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On the Relationship between Combinatorial Functions and Representation Theory*

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Abstract. The paper is devoted to the study of well-known combinatorial functions on the symmetric group $\mathfrak{S}_n$—the major index maj, the descent number des, and the inversion number inv—from the representation-theoretic point of view. We show that these functions generate the same ideal in the group algebra $\mathbb{C}[\mathfrak{S}_n]$, and the restriction of the left regular representation of the group $\mathfrak{S}_n$ to this ideal is isomorphic to its representation in the space of $n \times n$ skew-symmetric matrices. This allows us to obtain formulas for the functions maj, des, and inv in terms of matrices of an exceptionally simple form. These formulas are applied to find the spectra of the elements under study in the regular representation, as well as derive a series of identities relating these functions to one another and to the number fix of fixed points.

Key words: major index, descent number, inversion number, representations of the symmetric group, skew-symmetric matrices, dual complexity.

1. Introduction

In the representation theory of noncommutative locally compact groups one of the oldest ideas, which generalizes the idea of Fourier transform, is that the dual object of every element of a group algebra is an operator-valued function defined on the space of equivalence classes of irreducible representations so that its value at a given class is (up to equivalence) the type of the unitary operator corresponding to the given element in representations of this class.

Strange as it may seem, this idea, which has been fruitfully employed in the representation theory of Lie groups, has, to the authors' knowledge, gained little popularity in the representation theory of finite groups and, in particular, of the symmetric group $\mathfrak{S}_n$. Developing this idea, we suggest the following definition.

Definition 1. The *dual complexity* of an element a of a group algebra $\mathbb{C}[G]$ is the dimension of the cyclic subspace (ideal) $\mathrm{Ide}(a) = \mathbb{C}[G]a$ generated by all left translations of this element.

For example, the dual complexity of every group element (regarded as a δ-function) coincides with the order of the group, i.e., is maximum possible. What can be said of those elements of a group algebra which have low dual complexity? Is there some dependence, or rather opposition, between the size of the support of an element and its dual complexity, as in the classical theory of Fourier transform? Such questions for the symmetric group directly relate combinatorics to representation theory and have apparently not been studied. Our paper should be regarded as the first steps in this direction.

The following example was essentially observed (in a quite different context and in other terms) in [2]. Consider the distance from a permutation $g \in \mathfrak{S}_n$ to the identity element of $\mathfrak{S}_n$ in the word metric with respect to the Coxeter generators, or, in other words, the inversion number $\mathsf{inv}(g)$. The dual complexity of the corresponding element of $\mathbb{C}[\mathfrak{S}_n]$ is equal to $n(n-1)/2 + 1$, and the representation of $\mathfrak{S}_n$ in the ideal $\mathrm{Ide}(\mathsf{inv})$ is isomorphic to the sum of the identity representation and the representation in the space of $n \times n$ skew-symmetric matrices.

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We were interested in more complicated statistics* on the symmetric group, such as the major index $\mathsf{maj}(g)$ and the number $\mathsf{des}(g)$ of descents; see, e.g., [1]. The study of these statistics goes back to MacMahon [10], and by now there is an extensive literature devoted to various problems of enumerative combinatorics involving these functions, which play an important role in the combinatorics of permutations (see, e.g., [4]–[6], [8], [9], [11], and [13]). We also mention the recent unexpected appearance (in a somewhat different version, for Young tableaux) of the major index maj as a key element of a construction establishing a relationship between representations of the infinite symmetric group and the affine algebra $\widehat{\mathfrak{sl}_2}$; see [14]. This has become an additional motivation for investigating representation-theoretic, rather than purely combinatorial, properties of the statistics under consideration, which have received no attention so far.

An attempt to consider the complexity of these elements in the sense of Definition 1 has led to unexpected results: not only did the complexity turn out to equal the same humble value of $n(n-1)/2+1$, but it was also found that the corresponding ideals $\mathrm{Ide}(\mathsf{maj})$ and $\mathrm{Ide}(\mathsf{des})$ coincide with $\mathrm{Ide}(\mathsf{inv})$ and thus can be realized (up to the subspace of constants) in the space of skew-symmetric matrices, too.

Developing this idea, we have managed to obtain the surprisingly simple expressions (7)–(9) in terms of matrices of a very simple form for the functions maj, des, and inv, originally defined by nontrivial combinatorial conditions. The usefulness of these formulas is illustrated by the fact that they allow one to easily find the spectra of the elements under consideration in the regular representation (Theorems 4 and 5), as well as obtain a whole series of new simple identities relating these functions to each other and to another important function, the number fix of fixed points (Corollaries 2 and 4).

In Section 2 we briefly recall the definitions and basic properties of the statistics under study. The main results are presented in Section 3, which contains an explicit construction and a detailed study of an isomorphism between the representations of the group $\mathfrak{S}_n$ in the ideal $\mathrm{Ide}(\mathsf{maj}) = \mathrm{Ide}(\mathsf{des}) = \mathrm{Ide}(\mathsf{inv})$ and in the space of skew-symmetric matrices; expressions for these statistics as simple matrix elements are also obtained. In Section 4 these expressions are applied to obtain corollaries, namely, find the spectra of the functions under study in the regular representation and derive a series of identities involving them. Finally, in Section 5 we briefly describe a relationship of our results with investigations of the so-called Solomon descent algebra.

