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Relative Index Theorem in *K*-Homology

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Abstract. We prove an analog of Gromov–Lawson type relative index theorems for K-homology classes.

Key words: Fredholm module, cutting and pasting, relative index, K-homology.

Introduction. Let M and M' be two manifolds coinciding outside some subsets $Q \subset M$ and $Q' \subset M'$ (i.e., $M \setminus Q$ and $M' \setminus Q'$ are identified with each other), and let D and D' be elliptic operators on M and M', respectively, coinciding on $M \setminus Q \simeq M' \setminus Q'$. The difference $\operatorname{ind} D - \operatorname{ind} D'$ of their indices is called the *relative index* of D and D'. A *relative index theorem* is often understood in the literature as a statement of the following type: *the relative index is independent of the structure of M and M', as well as of D and D', on the set where they coincide, i.e., on $M \setminus Q$*; in other words, to compute the relative index of D and D', it suffices to know the structure of D and D' on Q and Q', respectively. Theorems of this kind are trivial for smooth closed manifolds (owing to the existence of a local index formula [1]), but in more general cases one has informative statements such as the relative index theorem for Dirac operators on complete noncompact Riemannian manifolds proved in the famous paper [2] by Gromov and Lawson. For further examples, we refer the reader to the paper [3], where the relative index theorem was proved in a rather general abstract framework that not only included many of the earlier known special cases but also permitted one to obtain a number of index formulas for elliptic differential operators and Fourier integral operators on manifolds with singularities (see [4]). Note, however, that the index is not the only homotopy invariant of elliptic operators, and hence it is of interest to obtain locality theorems for broader sets of invariants. There are various directions in which to generalize the relative index theorem. For example, Bunke [5] considered Dirac operators acting on sections of bundles of projective Hilbert B-modules over a complete noncompact Riemannian manifold, where B is a C^*-algebra, and obtained a relative index theorem for such operators, the index being an element of the K-group of B. Here we solve a different problem. Namely, if the elliptic operators in question are local with respect to some C^*-algebra A, then it is natural to ask how the corresponding classes in the K-homology of A vary under a “local” variation of the operators. Here the algebra A is not assumed to be commutative, and accordingly, localization is based on ideals in A. It turns out (this is the main result of the paper) that this variation obeys the same laws as the relative index in the “classical” theorems does. That is why we still refer to our theorem as a “relative index theorem,” even though it deals with K-homology classes rather than the index. All results are stated directly in terms of Fredholm modules; for the standard construction that assigns a Fredholm module to an elliptic operator, we refer the reader to the literature (e.g., see [6]).

1. *K*-homology. Recall the definition of K-homology groups of a C^*-algebra A (e.g., see [6, Chap. 8]). A *Fredholm module* over A is a triple $x = (\rho, H, F)$, where H is a Hilbert space, $\rho\colon A \to \mathfrak{B}(H)$ is a representation of A on H, and $F \in \mathfrak{B}(H)$ is an operator such that

$$[F, \rho(\varphi)] \sim 0 \quad \text{for any } \varphi \in A \quad (\textit{locality}), \qquad F \approx F^*, \qquad F^2 \approx 1, \tag{1}$$

where $\sim$ stands for equality modulo compact operators and $\approx$ for equality modulo *locally compact operators*, i.e., operators C such that the operators $\rho(\varphi)C$ and $C\rho(\varphi)$ are compact for every $\varphi \in A$. Two Fredholm modules (ρ, H, F_0) and (ρ, H, F_1) corresponding to one and the same representation ρ are said to be *homotopic* if they can be embedded in a family (ρ, H, F_t), $t \in [0, 1]$, of

Fredholm modules such that the function $t \mapsto F_t$ is operator norm continuous. A Fredholm module is said to be *degenerate* if all relations in (1) are satisfied exactly rather than modulo (locally) compact operators. We say that two Fredholm modules x and x' are *equivalent* if there exists a degenerate module x'' such that the modules $x \oplus x''$ and $x' \oplus x''$ are unitarily equivalent to homotopic Fredholm modules. The set of equivalence classes of Fredholm modules is denoted by $K^1(A)$; the direct sum of modules induces a structure of an abelian group on $K^1(A)$, which is called the (*odd*) *K-homology group of* A. The definition of the *even K-homology group* $K^0(A)$ is similar, but one considers *graded* Fredholm modules, i.e., ones equipped with the following additional structure: the space H is $\mathbb{Z}_2$-graded, $H = H_+ \oplus H_-$, the representation ρ is even (i.e., preserves the grading, $\rho(A)H_\pm \subset H_\pm$), and the operator F is odd (i.e., $FH_+ \subset H_-$ and $FH_- \subset H_+$).

