Functional Analysis and Its Applications, Vol. 47, No. 1, pp. 67–71, 2013
Translated from Funktsional′nyi Analiz i Ego Prilozheniya, Vol. 47, No. 1, pp. 82–87, 2013

On Pairs of Quadratically Related Operators*

V. L. Ostrovskyi and Yu. S. Samoilenko

Received February 28, 2011

Abstract. The problem of describing, up to similarity, pairs of quadratically related operators on a finite-dimensional complex linear space is studied.

Key words: pair of operators, quadratic relation.

1. Introduction. Consider a finite-dimensional linear space $\mathscr{V}$ over the field $\mathbb{C}$ of complex numbers and a set $(\mathscr{A}_1,\dots,\mathscr{A}_n)$ of linear operators on $\mathscr{V}$. Recall that the set $(\mathscr{A}_k)_{k=1}^n$ is said to be indecomposable if $\mathscr{V}$ cannot be decomposed into the direct sum of nontrivial subspaces $\mathscr{V}_1$ and $\mathscr{V}_2$ invariant with respect to all operators in this set. Sets $(\mathscr{A}_k)_{k=1}^n$ of operators on $\mathscr{V}$ and $(\widetilde{\mathscr{A}_k})_{k=1}^n$ of operators on $\widetilde{\mathscr{V}}$ are similar if there exists an invertible linear operator $\mathscr{C}\colon \mathscr{V}\to\widetilde{\mathscr{V}}$ such that $\widetilde{\mathscr{A}_k}=\mathscr{C}\mathscr{A}_k\mathscr{C}^{-1}$ for $k=1,\dots,n$.

In the case of one operator, the problem of classifying indecomposable operators up to similarity reduces to constructing a Jordan basis in the space $\mathscr{V}$; in this case, the Jordan cells exhaust all indecomposable operators. The problem of classifying indecomposable sets of linear operators up to similarity turns out to be very difficult already in the case $n=2$ (see, e.g., [1] and [2]). It is therefore natural to try to impose a condition on sets of operators under which this problem may be solvable. The simplest such condition is commutativity, but, as shown in [3] and [4], the problem of describing, up to similarity, pairs of commuting operators still remains difficult (even though there exists a basis in which the matrices of commuting operators are simultaneously triangular). Even under the additional constraint $\mathscr{A}^2=\mathscr{B}^3=\mathscr{A}\mathscr{B}^2=0$, the problem of describing, up to similarity, pairs of commuting operators $\mathscr{A}$ and $\mathscr{B}$ resists solving (that is, is wild [5]).

This note studies the problem of describing, up to similarity, the indecomposable pairs of operators $(\mathscr{A},\mathscr{B})$ satisfying a quadratic relation of the general form

$$P_2(\mathscr{A},\mathscr{B})=\alpha\mathscr{A}^2+\beta\mathscr{A}\mathscr{B}+\gamma\mathscr{B}\mathscr{A}+\delta\mathscr{B}^2+\epsilon\mathscr{A}+\zeta\mathscr{B}+\chi\mathscr{I}=0 \tag{i}$$

(here $\alpha,\beta,\gamma,\delta,\epsilon,\zeta,\chi\in\mathbb{C}$). We prove that, by means of nondegenerate affine changes of the generators, this quadratic relation can be reduced to one of nine forms (see Proposition 2); for each of the relations thus obtained, we either describe all indecomposable pairs up to similarity or show that the problem of classifying pairs of operators satisfying the relation under consideration is wild (see Theorem 2), i.e., contains the problem of describing, up to similarity, all pairs of operators subject to no constraints (for more details on wild problems, see [1] and [2]).

Remark 1. In this paper we consider *pairs of operators only on finite-dimensional linear spaces.* In the literature various categories of sets of operators on infinite-dimensional spaces satisfying various relations have also been studied.