2. The Combinatorial Functions maj, des, and inv

By $\mathfrak{S}_n$ we denote the symmetric group of order n, and by π_λ, the irreducible representation of $\mathfrak{S}_n$ corresponding to a Young diagram λ.

We consider the following functions of permutations $\sigma \in \mathfrak{S}_n$ (see, e.g., [1]): the *descent number*

$$\mathsf{des}(\sigma) = \#\,\mathsf{Des}(\sigma), \quad \text{where } \mathsf{Des}(\sigma) = \{i \in \{1,\dots,n-1\} : \sigma(i) > \sigma(i+1)\},$$

the *major index***

$$\mathsf{maj}(\sigma) = \sum_{i\in\mathsf{Des}(\sigma)} i,$$

the *inversion number*

$$\mathsf{inv}(\sigma) = \#\{(i,j) : i<j,\, \sigma(i) > \sigma(j)\},$$

and the *number of fixed points*

$$\mathsf{fix}(\sigma) = \#\{i \in \{1,\dots,n\} : \sigma(i) = i\}.$$

*According to a tradition coming from physics, functions on the symmetric group are often called statistics of permutations.

**The traditional Russian translation of this term means "main index," which is incorrect, since the statistic was named after Major Percy Alexander MacMahon.

The statistics maj and inv are *MacMahonian*, i.e., their generating functions coincide and equal

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathsf{maj}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathsf{inv}(\sigma)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1});$$

the statistic des is *Eulerian*, i.e., its generating function is given by the Euler polynomials:

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathsf{des}(\sigma)} = A_n(q), \quad \text{where } \sum_{n\geqslant 0} A_n(q)\frac{z^n}{n!} = \frac{(1-q)e^z}{e^{qz}-qe^z}.$$

It is not difficult to deduce that

$$\sum_{\sigma\in\mathfrak{S}_n} \mathsf{maj}(\sigma) = \sum_{\sigma\in\mathfrak{S}_n} \mathsf{inv}(\sigma) = n!\cdot\frac{n(n-1)}{4}, \qquad \sum_{\sigma\in\mathfrak{S}_n} \mathsf{des}(\sigma) = n!\cdot\frac{n-1}{2}. \tag{1}$$

For each statistic $\varepsilon = \mathsf{maj}, \mathsf{des}, \mathsf{inv}$, we denote the corresponding element of the group algebra $\mathbb{C}[\mathfrak{S}_n]$ by u_ε:

$$u_\varepsilon = \sum_{g\in\mathfrak{S}_n} \varepsilon(g)g \in \mathbb{C}[\mathfrak{S}_n].$$

We also need the *centered* (i.e., orthogonal to the constants) versions $\widetilde{\varepsilon}$ of the statistics under consideration, for which $\sum_{\sigma\in\mathfrak{S}_n}\widetilde{\varepsilon}(\sigma)=0$, namely,

$$\widetilde{\mathsf{des}}(\sigma) = \mathsf{des}(\sigma) - (n-1)/2, \qquad \widetilde{\mathsf{maj}}(\sigma) = \mathsf{maj}(\sigma) - n(n-1)/4,$$
$$\widetilde{\mathsf{inv}}(\sigma) = \mathsf{inv}(\sigma) - n(n-1)/4,$$

and the corresponding elements $u_{\widetilde{\varepsilon}} = \sum_{\sigma\in\mathfrak{S}_n}\widetilde{\varepsilon}(\sigma)\sigma$ of the group algebra.

MacMahon studied four fundamental statistics of permutations: maj, des, inv, and also the *excedance number*

$$\mathsf{exc}(\sigma) = \#\{i\in\{1,\dots,n-1\} : \sigma(i) > i\}.$$

This statistic is Eulerian as well, i.e., it has the same distribution as des. Thus, the four functions maj, inv and des, exc form two pairs of identically distributed statistics. However, in contrast to maj, des, and inv, the function exc generates another ideal, which coincides with the primary component of the natural representation (plus the subspace of constants). Hence its dual complexity is equal to $(n-1)^2+1$. It is not difficult to see that the same ideal is also generated by the function fix.

We also mention that sometimes, instead of the major index, the so-called *comajor index* comaj is used, where $\mathsf{comaj}(\sigma) = \sum_{i\in\mathsf{Des}(\sigma)}(n-i)$. Since, obviously, $\mathsf{comaj}(\sigma) = n\,\mathsf{des}(\sigma) - \mathsf{maj}(\sigma)$, all results obtained below for des and maj can easily be transferred to comaj.

3. Realization of the Combinatorial Functions maj, des, and inv in the Space of Skew-Symmetric Matrices

Let $\mathcal{M}$ be the space of $n\times n$ skew-symmetric matrices. There is a natural action of the symmetric group $\mathfrak{S}_n$ on $\mathcal{M}$ by simultaneous permutations of rows and columns. It is well known that this representation ϱ of $\mathfrak{S}_n$ decomposes into the sum of two irreducible representations $\pi_{(n-1,1)}$ (the natural representation) and $\pi_{(n-1,1^2)}$, which we denote by π_1 and π_2, respectively. Also, we denote the orthogonal projections of $\mathcal{M}$ onto the spaces of π_1 and π_2 by P_1 and P_2. We equip $\mathcal{M}$ with the inner product

$$\langle A,B\rangle = \frac{1}{2}\operatorname{Tr}(AB) = \sum_{i<j} a_{ij}b_{ij}, \qquad A=(a_{ij}),\ B=(b_{ij}). \tag{2}$$

Obviously, the representation ϱ is unitary.

