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Hyperquasipolynomials and Their Applications*

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Abstract. For a given nonzero entire function $g:\mathbb{C}\to\mathbb{C}$, we study the linear space $\mathscr{F}(g)$ of all entire functions f such that

$$f(z+w)g(z-w)=\varphi_1(z)\psi_1(w)+\dots+\varphi_n(z)\psi_n(w),$$

where $\varphi_1,\psi_1,\dots,\varphi_n,\psi_n\colon\mathbb{C}\to\mathbb{C}$. In the case of $g\equiv 1$, the expansion characterizes quasipolynomials, that is, linear combinations of products of polynomials by exponential functions. (This is a theorem due to Levi-Civita.) As an application, all solutions of a functional equation in the theory of trilinear functional equations are obtained.

Key words: quasipolynomial, Weierstrass sigma function, trilinear functional equation.

1. Introduction

Let f be an entire function that is not identically zero, and assume that

$$f(z+w)=\sum_{i=1}^{n}\varphi_i(z)\psi_i(w) \tag{1}$$

with some functions $\varphi_i,\psi_i\colon\mathbb{C}\to\mathbb{C}$ and with minimum possible $n=R(f)$. (The number n is called the *rank* of f). Levi-Civita [1] proved that

$$f(z)=\sum_{i=1}^{k}P_i(z)\exp(\alpha_i z) \tag{2}$$

with pairwise distinct complex numbers α_i and polynomials P_i such that

$$\sum_{i=1}^{k}(1+\deg P_i)=n.$$

One most remarkable property of this function family is that it is the set of all solutions of nth-order linear homogeneous differential equations with constant coefficients.

By analogy, if we have in mind the theory of bilinear homogeneous differential equations with constant coefficients, then we arrive at the following construction.

Definitions. Let g be an entire function that is not identically zero. A *hyperquasipolynomial with respect to* g is an arbitrary entire function f such that

$$f(z+w)g(z-w)=\varphi_1(z)\psi_1(w)+\dots+\varphi_n(z)\psi_n(w) \tag{3}$$

with some functions $\varphi_i,\psi_i\colon\mathbb{C}\to\mathbb{C}$.

Remark 1.1. The positive integer n, as well as the functions φ_i and ψ_i, in general depends on f and can be chosen in various ways. The set of all hyperquasipolynomials with respect to g is a complex linear space, which will be denoted by $\mathscr{F}(g)$. As was already mentioned, the space $\mathscr{F}(\mathbf{1})$ consists of quasipolynomials of the form (2) and the function identically equal to zero by the Levi-Civita theorem. Here and in the following, the symbol $\mathbf{1}$ stands for the function identically equal to 1.

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Remark 1.2. Set

$$\mathscr{D}_{2k} f(z) = \frac{\partial^{2k}}{\partial w^{2k}} f(z+w) f(z-w) \bigg|_{w=0}.$$

Then each function $f(z)$ satisfying the functional equation (3) with $f(z) = g(z)$ is a solution of the following $2n$th-order nonlinear homogeneous differential equation:

$$c_n \mathscr{D}_{2n} f(z) + \cdots + c_1 \mathscr{D}_2 f(z) + c_0 f^2(z) = 0,$$

where $c_n, \dots, c_0$ are complex constants.

Our aim is to study the structure of the spaces $\mathscr{F}(g)$. As an application, we find all solutions of the functional equation

$$f_1(z_1+w) f_2(z_2+w) f_3(z_1+z_2-w) = \varphi_1(z_1,z_2)\psi_1(w) + \varphi_2(z_1,z_2)\psi_2(w) + \varphi_3(z_1,z_2)\psi_3(w), \quad (4)$$

which was introduced in [2] and plays an important role in the theory of trilinear equations (see [3] and [4]).

2. Some Properties of the Spaces $\mathscr{F}(g)$

Take an expansion of the form (3) with minimum possible n for a given hyperquasipolynomial f with respect to g. Then each of the n-tuples $\{\varphi_1, \dots, \varphi_n\}$ and $\{\psi_1, \dots, \psi_n\}$ of functions is linearly independent. Hence there exist two n-tuples $\{z^{(1)}, \dots, z^{(n)}\}$ and $\{w^{(1)}, \dots, w^{(n)}\}$ of complex numbers such that $\det(\varphi_i(z^{(j)})) \neq 0$ and $\det(\psi_i(w^{(j)})) \neq 0$. By substituting $w^{(1)}, \dots, w^{(n)}$ for w and $z^{(1)}, \dots, z^{(n)}$ for z into the expansion (3), we obtain two systems of n linear equations with n unknowns, from which we find that

$$\varphi_i(z) = \sum_{j=1}^{n} a_i^{(j)} f(z+w^{(j)}) g(z-w^{(j)}), \quad (5)$$

$$\psi_i(w) = \sum_{j=1}^{n} b_i^{(j)} f(z^{(j)}+w) g(z^{(j)}-w) \quad (6)$$

with some complex numbers $a_i^{(j)}$ and $b_i^{(j)}$ $(1 \leqslant i, j \leqslant n)$.

Remark 2.1. In view of the preceding, we can assume that the functions φ_i and ψ_i in the expansion (3) are entire functions.