Remark 2. The authors have studied the problem of describing, up to unitary equivalence, the pairs of self-adjoint operators $(\mathscr{A},\mathscr{B})$ on a Hilbert space satisfying the quadratic relation

$$P_2(\mathscr{A},\mathscr{B})=\alpha\mathscr{A}^2+\beta\{\mathscr{A},\mathscr{B}\}+i\gamma[\mathscr{A},\mathscr{B}]+\delta\mathscr{B}^2+\epsilon\mathscr{A}+\zeta\mathscr{B}+\chi\mathscr{I}=0$$

with real coefficients α, β, γ, δ, ϵ, ζ, and χ in the cases of finite-dimensional, bounded, and unbounded operators [6]–[8].

*This work was supported in part by DFG grant no. SCHM1009/4-1 and by SFBR-RFBR joint grant no. F40.01/008.

2. Examples. 1. The problem of classifying pairs of operators $(\mathscr{A}^2 = 0, \mathscr{B})$ is wild. Indeed, the classification of pairs of operators

$$\mathscr{A} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

on the space $V \dotplus V$ is equivalent to the classification of pairs (A, B) of operators on V.

2. The problem of classifying pairs of operators $(\mathscr{A}^2 = \mathscr{I}, \mathscr{B})$ is wild. Indeed, the classification of pairs of operators

$$\mathscr{A} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} A & I \\ 0 & B \end{pmatrix}$$

on the space $V \dotplus V$ is equivalent to the classification of pairs (A, B) of operators on V.

3. The problem of classifying pairs of operators $(\mathscr{A}, \mathscr{B} = \mathscr{A}^2)$ is not wild. The structure of such a pair is determined by the normal form of the matrix of the operator $\mathscr{A}$.

4. The problem of classifying pairs of q-commuting operators $\mathscr{A}$ and $\mathscr{B}$ (i.e., pairs of operators related by $\mathscr{A}\mathscr{B} = q\mathscr{B}\mathscr{A}$, where $q \in \mathbb{C}$) is wild. Given any pair of operators A and B on V, we construct the operators

$$\mathscr{A} = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & A \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathscr{B} = \begin{pmatrix} 0 & qI & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & 0 & A \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

on the direct sum $\mathscr{V} = V \dotplus V \dotplus V \dotplus V$. Similarly, given a pair $\widetilde{A}, \widetilde{B} \colon \widetilde{V} \to \widetilde{V}$, we construct operators $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$ on the direct sum of four copies of the space $\widetilde{V}$.

Proposition 1 (see [3] and [9]). *A pair $(\mathscr{A}, \mathscr{B})$ of operators on $\mathscr{V}$ is indecomposable if and only if so is the pair (A, B) of operators on V. A pair $(\mathscr{A}, \mathscr{B})$ is similar to a pair $(\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})$ if and only if the pair (A, B) is similar to $(\widetilde{A}, \widetilde{B})$.*

Remark 3. In the construction described above both operators $\mathscr{A}$ and $\mathscr{B}$ are nilpotent and $\mathscr{A}^3 = \mathscr{B}^3 = 0$.

Let us show that the classification problem is also wild if $\mathscr{A}$ is invertible and $\mathscr{B}^2 = 0$. Consider nilpotent operators A_1 and A_2 on spaces V_1 and V_2, respectively. We set $\mathscr{V} = V_1 \dotplus V_2$,

$$\mathscr{A} = \begin{pmatrix} I + A_1 & 0 \\ 0 & -I + A_2 \end{pmatrix}, \qquad \text{and} \qquad \mathscr{B} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}. \tag{ii}$$

Since the spectra of the operators $I + A_1$ and $-I + A_2$ are disjoint, it follows that any idempotent operator $\mathscr{P}$ commuting with $\mathscr{A}$ has the form

$$\mathscr{P} = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix},$$

where P_1 and P_2 are idempotent operators on the spaces V_1 and V_2, respectively. In this case, the reduction problem for operators $\mathscr{A}$ and $\mathscr{B}$ of the form (ii) reduces to the reduction of a triple of operators A_1, A_2, and B on two spaces V_1 and V_2 such that $A_1 \colon V_1 \to V_1$, $A_2 \colon V_2 \to V_2$, $B \colon V_2 \to V_1$, $A_1 B = B A_2$, and A_1 and A_2 are nilpotent. The wildness of this problem was proved in [10] and [11].

