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Abelianization of the BGG Resolution of Representations of the Virasoro Algebra*

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To the memory of V. I. Arnold

Abstract. We construct a resolution that permits computing the t-character of representations of the Virasoro algebra from the $(2, 2p+1)$-models, i.e., the characters of the associated graded spaces with respect to the Poincaré–Birkhoff–Witt filtration.

Key words: Virasoro algebra, t-characters of irreducible representations, abelianization.

1. Introduction

Let $\mathfrak{a}$ be a Lie algebra. The universal enveloping algebra $U(\mathfrak{a})$ bears the Poincaré–Birkhoff–Witt filtration, and the associated graded algebra is the polynomial ring $S^\bullet(\mathfrak{a})$. There exists a similar filtration on each representation π of $\mathfrak{a}$ with a given cyclic vector v. Namely, $F_0 = \{v\}$ and $F_{i+1} = F_i + \mathfrak{a}F_i$. The associated graded space π^{ab} can naturally be equipped with the structure of a representation over the algebra $S^\bullet(\mathfrak{a})$. This construction, i.e., the passage from π to π^{ab}, will be referred to as *abelianization*. It is usually a rather difficult task to reveal the structure of the space π^{ab}. The case of $\mathfrak{a} = \mathfrak{sl}_n$ was analyzed in [6].

Let $\mathfrak{a}$ be the Virasoro algebra, and let π be an irreducible representation from the $(2, 2p+1)$-minimal model, where p is a positive integer. Then the representation π^{ab} has a simple explicit description (see [4]). Fix a base L_i, $i \in \mathbb{Z}$, of the Virasoro algebra. After the abelianization, the L_i^{ab} commute; let $a(z) = \sum L_i^{\mathrm{ab}} z^{-i}$. The current $a(z)$ on π^{ab} satisfies the relation $a(z)^p = 0$. More precisely, the operators L_i^{ab} act on π^{ab} as zero for $i \geqslant 0$, and $\pi^{\mathrm{ab}} \simeq \mathbb{C}[L_{-1}^{\mathrm{ab}}, L_{-2}^{\mathrm{ab}}, \dots]/J$; here J is the ideal generated by the elements $(L_{-1}^{\mathrm{ab}})^s$, $s < p$, and $\sum_{\alpha_1+\dots+\alpha_p=r} L_{\alpha_1}^{\mathrm{ab}} \cdots L_{\alpha_p}^{\mathrm{ab}}$, $r \in \mathbb{Z}$, where the number s depends on the choice of an irreducible representation from the minimal model. (There are p of these.)

In representation theory, the space π^{ab} shows up unexpectedly when studying integrable $\widehat{\mathfrak{sl}}_2$-modules. Take the standard base $\{e, h, f\}$ in $\mathfrak{sl}_2$ and the base $e(z)$, $h(z)$, $f(z)$ in $\widehat{\mathfrak{sl}}_2$, where $e(z) = \sum e_i z^{-i}$. Let R be an irreducible integrable representation of level $k \geqslant 0$ of the algebra $\widehat{\mathfrak{sl}}_2$. On R, the current $e(z)$ satisfies the relation $e(z)^{k+1} = 0$. Let R^{sub} be the $\mathbb{C}[e_i]$-submodule of R generated by the highest vector. It turns out that $R^{\mathrm{sub}} \simeq \mathbb{C}[e_{-1}, e_{-2}, \dots]/J$, where J is the very same ideal used above in the construction of the quotient of $\mathbb{C}[L_{-1}^{\mathrm{ab}}, L_{-2}^{\mathrm{ab}}, \dots]$ (L_{-i}^{ab} should be replaced by e_i); the number s depends on the choice of an integrable module of level k.

The representations of the Virasoro algebra from minimal models are in many respects similar to integrable modules over Kac–Moody algebras. In particular, they have resolutions of Bernstein–Gelfand–Gelfand (BGG) type consisting of Verma modules. Given a resolution, one can write out a formula for the character of an irreducible representation (see [7]). For example, the formula for the character of the vacuum representation π from a minimal model reads

$$\chi(\pi) = \sum_{i \geqslant 0} (-1)^i \frac{q^{2i+(p+2)i(i-1)/2}(1-q^{2i+1})}{(1-q)(1-q^2)\dots}.$$