For $1\leqslant i<j\leqslant n$, by E_{ij} we denote the skew-symmetric matrix with entry (i,j) equal to 1 and entry (j,i) equal to -1, all other entries being zero. Obviously, the matrices E_{ij} form an

orthonormal basis in $\mathcal{M}$ with respect to the inner product (2). By abuse of language, we call them the *matrix units.*

Consider (for $1 \leqslant i < j \leqslant n$ again) the following elements of the group algebra $\mathbb{C}[\mathfrak{S}_n]$:

$$e_{ij} = \frac{1}{\sqrt{n!}} \sum_{\sigma \in \mathbb{C}[\mathfrak{S}_n]} \varepsilon_{ij}(\sigma)\sigma, \quad \text{where } \varepsilon_{ij}(\sigma) = \begin{cases} 1 & \text{if } \sigma(i) < \sigma(j), \\ -1 & \text{if } \sigma(i) > \sigma(j). \end{cases}$$

Let $\mathcal{H}$ denote the subspace in $\mathbb{C}[\mathfrak{S}_n]$ spanned by these elements, and let Reg_l be the left regular representation of the group $\mathfrak{S}_n$ in $\mathbb{C}[\mathfrak{S}_n]$.

Lemma 1. *The subspace $\mathcal{H}$ is invariant under* Reg_l, *and the corresponding subrepresentation is isomorphic to the representation ϱ of $\mathfrak{S}_n$ in the space of skew-symmetric matrices $\mathcal{M}$.*

Proof. Consider the operator $\widehat{T}\colon \mathcal{H} \to \mathcal{M}$ defined by $\widehat{T}e_{ij} = E_{ij}$. It is easy to check that it determines a (nonunitary!) isomorphism of the representations under study. $\square$

In view of Lemma 1, we refer to the elements e_{ij} as the *pseudomatrix units* in $\mathcal{H}$. It is also convenient to put $e_{ii} = 0$ and $e_{ij} = -e_{ji}$ for $i > j$.

Obviously, the pseudomatrix units are not orthogonal with respect to the standard inner product in $\mathbb{C}[\mathfrak{S}_n]$. Namely, it is not difficult to obtain the following relations for $i < j$ and $k < l$:

$$\langle e_{ij}, e_{kl} \rangle = \begin{cases} 1, & i = k,\ j = l, \\ 1/3, & i = k,\ j \neq l \ \text{or}\ j = l,\ i \neq k, \\ -1/3, & i = l \ \text{or}\ j = k, \\ 0, & \{k, l\} \cap \{i, j\} = \varnothing. \end{cases}$$

Let C_1 and C_2 denote the normalized central Young symmetrizers (central idempotents in $\mathbb{C}[\mathfrak{S}_n]$) corresponding to the Young diagrams $(n-1, 1)$ and $(n-2, 1^2)$, respectively.

Lemma 2. *The following relation holds*:

$$C_1 e_{ij} = \frac{1}{n} \sum_{k=1}^{n} (e_{ik} + e_{kj}). \tag{3}$$

Proof. Given $k \neq l$, let c_{kl} denote the Young symmetrizer corresponding to a Young tableau of the form $\begin{array}{|c|c|c|c|}\hline k & & \cdot & \\ \hline l \\ \cline{1-1}\end{array}$ (the order of the other elements in the first row does not matter). It is not difficult to see that

$$C_1 = \frac{n-1}{n \cdot n!} \sum_{k \neq l} c_{kl}.$$

Now one can deduce from the properties of the pseudomatrix units e_{ij} that $c_{kl}e_{ij} = 0$ for $\{k, l\} \cap \{i, j\} = \varnothing$ and

$$\sum_{k \neq i} c_{ki} e_{ij} = n \cdot (n-2)! \sum_{k=1}^{n} e_{ik}, \qquad \sum_{k \neq j} c_{kj} e_{ij} = n \cdot (n-2)! \sum_{k=1}^{n} e_{kj},$$

which implies the desired formula. $\square$

Since the representation of $\mathfrak{S}_n$ in the space $\mathcal{M}$ is the sum of two irreducible representations, a unitary isomorphism between the representations in $\mathcal{H}$ and $\mathcal{M}$ can differ from the intertwining operator $\widehat{T}$ only in that the projections of the elements e_{ij} to each of the two irreducible components are multiplied by different coefficients, and these coefficients can be found from the condition that the images of the pseudomatrix units should be orthonormal. Namely, consider the operator

$$A = \frac{\sqrt{3}}{\sqrt{n+1}} (C_1 + \sqrt{n+1}\, C_2)$$

in $\mathbb{C}[\mathfrak{S}_n]$ and let

$$e'_{ij} = Ae_{ij}.$$

Theorem 1. *The operator $T\colon \mathcal{H} \to \mathcal{M}$ defined by*

$$Te'_{ij} = E_{ij}, \qquad 1 \leqslant i < j \leqslant n,$$

is a unitary isomorphism of the representations of the symmetric group $\mathfrak{S}_n$ in the spaces $\mathcal{H}$ and $\mathcal{M}$.