The results stated below hold for $K^0(A)$ as well as $K^1(A)$, and it is tacitly assumed throughout that all Fredholm modules involved are graded in the first case. For brevity, we often write φ rather than $\rho(\varphi)$ for $\varphi \in A$; which representation is meant is always clear from the context.

2. Fredholm modules agreeing on an ideal. Let $x = (\rho, H, F)$ and $\widetilde{x} = (\widetilde{\rho}, \widetilde{H}, \widetilde{F})$ be Fredholm modules over A, and let $J \subset A$ be an ideal. The orthogonal projections* $P\colon H \to H_0$, where $H_0 = JH \subset H$, and $\widetilde{P}\colon \widetilde{H} \to \widetilde{H}_0$, where $\widetilde{H}_0 = J\widetilde{H}$, commute with the action of A.

Definition. Given an operator $T\colon H_0 \to \widetilde{H}_0$ intertwining the representations ρ and $\widetilde{\rho}$, preserving the grading in the graded case, and satisfying $TPFPT^* \approx \widetilde{P}\widetilde{F}\widetilde{P}$, we say that x and $\widetilde{x}$ *agree on the ideal* J.

3. Cutting and pasting. Let $J_1, J_2 \subset A$ be ideals such that $J_1 + J_2 = A$. Let x and $\widetilde{x}$ be Fredholm modules over A agreeing on the ideal $J = J_1 \cap J_2$, and assume that the representations ρ and $\widetilde{\rho}$ are nondegenerate (i.e., $AH = H$ and $A\widetilde{H} = \widetilde{H}$). Then one can define a Fredholm module $x \diamond \widetilde{x}$ obtained, informally speaking, by "pasting together over J the part of x corresponding to J_1 with the part of $\widetilde{x}$ corresponding to J_2." To this end, we represent F (and, in a similar way, $\widetilde{F}$) by a 3×3 matrix associated with the decomposition of H into the direct orthogonal sum of the A-invariant subspaces $H_0 = JH$, $H_1 = J_1H \ominus H_0$ (the orthogonal complement), and $H_2 = J_2H \ominus H_0$, $H = H_1 \oplus H_0 \oplus H_2$ (in this particular order!).** We denote the orthogonal projection onto H_j by P_j, $j = 0, 1, 2$. Note that*** $\varphi P_1FP_2 = \varphi_1 P_1FP_2 = P_1\varphi_1FP_2 \sim P_1F\varphi_1P_2 = 0$ for any $\varphi \in A$; i.e., $P_1FP_2 \approx 0$, and likewise $P_2FP_1 \approx 0$, so that the desired representation can be written out as

$$F \approx \begin{pmatrix} a & b & 0 \\ b^* & c & d \\ 0 & d^* & e \end{pmatrix}, \qquad \widetilde{F} \approx \begin{pmatrix} \widetilde{a} & \widetilde{b} & 0 \\ \widetilde{b}^* & \widetilde{c} & \widetilde{d} \\ 0 & \widetilde{d}^* & \widetilde{e} \end{pmatrix}, \qquad \begin{matrix} a = a^*, & c = c^*, & e = e^*, \\ \widetilde{a} = \widetilde{a}^*, & \widetilde{c} = \widetilde{c}^*, & \widetilde{e} = \widetilde{e}^*, \end{matrix} \tag{2}$$

where all entries are local. The condition that x and $\widetilde{x}$ agree on J acquires the form $TcT^* \approx \widetilde{c}$. To simplify the notation, we identify H_0 with $\widetilde{H}_0$ via T; then we no longer write out T explicitly, and the agreement condition on J becomes $c \approx \widetilde{c}$. Set

$$H \diamond \widetilde{H} = H_1 \oplus H_0 \oplus \widetilde{H}_2, \quad \rho \diamond \widetilde{\rho} = \rho|_{H_1 \oplus H_0} \oplus \widetilde{\rho}|_{\widetilde{H}_2}, \quad F \diamond \widetilde{F} = \begin{pmatrix} a & b & 0 \\ b^* & c & \widetilde{d} \\ 0 & \widetilde{d}^* & \widetilde{e} \end{pmatrix}. \tag{3}$$

Proposition. *The Fredholm module $x \diamond \widetilde{x} = (\rho \diamond \widetilde{\rho}, H \diamond \widetilde{H}, F \diamond \widetilde{F})$ over A is well defined by formulas* (3).