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In the abelianized case, one can also write out a formula for the character. The formula for the quotient space

$$\pi^{\mathrm{ab}} = \mathbb{C}[e_{-2}, e_{-3}, \dots]/J, \qquad J = \Big\{ \sum_{\alpha_1+\dots+\alpha_p=r} e_{\alpha_1} \cdots e_{\alpha_p} \Big\}, \quad r \in \mathbb{Z},$$

is as follows. Let us equip π^{ab} with two gradings by setting $\deg_1 e_{-i} = i$ and $\deg_2 e_{-i} = 1$; then

$$\pi^{\mathrm{ab}} = \bigoplus \pi^{\mathrm{ab}}[\alpha, \beta], \qquad \chi(\pi^{\mathrm{ab}}) = \sum q^\alpha t^\beta \dim \pi^{\mathrm{ab}}[\alpha, \beta],$$

$$\chi(\pi^{\mathrm{ab}}) = \sum_{i\geqslant 0} (-1)^i \frac{t^{2i} q^{2i+(k+3)i(i-1)/2}}{(1-q^{i+1}t)\cdots(1-q^{2i}t)} \frac{1}{(1-q)\cdots(1-q^i)} \times \frac{1}{(1-q^{2(i+1)}t)(1-q^{2(i+1)+1}t)\dots}. \tag{1.1}$$

We see that $\chi(\pi^{\mathrm{ab}})$ is a t-deformation of the character $\chi(\pi)$. By substituting $t = 1$ into $\chi(\pi^{\mathrm{ab}})$, we obtain $\chi(\pi)$.

There are various ways to prove formula (1.1). The best-known proof is based on Schur's ideas and can be found in Andrews' book [1]. There is another, geometric argument. The formula for $\chi(\pi^{\mathrm{ab}})$ can be interpreted as a Lefschetz formula computing the alternating sum of characters of an action of a torus on the cohomology spaces of some linear bundle. Let Gr be the affine Grassmannian corresponding to the algebra $\widehat{\mathfrak{sl}}_2$, let m be the fixed point lying in the maximal torus and corresponding to the identity element of the Weyl group, let Sh be the orbit of m under the action of the current group in the group of nilpotent matrices (the Lie algebra of the latter group is generated by e_i), and let $\overline{Sh}$ be the closure of Sh in Gr. Then π^{ab} is the dual space of the space of sections of the bundle ξ^k restricted to $\overline{Sh}$, where ξ is the standard line bundle on Gr. The restriction of ξ^k to $\overline{Sh}$ has trivial higher cohomology. The Lefschetz formula for the character of π^{ab} is a sum over fixed points, and the contributions coincide with the individual terms in (1.1) ([2], [3]).

In the present paper, we suggest an interpretation of the character of the representation π^{ab} as the Euler characteristic of some complex, which is none other than the abelianized BGG resolution.

Consider the special case in which $p = 2$ and $s = 0$; i.e.,

$$\pi^{\mathrm{ab}} = \mathbb{C}[e_{-2}, e_{-3}, \dots]/\{S_j\}, \qquad S_j = \sum_{\alpha+\beta=j} e_\alpha e_\beta.$$

Thus, π^{ab} is the quotient of the polynomial ring in infinitely many variables by the ideal generated by a set of quadratic polynomials. Should the system $\{S_j\}$ be regular, one could obtain a resolution of the representation π^{ab} by the standard construction known in commutative algebra as the Koszul complex. The set $\{S_j\}$ is very far from being a regular sequence. Nevertheless, the construction of the Koszul complex can be modified as follows. Let us supplement the algebra $\mathbb{C}[e_{-2}, e_{-3}, \dots]$ by fermions $\psi_{-4}, \psi_{-5}, \dots$. Define a differential Q on the algebra $\mathbb{C}[\psi_{-4}, \psi_{-5}, \dots; e_{-2}, e_{-3}, \dots]$ by setting $Q(\psi_\alpha) = S_\alpha$. The zero homology of this complex is isomorphic to π^{ab}, but we also have higher homology. Take the quotient of $\mathbb{C}[\psi_{-4}, \psi_{-5}, \dots; e_{-2}, e_{-3}, \dots]$ by the quadratic relations

$$\psi(z)\psi'(z), \quad \psi(z)\psi'''(z), \quad e(z)\psi'(z) - 2e'(z)\psi(z).$$

Here $\psi(z) = \sum \psi_i z^{-i}$, and the symbol $'$ stands for the derivative with respect to z. The passage to the quotient does not affect the zero homology, while the higher homology disappears. Our main result is that this construction coincides with the abelianization of the BGG resolution. For $p = 2$, formula (1.1) can be obtained as the Euler characteristic of the quotient Koszul complex.

The minimal models of type $(2, 2p+1)$ are in many respects simpler that the general (p, q)-models. All representations from minimal theories of the Virasoro algebra admit two-sided Felder resolutions [5]. The terms of these resolutions are Fock spaces, and the differentials are constructed with the use of screenings. This construction is simplified dramatically for the $(2, 2p + 1)$-model, and we present it in Section 2. Such complexes are representations of some lattice vertex operator algebra V_p. We single out a subalgebra $W_p \subset V_p$ generated by the Virasoro algebra and an anticommuting current. BGG type resolutions are representations of this subalgebra. Abelianization takes the representations of W_p to the quotient Koszul complexes.