Proof. It is obvious from the above that T is an intertwining operator. It remains to show that it is unitary, i.e., that the system of elements $\{e'_{ij}\}$ in $\mathbb{C}[\mathfrak{S}_n]$ is orthonormal. Using Lemma 2, we easily find that, for two different pseudomatrix units e_{ij} and e_{kl}, we have

$$\langle C_1 e_{ij}, C_1 e_{kl}\rangle = \begin{cases} 0, & \{k,l\} \cap \{i,j\} = \varnothing, \\ (n+1)/(3n), & i=k,\ j \neq l \ \text{ or } \ j=l,\ i\neq k, \\ -(n+1)/(3n), & i=l,\ j\neq k \ \text{ or } \ j=k,\ i \neq l. \end{cases}$$

Since $\langle C_2 e_{ij}, C_2 e_{kl}\rangle = \langle e_{ij}, e_{kl}\rangle - \langle C_1 e_{ij}, C_1 e_{kl}\rangle$, we obtain

$$\langle C_2 e_{ij}, C_2 e_{kl}\rangle = \begin{cases} 0, & \{k,l\} \cap \{i,j\} = \varnothing, \\ -1/(3n), & i=k,\ j \neq l \ \text{ or } \ j=l,\ i\neq k, \\ 1/(3n), & i=l,\ j\neq k \ \text{ or } \ j=k,\ i \neq l, \end{cases}$$

and the fact that the system $\{e'_{ij}\}$ is orthonormal follows by a direct calculation. ☐

Theorem 2. *The centered vectors $u_{\widetilde{\mathsf{des}}}$, $u_{\widetilde{\mathsf{maj}}}$, and $u_{\widetilde{\mathsf{inv}}}$ lie in the space $\mathcal{H}$; namely,*

$$u_{\widetilde{\mathsf{des}}} = -\frac{\sqrt{n!}}{2}\sum_{k=1}^{n-1} e_{k,k+1}, \quad u_{\widetilde{\mathsf{maj}}} = -\frac{\sqrt{n!}}{2}\sum_{k=1}^{n-1} k e_{k,k+1}, \quad u_{\widetilde{\mathsf{inv}}} = -\frac{\sqrt{n!}}{2}\sum_{1\leqslant i<j\leqslant n} e_{ij}.$$

Proof. The theorem is proved by a straightforward calculation. ☐

Corollary 1. *Each of the vectors u_ε, where $\varepsilon = \mathsf{des}, \mathsf{maj}, \mathsf{inv}$, generates the same principal right ideal $\mathcal{I} = \mathcal{H} \oplus \{\text{const}\}$ in $\mathbb{C}[\mathfrak{S}_n]$. The restriction of the left regular representation* Reg_l *to $\mathcal{I}$ is the sum $\pi_{(n)} \oplus \pi_{(n-1,1)} \oplus \pi_{(n-2,1^2)}$ of three irreducible representations.*

In $\mathcal{M}$ consider the matrices

$$h_{\mathsf{des}} = \sum_{k=1}^{n-1} E_{k,k+1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ & 0 & 1 & \dots & 0 \\ & & 0 & \dots & 0 \\ & & & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \qquad h_{\mathsf{maj}} = \sum_{k=1}^{n-1} k E_{k,k+1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ & 0 & 2 & \dots & 0 \\ & & 0 & \dots & 0 \\ & & & \ddots & \\ & & & 0 & n-1 \\ & & & & 0 \end{pmatrix},$$

$$h_{\mathsf{inv}} = \sum_{i<j} E_{ij} = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ & 0 & 1 & \dots & 1 \\ & & 0 & \dots & 1 \\ & & & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

(when writing matrices, we always indicate only their upper triangular parts, since the lower triangular parts can be recovered by skew symmetry). We set $c_n = -\sqrt{n!}/2$. It follows from Theorems 2 and 1 that

$$\widehat{T}u_{\widetilde{\varepsilon}} = c_n h_\varepsilon, \qquad Tu_{\widetilde{\varepsilon}} = \frac{c_n}{\sqrt{3}}(\sqrt{n+1}\,P_1 h_\varepsilon + P_2 h_\varepsilon). \tag{4}$$

Lemma 3. *For $k<m$, the projection of the matrix unit E_{km} to the natural representation is given by the formula*

$$P_1 E_{km} = (a_{ij})_{i,j=1}^n, \quad \text{where } a_{ij} = \alpha_i^{(km)} - \alpha_j^{(km)} \text{ and } \alpha_j^{(km)} = \begin{cases} 1/n, & j=k, \\ -1/n, & j=m, \\ 0, & j\neq k,m. \end{cases}$$

Proof. It is well known that

$$\begin{aligned} &P_1\mathcal{M} = \{M=(a_{ij}),\ \text{where } a_{ij} = \alpha_i - \alpha_j \text{ and } (\alpha_j) \in \mathbb{R}^n,\ \textstyle\sum \alpha_j = 0\}, \\ &P_2\mathcal{M} = \{M = (a_{ij}) : a_{ji} = -a_{ij},\ \textstyle\sum_i a_{ij} = 0 \text{ for every } j\}. \end{aligned} \tag{5}$$

