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# Relative Version of the Titchmarsh Convolution Theorem*

**E. A. Gorin and D. V. Treschev**

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Abstract. We consider the algebra $C_u = C_u(\mathbb{R})$ of uniformly continuous bounded complex functions on the real line $\mathbb{R}$ with pointwise operations and sup-norm. Let $I$ be a closed ideal in $C_u$ invariant with respect to translations, and let $\mathrm{ah}_I(f)$ denote the minimal real number (if it exists) satisfying the following condition. If $\lambda > \mathrm{ah}_I(f)$, then $(\hat{f} - \hat{g})|_V = 0$ for some $g \in I$, where $V$ is a neighborhood of the point $\lambda$. The classical Titchmarsh convolution theorem is equivalent to the equality $\mathrm{ah}_I(f_1 \cdot f_2) = \mathrm{ah}_I(f_1) + \mathrm{ah}_I(f_2)$, where $I = \{0\}$. We show that, for ideals $I$ of general form, this equality does not generally hold, but $\mathrm{ah}_I(f^n) = n \cdot \mathrm{ah}_I(f)$ holds for any $I$. We present many nontrivial ideals for which the general form of the Titchmarsh theorem is true.

Key words: Titchmarsh's convolution theorem, estimation of entire functions, Banach algebra.

**1.** The classical Titchmarsh convolution theorem for a pair of compactly supported Lebesgue integrable functions on the real axis $\mathbb{R}$ says that the convex hull of the convolution support coincides with the arithmetic sum of the convex hulls of the supports of the functions. This theorem can easily be extended to Schwartz distributions over the space $S = S(\mathbb{R})$ of rapidly decreasing functions and has multidimensional versions. In a form specified below the convolution theorem is extended to functions supported on equally directed half-axes. Most of the generalizations can be reduced to the case when both functions are smooth and have compact supports (although such a reduction may obscure the simple gist of the matter).

In [1] an analysis of the final dynamics of an oscillator interacting with a thermostat required a "relative version" of Titchmarsh's theorem, where in the definition of the complement to a support local coincidence with the zero function is replaced by local membership in a certain ideal (we give the details below). Our aim is to present mathematical results of [1] in a simpler and more general form. In principle, we might use the term "singular supports," following the terminology of distribution theory related to deviations from smoothness or analyticity. However, from the algebraic viewpoint which we adopt here, it is more appropriate to talk about "relative" versions.

Titchmarsh's theorem is far from trivial, although its discrete cases (which deal, for example, with finite linear combinations of shifted Dirac $\delta$-functions) are obvious. It has numerous applications. Many various proofs and generalizations of this theorem have been found. We believe that one of the most transparent proofs is that presented in lectures by B. Ya. Levin [2, Lecture 16], although it uses not only classical (dated to the beginning of the past century) facts of the theory of analytic functions but also facts which were discovered later than the Titchmarsh theorem. With minor modifications similar methods work in the situation we are interested in, although the argument cannot be transferred word for word (because the relative version of the convolution theorem does not hold in full generality).

**2.** Let $M = M(\mathbb{R})$ be the complex Banach algebra of all complex (regular) Borel measures of finite total variation on $\mathbb{R}$ with the total variation norm. The multiplication is convolution, which is defined by

$$\int_{\mathbb{R}} g(t)\,(\mu_1 * \mu_2)(dt) = \int_{\mathbb{R}\times\mathbb{R}} g(s+t)\,\mu_1(ds)\,\mu_2(dt).$$

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In the definition of the convolution of measures, this relation must hold for all continuous functions $g$ vanishing at infinity. Therefore, it is also valid for all bounded continuous functions (a detailed discussion of this and other Banach algebras, as well as other basic facts of harmonic analysis, can be found in, e.g., [3]).

By using the standard pairing $\langle g, \mu\rangle = \int_{\mathbb{R}} g(t)\,\mu(dt)$ and the notation $(g \star \mu)(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} g(x + t)\,\mu(dt)$, it is possible to represent the definition of convolution in the form $\langle g, \mu_1 * \mu_2\rangle = \langle g\star\mu_1, \mu_2\rangle$.

We also use the notation $(g * \mu)(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} g(x - t)\,\mu(dt)$, which is compatible with all other standard notations.