Let f_1 and f_2 be two hyperquasipolynomials with respect to g_1 and g_2 with the expansions

$$f_1(z+w) g_1(z-w) = \sum_{i=1}^{n_1} \varphi_{1,i}(z) \psi_{1,i}(w), \quad (7)$$

$$f_2(z+w) g_2(z-w) = \sum_{j=1}^{n_2} \varphi_{2,j}(z) \psi_{2,j}(w). \quad (8)$$

If $g_1 = g_2 = g$, then for arbitrary complex numbers c_1 and c_2 one has

$$(c_1 f_1(z+w) + c_2 f_2(z+w)) g(z-w) = \sum_{i=1}^{n_1} (c_1 \varphi_{1,i}(z)) \psi_{1,i}(w) + \sum_{j=1}^{n_2} (c_2 \varphi_{2,j}(z)) \psi_{2,j}(w). \quad (9)$$

It readily follows that $\mathscr{F}(g)$ is a complex linear space with respect to the pointwise addition of functions and multiplication by complex numbers.

By multiplying the left- and right-hand sides of the expansion (7) by the respective sides of (8), we obtain the new expansion

$$(f_1(z+w) f_2(z+w))(g_1(z-w) g_2(z-w)) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (\varphi_{1,i}(z) \varphi_{2,j}(z)) (\psi_{1,i}(w) \psi_{2,j}(w)). \quad (10)$$

Remark 2.2. It follows from the representation (10) that if f_1 and f_2 are elements of $\mathscr{F}(g_1)$ and $\mathscr{F}(g_2)$, respectively, then the product $f_1 f_2$ belongs to $\mathscr{F}(g_1 g_2)$.

Remark 2.3. The set $\mathscr{F}(\mathbf{1})$ consisting of the quasipolynomials of the form (2) and the zero function is a ring with respect to the pointwise addition and multiplication of functions. Hence it follows from Remark 2.2 that for each entire function g that is not zero identically the space $\mathscr{F}(g)$ is an $\mathscr{F}(\mathbf{1})$-module.

Remark 2.4. Let f and g be nonzero entire functions. By using the change of variables $w \to -w$, one can readily verify that if $f \in \mathscr{F}(g)$, then $g \in \mathscr{F}(f)$.

Proposition 1. *Let $f \in \mathscr{F}(g)$. Then each of the functions determined by the transformations*

$$f(z) \to f(-z), \quad f(z+z_0), \quad \exp(\alpha+\beta z) f(z), \quad \frac{df}{dz} = f'$$

with any complex numbers z_0, α, and β lies in $\mathscr{F}(g)$ as well.

Proof. According to the expansion (3), Proposition 1 is a straightforward consequence of the relations

$$\begin{aligned}
f(-(z+w))g(z-w) &= f((-w)+(-z))g((-w)-(-z)),\\
f(z+w+z_0)g(z-w) &= f((z+\tfrac{1}{2}z_0)+(w+\tfrac{1}{2}z_0))g((z+\tfrac{1}{2}z_0)-(w+\tfrac{1}{2}z_0)),\\
f'(z+w)g(z-w) &= \frac{1}{2}\frac{\partial}{\partial z}(f(z+w)g(z-w)) + \frac{1}{2}\frac{\partial}{\partial w}(f(z+w)g(z-w)).
\end{aligned}$$

Remark 2.5. Let $f \in \mathscr{F}(g)$. Then for any complex numbers γ and $\delta \neq 0$ the function $\tilde{f}$ given by the formula $\tilde{f}(z) = \exp(\gamma z^2) f(\delta z)$ lies in $\mathscr{F}(\tilde{g})$ with $\tilde{g}(z) = \exp(\gamma z^2) g(\delta z)$.

Theorem 1. *One has $\mathscr{F}(g) = \mathscr{F}(\mathbf{1})$ for any quasipolynomial g.*

Proof. Since the expansions

$$\exp\alpha(z \pm w) = \exp\alpha z \cdot \exp(\pm\alpha w), \qquad (z \pm w)^n = \sum_{k=0}^{n} C_n^k z^{n-k}(\pm w)^k$$

hold for the generators of the ring $\mathscr{F}(\mathbf{1})$, it follows that $\mathscr{F}(\mathbf{1}) \subset \mathscr{F}(g)$. Let us prove the opposite inclusion. Let

$$g(z) = P(z)\exp\alpha z,$$

where $P(z)$ is a polynomial. Then it follows from an expansion of the form (3) that

$$f(z+w)P(z-w) = \sum_{i=0}^{n} (\exp(-\alpha z)\varphi_i(z))(\exp(\alpha w)\psi_i(z)).$$

By applying Proposition 1 and by differentiating $\deg P$ times (with regard to Remark 2.4), we obtain an expansion of the form (1). Thus, f is a quasipolynomial. By carrying out this procedure for each term in the representation of g as a sum of products of exponential functions by polynomials, we obtain the inclusion $\mathscr{F}(g) \subset \mathscr{F}(\mathbf{1})$. The proof of Theorem 1 is complete. □

The following theorem presents a very important statement proved in [5].

Theorem 2. *Let f be a hyperquasipolynomial with respect to g. Then f and g lie in $\mathscr{F}(f)$ and $\mathscr{F}(g)$ simultaneously.*

The proof is based on the following fairly obvious observations.