Using this fact, we easily find the desired projection. ☐

Lemma 4. *The following relations hold*:

$$P_1 h_{\mathsf{des}} = \frac{1}{n}\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 2 \\ & 0 & 0 & \cdots & 0 & 1 \\ & & 0 & \cdots & 0 & 1 \\ & & & \ddots & & \\ & & & 0 & 0 & 1 \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}, \qquad P_2 h_{\mathsf{des}} = \frac{1}{n}\begin{pmatrix} 0 & n-1 & -1 & -1 & \cdots & -1 & -2 \\ & 0 & n & 0 & \cdots & 0 & -1 \\ & & 0 & n & \cdots & 0 & -1 \\ & & & & \ddots & & \\ & & & & 0 & n & -1 \\ & & & & & 0 & n-1 \\ & & & & & & 0 \end{pmatrix};$$

$$P_1 h_{\mathsf{maj}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ & 0 & \cdots & 0 & 1 \\ & & \ddots & 0 & 1 \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \qquad P_2 h_{\mathsf{maj}} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & -1 \\ & 0 & 2 & 0 & \cdots & 0 & -1 \\ & & 0 & 3 & \cdots & 0 & -1 \\ & & & & \ddots & & \\ & & & & 0 & n-2 & -1 \\ & & & & & 0 & n-2 \\ & & & & & & 0 \end{pmatrix};$$

$$P_1 h_{\mathsf{inv}} = A, \qquad P_2 h_{\mathsf{inv}} = B,$$

where $A = (a_{ij})$ *and* $B = (b_{ij})$ *are the Toeplitz matrices with entries*

$$a_{i,i+k} = 2k/n, \qquad b_{i,i+k} = 1 - 2k/n. \tag{6}$$

Proof. The required relations easily follow from Lemma 3 or directly from (5). ☐

Let P denote the orthogonal projection of $\mathbb{C}[\mathfrak{S}_n]$ onto the subspace $\mathcal{H}$, and let $h_0 = TP\delta_e$.

Lemma 5. *The following relation holds*:

$$h_0 = \sqrt{\frac{3}{(n+1)!}}\,(A + \sqrt{n+1}\cdot B),$$

where the Toeplitz matrices A *and* B *are given by* (6).

Proof. Since $\{e'_{km}\}$ is an orthonormal system in $\mathcal{H}$, we can write $P\delta_e = \sum_{km}\langle \delta_e, e'_{km}\rangle e'_{km}$. The results obtained above imply

$$\begin{aligned} e'_{km} &= \sqrt{\frac{3}{n+1}}\,\widehat{T}^{-1}(P_1 E_{km} + \sqrt{n+1}\,P_2 E_{km}) \\ &= \sqrt{\frac{3}{n+1}}\Big(\sum_{i<j}(\alpha_i^{km} - \alpha_j^{km})E_{ij} + \sqrt{n+1}\sum_{i<j}(\delta_{km,ij} - \alpha_i^{km} + \alpha_j^{km})E_{ij}\Big) \\ &= \sqrt{\frac{3}{n+1}}\Big(\sqrt{n+1}\,e_{km} - \sum_{i<j}(\sqrt{n+1} - 1)(\alpha_i^{km} - \alpha_j^{km})e_{ij}\Big). \end{aligned}$$

It remains to observe that $\langle \delta_e, e_{ij}\rangle = 1/\sqrt{n!}$ for $i < j$, and the desired result follows by straightforward calculations. ☐

Now we are ready to prove our key theorem on the representation-theoretic meaning of the combinatorial statistics maj, des, and inv.

Theorem 3. *Let* $f = \sum_{\sigma\in\mathfrak{S}_n} c_\sigma \sigma$ *be an arbitrary element of* $\mathcal{H}$. *Then*

$$c_\sigma = \frac{1}{\sqrt{n!}}\langle \varrho(\sigma)h_{\mathsf{inv}}, \widehat{T}f\rangle.$$

In particular, for every permutation $\sigma \in \mathfrak{S}_n$, *the major index* $\mathsf{maj}(\sigma)$, *the descent number* $\mathsf{des}(\sigma)$, *and the inversion number* $\mathsf{inv}(\sigma)$ *can be calculated by the following "matrix" formulas*:

$$\mathsf{maj}(\sigma) = \frac{n(n-1)}{4} - \frac{1}{2}\langle \varrho(\sigma)h_{\mathsf{inv}}, h_{\mathsf{maj}}\rangle, \tag{7}$$

$$\mathsf{des}(\sigma) = \frac{n-1}{2} - \frac{1}{2}\langle \varrho(\sigma)h_{\mathsf{inv}}, h_{\mathsf{des}}\rangle, \tag{8}$$

$$\mathsf{inv}(\sigma) = \frac{n(n-1)}{4} - \frac{1}{2}\langle \varrho(\sigma)h_{\mathsf{inv}}, h_{\mathsf{inv}}\rangle, \tag{9}$$

where ϱ *is the representation of the group* $\mathfrak{S}_n$ *in the space* $\mathcal{M}$ *of* $n\times n$ *skew-symmetric matrices.*

Proof. Let $f = \sum_{\sigma\in\mathfrak{S}_n} c_\sigma \sigma \in \mathcal{H}$ be an arbitrary element of $\mathcal{H}$. For every permutation $\sigma \in \mathfrak{S}_n$, we have $c_\sigma = \langle \delta_\sigma, f\rangle = \langle \mathrm{Reg}_l(\sigma)\delta_e, f\rangle$. Since P is the orthogonal projection to the invariant subspace $\mathcal{H}$, we have $\langle \mathrm{Reg}_l(\sigma)\delta_e, f\rangle = \langle \mathrm{Reg}_l(\sigma)P\delta_e, f\rangle$. Acting on both factors by the unitary isomorphism T, we obtain

