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The Stability of Fixed Points of Discrete Dynamical Systems in the Space $\operatorname{conv}\mathbb{R}^n$

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Abstract. Conditions for the asymptotic Lyapunov stability of the fixed points of discrete dynamical systems in the space $\operatorname{conv}\mathbb{R}^n$ are established.

Key words: discrete dynamical systems, space $\operatorname{conv}\mathbb{R}^n$, Lyapunov stability.

Problems of the control theory of dynamical systems in space $\mathbb{R}^n$, in particular, the construction, approximations, and qualitative analysis of attainability sets [1], are closely related to the study of dynamical systems in the metric space $\operatorname{conv}\mathbb{R}^n$. Some results on qualitative analysis of discrete dynamical systems (DDS) in the space $\operatorname{conv}\mathbb{R}^n$ are given in [2].

Let $\operatorname{conv}\mathbb{R}^n$ be the metric space of convex compact sets in $\mathbb{R}^n$ endowed with the Hausdorff metric d_H. This space has the structure of a linear semigroup with Minkowski addition and multiplication by a nonnegative scalar. The difference between two elements is not always defined in this space. An embedding of $\operatorname{conv}\mathbb{R}^n$ into a linear normed space whose completion is isomorphic to a Banach space $C(S^{n-1})$ is described in [3, p. 967]. The correspondence $\operatorname{conv}\mathbb{R}^n \ni u \to h_u(p) \in C(S^{n-1})$, where $h_u(p)$ is the support function of the convex compact u, is an isometric isomorphism. Thus, we can define the difference between two elements in $\operatorname{conv}\mathbb{R}^n$ as an element of the space $C(S^{n-1})$.

Consider the following DDS in $\operatorname{conv}\mathbb{R}^n$:

$$\overline{u} = G(u, V[u]), \quad u \in \operatorname{conv}\mathbb{R}^n, \quad G\colon \operatorname{conv}\mathbb{R}^n \times \mathbb{R}_+ \to \operatorname{conv}\mathbb{R}^n, \tag{1}$$

where $V[u]$ is the volume of the compact set u and the map G satisfies the following condition: for any $(u_0, v_0) \in \operatorname{conv}\mathbb{R}^n \times \mathbb{R}_+$, there exists a $G'_u(u_0, v_0) \in L(C(S^{n-1}))$, a $G'_V(u_0, v_0) \in C(S^{n-1})$, and a neighborhood $\mathscr{U} \subset \operatorname{conv}\mathbb{R}^n \times \mathbb{R}$ such that, for all $(u, v) \in \mathscr{U}$, we have

$$G(u, v) = G(u_0, v_0) + G'_u(u_0, v_0)\Delta u + G'_v(u_0, v_0)\Delta v + o(\|(\Delta u, \Delta v)\|),$$

where $\Delta u = u - u_0$, $\Delta v = v - v_0$, $\|(u, v)\| = \|u\|_{C(S^{n-1})} + |v|$, and $L(X)$ is the Banach algebra of bounded linear operators on a Banach space X.

Suppose that there is a fixed point $u^* \in \operatorname{conv}\mathbb{R}^n$ of the DDS (1), that is, $u^* = G(u^*, V[u^*])$. Following [3, p. 968], we define the variations $\delta u = u - u^*$ and $\delta\overline{u} = \overline{u} - u^*$ as elements of the space $C(S^{n-1})$; then, using Steiner's formulas [3, p. 969], properties of the mixed volume functional, the assumptions about the mapping G, and the integral representation for the mixed volume functional [3, p. 968], we obtain the variational equation $\delta\overline{u} = \mathscr{L}\delta u$, where $\mathscr{L} \in L(C(S^{n-1}))$, $\mathscr{L}h = G'_u h + G'_V \int_{S^{n-1}} hF[u^*, d\omega]$, $G'_u = G'_u(u^*, v^*)$, $G'_V = G'_V(u^*, v^*)$, $v^* = V[u^*]$, and $F[u^*, d\omega]$ is the surface function of u^* [3, p. 963].