5. Consider the classification problem for pairs of operators satisfying the relation q-CCR, that is, such that $\mathscr{A}\mathscr{B} - q\mathscr{B}\mathscr{A} = \mathscr{I}$, where $q \in \mathbb{C}$.

First, consider the cases $q = 0$ and $q = 1$.

For $q = 0$, we have $\mathscr{A}\mathscr{B} = \mathscr{I}$; therefore, the operators under consideration are nonsingular and $\mathscr{B} = \mathscr{A}^{-1}$. The corresponding classification problem is not wild: the structure of any such pair is determined by the normal form of the matrix of $\mathscr{A}$.

For $q = 1$, we have $[\mathscr{A}, \mathscr{B}] = \mathscr{I}$. Since the trace of the commutator of any two matrices vanishes, there exist no such pairs on a finite-dimensional space (recall that we consider only operators on finite-dimensional spaces in this note).

Now, suppose that $q^2 \neq q$. In this case, the problem of classifying pairs of operators satisfying the condition $\mathscr{A}\mathscr{B} - q\mathscr{B}\mathscr{A} = \mathscr{I}$ is wild (see, e.g., [12]). Indeed, consider the pairs satisfying the additional condition $\mathscr{C} = \mathscr{A}\mathscr{B} = (1-q)^{-1}\mathscr{I} + \mathscr{N}$, where $\mathscr{N}$ is a nilpotent operator. Then we have the relation $\mathscr{A}\mathscr{N} = q\mathscr{N}\mathscr{A}$, where $\mathscr{A}$ is an invertible operator and $\mathscr{N}$ is a nilpotent operator; the wildness of this problem was proved above in Remark 3.

The subsequent examples 6–9 deal with relations of the form $[\mathscr{A}, \mathscr{B}] = P(\mathscr{A})$, where $P(\,\cdot\,)$ is a nonconstant polynomial.

6. The problem of classifying pairs of operators $(\mathscr{A}, \mathscr{B})$ satisfying the relation $[\mathscr{A}, \mathscr{B}] = \mathscr{A}$ is wild. Indeed, let B_1 and B_2 be nilpotent operators on spaces V_1 and V_2, respectively. We set $\mathscr{V} = V_1 \dotplus V_2$,

$$\mathscr{A} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad \mathscr{B} = \begin{pmatrix} B_1 & 0 \\ 0 & I + B_2 \end{pmatrix}.$$

The relation $[\mathscr{A}, \mathscr{B}] = \mathscr{A}$ is equivalent to $B_1 A = A B_2$. The rest of the proof is similar to that in Remark 3.

7. The problem of classifying pairs of operators $(\mathscr{A}, \mathscr{B})$ satisfying the relation $[\mathscr{A}, \mathscr{B}] = \mathscr{A}^2$ is wild. Indeed, imposing the additional condition $\mathscr{A}^2 = 0$, we obtain the problem of classifying pairs of commuting operators satisfying the relation $\mathscr{A}^2 = 0$, which is wild [5].

8. The problem of classifying pairs of operators $(\mathscr{A}, \mathscr{B})$ satisfying the relation $[\mathscr{A}, \mathscr{B}] = \mathscr{A}^2 + \mathscr{I}$ is wild. The proof is similar to that given in Example 6. We take nilpotent operators B_1 and B_2 on V_1 and V_2 and set $\mathscr{V} = V_1 \dotplus V_2$,

$$\mathscr{A} = \begin{pmatrix} iI & A \\ 0 & iI \end{pmatrix}, \qquad \text{and} \qquad \mathscr{B} = \begin{pmatrix} -iI + B_1 & 0 \\ 0 & iI + B_2 \end{pmatrix}.$$

Then $[\mathscr{A}, \mathscr{B}] = \mathscr{A}^2$ is equivalent to $B_1 A = A B_2$, and the wildness of the problem is proved by an argument similar to that used in Remark 3.

