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Generalized Analytic Functions, Moutard-Type Transforms, and Holomorphic Maps*

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Abstract. We continue the study of a Moutard-type transform for generalized analytic functions, which was initiated in [1]. In particular, we suggest an interpretation of generalized analytic functions as spinor fields and show that, in the framework of this approach, Moutard-type transforms for such functions commute with holomorphic changes of variables.

Key words: generalized analytic functions, spinors, Moutard transforms.

We study the basic pair of conjugate equations of generalized analytic function theory, which are

$$\partial_{\bar z}\psi = u\bar\psi \qquad \text{in } D, \tag{1}$$

$$\partial_{\bar z}\psi^+ = -\bar u\bar\psi^+ \quad \text{in } D, \tag{2}$$

where D is an open domain in $\mathbb{C}$, $u = u(z)$ is a given function in D, and $\partial_{\bar z} = \partial/\partial\bar z$ (see [4]). Here and in what follows, the expression $f = f(z)$ does not mean that f is holomorphic.

Quite recently, new progress in the theory of generalized analytic functions was made in [1], where it was shown that Moutard-type transforms can be applied to the pair of equations (1), (2). Moutard-type transforms were developed and successfully used in differential geometry, soliton theory in dimension $2+1$, and spectral theory in dimension 2; see [1] for further references. In particular, our work [1] was essentially stimulated by Taimanov's recent articles [2] and [3] on Moutard-type transforms for the Dirac operators in the framework of soliton theory in dimension $2+1$. On the other hand, our research was also strongly motivated by certain open problems of two-dimensional inverse scattering at fixed energy, where Eq. (1) arises as the $\bar\partial$-equation in spectral parameter.

A simple Moutard-type transform $\mathscr{M} = \mathscr{M}_{u,f,f^+}$ for the pair of conjugate equations (1), (2) is given by the formulas (see [1])

$$\tilde u = \mathscr{M}u = u + \frac{f\overline{f^+}}{\omega_{f,f^+}}, \tag{3}$$

$$\tilde\psi = \mathscr{M}\psi = \psi - \frac{\omega_{\psi,f^+}}{\omega_{f,f^+}}\, f, \qquad \tilde\psi^+ = \mathscr{M}\psi^+ = \psi^+ - \frac{\omega_{f,\psi^+}}{\omega_{f,f^+}}\, f^+, \tag{4}$$

where f and f^+ are some fixed solutions of Eqs. (1) and (2), respectively, ψ and ψ^+ are arbitrary solutions of (1) and (2), respectively, and $\omega_{\psi,\psi^+} = \omega_{\psi,\psi^+}(z)$ denotes the imaginary-valued function given by

$$\partial_z\omega_{\psi,\psi^+} = \psi\psi^+, \quad \partial_{\bar z}\omega_{\psi,\psi^+} = -\overline{\psi\psi^+} \quad \text{in } D; \tag{5}$$

this function is well defined, at least, for simply connected D, whereas the purely imaginary integration constant may depend on the particular situation. The point is that the functions $\tilde\psi$ and

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$\tilde\psi^+$ defined by (4) satisfy the following Moutard-transformed pair of conjugate equations:

$$\begin{aligned} \partial_{\bar z}\tilde\psi &= \tilde u\,\overline{\tilde\psi} \quad \text{in } D,\\ \partial_{\bar z}\tilde\psi^+ &= -\overline{\tilde u}\,\overline{\tilde\psi^+} \quad \text{in } D. \end{aligned}$$

We also obtain the following important new result.

Theorem 1. *For the simple Moutard transform* (3), (4),

$$\omega_{\tilde\psi,\tilde\psi^+} = \frac{\omega_{\psi,\psi^+}\omega_{f,f^+} - \omega_{\psi,f^+}\omega_{f,\psi^+}}{\omega_{f,f^+}} + c_{\tilde\psi,\tilde\psi^+},$$

where $c_{\tilde\psi,\tilde\psi^+}$ is a purely imaginary constant.

Applying Moutard-type transforms to study generalized analytic functions with contour poles requires investigating, in particular, compositions of these transforms and holomorphic maps.

Consider a holomorphic bijection W:

$$\begin{aligned} W\colon D &\to D_*, \quad z \mapsto \zeta(z),\\ W^{-1}\colon D_* &\to D, \quad \zeta \mapsto z(\zeta), \end{aligned} \tag{6}$$

where D is the same domain as in (1) and (2).

If $\psi(z)$ and $\psi^+(z)$ in Eqs. (1) and (2) are treated as scalar fields, then the conjugacy of these equations is not invariant with respect to holomorphic bijections. In the following theorem we give correct transformation formulas for the conjugate pair of equations (1), (2) with respect to holomorphic bijections.

Theorem 2. *Let W be a holomorphic bijection of the form* (6). *Consider*

$$u_*(\zeta) = u(z(\zeta))\sqrt{\frac{\partial z}{\partial\zeta}\,\frac{\partial\bar z}{\partial\bar\zeta}} = u(z(\zeta))\left|\frac{\partial z}{\partial\zeta}\right|, \tag{7}$$

$$\psi_*(\zeta) = \psi(z(\zeta))\sqrt{\frac{\partial z}{\partial\zeta}}, \quad \psi_*^+(\zeta) = \psi^+(z(\zeta))\sqrt{\frac{\partial z}{\partial\zeta}}, \tag{8}$$

where $u(z)$, $\psi(z)$, and $\psi^+(z)$ are the same as in Eqs. (1) *and* (2). *Then*

$$\partial_{\bar\zeta}\psi_* = u_*\bar\psi_* \quad \text{in } D_*, \tag{9}$$

$$\partial_{\bar\zeta}\psi_*^+ = -\bar u_*\bar\psi_*^+ \quad \text{in } D_*. \tag{10}$$

Moreover,

$$\omega_{\psi_*,\psi_*^+}(\zeta) = \omega_{\psi,\psi^+}(z(\zeta)), \tag{11}$$

where ω is defined by (5).

Remark. Formulas (7) and (8) admit the following natural interpretation: $\psi(z)$ and $\psi^+(z)$ can be treated as spinors, i.e., differential forms of type $(1/2, 0)$, and u can be treated as a differential form of type $(1/2, 1/2)$. The corresponding forms can be written as

$$u = u(z)\sqrt{dz\,d\bar z}, \quad \psi = \psi(z)\sqrt{dz}, \quad \psi^+ = \psi^+(z)\sqrt{dz}.$$

This is very natural, because Eq. (1) can be viewed as a special reduction of the two-dimensional Dirac system (see, e.g., [1]).

Theorem 2 implies that W in (6) generates a map of the conjugate pair of equations (1), (2) to the conjugate pair of equations (9), (10). We denote this map by the same symbol W. Using this interpretation of W, we obtain the following result.

Theorem 3. *The relation*

$$\mathscr{M}_{u_*,f_*,f_*^+} \circ W = W \circ \mathscr{M}_{u,f,f^+}$$

holds, where $\mathscr{M}_{u,f,f^+}$ *and* $\mathscr{M}_{u_*,f_*,f_*^+}$ *are defined by* (3) *and* (4) *and* u_*, f_*, f_*^+, *and* ω_{ψ_*,ψ_*^+} *are defined by* (7), (8), *and* (11).

Theorems 1, 2, and 3 can be proved by direct calculations.

In the framework of the approach based on the Moutard transform, using Theorem 3, we reduce the local study of generalized analytic functions with a contour pole on a real-analytic curve to the case of a contour pole on a straight line. This study will be continued in a subsequent paper.

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