The Fourier transform of a measure $\mu$ is the function on the dual axis determined by the equation

$$\widehat{\mu}(\lambda) = \int_{\mathbb{R}} e^{i\lambda t}\mu(dt).$$

Below $M_0 = M_0(\mathbb{R})$ denotes the set of measures $\mu \in M$ absolutely continuous with respect to the Lebesgue measure. It is well known (and easy to check) that $M_0$ is a closed ideal in $M$.

Each measure $\mu \in M_0$ has a density $h \in L^1(\mathbb{R})$ ($L^1(\mathbb{R})$ is the Lebesgue space of equivalence classes of integrable functions). Usually densities are identified with measures. This allows us to talk about convolutions, Fourier transforms, and other operations for $L^1$-functions (some time ago only such measures were considered, which required constantly repeating the words "almost everywhere"). We also use "densities" (and talk about the corresponding functions, usually smooth) keeping in mind, however, that, formally, these are measures. As usual, the term "function" will be used in a wider sense as well.

Any functional $\mu \to \widehat{\mu}(\lambda_0)$ determines a continuous unit-preserving homomorphism from $M$ to the field $\mathbb{C}$ of complex numbers. The kernel of such a functional is a maximal ideal, but, as is well known, such ideals do not exhaust even the Shilov boundary of the maximal ideal space. On the other hand, for $\widehat{M_0}$, there are no other homomorphisms to $\mathbb{C}$ except point ones. Moreover, $\widehat{M}$ and $\widehat{M_0}$ are locally isomorphic to each other (as algebras of functions on the dual axis).

**3.** Let $C_u = C_u(\mathbb{R})$ be the algebra of all uniformly continuous bounded complex functions on the real axis. This is a $C^*$-algebra with respect to the pointwise operations and the sup-norm. So any closed ideal in $C_u$ is symmetric. By the Gelfand–Naimark theorem $C_u$ is isometrically isomorphic to the algebra of all continuous functions on some compact set. This set is a compact (=bicompact) extension* of the real axis. We denote it by $\alpha\mathbb{R}$ (to emphasize that it is close to the Stone–Čech extension $\beta\mathbb{R}$ but is subordinate to it).

We put $(T_x f)(t) = f(x + t)$. The choice of the algebra $C_u$ instead of the algebra $C_b$ of all bounded continuous functions is explained by the fact that, for $C_u$, the representation $x \to T_x$ is strongly continuous, i.e., for any fixed $f$, the function $x \to T_x f$ is continuous.

The space $C_u$, together with the vector structure and the norm, naturally submerges into the space dual to $M_0$. Therefore, we can define the Fourier transform $\widehat{g}$ as a functional on $\widehat{M_0}$ by

$$\langle\widehat{g}, \widehat{\mu}\rangle = 2\pi\langle g, \mu\rangle.$$

The topology on $\widehat{M_0}$ is induced by $M_0$.

In distribution theory a similar definition is used. Since $S \subset M_0$, these definitions are compatible. Hence with (or without) the help of distribution theory we can discuss local properties of Fourier transforms of functions from $C_u$.

**4.** For any $\sigma > 0$, by $\mathbf{B}_\sigma$ we denote the Bernstein space, which includes all entire functions of exponential type $\leqslant \sigma$ bounded on the real axis. This is a Banach space with respect to the

*In the middle of the past century such extensions were systematically used by Yu. M. Smirnov to study the Efremovich proximity spaces by methods of general topology.

sup-norm (on $\mathbb{R}$). By the Bernstein inequality*,

$$|f(x+h) - f(x)| \leqslant 2\sin(\sigma h/2) \cdot \sup_{\mathbb{R}} |f|, \qquad 0 < \sigma h < \pi,$$

for all $f \in \mathbf{B}_\sigma$.

Let $\mathbf{B} = \bigcup_{\sigma>0} \mathbf{B}_\sigma$. Then, obviously, $\mathbf{B} \subset C_u$. Moreover, by (another) Bernstein theorem $\mathbf{B}$ is dense in $C_u$. To prove this theorem, it is convenient to consider the convolutions $f_\rho = f * v_\rho$, where $\{v_\rho\}$ is the Vallée-Poussin kernel and $\rho > 0$. The function $\widehat{v}_\rho$ is supported on the closed interval $|\lambda| \leqslant 2\rho$, equals 1 if $|\lambda| \leqslant \rho$, and is linear on the other two intervals.

