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Inverse Problems for Sturm–Liouville Operators with Potentials in Sobolev Spaces: Uniform Stability*

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Abstract. Two inverse problems for the Sturm–Liouville operator $Ly = -y'' + q(x)y$ on the interval $[0, \pi]$ are studied. For $\theta \geqslant 0$, there is a mapping $F\colon W_2^\theta \to l_B^\theta$, $F(\sigma) = \{s_k\}_1^\infty$, related to the first of these problems, where $W_2^\theta = W_2^\theta[0, \pi]$ is the Sobolev space, $\sigma = \int q$ is a primitive of the potential q, and l_B^θ is a specially constructed finite-dimensional extension of the weighted space l_2^θ, where we place the regularized spectral data $\mathbf{s} = \{s_k\}_1^\infty$ in the problem of reconstruction from two spectra. The main result is uniform lower and upper bounds for $\|\sigma - \sigma_1\|_\theta$ via the l_B^θ-norm $\|\mathbf{s} - \mathbf{s}_1\|_\theta$ of the difference of regularized spectral data. A similar result is obtained for the second inverse problem, that is, the problem of reconstructing the potential from the spectral function of the operator L generated by the Dirichlet boundary conditions. The result is new even for the classical case $q \in L_2$, which corresponds to $\theta = 1$.

Key words: Inverse Sturm-Liouville problem, singular potentials, stability for inverse problems.

In the present paper we study two classical inverse problems for the Sturm–Liouville operator

$$Ly = -y'' + q(x)y, \qquad x \in [0, \pi], \tag{0.1}$$

on a finite interval. The first is the problem of reconstructing the potential from the two spectra of the operator (0.1) with Dirichlet and Dirichlet–Neumann boundary conditions, respectively. (We refer to it as Borg's problem.) The second is the problem of reconstructing the potential from the spectral function of the operator (0.1) with Dirichlet boundary conditions. (This operator will be called the Dirichlet operator.) The solution of these problems has long been known for the case of real potentials $q \in L_2$; in particular, a complete characterization of spectral data for potentials q of this class has been obtained. Our aim is to solve these problems for potentials q in the scale of Sobolev spaces W_2^α for any given $\alpha \geqslant -1$, including the case of $\alpha \in [-1, 0)$, where the potential is a singular function (a distribution). Here an important role is played by special Hilbert spaces that we construct to solve these problems. These spaces are needed to define and study mappings which we associate with these problems, as well as to completely describe (characterize) spectral data for potentials with primitive $\sigma = \int q(t)\,dt$ ranging over the set of real functions in $W_2^{\alpha+1}$.

Once the inverse problems are solved, there arises an important question about a priori estimates, namely, the question of how small the change in a primitive of the potential q is in the norm of $W_2^{\alpha+1}$ when the spectral data undergo a change small in the norm of the corresponding Hilbert space, in which these data are placed. Earlier, a priori estimates have been obtained in the classical case (for $\alpha = 0$). But these are estimates of local type, in which the constants, as well as the radius of the neighborhood where the estimates hold, depend on the potential q. *The main goal of the present paper is to obtain uniform two-sided a priori estimates not only for the classical case* $\alpha = 0$ *but also for all* $\alpha > -1$. The case $\alpha = -1$ is exceptional. Our method fails for $\alpha = -1$. Simultaneously, we find out for what spectral data the constants in the a priori estimate may "deteriorate" (i.e., become large or small). We show that this can only be due to the following causes:

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(i) The norm of the regularized spectral data is large; i.e., there are large deviations of the spectral data from zero values (which correspond to the zero potential q).

(ii) The gap (distance) between neighboring eigenvalues is small, or one of the normalizing constants approaches zero (the number h occurring in the definitions of the sets $\Omega^\theta_B(h,r)$ and $\Omega^\theta_D(h,r)$ of regularized spectral data, which are given in Sections 2 and 3, is small).

We once more point out that *these estimates are new even for the classical case of potentials* $q \in L_2$, but the method used to prove the estimates heavily relies on preliminary results obtained when studying the inverse problems for potentials q in the entire scale of Sobolev spaces W^α_2.