Let $D(\lambda) = 1 - \int_{S^{n-1}} R(\lambda, G'_u)G'_V F[u^*, d\omega]$, $\lambda \in \rho(G'_u)$, where $R(\lambda, \cdot\,)$ is the resolvent of the corresponding operator, $\rho(\cdot\,)$ is its resolvent set, and $\sigma(\cdot\,)$ is its spectrum.

Theorem 1. *Assume that*

$$\sigma(G'_u) \cup \{\lambda \in \rho(G'_u) \mid D(\lambda) = 0\} \subset \{\lambda \in \mathbb{C} \,|\, |\lambda| < 1\}.$$

Then the fixed point u^* of the DDS (1) *is asymptotically stable.*

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Proof. If $|\lambda| > 1$ and $f \in C(S^{n-1})$, then the linear equation $(\lambda I - \mathscr{L})g = f$ has the unique solution

$$g = \frac{1}{D(\lambda)} R(\lambda, G'_u) G'_V \int_{S^{n-1}} R(\lambda, G'_u) f F[u^*, d\omega] + R(\lambda, G'_u) f,$$

which continuously depends on the function f; therefore, $\lambda \in \rho(\mathscr{L})$. Thus, $r(\mathscr{L}) < 1$, which proves the asymptotic stability of u^*.

In the particular case where $n = 2$ and $G(u,s) = \varphi(s)\mathbf{A}u + \psi(s)b$, $b \in \operatorname{conv}\mathbb{R}^2$, $\varphi, \psi \in C^1(\mathbb{R}_+; \mathbb{R}_+)$, $\mathbf{A} \in L(\mathbb{R}^2)$, the question of whether the point u^* is stable under the assumption that $h_{u^*} \in C^2(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$ reduces to the localization of the spectrum of the operator

$$\mathscr{L} \in L(C[0, 2\pi]), \qquad \mathscr{L}H(\theta) = \varphi(s^*)a(\theta)H(\gamma(\theta)) + \Gamma(\theta)\int_0^{2\pi} N(\tau)H(\tau)\,d\tau,$$

where $s^* = S[u^*]$, $\Gamma(\theta) = \varphi'(s^*)a(\theta)H_{u^*}(\gamma(\theta)) + \psi'(s^*)H_b(\theta)$, $N(\theta) = H_{u^*}(\theta) + H''_{u^*}(\theta)$, $H_u(\theta) = h_u(\cos\theta, \sin\theta)$,

$$a(\theta) = \sqrt{(a_{11}\cos\theta + a_{21}\sin\theta)^2 + (a_{12}\cos\theta + a_{22}\sin\theta)^2},$$
$$\cos\gamma(\theta) = \frac{a_{11}\cos\theta + a_{21}\sin\theta}{a(\theta)}, \quad \sin\gamma(\theta) = \frac{a_{12}\cos\theta + a_{22}\sin\theta}{a(\theta)},$$

and $A = [a_{ij}]_{i,j=1}^2$ is the matrix of the operator $\mathbf{A}$ in the canonical basis.

If $\operatorname{tr}^2\mathbf{A} - 4\det\mathbf{A} < 0$, then $a(\theta) = \sqrt{\det\mathbf{A}}$ and $\gamma(\theta) = \theta + \alpha$, where α is determined from the relations

$$\cos\alpha = \frac{\operatorname{tr}\mathbf{A}}{2\sqrt{\det\mathbf{A}}} \qquad \text{and} \qquad \sin\alpha = \frac{\sqrt{4\det\mathbf{A} - \operatorname{tr}^2\mathbf{A}}}{2\sqrt{\det\mathbf{A}}}.$$