Uniform continuity easily implies the uniform convergence $f_\rho \to f$ as $\rho \to \infty$. Furthermore, $\hat{f}_\rho = \hat{f}$ if $|\lambda| < \rho$. Finally, if $I$ is a translation-invariant ideal in the algebra $C_u$ (with respect to multiplication) and $f \in I$, then $f_\rho \in I$ for any $\rho > 0$.

These facts, as mentioned, make it possible to simplify some proofs, but in many cases it is better to do without them.

**5.** Suppose that $f$ is a continuous function in the closed upper half-plane $y = \operatorname{Im} z \geqslant 0$ of the complex plane $\mathbb{C}$. We assume also that $f$ is analytic for $y > 0$. If $f$ is bounded on the real axis $\mathbb{R}$ and has exponential type $\sigma > 0$ in the upper half-plane**, then, according to the classical Phragmen–Lindelöf theorem, we have

$$|f(z)| \leqslant e^{\sigma y} \sup_{\mathbb{R}} |f|. \tag{1}$$

The following asymptotic relation (see [2, Lecture 16]) supplements and, in a certain sense, sharpens this inequality:

$$\log|f(z)| = \sigma y + o(y). \tag{2}$$

It is valid on almost all half-lines $0 < \arg z < \pi$ and on each half-line from which a set of finite logarithmic length is removed. Recall that, by definition, a measurable set $E$ is of finite logarithmic length on the half-line $r > 0$ if $\int_E (1+r)^{-1} dr < \infty$.

Note that equation (2) remains valid if the function $f$ is of relatively weak growth on the real axis, in particular, of at most power growth.

**6.** Suppose that $f \in C_u$ and the support $\operatorname{supp}(\hat{f})$ of the Fourier transform of the function $f$ lies on a half-line $\lambda \leqslant \lambda_0 < \infty$. The least such $\lambda_0$ is denoted by $\operatorname{ah}(f)$ and called the *harmonic abscissa* of $f$. If there are no finite $\lambda_0$ with this property, we say that $\operatorname{ah}(f) = +\infty$. The following lemma (we omit its proof) can be obtained by standard analytic means (including (2)).

**Lemma 1.** *Suppose that $f \in C_u$ and $\sigma > 0$. Then the condition $\operatorname{ah}(f) \leqslant \sigma$ holds if and only if the Poisson extension of the function $x \to f(x)e^{i\sigma x}$ to the half-plane $\operatorname{Im} z > 0$ is a bounded analytic function. Furthermore, if $0 < \operatorname{ah}(f) < \infty$, then*

$$\operatorname{ah}(f) = \lim_{y\to\infty}{}^{*}\, y^{-1} \log|f(iy)|, \tag{3}$$

*where the asterisk means that the limit (which coincides with the upper limit) exists if from the $y$-axis an appropriate set of finite logarithmic length is removed.*

Suppose that $f \in C_u$ and $\operatorname{ah}(f) < \infty$. We put $(T_z f)(t) = f(z+t)$. It is clear that $T_z f \in C_u$. The next lemma follows from the previous one (we again omit the proof).

**Lemma 2.** *Let $f \in C_u$ be a function with $\operatorname{ah}(f) < \infty$. Then $z \to T_z f$ is a continuous function in the closed upper half-plane $\operatorname{Im} z \geqslant 0$ with values in $C_u$ which is analytic for $\operatorname{Im} z > 0$.*

Relation (3) immediately implies the relation

$$\operatorname{ah}(f_1 \cdot f_2) = \operatorname{ah}(f_1) + \operatorname{ah}(f_2) \qquad (\operatorname{ah}(f_1), \operatorname{ah}(f_2) < \infty), \tag{4}$$

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*This inequality means that the norm of the translation operator in $\mathbf{B}_\sigma$ coincides with the spectral radius. Some conditions implying such a coincidence for operators which are far from normal were considered in [4].

**The condition $\sigma > 0$ can always be achieved by a real translation of the function $f$. Hence this condition does not restrict generality, but it simplifies considerably some formulations and arguments.

which is equivalent to the classical Titchmarsh theorem.

**7.** Let $f \in C_u$, and let $I$ be a closed translation-invariant ideal in $C_u$. Then $f * \mu \in I$ for any measure $\mu \in M$. We shall use this fact below.

