^{1/3}x)^N + \sum_{k=0}^{N-1} a_{j,k}(\mathrm{w} + c^{1/3}x)^k \right) \mathrm{Ai}(\mathrm{w} + c^{1/3}x) + \left(\sum_{k=0}^{N-2} b_{j,k}(\mathrm{w} + c^{1/3}x)^k \right) \mathrm{Ai}'(\mathrm{w} + c^{1/3}x) \right) d\mu \equiv 0,$$

whence it follows by the induction assumption that

$$\int_0^\infty x^N\,\mathrm{Ai}(\mathrm{w} + c^{1/3}x)\, d\mu \equiv 0.$$

Now we write out Eq. (37) for $j = 2N+1$ and again take into account (36) to obtain

$$\int_0^\infty \Biggl(\biggl(\sum_{k=0}^{N-1} a_{j,k}(\mathrm{w}+c^{1/3}x)^k\biggr)\operatorname{Ai}(\mathrm{w}+c^{1/3}x) + \biggl((\mathrm{w}+c^{1/3}x)^N + \sum_{k=0}^{N-1} b_{j,k}(\mathrm{w}+c^{1/3}x)^k\biggr)\operatorname{Ai}'(\mathrm{w}+c^{1/3}x)\Biggr)\,d\mu \equiv 0,$$

whence, by the induction assumption and the step already carried out, we obtain

$$\int_0^\infty x^N \operatorname{Ai}'(\mathrm{w}+c^{1/3}x)\,d\mu \equiv 0.$$

The proof of Eqs. (38) is complete.

Now we use a standard trick to derive the relation $f(x) \equiv 0$ from (38). Consider the Fourier transform

$$h(\lambda) = \int_0^\infty e^{-i\lambda x}\operatorname{Ai}(c^{1/3}x)f(x)\,dx.$$

By (5), this function is an entire function of λ. Note that

$$h^{(n)}(0) = (-i)^n \int_0^\infty x^n \operatorname{Ai}(c^{1/3}x)f(x)\,dx = 0, \qquad n = 0, 1, \dots,$$

and hence $h(\lambda) \equiv 0$. Since the Fourier transform is injective, it follows that the function $\operatorname{Ai}(c^{1/3}x)f(x)$ is identically zero as well, and so $f(x) \equiv 0$. This completes the proof of Theorem 1. □

3. Completeness Theorem for the Operator L

Theorem 2. *Let the operator L be given by the differential expression* (3) *on the domain*

$$\mathfrak{D}(L) = \{y \in L_2(\mathbb{R}_+) : y, y' \in AC_{\mathrm{loc}},\ l(y) \in L_2(\mathbb{R}_+),\ y(0) = 0\},$$

where AC_{loc} is the space of locally absolutely continuous functions, and let conditions (4) *be satisfied. If*

$$\arg(c_0 + i) =: \gamma < 2\pi\alpha/(2+\alpha),$$

then the system of root functions of L is complete in $L_2(\mathbb{R}_+)$. Further, this system is a basis for Abel's summation method of order β for any $\beta \in ((2+\alpha)/(2\alpha), \pi/\gamma)$.

Proof. First, let us explain what the notion of basis means for summation by Abel's method. For simplicity, we assume that all eigenvalues are simple (see [2, Sec. 5] for the general case). Consider the series

$$S(t,f) = \sum_{k=0}^{\infty} \exp\{(-e^{-i\gamma/2}\lambda_k)^\beta t\}(f, z_k)y_k,$$

where $\{y_k\}$ is the eigenfunction system of L corresponding to the eigenvalues λ_k and $\{z_k\}$ is the biorthogonal system. The second claim of the theorem means that, for each function $f \in L_2(\mathbb{R}_+)$, the series $S(t,f)$ converges for every $t > 0$ in the norm of $L_2(\mathbb{R}_+)$, and there exists a strong limit $S(t,f) \to f$ as $t \to +0$. Needless to say, if the system $\{y_k\}$ is a basis for Abel's summation method, then it is complete, because it follows from the definition that any function f can be approximated by finite linear combinations of this system with arbitrary accuracy.

Without loss of generality, we assume that the constants M_0 and M_1 in conditions (4) are positive. Let $\mathfrak{D}_0(L)$ be the subspace of $\mathfrak{D}(L)$ formed by the compactly supported functions. By reproducing Lidskii's argument in [3], we find that the case of Weyl limit point holds under the assumptions of the theorem; i.e., only one solution of the equation $l(y) = 0$ belongs to $L_2(\mathbb{R}_+)$. In this case, the operator L has a bounded inverse. Further, $\mathfrak{D}_0(L)$ is an essential domain of L in this case; i.e., the closure of the restriction of L to $\mathfrak{D}_0(L)$ coincides with L. For each $f \in \mathfrak{D}_0(L)$, the number (Lf, f) lies in the sector bounded in the upper half-plane by the rays $\arg\lambda = \gamma$ and

$\mathbb{R}_+$. This follows from (4) after integration by parts. It follows from the invertibility of L that the operator $T = e^{-i\gamma/2}L$ is m-sectorial with opening angle γ. Then the operator $H = \operatorname{Re} T$ is given on the domain $\mathfrak{D}(H) = \mathfrak{D}(L)$ by the differential expression

$$l(y) = -y'' + \widetilde{r}(x)y,$$

and conditions (4) imply the estimate $\widetilde{r}(x) \geqslant Mr(x) \geqslant Max^{\alpha}$. It was shown in Lidskii's paper that if this inequality is satisfied, then the eigenvalues s_n of the operator H satisfy the estimate

$$s_n \geqslant M\, n^{2\alpha/(2+\alpha)}, \qquad n = 1, 2, 3, \dots,$$

i.e., the orders of the operators H and L are not less than $2\alpha/(2+\alpha)$. Now the assertion of the theorem follows from the Lidskii–Matsaev theorem; see [2, Theorem 5.1]. □

Remark. In the case of an even potential, the completeness of the system of eigenfunctions of the Schrödinger operator on the half-line with the Dirichlet and Neumann conditions, respectively, implies the completeness of the system of eigenfunctions of this operator on the entire line. Indeed, odd and even extensions of the eigenfunctions of the Dirichlet and Neumann operators on the half-line give the eigenfunctions of the operator on the entire line.

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