Let $f \in \mathscr{F}(g)$ have the expansion (3). Then it follows from (5), (6), and Proposition 1 that $\varphi_i \in \mathscr{F}(\varphi_i)$ and $\psi_i \in \mathscr{F}(\psi_i)$ for all $i = 1, \dots, n$. Further,

$$\begin{aligned} f(z+w)f(z-w)g^2(w) &= f(z+\tfrac{1}{2}w+\tfrac{1}{2}w)g(z+\tfrac{1}{2}w-\tfrac{1}{2}w)f(z-\tfrac{1}{2}w-\tfrac{1}{2}w)g(z-\tfrac{1}{2}w+\tfrac{1}{2}w) \\ &= \biggl(\sum_{i=0}^{n} \varphi_i(z+\tfrac{1}{2}w)\psi_i(\tfrac{1}{2}w)\biggr)\biggl(\sum_{j=0}^{n} \varphi_j(z-\tfrac{1}{2}w)\psi_j(-\tfrac{1}{2}w)\biggr) \\ &= \sum_{i,j=0}^{n} \varphi_i(z+\tfrac{1}{2}w)\varphi_j(z-\tfrac{1}{2}w)\psi_i(\tfrac{1}{2}w)\psi_j(-\tfrac{1}{2}w). \end{aligned}$$

Let $f \in \mathscr{F}(g)$, and assume that

$$M_{f,g}(R) = \sup_{|z| \leqslant R} |f(z)| + \sup_{|w| \leqslant R} |g(w)|$$

for some real $R \geqslant 1$. If $f(0) \neq 0$ and $g(0) \neq 0$, then the expansions (3), (5), and (6) imply the inequality

$$M_{f,g}(R) \leqslant C_0 M_{f,g}^4(\tfrac{1}{2}(R+R_0))$$

with some positive constants C_0 and R_0 depending on f and g alone. By passing from $f(z)$ and $g(w)$ to $f(z+z_0)$ and $g(w+w_0)$ with $f(z_0) \neq 0$ and $g(w_0) \neq 0$ if necessary, we arrive at another statement in [5] by iterating this inequality.

Theorem 3. *Let $f \in \mathscr{F}(g)$. Then*

$$M_{f,g}(R) \leqslant \exp(C(f,g)R^2)$$

with some absolute constant $C = C(f,g)$ for any $R \geqslant 1$.

Corollary 1. *Every hyperquasipolynomial is an entire function of order $\leqslant 2$.*

Corollary 2. *If a hyperquasipolynomial f has no zeros, then*

$$f(z) = \exp(\alpha + \beta z + \gamma z^2)$$

for some complex numbers α, β, and γ.

Corollary 3. *If a hyperquasipolynomial f vanishes only at points $z_1, \dots, z_k$ with multiplicities $l_1, \dots, l_k$, respectively, then*

$$f(z) = \biggl(\prod_{i=1}^{k} (z-z_i)^{l_i}\biggr) \exp(\alpha + \beta z + \gamma z^2)$$

for some complex numbers α, β, and γ.

Corollary 4. *For every entire function g of order > 2, the space $\mathscr{F}(g)$ contains only the zero function.*

3. Rank of a Pair of Hyperquasipolynomials

Let f and g be arbitrary entire functions other than identical zero. Assume that the expansion (3) holds for some positive integer n. The minimum positive integer n for which there exists an expansion of this form will be called the *rank of the pair* (f, g) and denoted by $R(f, g)$. If there does not exist a positive integer n with this property, then we set $R(f, g) = \infty$ (the infinite rank).

Remark 3.1. It readily follows from the definition that if $R(f, g) = n$ in the expansion (3), then each of the n-tuples $\{\varphi_1, \dots, \varphi_n\}$ and $\{\psi_1, \dots, \psi_n\}$ consists of linearly independent functions. Further, the first factors in any other representation of this form can be linearly expressed via $\varphi_1, \dots, \varphi_n$, and the second factors, via $\psi_1, \dots, \psi_n$.

Proposition 2. *Let (f_1, g_1) and (f_2, g_2) be finite rank pairs of hyperquasipolynomials.*

(i) *If $g_1 = g_2 = g$, then*

$$R(C_1 f_1 + C_2 f_2, g) \leqslant R(f_1, g) + R(f_2, g)$$

for any complex numbers C_1 and C_2.

(ii) $R(f_1f_2, g_1g_2) \leqslant R(f_1, g_1)R(f_2, g_2)$.
(iii) *If* $f_1 = f_2 = f$ *and* $g_1 = g_2 = g$, *then*

$$R(f, f) \leqslant R^6(f, g), \quad R(g, g) \leqslant R^6(f, g), \quad R(df/dz, g) \leqslant 2R(f, g).$$

The claims in Proposition 2 can readily be derived from the corresponding expansions in the preceding section.

One can readily verify that for any complex numbers $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma, \delta \neq 0, z_1$, and z_2 the pairs (f_1, g_1) and (f_2, g_2) of entire functions related by the formulas

$$f_2(z) = \exp(\alpha_1 + \beta_1 z + \gamma z^2) f_1(\delta z + z_1), \qquad g_2(z) = \exp(\alpha_2 + \beta_2 z + \gamma z^2) g_1(\delta z + z_2)$$

have the same rank. Hence we say that such pairs are *equivalent* and write $(f_1, g_1) \sim (f_2, g_2)$.

Obviously, $R(f, \mathbf{1}) = n$ for quasipolynomials of the form (2). The following proposition is a consequence of what was proved above.

Proposition 3. *Any rank* n *pair* (f, g) *of hyperquasipolynomials in which* g *has no zeros is equivalent to a pair* $(h, \mathbf{1})$, *where* h *is a quasipolynomial of the form* (2).