Consider points $\lambda_0 \in \mathbb{R}$ (here $\mathbb{R}$ is the dual copy of the real line) such that in some neighborhood of the point $\lambda_0$ the distribution $\hat{f}$ coincides with $\widehat{g}$, where $g \in I$. It is easy to see that (like in the trivial case) the set of all such points $\lambda_0$ is open. It is natural to use the notation $\operatorname{supp}_I(\hat{f})$ for the *complement* to this open set (in the dual axis). We refer to this complement as the *relative harmonic support of* $f$. The (absolute) *harmonic support* of $f$ is $\operatorname{supp}(\hat{f})$. Note that from the viewpoint of classical harmonic analysis the harmonic support of $f$ is the Björling spectrum of $f$, while from the viewpoint of Gelfand–Shilov theory it coincides with $\operatorname{hull}(K)$, where $K = \operatorname{Ann}(f)$ (= annihilator of the functional $f$ in the algebra of integrable functions under convolution).

In what follows, unless otherwise specified, we consider only functions $f \in C_u$ for which $\operatorname{ah}(f) < \infty$ and only translation-invariant ideals. On the other hand, mentioning these properties does not carry any additional meaning.

*The relative harmonic abscissa* $\operatorname{ah}_I(f)$ is the least upper bound of $\operatorname{supp}_I(\hat{f})$. It is clear that $\operatorname{ah}_I(f) \leqslant \operatorname{ah}(f)$ for all ideals $I$.

We shall try to understand to what extent (4) applies to relative harmonic abscissas. We mention at once that this relation is generally violated, and counterexamples are very simple.

**8.** The topology on the extension $\alpha\mathbb{R}$ comes together with homomorphisms $\varphi\colon C_u \to \mathbb{C}$. To any closed ideal $I \subset C_u$ there corresponds a unique closed set $Q \subset \alpha\mathbb{R}$ such that $I = \{f\colon \varphi(f) = 0$ for all $\varphi \in Q\}$. Moreover, the quotient algebra is (naturally) isometrically isomorphic to the algebra $C(Q)$ with the sup-norm. The norm of the image of an element $f \in C_u$ in the image of $C_u \to C_u/I$ is denoted by $\|f\|_{C_u/I}$. From the algebraic viewpoint, $Q = \operatorname{hull}(I)$ is the *algebraic hull of the ideal* $I$.

The operators $T_x^*$ conjugate to the automorphisms $T_x$ naturally generate homeomorphisms of the compact set $\alpha\mathbb{R}$, so that $(T_x^*\varphi)(f) = \varphi(T_x f)$. The translation invariance of the ideal $I = I(Q)$ is equivalent to the fact that $\varphi$ and $T_x^*\varphi$ lie or do not lie in $Q$ simultaneously.

It is easy to check that the above action of $\mathbb{R}$ on $\alpha\mathbb{R}$ is free.

For any homomorphism $\varphi$ from $C_u$ to $\mathbb{C}$, let $Q_\varphi$ denote the *closure* of the orbit $\{T_x^*\varphi\}$. If $\varphi$ corresponds to a point of $\mathbb{R}$, then $Q_\varphi = \alpha\mathbb{R}$. Otherwise $Q_\varphi \cap \mathbb{R} = \varnothing$.

The following lemma is one of our main results.

**Lemma 3.** *Let $I \subset C_u$ be an invariant closed ideal, and let $f \in C_u$ be a function with harmonic spectrum bounded on the right. Then*

$$\operatorname{ah}_I(f) = \varlimsup_{y\to\infty} y^{-1} \log \|T_{iy} f\|_{C_u/I}. \tag{5}$$

**Proof.** Let $\alpha$ denote the left-hand side in (5), and let $\beta$ denote the right-hand side. Without loss of generality we can assume that both these numbers are positive. Note that then $\operatorname{ah}(f) > 0$ as well.

In the case $I = \{0\}$ Lemma 3, in fact, follows from Lemma 1.

First, let us show that $\beta \leqslant \alpha$.