The history of analysis of inverse problems for the Sturm–Liouville operator goes back to Ambarzumian's paper [3]. However, the result in that paper proved to be atypical of the theory. A pioneering role was played by Borg's fundamental paper [4], whose main result is a uniqueness theorem for the reconstruction of the potential from two spectra. Another interpretation of Borg's results was suggested by Levinson [27]. Tikhonov [49] showed that the potential is uniquely determined by the Weyl–Titchmarsh function. Marchenko ([32], [33]) was the first to use the transformation operator in the study of inverse problems and prove the uniqueness of the solution of the inverse problem with given spectral function for Sturm–Liouville operators on a finite interval, as well as on the entire line. Gelfand and Levitan [13] found necessary and sufficient conditions for the reconstruction of the potential from the spectral function and presented explicit equations for the solution of the reconstruction problem. Levitan [28] and Gasymov with Levitan [12] obtained similar results for Borg's problem of reconstructing the potential from two spectra. A complete solution of Borg's problem for potentials in L_2 was obtained by Marchenko [34]. Krein ([25], [26]) suggested other formulas for solving inverse problems. In a series of papers, Trubowitz and his coauthors suggested a method for solving some inverse problems on a finite interval using the language of analytic mappings. A detailed exposition can be found in the book [40] by Pöschel and Trubowitz. Among the recent papers developing this method we mention Korotyaev and Chelkak's paper [24]. The inverse scattering problem studied by Faddeev ([9], [10]), Deift and E. Trubowitz [6], and Marchenko [34] (see [34] for more comprehensive information) played an important role in the solution of nonlinear equations. There are quite a few papers dealing with direct and inverse problems for Sturm–Liouville operators in impedance form. Note that there is a relationship between such operators and the usual Sturm–Liouville operators with singular potentials. The paper [1] by Albeverio, Hryniv, and Mykytyuk is one of the most recent papers on the topic; it contains numerous references.

In [43] the present authors suggested a regularization method for determining the Sturm–Liouville operator with distribution potentials $q \in W^{-1}_2$. Hryniv and Mykytyuk ([18], [21]) proved the existence of a transformation operator for equations with such potentials and gave solutions of the classical inverse problems for potentials $q \in W^{-1}_2$ (see, e.g., [19], [20], and [22]). Marchenko and Ostrovskii ([36], [34]) described the spectral data of Borg's problem for potentials q in the Sobolev spaces W^α_2 with integer smoothness indices $\alpha = 0, 1, 2, \dots$. Similar results for inverse problems with given spectral functions were obtained by Freiling and Yurko [11]. The present authors [45] introduced the scale of spaces $l^{\alpha+1}_B$ for studying the spectral data of Borg's problem and analyzed the problem in terms of these spaces for all smoothness indices $\alpha \geqslant -1$. Hryniv and Mykytyuk [23] used different terms and a different method to study Borg's problem and also the inverse problem with given spectral function for smoothness indices $\alpha \in [-1, 0]$.

Various a priori estimates of local character for inverse problems were obtained by numerous authors. Without going into much detail, we mention that results in this direction were obtained by Marchenko and Maslov [35], Ryabushko ([41], [42]), Hochstadt [17], Hald [15], Yurko [50] (see also [11]), Mizutani [39], Alekseev [2], Mclaughlin [30], Hitrik [16], Marletta and Weikard [38], and Malamud [31].

Speaking of inverse problems on a finite interval, one should mention the problem of reconstructing the potential from the two spectra of the periodic and antiperiodic problems. Naturally, it is related to the study of the Hill operator on the entire line. There are many interesting papers concerned with this problem, and it has been studied most comprehensively. We emphasize important results due to Marchenko and Ostrovskii ([36], [37]). The inverse problem for the periodic case is the only one for which uniform a priori estimates of the difference of potentials via the difference of spectral data have been obtained (see [37]). Among the recent publications on the periodic problem we mention the paper [7] by Djakov and Mityagin, where, along with new important results, detailed information and a bibliography are given for the case of classical potentials, and also their paper [8], where singular potentials are considered. For more details concerning the inverse problems considered there, as well as other inverse problems, we refer the reader to the monographs [34] by Marchenko, [29] by Levitan, and [11] by Freiling and Yurko, as well as to Gesztesy's survey paper [14].