*The subspace JH, as well as the subspaces J_1H and J_2H considered below, is closed. This is a special case of the general assertion that the subspace BH of a Hilbert space H equipped with a representation of a C^*-algebra B is closed (see [6, pp. 25–26, Sec. 1.9.17]).

**In specific examples, some of the subspaces H_0, H_1, and H_2 may prove to be trivial (zero). Our argument remains valid in this case, but the result is not of much interest.

***From now on, for an arbitrary $\varphi \in A$, by $\varphi_1 \in J_1$ and $\varphi_2 \in J_2$ we denote arbitrary elements such that $\varphi = \varphi_1 + \varphi_2$.

Proof. In terms of the matrix in (2), the condition $F^2 \approx 1$ becomes*

$$a^2 + bb^* \approx 1, \quad ab + bc \approx 0, \quad cd + de \approx 0, \quad d^*d + e^2 \approx 1, \qquad bd \approx 0, \tag{4}$$

$$\varphi b^*b \sim \varphi_1(1 - c^2), \quad \varphi dd^* \sim \varphi_2(1 - c^2) \qquad \text{for any } \varphi \in A, \tag{5}$$

and the last condition in (4) is satisfied automatically ($\varphi bd = (\varphi_1 b)d \sim (b\varphi_1)d = b(\varphi_1 d) = 0$), while condition (5) follows from the fact that $b^*b + dd^* + c^2 \approx 1$. Similar relations hold for $\widetilde{F}$. To prove the proposition, it suffices to verify that $(F \diamond \widetilde{F})^2 \sim 1$. (The other conditions in (1) obviously hold for $x \diamond \widetilde{x}$.) The verification, after squaring the matrix, is reduced to routine calculations using the relation $c \approx \widetilde{c}$ and also relations (4)–(5) for F and $\widetilde{F}$. For example, for the entry in the second line and second row, we obtain $\varphi((F \diamond \widetilde{F})^2)_{22} = \varphi(b^*b + c^2 + \widetilde{d}\widetilde{d}^*) \sim \varphi_1(1 - c^2) + \varphi c^2 + \varphi_2(1 - c^2) = \varphi 1$, $\varphi \in A$. □

The Fredholm module $\widetilde{x} \diamond x$ is defined in a similar way.

4. Relative index theorem. Now we are in a position to state our main result.

Theorem. *Under the assumptions of Sec.* 3, *one has*

$$[x \diamond \widetilde{x}] - [x] = [\widetilde{x}] - [\widetilde{x} \diamond x] \tag{6}$$

in the K-homology of A, where $[x] \in K^*(A)$ is the element defined by the Fredholm module x.

Identity (6) means that the difference of K-homology classes resulting from the nonagreement of Fredholm modules over the ideal J_2 is independent of the structure of these modules over the ideal J_1 (where they agree).

Remark. As far as the author is aware, the result is new not only for a noncommutative algebra but even for a commutative algebra A in that relation (6) is established for elements of the K-homology group rather than for the indices of the operators in question. (Note, however, that this was essentially done “behind the scenes” in [5] for the case in which A is a function algebra on a complete noncompact Riemannian manifold and the Fredholm modules correspond to some Dirac type operators.) The classical relative index theorems can be obtained from our result if one assumes that A is a unital function algebra: it suffices to use the homomorphism ind: $K^0(A) \longrightarrow K^0(\mathbb{C}) \simeq \mathbb{Z}$ corresponding to the natural embedding of $\mathbb{C}$ in A. Thus, the theorem stated above can be viewed as a natural generalization of relative index theorems in the framework of noncommutative geometry. Note also that a similar theorem holds in Kasparov’s KK-theory. This theorem will be published elsewhere.

Outline of the proof. It suffices to deform the Fredholm module $x \oplus \widetilde{x}$ to a module that is unitarily equivalent to the module $(x \diamond \widetilde{x}) \oplus (\widetilde{x} \diamond x)$. The homotopy is given by the family of Fredholm modules $(\rho \oplus \widetilde{\rho}, H \oplus \widetilde{H}, \mathscr{F}_t)$, $t \in [0, \pi/2]$, where the operator $\mathscr{F}_t$ is specified in the direct sum decomposition $H \oplus \widetilde{H} = H_1 \oplus H_0 \oplus H_2 \oplus \widetilde{H}_1 \oplus H_0 \oplus \widetilde{H}_2$ by the 6×6 block matrix

$$\mathscr{F}_t = \begin{pmatrix} a & b & 0 & 0 & 0 & 0 \\ b^* & c & d\cos t & 0 & 0 & -\widetilde{d}\sin t \\ 0 & d^*\cos t & e & 0 & d^*\sin t & 0 \\ 0 & 0 & 0 & \widetilde{a} & \widetilde{b} & 0 \\ 0 & 0 & d\sin t & \widetilde{b}^* & c & \widetilde{d}\cos t \\ 0 & -\widetilde{d}^*\sin t & 0 & 0 & \widetilde{d}^*\cos t & \widetilde{e} \end{pmatrix}. \tag{7}$$

*Here 1 stands for the identity operators on relevant subspaces.