9. The problem of classifying pairs of operators $(\mathscr{A}, \mathscr{B})$ satisfying the condition $q[\mathscr{A}, \mathscr{B}] = \mathscr{A}^2 + \mathscr{B}$ with $q \neq 0$ is wild. Indeed, let $\mathscr{B}' = -q\mathscr{A}$, and let $\mathscr{A}' = \mathscr{A}^2 + \mathscr{B}$. Then the problem of classifying pairs of operators $(\mathscr{A}, \mathscr{B})$ satisfying the above relation is equivalent to the problem of classifying pairs of operators $(\mathscr{A}', \mathscr{B}')$ for which $q[\mathscr{A}, \mathscr{B}] = \mathscr{A}^2 + \mathscr{B}$ implies $[\mathscr{A}', \mathscr{B}'] = \mathscr{A}'$.

The arguments used in Examples 6–8 remain valid in a more general situation.

Theorem 1. *The problem of classifying pairs of operators $\mathscr{A}$ and $\mathscr{B}$ on a finite-dimensional linear space $\mathscr{V}$ which satisfy the relation $[\mathscr{A}, \mathscr{B}] = P(\mathscr{A})$, where $P(\,\cdot\,)$ is a nonconstant polynomial, is wild.*

Proof. First, suppose that the polynomial $P(\,\cdot\,)$ has a multiple root, which we denote by λ. Then this polynomial can be represented in the form $P(x) = (x - \lambda)^2 Q(x)$, where $Q(x)$ is some other polynomial, and the operators $\mathscr{A}$ and $\mathscr{B}$ must satisfy the relation $[\mathscr{A}, \mathscr{B}] = (\mathscr{A} - \lambda I)^2 Q(\mathscr{A})$. Let us introduce the operator $\mathscr{A}' = \mathscr{A} - \lambda\mathscr{I}$. Then $[\mathscr{A}', \mathscr{B}] = \mathscr{A}'^2 Q(\mathscr{A}' + \lambda I)$. Assuming that $\mathscr{A}'^2 = 0$, we obtain $[\mathscr{A}', \mathscr{B}] = 0$, and the wildness of the problem follows from that of the problem for pairs of commuting operators A and B satisfying $\mathscr{A}^2 = \mathscr{B}^3 = 0$ [5].

Now, suppose that λ is a simple root of the polynomial $P(\,\cdot\,)$. Consider a direct sum decomposition $\mathscr{V} = V_1 \dotplus V_2$ and block matrices $\mathscr{A}$ and $\mathscr{B}$ of the form

$$\mathscr{A} = \begin{pmatrix} \lambda I & A \\ 0 & \lambda I \end{pmatrix} \qquad \text{and} \qquad \mathscr{B} = \begin{pmatrix} B_1 + \mu_1 I & 0 \\ 0 & B_2 + \mu_2 I \end{pmatrix},$$

where B_1 and B_2 are nilpotent operators on the spaces V_1 and V_2, respectively. Since λ is a root of $P(\,\cdot\,)$, we have

$$P(\mathscr{A}) = \begin{pmatrix} 0 & P'(\lambda)A \\ 0 & 0 \end{pmatrix},$$

and the condition $[\mathscr{A},\mathscr{B}] = P(\mathscr{A})$ reduces to $AB_2 - B_1A + (\mu_2 - \mu_1)A = P'(\lambda)A$. Choose μ_1 and μ_2 so that $\mu_2 - \mu_1 = P'(\lambda)$; since λ is a simple root, it follows that $P'(\lambda) \neq 0$ and $\mu_2 \neq \mu_1$. Thus, any idempotent operator $\mathscr{J}$ commuting with $\mathscr{B}$ has the following block form in the decomposition under consideration:

$$\mathscr{J} = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix},$$

where J_1 and J_2 are idempotent operators on V_1 and V_2, respectively. The reduction problem for operators $\mathscr{A}$ and $\mathscr{B}$ of this form reduces by an argument similar to that used in Remark 3 to the wild problem of describing triples of operators A, B_1, and B_2 on two spaces V_1 and V_2 such that $B_1\colon V_1 \to V_1$, $B_2\colon V_2 \to V_2$, $A\colon V_2 \to V_1$, $AB_2 = B_1A$, and B_1 and B_2 are nilpotent. ☐