The present paper is a continuation of the authors' series of papers [45]–[47], which deal with the solution of inverse problems with potentials in Sobolev spaces. In these papers, the spaces where the regularized spectral data of the two problems under consideration should be placed were constructed and properties of the mappings taking a primitive $\sigma = \int q(t)\,dt$ of the potential to the regularized spectral data were studied. The key assertion is Theorem 1.3 (stated below in a convenient form) that these mappings are weakly nonlinear. (This theorem was proved for various problems in [46] and [47].) As we have mentioned already, the solution of Borg's problem for potentials $q \in W_2^\alpha$ in the entire scale $\alpha \geqslant -1$ was given by the authors in [45]. Injectivity was proved by a modification of Borg's method, and ideas due to Trubowitz and his coauthors were developed to describe the range and the reconstruction procedure. In the present paper we supplement the studies in [45] of Borg's problem; in particular, we prove local stability for all smoothness indices $\alpha \geqslant -1$. We also show that the solution in Sobolev spaces of the inverse problem with given spectral function of the Dirichlet operator can be carried out for all $\alpha \geqslant -1$ by the same scheme as the solution of Borg's problem. However, when implementing this scheme, the proofs of some similar claims require new approaches. *Our main goal is uniform a priori estimates*, *which we obtain for* $\alpha > -1$. To prove these, we develop a new method based on weak nonlinearity theorems for mappings which we construct.

The first section is auxiliary. We recall the main definitions and constructions of spaces and present the results of [45]–[47] needed in what follows in a convenient form. Section 2 is the main part of the paper. Here we supplement the results in [45] concerning Borg's problem and prove local and uniform a priori estimates for this problem. In Section 3 all results for Borg's problem are extended to the inverse problem with given spectral function of the Dirichlet operator.

1. Definitions of Spaces and Nonlinear Mappings Related to Inverse Problems. Theorems on Properties of Such Mappings

First, recall that the definition of the Sturm–Liouville operator with classical potential $q \in L_1[0,\pi]$ can be extended to distribution potentials in the Sobolev space $W_2^{-1}[0,\pi]$. Assume that a complex-valued potential q belongs to the Sobolev space $W_2^\alpha[0,\pi]$ for some $\alpha \geqslant -1$. Let $\sigma(x) = \int q(x)\,dx$, where the primitive is understood in the sense of distributions. Following [43] (see also [44], where alternative definitions are given), we define the Dirichlet operator by the formula

$$L_D y = Ly = -(y^{[1]})' - \sigma(x)y^{[1]} - \sigma^2(x)y, \qquad y^{[1]}(x) := y'(x) - \sigma(x)y(x), \tag{1.1}$$

with domain

$$\mathscr{D}(L_D) = \{y,\, y^{[1]} \in W_1^1[0,\pi] \mid Ly \in L_2[0,\pi],\, y(0) = y(\pi) = 0\}.$$

The Dirichlet–Neumann operator is defined in the same way, by $L_{DN}y = Ly$, on the domain

$$\mathscr{D}(L_{DN}) = \{y,\, y^{[1]} \in W_1^1[0,\pi] \mid Ly \in L_2[0,\pi],\, y(0) = y^{[1]}(\pi) = 0\}.$$

For smooth functions σ, the right-hand sides in (0.1) and (1.1) coincide, and we obtain the classical Sturm–Liouville operator with Dirichlet boundary conditions in the first case and the operator with boundary condition $y(0) = 0$ and the Robin boundary condition $y'(\pi) - hy(\pi) = 0$, where $h = \sigma(\pi)$, in the second case. The operator is independent of the choice of the constant in the definition of the primitive σ of the potential q in the first case but depends on it in the second case. If this constant is chosen so that $\sigma(\pi) = 0$, then we obtain the classical Dirichlet–Neumann operator.

Now, let us define spectral data for the problems considered in the paper. Let $s(x, \lambda)$ be the unique solution of the equation $Ly - \lambda y = 0$ with conditions $s(0, \lambda) = 0$ and $s^{[1]}(0, \lambda) = \sqrt{\lambda}$. (The existence and uniqueness of such a solution is known [43].) Obviously, the zeros $\{\lambda_k\}_1^\infty$ and $\{\mu_k\}_1^\infty$ of the entire functions $s(\pi, \lambda)/\sqrt{\lambda}$ and $s^{[1]}(\pi, \lambda)/\sqrt{\lambda}$ are the eigenvalues of the operators L_D and L_{DN}, respectively. For a real-valued potential q, all zeros of these functions are simple and real, and we assume that they are numbered in strictly ascending order. For complex-valued q, one can number them so that the sequences $\{|\lambda_k|\}_1^\infty$ and $\{|\mu_k|\}_1^\infty$ are nondecreasing.