Let $\varepsilon > 0$. By an argument similar to that used in Section 4, there exists a function $g \in \mathbf{B} \cap I$ depending only on $f$ and $\varepsilon$ and such that $\operatorname{ah}(f-g) < \alpha+\varepsilon$. We have $T_{iy} g \in I$. Indeed, for example, an "upward" translation of functions $g$ bounded in the upper half-plane is given by convolution with a Poisson kernel, and convolutions do not take functions away from the ideal, because it is invariant with respect to (real) translations. Therefore,

$$\|T_{iy} f\|_{C_u/I} = \|T_{iy}(f-g)\|_{C_u/I} \leqslant \|T_{iy}(f-g)\| \leqslant \text{const} \cdot e^{(\alpha+\varepsilon)y}.$$

Hence, $\beta \leqslant \alpha$.

Now, assuming that $\beta + \varepsilon < \alpha$ for some $\varepsilon > 0$, we shall obtain a contradiction.

We take a function $h \in S$ such that

$$\widehat{h}(\lambda) = \begin{cases} 1 & \text{if } |\lambda - \alpha| \leqslant \varepsilon/2, \\ 0 & \text{if } |\lambda - \alpha| \geqslant \varepsilon. \end{cases}$$

Obviously,

$$f * h \notin I. \tag{6}$$

Let $\psi$ be a continuous linear functional on $C_u$ such that $\psi|I = 0$ and $\|\psi\| = 1$. For any $y \geqslant 0$, we put $g(z) = \psi(T_z f)$. This function is continuous for $\operatorname{Im} z \geqslant 0$, bounded and uniformly continuous on any horizontal axis in the closed upper half-plane, and analytic for $\operatorname{Im} z > 0$ (see Lemma 2).

Since $\psi|I = 0$, we have $|g(iy)| \leqslant \text{const} \cdot e^{\beta y}$ for any $y \geqslant 0$. Hence $\operatorname{ah}(g) \leqslant \beta$. This implies $g * h = 0$. Therefore,

$$0 = (g * h)(0) = \int_{\mathbb{R}} \psi(T_{-x} f) h(x)\, dx = \psi\biggl( \int_{\mathbb{R}} (T_{-x} f) h(x)\, dx \biggr) = \psi(f * h).$$

Since the functional $\psi$ (satisfying the above conditions) is arbitrary, the Hahn–Banach theorem implies $f * h \in I$. This contradicts condition (6), which proves the lemma.

As an obvious corollary we obtain the inequality

$$\operatorname{ah}_I(f_1 \cdot f_2) \leqslant \operatorname{ah}_I(f_1) + \operatorname{ah}_I(f_2). \tag{7}$$

Below we give a simple example, which shows that inequality (7) may be strict. Take the unique analytic branch of the function $\sqrt{z}$ in the upper half-plane such that $\sqrt{1} = 1$. We put $f_1(z) = \exp(-\sqrt{z})$. Let $f_2$ denote a similar function in the upper half-plane which tends to zero on the negative half-axis. Let $C_0$ be the ideal of functions $f$ such that $|f(x)| \to 0$ as $|x| \to \infty$ on the real axis. Obviously, $f_1 f_2 \in C_0$, while none of the factors lies in $C_0$.

For products of coinciding multipliers, the equality remains true (and immediately follows from Lemma 3).

**Theorem 1.** $\operatorname{ah}_I(f^n) = n \cdot \operatorname{ah}_I(f)$.

**Remark.** The general case of Theorem 1 can be reduced to the case $n = 2$ by standard means. Indeed, if the required relation holds for $n = 2$, then, by induction, it holds for all $n = 2^k$. For other $n$, by choosing $m$ from the condition $2^n = m + n$ and using (7) (which can easily be proved directly), we obtain Theorem 1 in the general case.

It is interesting that in the classical situation there are arguments (discovered by Mikusinski) which make it possible to pass from squares to products of arbitrary pairs (see, e.g., [5, Sec. 4.3]). The above example shows that in our case such a "jump" is impossible.

**9.** In the proof of Lemma 3 we used all continuous functionals $\psi$ whose kernel contains the ideal $I$, although we could take only multiplicative ones, because any closed ideal in $C_u$ is an intersection of maximal ones. Below we present a lemma, which is similar to Lemma 3, where such a restriction is of fundamental importance. Moreover, formally speaking, we shall need only one multiplicative functional.

Recall that, for any $\varphi \in \alpha\mathbb{R}$, $Q_\varphi$ denotes the closure of the orbit of the "point" $\varphi$. Below it is reasonable to assume at once that $\varphi \notin \mathbb{R}$. Like in Lemma 3, we assume that $\operatorname{ah}(f) < \infty$. The symbols $\alpha$ and $\beta$ have the same meaning as in Lemma 3, and the symbol $\lim^*$ has the same meaning as in Lemma 1.