In Section 3, we study the abelianization of the vertex operator algebra W_p, i.e., some supercommutative vertex algebra, and its representations. We show that the quotient Koszul complexes have nontrivial homology only in the zeroth graded component. We do not deal with all representations from the $(2, 2p + 1)$-model, mainly restricting ourselves to the case of the vacuum representation. The general case can be considered in a similar way.

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2. Felder Complexes of the $(2, 2p + 1)$-minimal models of the Virasoro algebra

Recall that the Felder complexes of the $(2, 2p+1)$-minimal models are representations of some lattice vertex operator algebra V_p. It is easiest to construct V_p as a subalgebra of the Clifford algebra C_p. The generators of C_p are the currents $\psi_1(z), \dots, \psi_{2p+1}(z)$ and $\psi_1^*(z), \dots, \psi_{2p+1}^*(z)$ satisfying the standard relations

$$[\psi_\alpha(z), \psi_\beta(w)]_+ = 0, \quad [\psi_\alpha^*(z), \psi_\beta^*(w)]_+ = 0, \quad [\psi_\alpha(z), \psi_\beta^*(w)]_+ = \delta_{\alpha,\beta}(z/w),$$

where $[\,\cdot\,,\,\cdot\,]_+$ stands for the anticommutator. The algebra V_p is the subalgebra of C_p generated by the currents $\theta_p(z) = \psi_1(z) \cdots \psi_{2p+1}(z)$ and $\theta_p^*(z) = \psi_1^*(z) \cdots \psi_{2p+1}^*(z)$. It is easily seen that

$$\frac{\theta_p(z)\theta_p(w)}{(z-w)^{2p+1}} = \frac{\theta_p(w)\theta_p(z)}{(w-z)^{2p+1}},$$

the operators $\theta_p^*(z)$ satisfy a similar relation, and moreover,

$$\frac{1}{2p+1}\,\theta_p(z)\theta_p^*(w) = (z-w)^{-2p-1}I + (z-w)^{-2p}h(z) + (z-w)^{-2p+1}\bar{T}(z) + \dots, \tag{2.1}$$

where I is the identity operator, $h(z)$ is the current corresponding to the Heisenberg algebra, and $\bar{T}(z)$ is the current corresponding to the Virasoro algebra. The algebra V_p is naturally graded by $\deg \theta_p(z) = 1$ and $\deg \theta_p^*(z) = -1$. The category of its representations is semisimple; there are $2p+1$ irreducible representations $\mathscr{R}_0, \dots, \mathscr{R}_{2p}$. The representation $\mathscr{R}_j$ is characterized by the fact that it contains a vacuum vector v_j such that $\theta_p[s]v_j = 0$, $s \geqslant j$, where $\theta_p(z) = \sum_{s\in\mathbb{Z}} \theta_p[s]z^{-s}$; the representation $\mathscr{R}_0$ is called the *vacuum representation* of V_p. The irreducible representations $\mathscr{R}_j$ inherit the grading. Set $S = \theta_p^*[0]$. Clearly, $S^2 = 0$ and hence each $\mathscr{R}_j$ is a complex with differential of degree -1 equal to S. The algebra V_p has two important subalgebras, the Heisenberg and Virasoro algebras, which occur in the operator product (2.1); $h(z) = \frac{1}{2p+1}\sum_{i=1}^{2p+1} {:}\psi_i(z)\psi_i^*(z){:}$ and $\bar{T}(z) = \frac{1}{2p+1}\sum_{i=1}^{2p+1} {:}\psi_i'(z)\psi_i^*(z){:}$. The Virasoro algebra can be deformed as follows: $T(z) = \bar{T}(z) + \beta h'(z)$, where β is a complex number, so that the current $T(z)$ commutes with S. The central charge of this Virasoro algebra is $1-3(2p-1)/(2p+1)$. Felder's theorem states that each complex $\mathscr{R}_j$ has nonzero homology in only one degree; moreover, the homology space is an irreducible representation of the Virasoro algebra from the $(2, 2p+1)$-model. The spaces $\mathscr{R}_j$ equipped with the differential S are called the *Felder complexes*.

Consider the subalgebra $W_p \subset V_p$ generated by the Virasoro algebra $T(z)$ and the operator $\theta_p(z)$. Note that, in the terminology of conformal field theory, $\theta_p(z)$ is a primary field of type $\phi_{3,1}$. The algebra W_p is an extension of the Virasoro algebra by the field $\phi_{3,1}$. The vacuum representation of W_p is the subspace of $\mathscr{R}_0$ generated from the highest vector by the operators in W_p. We denote this vacuum representation by Vac_p. Felder's results readily imply the structure of Vac_p treated as a module over the Virasoro algebra. Let M_λ be the Verma module with highest weight λ over the Virasoro algebra. (The central charge is fixed and equal to $1-3(2p-1)/(2p+1)$.)