In Borg's problem, the potential should be reconstructed from the two spectra $\{\lambda_k\}$ and $\{\mu_k\}$ of the operators L_D and L_{DN}. Specifying these two spectra is equivalent to specifying the numbers

$$s_{2k-1} = \sqrt{\mu_k} - (k - 1/2), \quad s_{2k} = \sqrt{\lambda_k} - k, \qquad k = 1, 2, \dots .$$

i.e., the sequence $\{s_k\}_1^\infty = \{s_k(B)\}_1^\infty$. We say that such a sequence defines the *regularized spectral data* of Borg's problem. Here and in what follows, we assume that the branch of the square root is chosen in such a way that the argument of $\sqrt{\lambda}$ ranges in $(-\pi/2, \pi/2]$.

It is known [28, Chap. 3] that the spectral function of the Dirichlet operator can be uniquely reconstructed from the eigenvalues of this operator and the so-called *normalizing constants* determined by the formulas

$$\alpha_k = \int_0^\pi s^2(x, \lambda_k)\, dx.$$

We retain this definition of normalizing constants for complex-valued potentials as well. The sequences $\{\lambda_k\}_1^\infty \cup \{\alpha_k\}_1^\infty$ form the spectral data of the operator L_D. Specifying these data is equivalent to specifying the numbers

$$s_{2k} = \sqrt{\lambda_k} - k, \quad s_{2k-1} = \alpha_k - \pi/2, \qquad k = 1, 2, \dots, \tag{1.2}$$

We say that the sequence $\{s_k\}_1^\infty = \{s_k(D)\}_1^\infty$ defines the *regularized spectral data of the operator L_D*.

Thus, we come to two problems: to reconstruct a primitive of the potential q either from the regularized spectral data of the operator L_D or from the spectral data of Borg's problem. Clearly, the reconstruction of q is impossible in the singular case, and one should deal with its primitive $\sigma = \int q(x)\, dx$. For $q \in W_2^\alpha$, $\alpha \geqslant -1$, we have $\sigma \in W_2^\theta$, where $\theta = \alpha + 1 \geqslant 0$. The case of a classical potential $q \in L_2$ corresponds to the exponent $\theta = 1$. We also note that the passage to the reconstruction of the primitive changes the statement of the problem. For example, when reconstructing the differentiable function σ from the spectral data of Borg's problem, one reconstructs not only the potential $q = \sigma'$ but also the constant $h = \sigma(\pi)$ in the Robin boundary condition, while from the spectral data of the operator L_D the function σ can only be reconstructed up to a constant.

To use the language of the theory of mappings in what follows, we should understand in what spaces the above-defined spectral data lie as the primitive σ ranges over W_2^θ, $\theta \geqslant 0$. It turns out that, in both problems, for these spaces one can take finite-dimensional extensions of ordinary weighted l_2-spaces. The particular construction of extensions is determined by analyzing the asymptotic formulas for the eigenvalues λ_n and μ_n and the normalizing constants α_n. This is explained in detail in [46] and [47]. The construction of extensions can also be understood from Theorem 1.2 below, after integrating by parts the formulas defining the operators T_B and T_D.

Let us construct the space for the regularized spectral data of Borg's problem. Let l_2^θ be the weighted l_2-space of sequences $\mathbf{x} = \{x_1, x_2, \dots\}$ of complex numbers such that

$$\|\mathbf{x}\|_\theta^2 := \sum_1^\infty |x_k|^2 k^{2\theta} < \infty.$$

Consider the special sequences

$$\mathbf{e}_{2s-1} = \{k^{-(2s-1)}\}_{k=1}^{\infty}, \quad \mathbf{e}_{2s} = \{(-1)^k\, k^{-(2s-1)}\}_{k=1}^{\infty}, \qquad s = 1, 2, \dots .$$

Let $m = [\theta/2 + 3/4]$, where $[a]$ is the integer part of a number a. We set

$$l_B^\theta = l_2^\theta \oplus \operatorname{span}\{\mathbf{e}_k\}_{k=1}^{2m}.$$