**Lemma 4.** *If* $f \in I = I(Q_\varphi)$, *then*

$$\operatorname{ah}_I(f) = \operatorname*{lim^*}_{y \to \infty} y^{-1} \log |\varphi(T_{iy} f)|. \tag{8}$$

**Proof.** Let $\gamma$ denote the right-hand side of (8). It is obvious that $\gamma \leqslant \beta$. Since, by Lemma 3, $\alpha = \beta$, it is sufficient to check that $\beta \leqslant \gamma$.

Note that, for the subalgebra of functions $f \in C_u$ with $\operatorname{ah}(f) < \infty$, the map $f \to T_z f$ is an endomorphism for any fixed $z$ from the upper half-plane. Therefore, $f \to \varphi(T_z f)$ is a homomorphism to $\mathbb{C}$ (we assume that the conditions of Lemma 4 include the assumption $\mathrm{ah}(f) < \infty$). Finally, it is clear that

$$\|f\|_{C_u/I} = \sup_{x\in\mathbb{R}} |(T_x^*\varphi)(f)|. \tag{9}$$

Suppose that $\|f\| = 1$ (this changes nothing). Let $g(z) = \varphi(T_z f)$ for $\operatorname{Im} z \geqslant 0$. The function $g$ has exponential type in the upper half-plane, and on the real axis $|g(x)| \leqslant 1$. Hence, by inequality (1) (i.e., by the Phragmen–Lindelöf theorem), $|g(z)| \leqslant \exp \gamma y$.

This inequality combined with (9) implies the inequality $\beta \leqslant \gamma$, which proves the lemma.

**Theorem 2.** *If $I = I(Q_\varphi)$, then*

$$\mathrm{ah}_I(f_1 \cdot f_2) = \mathrm{ah}_I(f_1) + \mathrm{ah}_I(f_2). \tag{10}$$

Note that, formally speaking, Theorem 2 includes the Titchmarsh theorem.

Let $Q_1$ and $Q_2$ be invariant closed subsets in $\alpha\mathbb{R}$, and let $Q = Q_1 \cup Q_2$. The following theorem is a general form of the above example.

**Theorem 3.** *If $Q_1 \setminus Q_2 \neq \varnothing \neq Q_2 \setminus Q_1$, then, for the ideal $I = I(Q)$, relation* (10) *does not generally hold.*

**Proof.** Below $k = 1$ or 2. Let $\varphi_k \in Q \setminus Q_k$. There exist functions $f_k$ such that $f_k \in I(Q_k)$ and $\varphi_k(f_k) = 1$. Considerations in Section 4 imply the existence of functions $g_k \in \mathbf{B} \cap I(Q_k)$ such that $\|f_k - g_k\| < \varepsilon < 1/2$. Both functions $g_1$ and $g_2$ have compact harmonic spectra and do not lie in $I$. Hence their harmonic abscissas are finite real numbers. On the other hand, $g_1 \cdot g_2 \in I$. This proves the theorem.

**10.** It remains to make several remarks. Probably, Theorems 2 and 3 admit a synthesis which leads to a criterium, i.e., to a description of all ideals for which the Titchmarsh theorem holds in full generality. It is natural to expect that acomplishing this objective requires much more careful study of the "orbit space."

Obviously, such a criterium, as well as Theorem 1, is of interest in the multidimensional case too. In solving multidimensional problems, it seems to be promising to extend Lions's original approach [6]–[8] (a more contemporary description see, e.g., in [9, Sec. 16.3]) to the situation under consideration. The application of the above asymptotic relation (2) may be useful; it may simplify some arguments even in the classical situation. Of course, multidimensional versions of Theorem 1 are interesting as well. Moreover, it is known in what topological Abelian groups the classical Titchmarsh theorem holds (see, in particular, [10] and [11]). For example, $\mathbb{R}^n$ with the discrete topology will do. A similar question makes sense in the case considered in this paper.

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Moscow State Pedagogical University
e-mail: evgeny.gorin@mtu-net.ru

V. A. Steklov Mathematical Institute
e-mail: treschev@mi.ras.ru

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