If $\lambda = \lambda_i = pi(i+1) + i(i-1)/2$, then the module M_{λ_i} has a singular vector at the level $2i+1$. Let $\bar{M}_i$ be the quotient of M_{λ_i} by the submodule generated by this singular vector. Its character is

$$q^{pi(i+1)+i(i-1)/2} \frac{(1-q^{2i+1})}{\prod_{i>0}(1-q^i)} . \tag{2.2}$$

Proposition 2.1. *The representation* Vac_p *treated as a module over the Virasoro algebra is isomorphic to* $\bigoplus_{i>0} \bar{M}_i$.

The algebra W_p is a differential vertex operator algebra. The differential is the supercommutator with S. The supercommutation operation differentiates the operator product. Note that the supercommutator of S with $\theta_p(z)$ is a descendant of the current $T(z)$. This descendant corresponds to the singular vector in the vacuum representation from the $(2, 2p+1)$-model of the Virasoro algebra. It follows that W_p is a differential subalgebra of V_p.

Note that Vac_p is a differential module over W_p. The differential S commutes with the action of the Virasoro algebra, and we arrive at the complex

$$R_0 \colon 0 \leftarrow \bar{M}_0 \leftarrow \bar{M}_1 \leftarrow \bar{M}_2 \leftarrow \dots .$$

Proposition 2.2. *The complex* R_0 *has nontrivial homology only in the zeroth term. The homology space is isomorphic to the vacuum representation from the* $(2, 2p+1)$*-model.*

Remark 2.3. The complex R_0 is a version of the BGG resolution for the Virasoro algebra. Recall that integrable representations of Kac–Moody algebras have resolutions consisting of parabolic Verma modules. The complex R_0 is an analog of such a resolution.

Now let us equip the algebra W_p with a filtration similar to the Poincaré–Birkhoff–Witt filtration on universal enveloping algebras of Lie algebras. Set $F_0(W_p) \simeq \mathbb{C}$, $F_1(W_p) = \mathbb{C} \oplus \{T(z), \theta_p(z)\}$, and $F_i(W_p) = F_1(W_p) \cdot F_{i-1}(W_p)$. (Here the dot stands for the operator product.) We denote the associated graded algebra by W_p^{ab}; this is a vertex operator algebra in the sense that W_p^{ab} is equipped with the structure of an operator product.

3. Supercommutative Differential Vertex Operator Algebras

The simplest example of a supercommutative differential vertex operator algebra is a semi-infinite Koszul complex K. This complex (the corresponding vertex algebra) has two generators $\psi(z)$ and $a(z)$ such that $[\psi(z)\psi(w)]_+ = 0$, $[a(z), a(w)] = 0$, and $[\psi(z), a(w)] = 0$, and the derivation Q is given by the formulas $Q\psi(z) = a(z)$ and $Qa(z) = 0$. Note that K has a contracting homotopy, which is given by the "inverse differential," i.e., a derivation B such that $Ba(z) = \psi(z)$, $B\psi(z) = 0$, and $[Q, B]_+ = 1$. Define a representation $V(K)$ of the algebra K as the representation induced from the trivial representation of the subalgebra $\{a_i, \psi_i, i \geqslant -1\}$, $a(z) = \sum a_i z^{-i}$, $\psi(z) = \sum \psi_i z^{-i}$. The representation $V(K)$ will be referred to as the vacuum representation. The operators Q and B act on $V(K)$, the homology of Q is isomorphic to $\mathbb{C}$, and the homology of B is one-dimensional as well.

One can pass to the quotient of K by a differential ideal without changing the homology. Namely, set

$$K_{(1)} = K/\{\psi(z)\psi'(z), a(z)\psi'(z) - a'(z)\psi(z)\}.$$

Note that $Q(\psi(z)\psi'(z)) = a(z)\psi'(z) - a'(z)\psi(z)$. The ideal in K generated by the elements $\psi(z)\psi'(z)$ and $a(z)\psi'(z) - a'(z)\psi(z)$ is invariant with respect to the derivation B, and hence the homology of the algebra $K_{(1)}$ is isomorphic to $\mathbb{C}$.