In the case where $\alpha q \in 2\pi\mathbb{Z}$, $q > 0$, the operator $\mathscr{L}^q$ has the form

$$\mathscr{L}^q H(\theta) = \mu^q H(\theta) + \int_0^{2\pi} \sum_{k=1}^{q} \mathrm{A}_{kq}(\theta)\mathrm{B}_{kq}(\tau)H(\tau)\,d\tau, \qquad \mu = \sqrt{\det\mathbf{A}}\varphi(s^*),$$

where the functions A_{kq} and B_{kq} are successively defined as

$$\mathrm{A}_{k,i+1}(\theta) = \mathrm{A}_{ki}(\theta), \quad \mathrm{B}_{k,i+1}(\tau) = \mu\mathrm{B}_{ki}(\tau - \alpha) + N(\tau)\int_0^{2\pi} \mathrm{B}_{ki}(\xi)\Gamma(\xi)\,d\xi,$$
$$\mathrm{A}_{i+1,i+1}(\theta) = \mu^i\Gamma(\theta + i\alpha), \quad \mathrm{B}_{i+1,i+1}(\tau) = N(\tau), \qquad k = 1, \dots, i,\ i = 1, \dots, q;$$

the initial values are $\mathrm{A}_{11}(\theta) = \Gamma(\theta)$ and $\mathrm{B}_{11}(\tau) = N(\tau)$. Consider the polynomial $D(\lambda) = \det[(\lambda - \mu^q)\delta_{mn} - \int_0^{2\pi} \mathrm{A}_{mq}(\tau)\mathrm{B}_{nq}(\tau)\,d\tau]_{m,n=1}^q$.

In the case $\alpha \notin 2\pi\mathbb{Q}$, we introduce the function

$$D(\lambda) = \begin{cases} 1 + \sum_{k=0}^{\infty} \lambda^k \mu^{-k-1} \int_0^{2\pi} N(\theta)\Gamma(\theta - (k+1)\alpha)\,d\theta, & |\lambda| < \mu, \\ 1 - \sum_{k=0}^{\infty} \lambda^{-k-1} \mu^k \int_0^{2\pi} N(\theta)\Gamma(\theta + k\alpha)\,d\theta, & |\lambda| \geqslant \mu. \end{cases}$$

Proposition 1. *Suppose that* $\operatorname{tr}^2\mathbf{A} - 4\det\mathbf{A} < 0$, $\omega < 1$, *and all roots of the equation* $D(\lambda) = 0$ *are less than* 1 *in absolute value. Then the fixed point* u^* *of the DDS* (1) *is Lyapunov asymptotically stable.*

Proof. If $\alpha q \in 2\pi\mathbb{Z}$, then, by Dunford's spectral mapping theorem [4, p. 32], the condition $r(\mathscr{L}) < 1$ is equivalent to the inequality $r(\mathscr{L}^q) < 1$. The operator $\mathscr{L}^q$ is the sum of a scalar operator and an integral operator with degenerate kernel; therefore, it is easy to calculate its spectrum. If $\alpha \notin 2\pi\mathbb{Q}$, then the assertion of Proposition 1 is derived from Theorem 1.

Now suppose that $\operatorname{tr}^2\mathbf{A} - 4\det\mathbf{A} \geqslant 0$; for all $\lambda \in \mathbb{C}$, $|\lambda| > \varphi(s^*)\|\mathbf{A}\|$, we define the function

$$D(\lambda) = 1 - \sum_{k=0}^{\infty} \lambda^{-k-1} \int_0^{2\pi} \varphi^k(S[u^*])a^k(\theta)\Gamma(\gamma^{[k]}(\theta))N(\theta)\,d\theta,$$

where $\gamma^{[k]}$ is the kth iteration of the mapping $\gamma\colon S^1 \to S^1$.

Proposition 2. *Suppose that* $\operatorname{tr}^2 \mathbf{A} - 4\det \mathbf{A} \geqslant 0$, $\varphi(s^*)\|\mathbf{A}\| < 1$, *and the equation* $D(\lambda) = 0$ *has no solutions for which* $|\lambda| > \varphi(s^*)\|\mathbf{A}\|$. *Then the fixed point* u^* *of the DDS* (1) *is Lyapunov asymptotically stable.*

The proof is similar to that of Theorem 1.

It is of interest to obtain necessary and sufficient conditions for the stability of the fixed points of a DS in the space $\operatorname{conv}\mathbb{R}^n$ in the case of any positive integer n.

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