Let a pair (f, g) of hyperquasipolynomials have rank 1. Then

$$f(z + w)g(z - w) = \varphi_1(z)\psi_1(w).$$

If $\varphi_1(z_0) = 0$, then $f(z_0 + w)g(z_0 - w) = 0$ for all w. But then f or g is zero identically, which contradicts the relation $R(f, g) = 1$. By the same argument, ψ_1 does not vanish anywhere. Hence both f and g do not vanish anywhere. In view of Proposition 3, we arrive at the following assertion.

Proposition 4. *All pairs of hyperquasipolynomials with* $R(f, g) = 1$ *are given by the formulas*

$$f(z) = \exp(\alpha + \beta z + \gamma z^2), \qquad g(z) = \exp(\eta + \delta z + \gamma z^2).$$

4. Determinant Functional Equations for Finite Rank Pairs of Hyperquasipolynomials

Let $R(f, g) = n$, and let the expansion (3) hold. By successively setting $w = w_0, w_1, \dots, w_n$, we construct a function $F\colon \mathbb{C} \to \mathbb{C}^{n+1}$ by the formula

$$F(z) = \begin{pmatrix} f(z + w_0)g(z - w_0) \\ \vdots \\ f(z + w_n)g(z - w_n) \end{pmatrix}$$
$$= \alpha_1(z) \begin{pmatrix} \beta_1(w_0) \\ \vdots \\ \beta_1(w_n) \end{pmatrix} + \dots + \alpha_j(z) \begin{pmatrix} \beta_j(w_0) \\ \vdots \\ \beta_j(w_n) \end{pmatrix} + \dots + \alpha_n(z) \begin{pmatrix} \beta_n(w_0) \\ \vdots \\ \beta_n(w_n) \end{pmatrix}.$$

For $z = z_0, z_1, \dots, z_n$, it follows that the vectors $F(z_0), F(z_1), \dots, F(z_n)$ lie in an n-dimensional subspace of $\mathbb{C}^{n+1}$. Hence the determinant

$$\mathscr{D}_{f,g}\begin{pmatrix} z_0, z_1, \dots, z_n \\ w_0, w_1, \dots, w_n \end{pmatrix} = \det \begin{pmatrix} \dots & \dots & \dots \\ \dots & f(z_i + w_j)g(z_i - w_j) & \dots \\ \dots & \dots & \dots \end{pmatrix} \tag{11}$$

(where i is the row number and j is the column number) is zero. If

$$\mathscr{D}_{f,g}\begin{pmatrix} z_1, \dots, z_n \\ w_1, \dots, w_n \end{pmatrix} \neq 0 \tag{12}$$

for some complex numbers $z_1, \dots, z_n$ and $w_1, \dots, w_n$, then the relation

$$\begin{aligned}\mathscr{D}_{f,g}\begin{pmatrix} z, z_1, \dots, z_n \\ w, w_1, \dots, w_n \end{pmatrix} &= f(z+w)g(z-w)\mathscr{D}_{f,g}\begin{pmatrix} z_1, \dots, z_n \\ w_1, \dots, w_n \end{pmatrix} \\ &\quad - f(z+w_1)g(z-w_1)\mathscr{D}_{f,g}\begin{pmatrix} z_1, z_2, \dots, z_n \\ w, w_2, \dots, w_n \end{pmatrix} + \dots \\ &\quad + (-1)^n f(z+w_n)g(z-w_n)\mathscr{D}_{f,g}\begin{pmatrix} z_1, z_2, \dots, z_n \\ w, w_1, \dots, w_{n-1} \end{pmatrix} = 0 \end{aligned} \tag{13}$$

implies the expansion (3) with minimum possible n by virtue of (1).

Remark 4.1. It follows from the preceding that a pair (f, g) of hyperquasipolynomials has finite rank n if and only if the function of $2n+2$ complex variables on the left-hand side in (11) is zero identically and the function of $2n$ variables on the left-hand side in (12) is not zero identically.

One can readily verify that if g is an odd function, then

$$\mathscr{D}_{f,g}\begin{pmatrix} z_1, z_2, z_3 \\ z, z_2, z_3 \end{pmatrix} = f(z_2+z_3)g(z_2-z_3)\mathscr{W}_{f,g}(z, z_1, z_2, z_3) \tag{14}$$

for any complex numbers z, z_1, z_2, and z_3, where

$$\begin{aligned}\mathscr{W}_{f,g}(z, z_1, z_2, z_3) &= f(z+z_1)g(z-z_1)f(z_2+z_3)g(z_2-z_3) \\ &\quad + f(z+z_2)g(z-z_2)f(z_3+z_1)g(z_3-z_1) \\ &\quad + f(z+z_3)g(z-z_3)f(z_1+z_2)g(z_1-z_2).\end{aligned}$$

Further,

$$\mathscr{D}_{f,g}\begin{pmatrix} z_1, z_2, z_3, z_4 \\ z_0, z_2, z_3, z_4 \end{pmatrix} = \mathscr{W}_{f,g}(z_0, z_2, z_3, z_4)\mathscr{W}_{f,g}(z_1, z_2, z_3, z_4) \tag{15}$$

for any complex numbers z_0, z_1, z_2, z_3, and z_4.