The first and second conditions in (1) are obvious for $\mathscr{F}_t$, and the third condition ($\mathscr{F}_t^2 = 1$) can be verified by routine computations. Next, $\mathscr{F}_0 = F \oplus \widetilde{F}$ and

$$\mathscr{F}_{\pi/2} = \begin{pmatrix} a & b & 0 & 0 & 0 & 0 \\ b^* & c & 0 & 0 & 0 & -\widetilde{d} \\ 0 & 0 & e & 0 & d^* & 0 \\ 0 & 0 & 0 & \widetilde{a} & \widetilde{b} & 0 \\ 0 & 0 & d & \widetilde{b}^* & c & 0 \\ 0 & -\widetilde{d}^* & 0 & 0 & 0 & \widetilde{e} \end{pmatrix} = U^* \begin{pmatrix} a & b & 0 & 0 & 0 & 0 \\ b^* & c & \widetilde{d} & 0 & 0 & 0 \\ 0 & \widetilde{d}^* & \widetilde{e} & 0 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{a} & \widetilde{b} & 0 \\ 0 & 0 & 0 & \widetilde{b}^* & c & d \\ 0 & 0 & 0 & 0 & d^* & e \end{pmatrix} U, \tag{8}$$

where the unitary operator

$$U\colon H \oplus \widetilde{H} \equiv H_1 \oplus H_0 \oplus H_2 \oplus \widetilde{H}_1 \oplus H_0 \oplus \widetilde{H}_2 \longrightarrow H_1 \oplus H_0 \oplus \widetilde{H}_2 \oplus \widetilde{H}_1 \oplus H_0 \oplus H_2 \equiv (H \diamond \widetilde{H}) \oplus (\widetilde{H} \diamond H)$$

interchanges the third and sixth components and then multiplies the third component by -1. Thus, $\mathscr{F}_{\pi/2} = U^*((F \diamond \widetilde{F}) \oplus (\widetilde{F} \diamond F))U$, as desired. □

5. Example. Let $\Gamma \subset O(n,\mathbb{R})$ be a discrete subgroup of polynomial growth, and let $A = C_0^+(\mathbb{R}^n) \rtimes \Gamma$ be the C^*-crossed product of the unitization $C_0^+(\mathbb{R}^n)$ of the algebra $C_0(\mathbb{R}^n)$ by the group Γ (whose action on $C_0^+(\mathbb{R}^n)$ is induced by its standard action on $\mathbb{R}^n$). Next, let $g \in O(n,\mathbb{R})$ be a given element commuting with elements of Γ. Consider a nonlocal Callias type operator* in $\mathbb{R}^n$ defined as the crossed product $D = \not{\partial} \# a(r) T_g$, where $\not{\partial}$ is the Dirac operator, $a(r)$ is a smooth function of $r = \sqrt{x_1^2 + \cdots + x_n^2}$ equal to a nonzero constant in a neighborhood of infinity, and T_g is a bundle isomorphism over $g\colon \mathbb{R}^n \to \mathbb{R}^n$ commuting with the action of Γ. The unbounded operator D is Fredholm in L^2, almost commutes with the action of A, and hence defines an element $[D] \in K^*(A)$. Take two copies of the space $\mathbb{R}^n$ with the operator D and carry out the following cutting and pasting operation: cut each of the copies of $\mathbb{R}^n$ along the sphere S^{n-1}, paste the two balls together along the boundary to obtain the sphere S^n, and paste the remaining two noncompact parts together to form the cylinder $S^{n-1} \times \mathbb{R}$. The algebra A acts in a natural way on the sphere as well as on the cylinder, and the two copies of D produce an operator $\mathscr{D}$, the double of D, on the sphere S^n and an operator $\mathscr{D}_0$ on the cylinder. An application of the theorem shows that $\tau^*[D] + [D] = [\mathscr{D}] + [\mathscr{D}_0]$ in $K^*(A)$, where $\tau^*\colon K^*(A) \to K^*(A)$ is the involution induced by the mapping $\tau\colon x \mapsto x/r^2$ of the one-point compactification of $\mathbb{R}^n$ into itself.

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*For classical Callias type operators, e.g., see [5].