Theorem 3.1. *The algebras* $K_{(1)}$ *and* W_1^{ab} *are isomorphic.*

For $p = 1$, the central charge of the Virasoro algebra is 0. The homology of the representation Vac_1, i.e., of the vacuum representation of W_1, is isomorphic to $\mathbb{C}$. Note that in this case the operators $\theta_1(z)$ and $\theta_1^*(z)$ generate an $(N=2)$-superalgebra. The supercommutator of the operators $\theta_1(z)$ and $\theta_1^*(z)$ is a linear combination of operators from the Heisenberg algebra and the Virasoro algebra, which lie in the $(N=2)$-superalgebra as well. The supercommutator of $\theta_1(z)$ with S is exactly the current $T(z)$ corresponding to the Virasoro algebra. The current $\theta_1(z)$ is a product of three fermions; consequently, $\theta_1(z)\theta_1'(z)=0$ and hence $[S,\theta_1(z)\theta_1'(z)]=T(z)\theta_1'(z)-\theta_1(z)T'(z)=0$. This formula relating the operators $T(z)$ and $\theta_1(z)$ follows from the fact that $\theta_1(z)$ is a primary field of type $\phi_{3,1}$. The passage to the associated graded algebra takes the relations satisfied by $\theta_1(z)$ and $T(z)$ to relations in the algebra $K_{(1)}$ between the operators $\psi(z)$ and $a(z)$. Now for the algebra $K_{(1)}$ we defined an analog of the vacuum representation $V(K_{(1)})$, which is the quotient of the representation $V(K)$ by the subspace $\psi(z)\psi'(z)V(K)+Q(\psi(z)\psi'(z))V(K)$. To prove the theorem, it suffices to show that the characters of the representations Vac_1 and $V(K_{(1)})$ coincide. To this end, we construct a monomial base in the space $V(K_{(1)})$. The algebra $K_{(1)}$ is graded with $\deg\psi(z)=1$ and $\deg a(z)=0$, and so the representation $V(K_{(1)})=\bigoplus_{j\geqslant 0}V[j]$ is graded as well.

Lemma 3.2. *The following set of vectors generates the space* $V[j]$:

$$\psi_{\alpha_{j-1}}\psi_{\alpha_{j-2}}\cdots\psi_{\alpha_1}\psi_{-2}\cdot a_{-2-j+1}^{s_2}a_{-3-j+1}^{s_3}\cdots a_{-l-j+1}^{s_l}v,$$

where $l>2,\ s_j\geqslant 0,\ \alpha_1\leqslant -5,$ *and* $\alpha_i-\alpha_{i-1}\leqslant -3,$ *and*

$$\psi_{\beta_j}\psi_{\beta_{j-1}}\cdots\psi_{\beta_1}\cdot a_{-2-j}^{s_2}a_{-3-j}^{s_3}\cdots a_{-l-j}^{s_l}v,$$

where $l>2,\ s_j\geqslant 0,\ \beta_1\leqslant -3,$ *and* $\beta_i-\beta_{i-1}\leqslant -3.$

To prove this, one should use the relations in the algebra $K_{(1)}$: every monomial can be expressed via those indicated in the lemma. The vacuum representation Vac_1 is graded as well, and the character of the component $\mathrm{Vac}_1[j]$ is $q^{(3j^2+j)/2}(1-q^{2j+1})/\prod_{i>0}(1-q^i)$ (see (2.2) for $p=1$). The character of the space $V[j]$ cannot be less than the character of the space $\mathrm{Vac}_1[j]$. (More precisely, each coefficient in the character $\sum c_mq^m$ of $V[j]$ is greater than or equal to the respective coefficient in the character of $\mathrm{Vac}_1[j]$.) On the other hand, by computing the character of the combinatorial data in the lemma, one obtains precisely the character of the space $\mathrm{Vac}_1[j]$.

Now let us proceed to the case of $p>1$. The algebra $K_{(p)}$ is defined as follows:

$$K_{(p)}=K/J_p,$$

where the ideal J_p is generated by the elements $\psi(z)\psi^{(r)}(z)$, $r<2p$, and $a(z)\psi'(z)-pa'(z)\psi(z)$. (Here $\psi^{(r)}(z)$ stands for the rth derivative.) We equip $K_{(p)}$ with a differential Q by setting $Qa(z)=0$ and $Q\psi(z)=a(z)^p$.

Lemma 3.3. *The differential Q is well defined; in other words, the ideal J_p is invariant under the action of Q.*

This lemma can be verified by a straightforward computation. For example, take the relation $\psi(z)\psi'(z)=0$ and apply the differential Q to it:

$$Q(\psi(z)\psi'(z))=a(z)^p\psi'(z)-\psi(z)(a^p(z))'=a^{p-1}(z)(a(z)\psi'(z)-p\psi(z)a'(z)).$$

This means that $Q(\psi(z)\psi'(z))\in J_p$.

Theorem 3.4. *$K_{(p)}$ and W_p^{ab} are isomorphic as differential graded supercommutative vertex operator algebras.*

First, let us show that the vacuum representation of $K_{(p)}$ is a complex having nontrivial homology only in the zeroth grading. The vacuum representation of $K_{(p)}$ is determined by the annihilation conditions for the vacuum vector; $V(K_{(p)})$ is generated by a vector v such that $a_iv=0$ for $i>-2$ and $\psi_iv=0$ for $i>-2p$. Similar annihilation conditions hold for the vacuum representation of W_p.

