Functional Analysis and Its Applications, Vol. 48, No. 4, pp. 295–297, 2014
Translated from Funktsional′nyi Analiz i Ego Prilozheniya, Vol. 48, No. 4, pp. 74–77, 2014

Two-Dimensional von Neumann–Wigner Potentials with a Multiple Positive Eigenvalue*

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Received August 2, 2013

Abstract. By the Moutard transformation method we construct two-dimensional Schrödinger operators with real smooth potentials decaying at infinity and having a multiple positive eigenvalue. These potentials are rational functions of spatial variables and their sines and cosines.

Key words: two-dimensional Schrödinger operator, Moutard transformation, positive eigenvalues.

Let H be a Schrödinger operator

$$H = -\Delta + U$$

with potential $U(x)$ on $\mathbb{R}^N$ decaying at infinity. The potential U is called a *von Neumann–Wigner potential* if H has a positive eigenvalue with an eigenfunction in $L_2(\mathbb{R}^N)$, i.e., there is a point of its discrete spectrum which is embedded in the absolutely continuous spectrum.

The first example of such a potential was constructed by von Neumann and Wigner [1]. They found a three-dimensional rotation-symmetric nonsingular potential $U(r)$ depending only on the distance r from the origin with the following asymptotic behavior (a computational mistake made in [1] and reported later is corrected here):

$$U(r) = -\frac{8\sin 2r}{r} + O(r^{-2}) \quad \text{as } r = |x| \to \infty,\ x \in \mathbb{R}^3.$$

The Schrödinger operators with $U(x) = o(1/|x|)$ as $x \to \infty$ have no positive eigenvalues [2].

In the present note we explicitly construct multiparameter families of two-dimensional potentials which decay as $1/|x|$ and have a multiple positive eigenvalue. To our knowledge, these are the first examples of such potentials.

We use the method first applied in [3] and [4] to construct two-dimensional Schrödinger operators with fast decaying potentials and multidimensional kernels. This method is based on the Moutard transformation of two-dimensional Schrödinger operators and is described as follows. Let ω be a formal solution of the equation

$$H\omega = (-\Delta + U(x,y))\omega = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{1}$$

The Moutard transformation corresponding to H and ω gives a new Schrödinger operator

$$\widetilde{H} = -\Delta + \widetilde{U}, \qquad \widetilde{U} = U - 2\Delta \log \omega,$$

such that if φ satisfies $H\varphi = 0$, then a function θ determined up to a summand const/ω by the consistent system

$$(\omega\theta)_x = -\omega^2(\varphi/\omega)_y, \qquad (\omega\theta)_y = \omega^2(\varphi/\omega)_x, \tag{2}$$

satisfies the equation

$$\widetilde{H}\theta = 0.$$

*The work was partially supported by Federal Targeted Program No. 14.A18.21.0866 of the Ministry of Education and Science of Russian Federation (R. G. N.), the interdisciplinary project "Geometrical and algebraic methods for finding explicit solutions to equations of mathematical physics and continuum mechanics" of Russian Academy of Sciences, Siberian Branch (I. A. T. and S. P. Ts.), by grant 1431/GF of the Ministry of Education and Science of Republic of Kazakhstan (I. A. T.), and by the 2014 research government order No. 1.1462.2014/K of the Ministry of Education and Science for Siberian Federal University (S. P. Ts.).

So, there are maps

$$U \to M_\omega(U) = U - 2\Delta \log \omega, \qquad \varphi \to S_\omega(\varphi) = \{\theta + C/\omega,\ C \in \mathbb{C}\}.$$

Let us consider an operator H and a pair of solutions to Eq. (1), ω_1 and ω_2. For every $\theta_1 \in S_{\omega_1}(\omega_2)$, there is a function (the result of a double iteration of the Moutard transformation)

$$\widehat{U} = M_{\theta_1} M_{\omega_1}(U) - U = -2\Delta \log(\theta_1 \omega_1).$$