Here we have taken into account the fact that the sequences $\mathbf{e}_k$ do not belong to the space l_2^θ for $k \leqslant 2m$ and belong to it for $k > 2m$. Thus, l_B^θ consists of the elements $\mathbf{x} + \sum_{k=1}^m c_k \mathbf{e}_k$, where $\mathbf{x} \in l_2^\theta$ and the $\{c_k\}_1^m$ are arbitrary complex numbers. The inner product of elements of l_B^θ is determined by the formula

$$\left(\mathbf{x} + \sum_{k=1}^m c_k \mathbf{e}_k, \mathbf{y} + \sum_{k=1}^m d_k \mathbf{e}_k\right) = (\mathbf{x}, \mathbf{y})_\theta + \sum_{k=1}^m c_k \overline{d_k}.$$

We associate the space thus constructed with the regularized spectral data of the operator L_B. Although this space is defined as a finite-dimensional extension of the weighted space l_2^θ, it is convenient to write its elements in the form of usual sequences. For example, if $3/2 \leqslant \theta < 5/2$, then l_B^θ consists of sequences $\mathbf{x} = \{x_k\}_1^\infty$ with coordinates

$$x_k = y_k + \alpha_1 k^{-1} + \alpha_2 (-1)^k k^{-1}, \quad \text{where } \{y_k\}_1^\infty \in l_2^\theta \quad \text{and} \quad \alpha_1, \alpha_2 \in \mathbb{C}.$$

It readily follows from this representation that l_D^η is compactly embedded in l_D^θ for $\eta > \theta$. (Here we have taken into account the compactness of the embedding $l_2^\eta \hookrightarrow l_2^\theta$, $\eta > \theta$.)

To construct the space l_D^θ of regularized spectral data for the Dirichlet operator, one should use the sequences

$$\widehat{\mathbf{e}}_{2s-1} = \{0, 2^{-(2s-1)}, 0, 4^{-(2s-1)}, 0, 6^{-(2s-1)}, \dots\}, \qquad \widehat{\mathbf{e}}_{2s} = \{2^{-(2s)}, 0, 4^{-(2s)}, 0, 6^{-(2s)}, \dots\}$$

instead of $\mathbf{e}_k$. We define l_D^θ by the formula $l_D^\theta = l_2^\theta \oplus \operatorname{span}\{\widehat{\mathbf{e}}_k\}_{k=1}^m$, where the number m is uniquely determined by the condition $m - 1/2 \leqslant \theta < m + 1/2$. Note that in [47] the space containing the regularized spectral data of L_D^θ was constructed in a space of two-sided sequences. Here we have implemented an equivalent construction in a space of one-sided sequences, so that the two spaces look similar.

We define the nonlinear operators

$$F_B(\sigma) = \{s_k(B)\}_1^\infty, \qquad F_D(\sigma) = \{s_k(D)\}_1^\infty. \tag{1.3}$$

It follows from results of [44] and [18] that the sequences formed by the regularized spectral data on the right-hand sides in (1.3) lie in l_2 for every primitive $\sigma = \int q(x)\,dx \in L_2(0, \pi)$. Hence both operators in (1.3) are well-defined as operators from L_2 to l_2. Moreover, according to [45] and [47], the ranges of the restrictions of these operators to the Sobolev spaces W_2^θ, $\theta > 0$, lie in l_B^θ and l_D^θ, respectively. It is for this purpose that we have extended the spaces l_2^θ. The corresponding result does not hold if we do not add special sequences to l_2^θ.

In the following, we use results of [45]–[47], which we give in the form needed for our purposes.

Theorem 1.1. *For each $\theta \geqslant 0$, the nonlinear operators F_B and F_D are well defined as operators from W_2^θ into l_B^θ and l_D^θ, respectively. These operators are Fréchet differentiable at each point (function) σ, provided that this function is real-valued and all eigenvalues $\lambda_k(\sigma)$ and $\mu_k(\sigma)$ are nonzero. (For the mapping F_D, it suffices that the $\lambda_k(\sigma)$ alone be nonzero.) In particular, these operators are Fréchet differentiable at the point $\sigma = 0$, and their Fréchet derivatives at this point* *are the linear operators T_B and T_D given by the formulas*

$$(T_B\sigma)_k = -\frac{1}{\pi}\int_0^\pi \sigma(t)\sin(kt)\,dt, \quad k = 1,2,\dots,$$

$$\begin{cases} (T_D\sigma)_{2k-1} = -\int_0^\pi (\pi - t)\sigma(t)\cos(2kt)\,dt, & k = 1,2,\dots, \\ (T_D\sigma)_{2k} = -\frac{1}{\pi}\int_0^\pi \sigma(t)\sin(2kt)\,dt, & k = 1,2,\dots. \end{cases}$$