3. Main theorem.

Proposition 2. *A nondegenerate affine change of variables reduces relation* (i) *to one of the nine relations*

$$(1)\ \mathscr{A}^2 = 0, \quad (2)\ \mathscr{A}^2 = \mathscr{I}, \quad (3)\ \mathscr{A}^2 = \mathscr{B},$$
$$(4)\ \mathscr{A}\mathscr{B} = q\mathscr{B}\mathscr{A}, \quad (5)\ \mathscr{A}\mathscr{B} - q\mathscr{B}\mathscr{A} = \mathscr{I}, \quad (6)\ [\mathscr{A},\mathscr{B}] = \mathscr{A},$$
$$(7)\ [\mathscr{A},\mathscr{B}] = \mathscr{A}^2, \quad (8)\ [\mathscr{A},\mathscr{B}] = \mathscr{A}^2 + \mathscr{I}, \quad (9)\ q[\mathscr{A},\mathscr{B}] = \mathscr{A}^2 + \mathscr{B},$$

where $q \in \mathbb{C}$.

Proof. We can rewrite relation (i) in the form

$$\alpha\mathscr{A}^2 + \beta'\{\mathscr{A},\mathscr{B}\} + \delta\mathscr{B}^2 + \epsilon\mathscr{A} + \zeta\mathscr{B} + \chi\mathscr{I} = \gamma'[\mathscr{A},\mathscr{B}],$$

where $\beta' = (\beta+\gamma)/2$ and $\gamma' = (\gamma-\beta)/2$ (as usual, $\{\mathscr{A},\mathscr{B}\} = \mathscr{A}\mathscr{B}+\mathscr{B}\mathscr{A}$ and $[\mathscr{A},\mathscr{B}] = \mathscr{A}\mathscr{B}-\mathscr{B}\mathscr{A}$).

The rest of the proof is based on an argument similar to the reduction of equations of quadratic curves to canonical form by using affine changes of variables and on the consideration of all possible constraints on the coefficients. ☐

Remark 4. Although relations (1)–(9) cannot be transformed into each other by affine changes, the quadratic change $\mathscr{B}' = -q\mathscr{A}$, $\mathscr{A}' = \mathscr{A}^2 + \mathscr{B}$ constructed in studying relation (9) establishes an isomorphism between the algebras generated by relations (6) and (9).

The reduction problem for a quadratically related pair of operators is usually wild. To be more precise, the following theorem is valid.

Theorem 2. *The problem of describing, up to similarity, indecomposable pairs* $(\mathscr{A},\mathscr{B})$ *of linear operators on a finite-dimensional complex space satisfying one of relations* (1)–(9) *is wild except in the following cases*:

the operators are related by (3); *in this case, the classification of pairs satisfying the relation* $\mathscr{A}^2 = \mathscr{B}$ *reduces to that of the Jordan forms of matrices of arbitrary operators* $\mathscr{A}$;

the operators are related by (5) *with* $q = 0$; *the classification of pairs satisfying the relation* $\mathscr{A}\mathscr{B} = \mathscr{I}$ *reduces to that of the Jordan forms of matrices of invertible operators* $\mathscr{B}$;

the operators are related by (5) *with* $q = 1$; *there exist no pairs satisfying the relation* $[\mathscr{A},\mathscr{B}] = \mathscr{I}$ (*see Example* 5).