5. Rank 2 Pairs of Hyperquasipolynomials

Let Γ be an arbitrary lattice on the complex plane. The product

$$z \cdot \prod_{w \in \Gamma}{}' \left(1 - \frac{z}{w}\right) \exp\left(\frac{z}{w} + \frac{1}{2}\left(\frac{z}{w}\right)^2\right)$$

defines the Weierstrass sigma function $\sigma = \sigma_\Gamma(z)$ of z associated with Γ. In the degenerate cases, one has

$$\sigma_\Gamma(z) = z \quad \text{for } \Gamma = \{0\};$$

$$\sigma_\Gamma(z) = z \prod_{n \in \mathbb{Z}}{}' \left(1 - \frac{z}{nw_0}\right) \exp\left(\frac{z}{nw_0} + \frac{1}{2}\left(\frac{z}{nw_0}\right)^2\right) = \frac{w_0}{\pi} \sin \frac{\pi z}{w_0} \exp\left(\frac{\pi^2}{6}\left(\frac{z}{w_0}\right)^2\right)$$

for $\Gamma = \{nw_0 \mid n \in \mathbb{Z}\}$ with an arbitrary complex number $w_0 \neq 0$. The prime on both products indicates that the factors with $w = 0$ or $n = 0$ are omitted.

The sigma function is an odd entire function of second order (for $\Gamma \neq \{0\}$) with simple zeros at the lattice points. Weierstrass [6] showed that for any lattice Γ and any complex numbers α and γ the function f given by the formula

$$f(z) = \sigma_\Gamma(z) \exp(\alpha + \gamma z^2) \tag{16}$$

satisfies the functional equation

$$\mathscr{W}_{f,f}(z, z_1, z_2, z_3) = 0. \tag{17}$$

In the same paper, he conjectured that this equation has no other nonzero solutions. This was later proved by Hurwitz (see [7] and also [8, Chap. 20, Example 38]).

The functional equation

$$\mathscr{W}_{f,g}(z, z_1, z_2, z_3) = 0 \tag{18}$$

with an odd function g can be viewed as a generalization of Eq. (17).

Remark 5.1. For $f = g$, there is no need to require that f be odd, because this property follows from the identity

$$\mathscr{W}_{f,f}(z, z_1, w + z_1, w + z_1) = f(z + w + z_1)f(z - w - z_1)f(w + 2z_1)(f(w) + f(-w)) = 0.$$

Remark 5.2. It follows from (14) that all rank 2 pairs (f, g) of hyperquasipolynomials with an odd function g form the solution set of the generalized Weierstrass functional equation (18).

Theorem 4. *There does not exist a rank* 3 *pair of hyperquasipolynomials in which at least one of the functions is odd.*

Proof. Let $R(f, g) = 3$. In view of Remark 2.4, we can assume that g is odd. It follows from identity (15) that

$$\mathscr{W}_{f,g}(z, z_2, z_3, z_4) = 0$$

for $z_0 = z_1 = z$. But then $R(f, g) = 2$ according to Remark 5.2, which contradicts our assumption. The proof of Theorem 4 is complete.

Remark 5.3. If $R(f, g) = 2$ and at least one of the functions in the pair has no zeros, then it follows from Proposition 3 that this pair is equivalent to one of the four pairs

$$(\sin, \mathbf{1}), \quad (\mathbf{1}, \sin), \quad (\mathbf{Id}, \mathbf{1}), \quad (\mathbf{1}, \mathbf{Id}),$$

where $\mathbf{Id}\colon \mathbb{C} \to \mathbb{C}$ is the identity mapping.

Now assume that f and g vanish at least at one point. By the preceding, we can assume without loss in generality that $f(0) = g(0) = 0$. Then

$$\begin{aligned} 0 = \mathscr{D}_{f,g}\begin{pmatrix} z_1, z_2, z_3 \\ z_1, z_2, z_3 \end{pmatrix} &= f(z_1 + z_2)g(z_1 - z_2)f(z_2 + z_3)g(z_2 - z_3)f(z_3 + z_1)g(z_3 - z_1) \\ &\quad + f(z_1 + z_3)g(z_1 - z_3)f(z_2 + z_1)g(z_2 - z_1)f(z_3 + z_2)g(z_3 - z_2) \\ &= f(z_1 + z_2)f(z_1 + z_3)f(z_2 + z_3) \\ &\quad \times (g(z_1 - z_2)g(z_2 - z_3)g(z_3 - z_1) + g(z_1 - z_3)g(z_2 - z_1)g(z_3 - z_2)). \end{aligned}$$

It follows (we set $z_1 - z_3 = z$ and $z_3 - z_2 = w$) that the function $\psi(z) = -g(z)/g(-z)$ satisfies the functional equation

$$\psi(z + w) = \psi(z)\psi(w).$$

Hence

$$\psi(z) = e^{2\lambda z}, \quad h(z) = g(z)e^{-\lambda z} = -g(-z)e^{\lambda z} = -h(-z), \quad g(z) = e^{\lambda z}h(z)$$

for some complex number λ, where $h(z)$ is an odd entire function. By the same argument,

$$f(z) = e^{\eta z}r(z)$$

with some complex number η and odd entire function r. Since $(f, g) \sim (r, h)$, we can assume in what follows that f and g are odd entire functions. By (14),

$$\begin{aligned} \mathscr{W}_{f,g}(-z_1, z_1, z_2, z_3) = f(-z_1 + z_2)g(-z_1 - z_2)f(z_3 + z_1)g(z_3 - z_1) \\ + f(-z_1 + z_3)g(-z_1 - z_3)f(z_1 + z_2)g(z_1 - z_2) = 0. \end{aligned}$$

By setting $\psi(z) = f(z)/g(z)$, we reduce this relation to the form

$$\psi(z_1 - z_2)\psi(z_3 + z_1) = \psi(z_3 - z_1)\psi(z_1 + z_2).$$

For $z_2 = 0$, we obtain

$$\psi(z_3 + z_1) = \psi(z_3 - z_1).$$

Since $z_3 + z_1$ and$z_3 - z_1$ can be arbitrary complex numbers, it follows that ψ is a constant, and hence $f(z) = cg(z)$ with some complex number c. By the Hurwitz theorem on the solutions of the Weierstrass functional equation (17), we arrive at the following result due to Rochberg and Rubel [9].