We need the following abelianization of the vacuum representation from the $(2,2p+1)$-model. The vertex algebra of the Virasoro algebra for $c=1-3(2p-1)/(2p+1)$ contains the field $T^{(p)}(z)$ corresponding to the singular vector at the level $2(p+1)$. We use the following filtration on the vertex algebra: $F_1=\{\mathbb{C}+T(z)+T^{(p)}(z)\}$. This induces a filtration on the vacuum representation. Now let us describe the associated graded algebra.

Let $A=\mathbb{C}[b_{-2},b_{-3},\dots]$ be the polynomial ring in infinitely many variables, and let $b(z)=\sum_{i\geqslant 2}b_{-i}z^i$, so that $b(z)^p=z^{2p}b_{-2}^p+\cdots=\sum_{\alpha\geqslant 2p}z^\alpha b_{-\alpha}^{[p]}$. We equip A with a filtration by setting

$F_0 \simeq \mathbb{C}$, $F_1 = F_0 + \{b_i, b_j^{[p]}\}$, and $F_s = F_1 \cdot F_{s-1}$; let A_p^{gr} be the associated graded algebra. The generators of A_p^{gr} are the currents $\bar{b}(z) = \sum_{i\geqslant 2} \bar{b}_{-i} z^i$ and $c(z) = \sum_{j\geqslant 2p} c_{-j} z^j$ corresponding to the elements $b(z)$ and $b(z)^p$ of the algebra A.

Lemma 3.5. *The determining relations in the algebra A_p^{gr} have the form*

$$\bar{b}(z)^p = 0, \qquad p\bar{b}'(z)c(z) = \bar{b}(z)c'(z).$$

First, note that these relations hold in A_p^{gr}. Now let us study the case of $p = 2$. The relations in A_2^{gr} are quadratic, and hence we can indicate a monomial base. This resembles the situation we have dealt with in the proof of Theorem 3.1, where the operators $\psi(z)$ and $a(z)$ in the algebra $K_{(1)}$ satisfy quadratic relations of similar form. Let us grade A_2^{gr} by setting $\deg c(z) = 0$, $\deg \bar{b}(z) = 1$, and $A_2^{\mathrm{gr}} = \bigoplus_{j\geqslant 0} A_2^{\mathrm{gr}}[j]$. The component $A_2^{\mathrm{gr}}[j]$ is generated by the following monomials:

$$\bar{b}_{-2}\bar{b}_{\alpha_1}\bar{b}_{\alpha_2}\cdots\bar{b}_{\alpha_{j-1}} \cdot c^{s_0}_{-2p-j+1} c^{s_1}_{-2p-1-j+1} \cdots c^{s_l}_{-2p-l-j+1},$$

where $l \geqslant 0$, $s_j \geqslant 0$, $\alpha_1 \leqslant -4$, and $\alpha_i - \alpha_{i-1} \leqslant -2$, and

$$\bar{b}_{\beta_1}\bar{b}_{\beta_2}\cdots\bar{b}_{\beta_j} \cdot c^{s_0}_{-2p-j} c^{s_1}_{-2p-1-j} \cdots c^{s_l}_{-2p-l-j},$$

where $l \geqslant 0$, $s_j \geqslant 0$, $\beta_1 \leqslant -3$, and $\beta_i - \beta_{i-1} \leqslant -2$. The character of the space A_2^{gr} cannot be less that $(1-q^2)^{-1}(1-q^3)^{-1}\ldots$. (This is the character of the space A.) On the other hand, the character of the above-indicated combinatorial data exactly coincides with the character of the space A. Indeed, if we compute the character of the space $A_2^{\mathrm{gr}}[j]$ assuming that all monomials are linearly independent, then we obtain

$$\frac{q^{j(j+1)}(1-q^j)(1-q^{j+1})(1-q^{2j+1})}{\prod_{i\geqslant 1}(1-q^i)}.$$

Our claim readily follows from the easy-to-verify identity

$$\sum_{j\geqslant 0} q^{j(j+1)}(1-q^j)(1-q^{j+1})(1-q^{2j+1}) = 1-q.$$

It is difficult to indicate a monomial base for general p. The proof uses the infinite Gordon identity (i.e., the Gordon identity in which the number of particles is set to infinity; see [1]). Let us introduce a filtration on A_2^{gr} by setting $F_0 = \mathbb{C}$, $F_1 = F_0 + \{\bar{b}(z), c(z), \bar{b}(z)c(z)\}$, and $F_i = F_1 \cdot F_{i-1}$. The associated graded algebra is generated by the current components $b_1(z)$, $b_2(z)$, and $c(z)$, which correspond to the currents $\bar{b}(z)$, $\bar{b}(z)c(z)$, and $c(z)$, respectively. The currents $b_1(z)$ and $b_2(z)$ satisfy the monomial quadratic relations

$$\begin{gathered} b_1(z)^2 = 0, \quad b_1(z)b_2(z) = 0, \quad b_1'(z)b_2(z) = 0, \quad b_2(z)^2 = 0, \\ (b_2'(z))^2 = 0, \quad b_1(z)c(z) = 0, \quad b_1'(z)c(z) = 0, \end{gathered} \tag{3.1}$$