Corollary. *Explicit conditions on the coefficients in relation* (i) *which make it possible to determine whether the corresponding problem is wild are as follows. The problem of describing, up to similarity, indecomposable pairs of linear operators on a finite-dimensional complex space which satisfy relation* (i) *is wild except in the following cases*:

- $(\beta+\gamma)^2 = 4\alpha\delta$, $\gamma = \beta$, *and* $\alpha\zeta - \beta\epsilon \neq 0$; *in this case, relation* (i) *reduces to* (3);
- $(\beta+\gamma)^2 \neq 4\alpha\delta$, $\beta\gamma = \alpha\delta$, *and* $\epsilon^2\delta - \epsilon\zeta(\beta+\gamma) + \zeta^2\alpha + 4\chi((\beta+\gamma)^2 - 4\alpha\delta) \neq 0$; *in this case, relation* (i) *reduces to* (5) *with* $q = 0$;
- $\alpha = \delta = \epsilon = \zeta = 0$, $\gamma = -\beta$, *and* $\chi \neq 0$; *in this case, relation* (i) *reduces to* (5) *with* $q = 1$.

Remark 5. The problem of classifying, up to similarity, pairs of operators related by two quadratic relations was considered by various authors. For example, in [13], the pairs of operators satisfying the relations $\mathscr{A}\mathscr{B} = \mathscr{B}\mathscr{A} = 0$ were described up to similarity.

The authors sincerely thank V. M. Bondarenko, Yu. A. Drozd, S. A. Kruglyak, and the referees for advice and useful comments.

References

[1] P. Donovan and M. R. Freislich, The Representation Theory of Finite Graphs and Associated Algebras, Carleton Math. Lecture Notes, vol. 5, Carleton Univ., Ottawa, 1973.
[2] Yu. A. Drozd, in: Representations and Quadratic Forms, Inst. Mat. Akad. Nauk Ukr. SSR, Kiev, 1979, 39–74.
[3] S. A. Kruglyak, Dokl. Akad. Nauk SSSR, **153**:6 (1963), 1253–1256.
[4] I. M. Gelfand and V. A. Ponomarev, Funkts. Anal. Prilozhen., **3**:4 (1969), 81–82; English transl.: Functional Anal. Appl., **3**:4 (1969), 325–326.
[5] Yu. A. Drozd, Funkts. Anal. Prilozhen., **6**:4 (1972), 41–43; English transl.: Functional Anal. Appl., **6**:4 (1972), 286–288.
[6] V. L. Ostrovskyi and Yu. S. Samoilenko, "Introduction to the theory of representations of finitely presented ∗-algebras, I: Representations by bounded operators," Rev. Math. Math. Phys., **11**:1 (1999).
[7] V. L. Ostrovskii and Yu. S. Samoilenko, Zap. Nauchn. Sem. LOMI, **172** (1989), 121–129; English transl.: J. Soviet Math., **59**:5 (1992), 1107–1113.
[8] V. L. Ostrovskyĭ and Yu. S. Samoĭlenko, Seminar Sophus Lie, **3** (1993), 185–218.
[9] M. V. Akhramovich and M. A. Muratov, Tavricheskii Vestn. Inf. Mat., 2010, No. 2, 17–25.
[10] M. Hoshino and J. Miyachi, Tsukuba J. Math., **12**:1 (1988), 65–93.
[11] Y. Han, J. Algebra, **247**:1 (2002), 57–77.
[12] V. Bavula and V. Bekkert, Comm. Algebra, **28**:11 (2000), 5067–5100.
[13] I. M. Gelfand and V. A. Ponomarev, Uspekhi Mat. Nauk, **23**:2 (1968), 3–59; English transl.: Russian Math. Surveys, **23**:2 (1968), 1–58.

Institute of Mathematics, National Academy of Sciences of Ukraine
e-mail: vo@imath.kiev.ua

Institute of Mathematics, National Academy of Sciences of Ukraine
e-mail: yurii_sam@imath.kiev.ua

Translated by O. V. Sipacheva