$$c_\sigma = \langle T\,\mathrm{Reg}_l(\sigma)P\delta_e, Tf\rangle = \langle \varrho(\sigma)TP\delta_e, Tf\rangle = \langle \varrho(\sigma)h_0, Tf\rangle$$
$$= \sqrt{\frac{3}{(n+1)!}}(\langle \varrho(\sigma)A, P_1Tf\rangle + \sqrt{n+1}\langle \varrho(\sigma)B, P_2Tf\rangle),$$

where we have used the orthogonality of the spaces of different representations. Note that $P_1Tf = \frac{\sqrt{n+1}}{\sqrt{3}}P_1\widehat{T}f$ and $P_2Tf = \frac{1}{\sqrt{3}}P_2\widehat{T}f$, whence

$$c_\sigma = \frac{1}{\sqrt{n!}}(\langle \varrho(\sigma)A, P_1\widehat{T}f\rangle + \langle \varrho(\sigma)B, P_2\widehat{T}f\rangle) = \frac{1}{\sqrt{n!}}\langle \varrho(\sigma)h_{\mathsf{inv}}, \widehat{T}f\rangle;$$

we again used orthogonality and also applied the relation $h_{\mathsf{inv}} = A + B$. The rest follows from (4). $\square$

4. Corollaries: Spectra and Convolutions

In this section we show how to easily obtain results on the spectra of elements (in the regular representation), convolutions, etc. by using the matrix formulas for combinatorial functions found above.

Let M_ε, where $\varepsilon = \mathsf{maj}, \mathsf{inv}, \mathsf{des}$, denote the operator of right multiplication by u_ε in $\mathbb{C}[\mathfrak{S}_n]$. Since the operators of left and right multiplication in $\mathbb{C}[\mathfrak{S}_n]$ commute, it follows that every eigenspace of M_ε is an invariant subspace for the left regular representation Reg_l.

Theorem 4. *Each of the operators M_ε, $\varepsilon = \mathsf{maj}, \mathsf{des}$, has two nonzero eigenvalues, $s_0^\varepsilon > 0$ and $s_1^\varepsilon < 0$, where*

$$s_0^{\mathrm{maj}} = n! \cdot \frac{n(n-1)}{4}, \qquad s_1^{\mathrm{maj}} = -\frac{n!}{2};$$
$$s_0^{\mathrm{des}} = n! \cdot \frac{n-1}{2}, \qquad s_1^{\mathrm{des}} = -(n-1)!.$$

The corresponding eigenspaces for both operators coincide. The subspace corresponding to s_0^ε is one-dimensional and coincides with the subspace of constants, i.e., with the subspace of the identity subrepresentation $\pi_{(n)}$ in Reg_l. The subspace corresponding to s_1^ε has dimension $n(n-1)/2$ and coincides with the subspace $\mathcal{H}$ introduced above (in particular, the representation in it is isomorphic to the sum $\pi_{(n-1,1)} + \pi_{(n-2,1^2)}$ of two irreducible representations).

Proof. Obviously, the constant element $\mathbf{1} = \sum_{g\in\mathfrak{S}_n} g$ in $\mathbb{C}[\mathfrak{S}_n]$ is an eigenvector for M_ε with eigenvalue $s_0^\varepsilon = \sum_{g\in\mathfrak{S}_n} \varepsilon(g)$, whose particular values can be found from (1). Hence in what follows we can consider the operators $M_{\widetilde{\varepsilon}}$ of multiplication by the centered (orthogonal to the constants) vectors $u_{\widetilde{\varepsilon}}$; we must prove that each of these operators has a single nonzero eigenvalue equal to s_1^ε and the corresponding eigenspace coincides with $\mathcal{H}$.

The fact that $\mathcal{H}$ is an invariant subspace for $M_{\widetilde{\varepsilon}}$ outside of which the operator vanishes immediately follows from Theorem 2. Each of the irreducible subspaces $H_1 = C_1\mathcal{H}$ and $H_2 = C_2\mathcal{H}$ is invariant as well. Let $f = \sum_{\sigma\in\mathfrak{S}_n} f(\sigma)\sigma \in H_k \subset \mathcal{H}$, where $k = 1, 2$. By Theorem 3 we have $f(\sigma) = \frac{1}{\sqrt{n!}}\langle \varrho(\sigma)h_{\mathsf{inv}}, \widehat{T}f\rangle$. Therefore,

$$(M_{\widetilde{\varepsilon}}f)(\sigma) = \sum_{g\in\mathfrak{S}_n} f(g)\widetilde{\varepsilon}(g^{-1}\sigma) = -\frac{1}{2\sqrt{n!}}\sum_{g\in\mathfrak{S}_n}\langle \varrho(g)h_{\mathsf{inv}}, \widehat{T}f\rangle\langle \varrho(g^{-1}\sigma)h_{\mathsf{inv}}, h_\varepsilon\rangle$$
$$= -\frac{1}{2\sqrt{n!}}\sum_{g\in\mathfrak{S}_n}\langle \varrho(g)h_{\mathsf{inv}}, \widehat{T}f\rangle\langle \varrho(g)h_\varepsilon, \varrho(\sigma)h_{\mathsf{inv}}\rangle = -\frac{\sqrt{n!}}{2\dim\pi_k}\langle P_kh_{\mathsf{inv}}, P_kh_\varepsilon\rangle\langle \widehat{T}f, \varrho(\sigma)h_{\mathsf{inv}}\rangle$$

by the orthogonality relations for matrix elements. But the last expression is equal to

$$-\frac{n!}{2\dim\pi_k}\langle P_k h_{\mathsf{inv}}, P_k h_\varepsilon\rangle f(\sigma),$$

which implies that f is an eigenvector of $M_{\widetilde{\varepsilon}}$ with eigenvalue

$$-\frac{n!}{2\dim\pi_k}\langle P_k h_{\mathsf{inv}}, P_k h_\varepsilon\rangle. \tag{10}$$