Theorem 5. *Let (f,g) be a rank 2 pair of hyperquasipolynomials with functions f and g vanishing at least at one point. Then $(f,g) \sim (\sigma_\Gamma, \sigma_\Gamma)$ for some lattice Γ.*

In other words, all pairs (f,g) of functions in Theorem 5 are determined by the formulas

$$f(z) = \exp(\alpha + \beta z + \gamma z^2)\sigma_\Gamma(z+u), \qquad g(z) = \exp(\delta + \eta z + \gamma z^2)\sigma_\Gamma(z+v),$$

where α, β, γ, δ, η, u, and v are arbitrary complex numbers.

6. Generalization of the Expansion (3)

Tuples of entire functions $f_1, \dots, f_{k-1}$, and g ($k \geqslant 2$) for which one has the expansion

$$f_1(z_1+w)\cdots f_{k-1}(z_{k-1}+w)g(z_1+\cdots+z_{k-1}-w) = \sum_{i=1}^{n} \varphi_i(z_1,\dots,z_{k-1})\psi_i(w) \tag{19}$$

play an important role in the theory of k-linear homogeneous differential equations with constant coefficients (see [3] and [4]). For $k=2$, one arrives at the expansion (3).

Remark 6.1. If one sets $z_j = 0$ for all $j \neq i$ in (19), then one obtains an expansion of the form (3) for the pair (f_i, g). Consequently, each function f_i in the expansion (19) is a hyperquasipolynomial with respect to g.

By analogy with the case of $k=2$, the minimum positive integer n for which the expansion (19) holds will be called the *rank* of the function tuple in question and will be denoted by $R(f_1,\dots,f_{k-1},g)$.

For given complex numbers $u_0, u_1, \dots, u_{k-1}$, the changes of variables

$$\begin{aligned} z_1 &\to z_1 - \frac{1}{k}(u_1+\cdots+u_{k-1}) + u_1 + \frac{1}{k}u_0 = z_1 + u_1', \\ &\cdots\cdots\cdots\cdots\cdots \\ z_{k-1} &\to z_{k-1} - \frac{1}{k}(u_1+\cdots+u_{k-1}) + u_{k-1} + \frac{1}{k}u_0 = z_{k-1} + u_{k-1}', \\ w &\to w + \frac{1}{k}(u_1+\cdots+u_{k-1}) - \frac{1}{k}u_0 = w + u_0' \end{aligned}$$

reduce the expansion (19) to the form

$$\begin{aligned} f_1(z_1+w+u_1)\cdots f_{k-1}(z_{k-1}+w+u_{k-1})g(z_1+\cdots+z_{k-1}-w+u_0) \\ = \sum_{i=1}^{n}\varphi_i(z_1+u_1',\dots,z_{k-1}+u_{k-1}')\psi_i(w+u_0'). \end{aligned}$$

Remark 6.2. It follows from the preceding that the rank of a tuple of entire functions is preserved under translations with respect to the argument.

Remark 6.3. Let τ be an arbitrary permutation on the set $\{1,\dots,k-1\}$ of positive integers, and let α_i, β_i, u_i ($i = 0,1,\dots,k-1$) and $\gamma, \delta \neq 0$ be complex numbers. Then the functions defined by the formulas

$$\widetilde{f}_i(z) = \exp(\alpha_i + \beta_i z + \gamma z^2) f_{\tau(i)}(\delta z + u_i), \qquad \widetilde{g}(z) = \exp(\alpha_0 + \beta_0 + \gamma z^2) g(\delta z + u_0)$$

satisfy the relation

$$R(\widetilde{f}_1,\dots,\widetilde{f}_{k-1},\widetilde{g}) = R(f_1,\dots,f_{k-1},g).$$

Remark 6.4. We say that the tuples $(f_1,\dots,f_{k-1},g)$ and $(\widetilde{f}_1,\dots,\widetilde{f}_{k-1},\widetilde{g})$ in Remark 6.3 are equivalent.

Proposition 5. *Let $R(f_1,\dots,f_{k-1},g) = 1$, and assume that the entire function g has no zeros. Then the function tuple $(f_1,\dots,f_{k-1},g)$ is equivalent to the tuple $(h_1,\dots,h_{n-1},\mathbf{1})$, where the h_i are quasipolynomials with $R(h_i) \leqslant n$.*

Proof. Since g is an entire function of order $\leqslant 2$ without zeros, it follows that

$$g(z) = \exp(\alpha + \beta z + \gamma z^2)$$

for some complex numbers α, β, and γ.

By Remark 6.3, the tuple $(f_1, \dots, f_{k-1}, g)$ is equivalent to the tuple $(h_1, \dots, h_{n-1}, \mathbf{1})$ of entire functions such that

$$h_1(z_1 + w) \cdots h_{k-1}(z_{k-1} + w) = \sum_{i=1}^{n} \theta_i(z_1, \dots, z_{k-1}) \chi_i(w).$$

By setting $z_j = 0$ for all $i \neq j$, we obtain an expansion of the form (1) for h_i. The proof of Proposition 5 is complete. □

Remark 6.5. For any quasipolynomials $f_1, \dots, f_{k-1}$, the rank of the tuple $(f_1, \dots, f_{k-1}, \mathbf{1})$ is finite, and one has an expansion of the form (19) with $g = \mathbf{1}$.