The result of this iteration depends on the choice of $\theta_1 \in S_{\omega_1}(\omega_2)$, i.e., on the integration constant C in (2). Moreover, the functions

$$\psi_1 = \frac{1}{\theta_1} \quad \text{and} \quad \psi_2 = \frac{\omega_2}{\omega_1 \theta_1}$$

satisfy the equation

$$(-\Delta + M_{\theta_1} M_{\omega_1}(U))\psi = 0.$$

Unlike in [3] and [4], where such a double iteration was applied to the case $U = 0$, we apply it to the constant potential $U = -k^2$, $k \in \mathbb{R}$. Therefore, ω_1 and ω_2 satisfy the Helmholtz equation

$$-\Delta\omega = k^2\omega.$$

A large set of solutions to this equation is given by functions of the form

$$\operatorname{Re}\left[\frac{\partial^m}{\partial\lambda^m} \exp\left(i\,\frac{k}{2}\left(\lambda z + \frac{\bar{z}}{\lambda}\right)\right)\right], \qquad z = x + iy,\ \lambda \in \mathbb{C},\ m = 0, 1, 2, \dots, \tag{3}$$

and their linear combinations.

For simplicity, we consider the case $k^2 = 1$ and demonstrate the method by an explicit example.

Theorem 1. *If* $U = -1$ *and*

$$\omega_1 = x^2 \cos y - y \sin y + y^2 \sin x + x \cos x, \quad \omega_2 = 4(y \cos x + x \sin y), \qquad x, y \in \mathbb{R},$$

then the two-dimensional potential $\widehat{U}$ *takes the form*

$$\widehat{U} = P/Q^2,$$

where

$$\begin{aligned} Q = \omega_1\theta_1 = {} & -x^4 - y^4 - 4x^2 y \sin x \sin y + x^2(-8\cos y \sin x - 2\sin^2 y - 1) \\ & + 4xy^2 \cos x \cos y - 16xy \cos x \sin y + 2x \cos x(-8\cos y - \sin x) \\ & + y^2(-8\cos y \sin x + 2\sin^2 x - 3) + 2y \sin y(\cos y + 8\sin x) \\ & + 16\cos y \sin x + \sin^2 x - \sin^2 y + 4C + 1, \end{aligned}$$

and P *is a polynomial in* x *and* y *and in the sines and cosines of* x *and* y:

$$P = 16(x^6 y \sin x \sin y - x^5 y^2 \cos x \cos y + x^2 y^5 \sin x \sin y - xy^6 \cos x \cos y) + \dots$$

(*the dots denote the lower order terms in* x *and* y). *The functions* ψ_1 *and* ψ_2 *take the forms*

$$\psi_1 = \omega_1/Q \quad \text{and} \quad \psi_2 = \omega_2/Q$$

and satisfy the equation

$$\widehat{H}\psi = \psi, \qquad \widehat{H} = -\Delta + \widehat{U}.$$

If C *is negative and* $|C|$ *is sufficiently large, then* Q *has no zeros and, therefore, the potential* $\widehat{U}$ *and the functions* ψ_1 *and* ψ_2 *are smooth. Moreover,*

$$\widehat{U} = O(r^{-1}), \quad \psi_1 = O(r^{-2}), \quad \psi_2 = O(r^{-3}) \quad \text{as } r = \sqrt{x^2 + y^2} \to \infty. \tag{4}$$

Therefore, ψ_1 *and* ψ_2 *lie in* $L_2(\mathbb{R}^2)$ *and are linearly independent eigenfunctions of the operator* $\widehat{H} = -\Delta + \widehat{U}$ *with eigenvalue* $E = 1$.

Using various linear combinations of the solutions (3), one can easily construct multiparameter families of similar two-dimensional potentials $\widehat{U}$ with asymptotics (4) and solutions ψ_i at the energy level $E = k^2$ and even increase the decay rate of the eigenfunctions ψ_i at infinity. However, the decay rate of potentials cannot be increased due to the already mentioned Kato theorem [2].

We believe that, by applying multiple iterations, one may obtain such potentials with positive eigenvalues of higher multiplicity.

Acknowledgement. This work was done during the visit of one of the authors (I. A. T.) to Centre de Mathématique Appliquées of Ecole Polytechnique in July, 2013.

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