and moreover, $3b_2'(z)c(z) - 2b_2(z)c(z) = 0$. These are determining relations in the associated graded algebra. To show this, we introduce a filtration on the algebra with generators $b_1(z)$, $b_2(z)$, and $c(z)$ by taking $\{\mathbb{C}, b_1(z), b_2(z), b_1(z)c(z), c(z)\}$ for F_1. The generators corresponding to these currents in the associated graded algebra will be denoted by $b_1(z)$, $b_2(z)$, $b_3(z)$, and $c(z)$. These generators satisfy quadratic relations similar to (3.1). This "refinement" procedure can be carried out infinitely many times, and in the end we obtain the algebra with generators $b_i(z)$, $i = 1, 2, \ldots$, satisfying the quadratic relations

$$b_i(z)b_j^{(n)}(z) = 0, \qquad i \leqslant j, n < 2i. \tag{3.2}$$

The currents $b_j(z)$ have the decompositions $b_j(z) = \sum_{l\leqslant -2j} b_j[l]z^{-l}$.

The (infinite) Gordon formula states that the algebra with generators $b_j[m]$ satisfying the quadratic relations (3.2) has the character $\chi = \prod_{j>1}(1-q^j)^{-1}$. More precisely, χ is the Hilbert series of this algebra, provided that the latter is equipped with the grading $\deg b_j[m] = -m$. As a consequence, we find that all associated graded algebras are algebras with quadratic relations. Now the lemma follows from the fact that the algebra A_p^{gr} admits a similar filtration. This filtration procedure can be applied infinitely many times, and the final associated graded algebra is the same algebra with generators $b_j[m]$ and relations (3.2). Let us show how to do this in the case of $p = 3$. We construct a filtration on A_3^{gr} by setting $F_1 = \{\mathbb{C}, \bar{b}(z), \bar{b}(z)^2, c(z)\}$. The corresponding generators in the associated graded algebra will be denoted by $b_1(z)$, $b_2(z)$, and $c(z)$. These generators satisfy the quadratic relations (3.1).

Proposition 3.6. *The complex $V(K_{(p)})$ has nonzero homology only in the zeroth grading, and the character of the homology space coincides with the character of the vacuum representation from the $(2, 2p+1)$-model.*

The algebra $K_{(p)}$ is generated by $\psi(z)$ and $a(z)$. We introduce a filtration on it by setting $F_1 = \{\mathbb{C}, \psi(z), a(z), a(z)^p\}$ and $F_i = F_{i-1} \cdot F_1$. The associated graded algebra has the generators $\psi(z)$ and $a(z)$ corresponding to the old generators and the new generator $c(z)$ corresponding to $a(z)^p$. The differential Q acts on $K_{(p)}^{\mathrm{gr}}$ as follows: $Q\psi(z) = c(z)$ and $Qc(z) = Qa(z) = 0$. The determining relations in the algebra $K_{(p)}^{\mathrm{gr}}$ have the form

$$\begin{gathered} a(z)^p = 0, \quad pa'(z)c(z) = a(z)c'(z), \quad pa'(z)\psi(z) = a(z)\psi'(z), \\ \psi(z)\psi^{(r)}(z) = 0, \quad c(z)\psi^{(r)}(z) - \psi(z)c^{(r)}(z) = 0, \quad r < 2p. \end{gathered} \tag{3.3}$$

Note that $Q(\psi(z)\psi^{(r)}(z)) = c(z)\psi^{(r)}(z) - \psi(z)c^{(r)}(z)$. The proof of this fact is a somewhat complicated version of the proof of Lemma 3.5, and we omit it.

We define an "inverse" differential B on the algebra $K_{(p)}^{\mathrm{gr}}$ by the formulas $B\psi(z) = 0$, $Ba(z) = 0$, and $Bc(z) = \psi(z)$. It follows from the form of relations in $K_{(p)}^{\mathrm{gr}}$ that B is well defined. Furthermore, the subalgebra generated by $\psi(z)$ and $c(z)$ is contractible; i.e., the homology of Q restricted to this subalgebra is isomorphic to $\mathbb{C}$. The homology of the entire complex is nontrivial only in the zeroth grading and is generated by the current $a(z)$.

All these considerations apply to the vertex algebra $K_{(p)}^{\mathrm{gr}}$ itself as well as to its representations, say, to $V(K_{(p)}^{\mathrm{gr}})$. We see that the homology of $V(K_{(p)}^{\mathrm{gr}})$ is a representation of the algebra with generator $a(z) = \sum a_i z^{-i}$, $a(z)^p = 0$, and it contains a highest (vacuum) vector v such that $a_i v = 0$, $i \geqslant -1$. Thus, the homology of the space $V(K_{(p)}^{\mathrm{gr}})$ coincides with the abelianization of the vacuum representation from the $(2, 2p+1)$-model (see the introduction).