Proposition 6. *Let* $R(f_1, \dots, f_{k-1}, g) = n$ *and* $g(0) = 0$. *Then* $R(f_i, f_j) \leqslant n - 1$ *for any* $1 \leqslant i < j \leqslant k - 1$.

Proof. For $w = 0$, the right-hand side of (19) does not vanish identically, and hence

$$(\psi_1(0), \dots, \psi_n(0)) \neq (0, \dots, 0).$$

Without loss of generality, we can assume that $\psi_n(0) = 1$. Note that

$$\begin{aligned} &\varphi_j(z_1, \dots, z_{k-1})\psi_j(w) + \varphi_n(z_1, \dots, z_{k-1})\psi_n(w) \\ &\quad = \varphi_j(z_1, \dots, z_{k-1})(\psi_j(w) - \psi_j(0)\psi_n(w)) \\ &\qquad + (\varphi_n(z_1, \dots, z_{k-1}) + \psi_j(0)\varphi_j(z_1, \dots, z_{k-1}))\psi_n(w). \end{aligned}$$

By the consecutive changes of variables

$$\begin{gathered} \varphi_n(z_1, \dots, z_{k-1}) \to \varphi_n(z_1, \dots, z_{k-1}) + \psi_j(0)\varphi_j(z_1, \dots, z_{k-1}), \\ \psi_j(w) \to \psi_j(w) - \psi_j(0)\psi_n(w) \end{gathered}$$

for $j = 1, \dots, n-1$, we obtain the expansion (19) with $\psi_1(0) = \dots = \psi_{n-1}(0) = 0$ and $\psi_n(0) = 1$. By setting $w = 0$ and $z_1 + \dots + z_{k-1} = 0$ in (19), we arrive at the equation

$$\varphi_n(z_1, \dots, z_{k-1}) = 0.$$

Hence the expansion (19) for these $z_1, \dots, z_{k-1}$ acquires the form

$$f_1(z_1 + w) \cdots f_{k-1}(z_{k-1} + w) = \sum_{i=1}^{n-1} \varphi_i(z_1, \dots, z_{k-1}) \frac{\psi_i(w)}{g(-w)}.$$

By setting $z_l = 0$ for all $l \neq i, j$, $z_i = z$, and $z_j = -z$, we obtain an expansion of the form (3) for f_i and f_j. The proof of Proposition 6 is complete. □

Theorem 6. *The set of all triples of entire functions that are not zero identically and satisfy and expansion of the form* (4) *consists of all triples* (f_1, f_2, g) *equivalent to one of the triples*

$$(\mathbf{Q}, \mathbf{1}, \mathbf{1}), \quad (\mathbf{1}, \mathbf{1}, \mathbf{Q}), \quad (\mathbf{Id}, \mathbf{Id}, \mathbf{1}), \quad (\sin, \sin, \mathbf{1}), \quad (\mathbf{1}, \mathbf{Id}, \mathbf{Id}), \quad (\mathbf{1}, \sin, \sin), \quad (\sigma_\Gamma, \sigma_\Gamma, \sigma_\Gamma),$$

where $\mathbf{Q}$ *is an arbitrary quasipolynomial of rank* $\leqslant 3$, $\mathbf{Id}(z) = z$, *and* Γ *is an arbitrary lattice in* $\mathbb{C}$.

Proof. (i) Assume that at least one of the functions f_1 and f_2, as well as g, does not vanish anywhere. Since we can interchange f_1 and f_2, we can assume that f_2 and g have no zeros. By setting $z_1 = 0$ in (4), we find from Proposition 5 that the triple (f_1, f_2, g) is equivalent to a triple $(\mathbf{Q}, \mathbf{1}, \mathbf{1})$ with an entire function $\mathbf{Q}$ for which an expansion of the form (1) with $n \leqslant 3$ holds. Hence $\mathbf{Q}$ is a quasipolynomial of rank $\leqslant 3$, and the proof of the Theorem for this case is complete.

(ii) Assume that f_1 and f_2 have no zeros. By Proposition 5, the triple (f_1, f_2, g) is equivalent to a triple of the form $(\mathbf{1}, \mathbf{1}, \mathbf{Q})$ with a three-term expansion (1) for $\mathbf{Q}$. Hence $\mathbf{Q}$ is a quasipolynomial of rank $\leqslant 3$, which completes the proof of the Theorem for this case.

(iii) Assume that f_1 and f_2 have zeros and g has no zeros. By Proposition 5, the triple (f_1, f_2, g) is equivalent to a triple of the form $(h_1, h_2, \mathbf{1})$ with quasipolynomials h_1 and h_2 of rank $\leqslant 3$ such that

$$h_1(z_1 + w)h_2(z_2 + w) = \sum_{i=1}^{3} \varphi_i(z_1, z_2)\psi_i(w). \tag{20}$$