It remains to observe that $\dim\pi_1 = n-1$ and $\dim\pi_2 = (n-1)(n-2)/2$ and calculate the inner products of matrices by using Lemma 4:

$$\langle P_1 h_{\mathsf{inv}}, P_1 h_{\mathsf{des}}\rangle = \frac{2(n-1)}{n}, \qquad \langle P_2 h_{\mathsf{inv}}, P_2 h_{\mathsf{des}}\rangle = \frac{(n-1)(n-2)}{n},$$
$$\langle P_1 h_{\mathsf{inv}}, P_1 h_{\mathsf{maj}}\rangle = n-1, \qquad \langle P_2 h_{\mathsf{inv}}, P_2 h_{\mathsf{maj}}\rangle = \frac{(n-1)(n-2)}{2}.$$

We see that in the subspaces H_1 and H_2 the eigenvalues coincide and are equal to the desired value. □

Corollary 2. *In the group algebra $\mathbb{C}[\mathfrak{S}_n]$ the following identities for convolutions of the combinatorial functions under consideration hold*:

$$u_{\widetilde{\mathsf{maj}}} * u_{\widetilde{\mathsf{des}}} = -(n-1)!\cdot u_{\widetilde{\mathsf{maj}}}, \qquad u_{\widetilde{\mathsf{maj}}} * u_{\widetilde{\mathsf{maj}}} = -\frac{n!}{2}\cdot u_{\widetilde{\mathsf{maj}}},$$
$$u_{\widetilde{\mathsf{des}}} * u_{\widetilde{\mathsf{des}}} = -(n-1)!\cdot u_{\widetilde{\mathsf{des}}}, \qquad u_{\widetilde{\mathsf{des}}} * u_{\widetilde{\mathsf{maj}}} = -\frac{n!}{2}\cdot u_{\widetilde{\mathsf{des}}},$$
$$u_{\widetilde{\mathsf{inv}}} * u_{\widetilde{\mathsf{des}}} = -(n-1)!\cdot u_{\widetilde{\mathsf{inv}}}, \qquad u_{\widetilde{\mathsf{inv}}} * u_{\widetilde{\mathsf{maj}}} = -\frac{n!}{2}\cdot u_{\widetilde{\mathsf{inv}}},$$

or, explicitly,

$$\sum_{g\in\mathfrak{S}_n}\widetilde{\mathsf{maj}}(g)\widetilde{\mathsf{des}}(g^{-1}\sigma) = -(n-1)!\cdot\widetilde{\mathsf{maj}}(\sigma)$$

and similarly for the other convolutions. In particular, $u_{\widetilde{\mathsf{des}}}$ and $u_{\widetilde{\mathsf{maj}}}$ are, up to normalization, idempotents in $\mathbb{C}[\mathfrak{S}_n]$.

Proof. By Theorem 4, each of the vectors $u_{\widetilde{\mathsf{maj}}}$, $u_{\widetilde{\mathsf{des}}}$, and $u_{\widetilde{\mathsf{inv}}}$ lies in eigenspaces of the operators of right multiplication by $u_{\widetilde{\mathsf{des}}}$ and $u_{\widetilde{\mathsf{maj}}}$ with known eigenvalues, which implies the desired identities. □

Theorem 5. *The operator M_{inv} has three nonzero eigenvalues, $s_0^{\mathsf{inv}} > 0$ and $s_1^{\mathsf{inv}}, s_2^{\mathsf{inv}} < 0$, where*

$$s_0^{\mathrm{inv}} = n!\cdot\frac{n(n-1)}{4}, \qquad s_1^{\mathrm{inv}} = -\frac{(n+1)!}{6}, \qquad s_2^{\mathrm{inv}} = -\frac{n!}{6}.$$

The subspace corresponding to s_0^{inv} is one-dimensional and coincides with the subspace of constants. The subspace corresponding to s_1^{inv} has dimension $n-1$ and coincides with $H_1 = C_1\mathcal{H}$. The subspace corresponding to s_2^{inv} has dimension $(n-1)(n-2)/2$ and coincides with $H_2 = C_2\mathcal{H}$.

Proof. The proof is similar to that of Theorem 4; we must only calculate

$$\|P_1 h_{\mathsf{inv}}\|^2 = \sum_{k=1}^{n-1}\frac{4k^2}{n^2}(n-k) = \frac{n^2-1}{3} \quad\text{and}\quad \|P_2 h_{\mathsf{inv}}\|^2 = \frac{n(n-1)}{2} - \frac{n^2-1}{3} = \frac{(n-1)(n-2)}{2}$$

and substitute these values into (10). □

Corollary 3. *The function* inv *on the group $\mathfrak{S}_n$ is conditionally nonpositive definite, i.e., for every collection of complex numbers $(x_\sigma)_{\sigma\in\mathfrak{S}_n}$ such that $\sum_{\sigma\in\mathfrak{S}_n} x_\sigma = 0$,*

$$\sum_{g,h\in\mathfrak{S}_n}\mathsf{inv}(g^{-1}h)x_g\bar{x}_h \leqslant 0. \tag{11}$$

Inequality (11) holds also for maj and des; however, these functions, in contrast to inv, are not symmetric on $\mathfrak{S}_n$ (i.e., do not satisfy the relation $f(\sigma^{-1}) = \overline{f(\sigma)}$), and hence they are not conditionally nonpositive definite according to the classical definition.