Now let us prove Theorem 3.4. It remains to verify that the characters of the spaces $V(K_{(p)})$ and Vac_p coincide. For $p = 1$, we have verified this claim by explicitly indicating a monomial base in $V(K_{(1)})$. In the general case, this is done exactly in the same way; the base consists of the vectors

$$\psi_{\alpha_{j-1}}\psi_{\alpha_{j-2}}\cdots\psi_{\alpha_1}\psi_{-2p}\cdot a_{-2-j+1}^{s_2}a_{-3-j+1}^{s_3}\cdots a_{-l-j+1}^{s_l}v,$$

where $l > 2$, $s_j \geqslant 0$, $\alpha_1 \leqslant -4p-1$, and $\alpha_i - \alpha_{i-1} \leqslant -2p-1$, and

$$\psi_{\beta_j}\psi_{\beta_{j-1}}\cdots\psi_{\beta_1}\cdot a_{-2-j}^{s_2}a_{-3-j}^{s_3}\cdots a_{-l-j}^{s_l}v,$$

where $l > 2$, $s_j \geqslant 0$, $\beta_1 < -2p$, and $\beta_i - \beta_{i-1} \leqslant -2p-1$.

The algebra $K_{(p)}$ can be deformed; namely, it has a "difference" version. Fix a number $t \neq 0$. For $p = 1$, set $K_{(1)}(t) = K/J$ (recall that K is generated by $a(z)$ and $\psi(z)$), where J is the ideal generated by the currents $\psi(z)\psi(zt)$ and $Q(\psi(z)\psi(zt)) = a(z)\psi(zt) - \psi(z)a(zt)$. The homology of $K_{(1)}(t)$ is one-dimensional, which can be proved with the use of the same contracting homotopy B. If $p \geqslant 1$, then $K_{(p)}(t) = K/J_p$, where the ideal J_p is generated by $\psi(z)\psi(zt), \psi(z)\psi(zt^2), \ldots, \psi(z)\psi(zt^p)$, and also $a(z)\psi(zt) - \psi(z)a(zt^p)$; $K_{(p)}(t)$ is a differential algebra , where the differential Q is given by the formulas $Q\psi(z) = a(z)a(tz)\cdots a(zt^p)$ and $Qa(z) = 0$.

Proposition 3.7. *The differential Q on $K_{(p)}(t)$ is well defined and has nontrivial homology only in the zeroth grading. The homology algebra is generated by a current $a(z)$ satisfying the relation $a(z)a(tz)\cdots a(zt^p) = 0$.*

Remark 3.8. The complex Vac_p (a subcomplex of the Felder complex) is a cyclic representation of the vertex algebra W_p. More precisely, Vac_p is an induced representation; i.e., it can be described as the quotient W_p/I, where I is the right ideal generated by the elements L_i, $i \geqslant -1$, and $\theta_p[i]$, $i \geqslant -2p+1$. (Here W_p is treated as an associative algebra, i.e., the algebra generated by components of currents of the vertex operator algebra.) The annihilation condition for the highest vector can be weakened. For example, set $Y_p = W_p/J$, where J is the right ideal generated by the elements L_i, $i \geqslant 0$, $\theta_p[i]$, $i \geqslant -p+1$, and $L_0 - h$, where $h = -(p-1)(p-3)/(8p)$. The number h is the highest weight of the "middle" representation T in the Kac table of highest weights of irreducible representations from the $(2, 2p+1)$-model; h is the least highest weight in this table. The differential Q acts on Y_p. The complex Y_p is a resolution of the representation T. One can show that Y_p is the standard BGG resolution of T. It follows that the complex

$$0 \leftarrow T \leftarrow Y_p[0] \leftarrow Y_p[1] \leftarrow \dots$$

is acyclic, and each component $Y_p[i]$, $i > 0$, is a direct sum of two Verma modules, while $Y_p[0]$ is isomorphic to a Verma module.

Conclusion. The phenomenon that holds for the representations of the Virasoro algebra from the $(2, 2p+1)$-minimal model occurs for many other vertex operator algebras. The first example where this is easy to verify is given by some classes of representations of the algebra $\widehat{\mathfrak{sl}}_2$. A more complicated case is given by representations of W-algebras, say, models of type $(3, p)$ of the algebra W_3. Abelianization of BGG type resolutions leads to quotients of the Koszul complex by fairly simple relations. From the viewpoint of commutative algebra, this means that for some algebras that are pretty far from the complete intersection there still exists an analog of the Koszul complex. The presence of such a complex leads to rather unexpected formulas for the Hilbert series. It would be highly desirable to reveal the cause of this. Furthermore, it would be of interest to interpret the complex constructed in this paper geometrically as some fact about the Grassmannian of the algebra $\widehat{\mathfrak{sl}}_2$.

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