Using transformations of the form

$$h(z) \to \exp(\alpha + \beta z)h(\Delta z + u)$$

with $\Delta \neq 0$, we can assume that h_1 coincides with one of the functions given by the formulas

$$r_1(z) = z, \quad r_2(z) = z^2 + A, \quad r_3(z) = C \sin z \exp(\lambda z) + D$$

with complex numbers A, $C \neq 0$, D, and λ. Assume that $h_1 = r_1$. By setting $z_1 = z$ and $z_2 = z$, we obtain $R(h_1, h_2) = 2$ by Theorem 4, and Theorem 5 with $\Gamma = \{0\}$ implies that

$$(h_1, h_2) \sim (\mathbf{Id}, \mathbf{Id}).$$

Hence we find that the triple (f_1, f_2, g) is equivalent to the triple $(\mathbf{Id}, \mathbf{Id}, \mathbf{1})$. Now let $h_1 = r_2$. By differentiating both parts of the expansion (20) with respect to z_1, we obtain a new expansion of the same type with $h_1(z)$ replaced by z. By the same argument as above, we have

$$(\mathbf{Id}, h_2) \sim (\mathbf{Id}, \mathbf{Id}).$$

One can readily verify that $R(h_1, \mathbf{Id}) = 4$. This contradicts the inequality $R(h_1, \mathbf{Id}, \mathbf{Id}) = R(f_1, f_2, g) \leqslant 3$. Hence the case of $h_2 = r_2$ is impossible. Finally, assume that $h = r_3$. We again differentiate both parts of (20) with respect to z_1. As a result, we obtain an expansion of the form (20) with $h_1(z)$ replaced by

$$C(\cos z + \lambda \sin z)\exp(\lambda z) = C_1 \sin(z + \theta_1)\exp(\lambda z).$$

By applying Theorems 4 and 5, we find that $R(h_1, h_2) = 2$ and $(h_1, h_2) \sim (h_1, \sin)$. Let us again apply Theorem 5. As a result, we see that the triple (f_1, f_2, g) is equivalent to the triple $(\sin, \sin, \mathbf{1})$, which completes the proof of Theorem 6 for this case.

(iv) Let only one of the functions f_1, f_2, and g have zeros. Then the triple (f_1, f_2, g) is equivalent to the triple $(\mathbf{1}, h_2, r)$. By Proposition 6, $R(\mathbf{1}, h_2) = 2$, and hence the pair $(\mathbf{1}, h_2)$ is equivalent to either $(\mathbf{1}, \mathbf{Id})$ or $(\mathbf{1}, \sin)$ by Theorem 5. It remains to use Theorem 4 and then, once more, Theorem 5. The proof of case (iv) is complete.

(v) Assume that each of the functions f_1, f_2, and g has zeros. By Proposition 6 and Theorem 5, the triple (f_1, f_2, g) is equivalent to $(\sigma_\Gamma, \sigma_\Gamma, h)$ for some lattice Γ. By applying Theorem 4 and, once more, Theorem 5, we finally establish that the triple (f_1, f_2, g) is equivalent to $(\sigma_\Gamma, \sigma_\Gamma, \sigma_\Gamma)$. The expansion (20) follows from the classical identity

$$\sigma_\Gamma(x + y + z)\sigma_\Gamma(x - y)\sigma_\Gamma(y - z)\sigma_\Gamma(z - x) = \frac{1}{2}\sigma_\Gamma^3(x)\sigma_\Gamma^3(y)\sigma_\Gamma^3(z) \det \begin{pmatrix} 1 & \wp_\Gamma(x) & \wp_\Gamma'(x) \\ 1 & \wp_\Gamma(y) & \wp_\Gamma'(y) \\ 1 & \wp_\Gamma(z) & \wp_\Gamma'(z) \end{pmatrix},$$

where $\wp_\Gamma(z) = -d^2\sigma_\Gamma(z)/dz^2$ is the Weierstrass elliptic function.

Conclusion

Let

$$F = \bigcup_g F(g)$$

be the set of all hyperquasipolynomials. It follows from Remark 2.2 that F is a multiplicative semigroup with respect to pointwise multiplication. Let us show that F, in contrast to the set of quasipolynomials, is not a linear space.

The functions z and $\exp(z^2)$ are hyperquasipolynomials with respect to themselves. Consider the sum

$$f(z) = z + \exp(z^2). \tag{21}$$

Assume that there exists an entire function g such that $R(f, g) < \infty$.

By Proposition 1, the functions determined by the formulas

$$z = \frac{1}{2}(f(z) - f(-z)), \quad \exp(z^2) = \frac{1}{2}(f(z) - f(-z)) \tag{22}$$

are hyperquasipolynomials with respect to g as well. By Theorem 1, g is a quasipolynomial. Then the functions (22) are quasipolynomials as well by the same theorem. But this is impossible. Hence the sum (21) of two hyperquasipolynomials is not a hyperquasipolynomial.

Theorem 1 completely determines the spaces $\mathscr{F}(g)$ with quasipolynomials g and $\mathscr{F}(\sigma_\Gamma)$ with a degenerate lattice Γ. It is of interest to ask what the structure of the space $\mathscr{F}(\sigma_\Gamma)$ is for nondegenerate lattices Γ. The answer is probably as follows.

The space $\mathscr{F}(\sigma_\Gamma)$ for a nondegenerate lattice Γ consists of linear combinations with quasipolynomial coefficients of translates of the function σ_Γ and its derivatives with respect to the argument; in other words, any element of $\mathscr{F}(\sigma_\Gamma)$ has the form

$$\sum_{i=1}^{n}\sum_{j=1}^{m_i} Q_{i,j}(z)\sigma_\Gamma^{(i)}(z + z_{ij}),$$

where the $Q_{i,j}$ are quasipolynomials and the z_{ij} are arbitrary complex numbers.

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