The techniques developed above allow one to obtain also other identities. Consider, for example, the character χ_{nat} of the natural representation $\pi_{(n-1,1)}$ of the symmetric group $\mathfrak{S}_n$. It is well known that $\chi_{\mathrm{nat}}(g) = \mathsf{fix}(g) - 1$.

Corollary 4.

$$\frac{1}{n!}\sum_{g\in\mathfrak{S}_n} \varepsilon(g)(\mathsf{fix}(g)-1) = \begin{cases} -1/2, & \varepsilon = \mathsf{maj}, \\ -1/n, & \varepsilon = \mathsf{des}, \\ -(n+1)/6, & \varepsilon = \mathsf{inv}\,. \end{cases} \tag{12}$$

Proof. Let $\{f_i\}_{i=1}^{n-1}$ be an orthonormal basis in the space $P_1\mathcal{M}$ of the natural representation. Then, for $\sigma \in \mathfrak{S}_n$, we have $\chi_{\mathrm{nat}}(\sigma) = \sum_{i=1}^{n-1}\langle \varrho(\sigma)f_i, f_i\rangle$. Thus, the left-hand side in (12) equals (again by the orthogonality relations for matrix elements)

$$-\frac{1}{2}\sum_i \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \langle \varrho(\sigma)f_i, f_i\rangle\langle \varrho(\sigma)h_{\mathsf{inv}}, h_\varepsilon\rangle = -\frac{1}{2(n-1)}\langle P_1h_{\mathsf{inv}}, P_1h_\varepsilon\rangle,$$

and (12) follows from the formulas for the inner products obtained in the proofs of Theorems 4 and 5. ☐

Note that identity (12) for the major index can also be obtained from known results on the joint distribution of the statistics maj and fix (see [9]), but this requires heavy analytical calculations, whereas the matrix formulas give the answer immediately and simultaneously for the three statistics maj, des, and inv.

5. The Solomon Algebra

Originally, we proved Theorem 4 by using the techniques developed in [7] in connection with the study of the so-called Solomon descent algebra [12]. However, it then turned out that the approach presented in Section 3 and based on the observations from Theorems 1 and 2 is much simpler and more efficient, and it yields more results. Nevertheless, we believe that the link to the study in [7] is important and worth further investigation, so we briefly describe it in this section.

Let $\mathrm{Comp}(n)$ denote the set of compositions of a positive integer n. For $p = (\alpha_1, \dots, \alpha_k) \in \mathrm{Comp}(n)$, we set

$$B_p = \sum_{\sigma:\mathsf{Des}(\sigma)\subset\{\alpha_1,\alpha_1+\alpha_2,\dots,\alpha_1+\dots+\alpha_{k-1}\}} \sigma \in \mathbb{C}[\mathfrak{S}_n].$$

In particular, $B_{(1^n)} = \sum_{\sigma\in\mathfrak{S}_n}\sigma$, and for $p_k = (1,\dots,1,2,1,\dots,1)$ (where 2 is in the kth position), $B_{p_k} = \sum_{\sigma:k\notin\mathsf{Des}(\sigma)}\sigma$.

The elements $\{B_p\}_{p\in\mathrm{Comp}(n)}$ generate a subalgebra Σ_n of the group algebra $\mathbb{C}[\mathfrak{S}_n]$, which is called the *Solomon descent algebra*. In the paper [7] devoted to the structure and representations of the Solomon algebra another important basis $\{I_p\}_{p\in\mathrm{Comp}(n)}$ of the algebra Σ_n was introduced and transition formulas between the bases were obtained. In particular, $B_{(1^n)} = I_{(1^n)}$ and $B_{p_k} = I_{p_k} + \frac{1}{2}I_{(1^n)}$.

Let $a = \sum_q a_qI_q \in \Sigma_n$, and let M_a be the operator of right multiplication by a in $\mathbb{C}[\mathfrak{S}_n]$. In [7, Theorem 4.4] it was (implicitly) proved that the eigenvalues s_λ of M_a are indexed by the partitions λ of n and $s_\lambda = b_\lambda\sum_{p:\lambda(p)=\lambda} a_p$; moreover, the multiplicity of the eigenvalue s_λ equals $n!/z_\lambda$.

Our results on the spectra of the elements u_{des} and u_{maj} are a consequence of the following simple observation.

Lemma 6.

$$u_{\mathsf{des}} = (n-1)B_{(1^n)} - \sum_{k=1}^{n-1} B_{p_k} = \frac{(n-1)}{2} I_{(1^n)} - \sum_{k=1}^{n-1} I_{p_k},$$

$$u_{\mathsf{maj}} = \frac{n(n-1)}{2} B_{(1^n)} - \sum_{k=1}^{n-1} k B_{p_k} = \frac{n(n-1)}{4} I_{(1^n)} - \sum_{k=1}^{n-1} k I_{p_k}.$$

The description of the eigenspaces can also be deduced from results of [7] (see also [3, Theorem 2.2 and Corollary 2.3]). However, our approach makes it possible to obtain Theorem 4 much easier. Note also that the vector u_{inv} does not lie in the Solomon algebra, and hence Theorem 5 cannot be obtained by the method described in this section.

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