Proof. The claim follows from [46, Proposition 1 and Theorem 6.1] for the operator F_B and from [47, Proposition 1 and Theorem 4.2] for the operator F_D. □

Theorem 1.2. *The spaces l^θ_B and l^θ_D form a scale of spaces compactly embedded in one another and closed with respect to interpolation; i.e., $[l^0, l^\theta]_\tau = l^{\theta\tau}$ for all $\theta \geqslant 0$ and $\tau \in [0,1]$. (Here we omit the subscripts B or D for brevity.) For each $\theta \geqslant 0$, the operator T_B isomorphically maps the space W^θ_2 onto l^θ_B. The operator T_D isomorphically maps the space $W^\theta_2 \ominus \{1\}$ onto l^θ_D.*

Proof. The first claim of the theorem was proved for the space l^θ_B in [46, Proposition 4]. The proof carries over verbatim to the space l^θ_D. The second claim was proved in [46, Lemma 1] for the operator T_B and in [47, Proposition 3] for the operator T_D. □

The following theorem is the most important point in the proof of the main results of the present paper. In particular, it says that the mappings F_B and F_D are weakly nonlinear; i.e., they are compact perturbations of linear mappings. The exact dependence $\tau = \tau(\theta)$, which characterizes the "quality" of compactness, is important as well.

Theorem 1.3. *For each $\theta \geqslant 0$, the operator F_B maps W^θ_2 into l^θ_B and admits a representation of the form*

$$F_B(\sigma) = T_B\sigma + \Phi_B(\sigma).$$

Here T_B is the linear operator defined in Theorem 1.1 and Φ_B maps W^θ_2 into l^τ_B, where

$$\tau = \begin{cases} 2\theta & \text{if } 0 \leqslant \theta \leqslant 1, \\ \theta + 1 & \text{if } 1 \leqslant \theta < \infty. \end{cases}$$

Moreover, the mapping $\Phi_B\colon W^\theta_2 \to l^\tau_B$ is bounded in every ball; i.e., $\|\Phi(\sigma)_B\|_\tau \leq C(R)$ whenever $\|\sigma\|_\theta \leqslant R$, where the constant C depends only on the radius R of the ball. A similar assertion holds for the operator F_D. Namely, $F_D(\sigma) = T_D\sigma + \Phi_D(\sigma)$, and the mapping $\Phi_D\colon W^\theta_2 \ominus \{1\} \to l^\tau_D$ has the same property as Φ_B.

Proof. This theorem was proved in [46] for the operator F_B and in [47] for the operator F_D. For $\theta > 0$, the compactness of the nonlinear terms in the representations of the operators F_B and F_D follows from the compactness of the embeddings $l^\eta \hookrightarrow l^\theta$ for $\eta > \theta$. (We omit the subscript B or D for brevity.) The case of $\theta = 0$ is exceptional; for $\theta = 0$, this theorem does not imply the compactness of the nonlinear terms. □

2. Borg's Problem. Characterization of Spectral Data for the Primitives σ of Real-Valued Potentials $q \in W^\alpha_2$. Uniform A Priori Estimates

We use the following notation in this section and in Section 3. We denote the set of real-valued functions in W^θ_2 by $W^\theta_{2,\mathbb{R}}$, the closed ball of radius R centered at the origin in $W^\theta_{2,\mathbb{R}}$ by $\mathscr{B}^\theta_{\mathbb{R}}(R)$, the set of all functions in $W^\theta_{2,\mathbb{R}}$ for which $\mu_1(\sigma) \geqslant 1/4$ by Γ^θ_B, and the intersection of Γ^θ_B with $\mathscr{B}^\theta_{\mathbb{R}}(R)$ by $\mathscr{B}^\theta_\Gamma(R)$. Here $\mu_1(\sigma)$ is the first eigenvalue of L_{DN}. The number $1/4$ has been taken for definiteness and simplicity; it can be replaced by any number $\eta > 0$, but then one should write $s_1 \geqslant \sqrt{\eta} - 1/2$ in (2.2) and (2.3).

The spectra $\{\lambda_k\}$ and $\{\mu_k\}$ of the operators L_D and L_{DN} are known to satisfy the interlacing condition

$$\mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \dots < \mu_n < \lambda_n < \mu_{n+1} < \dots, \tag{2.1}$$

provided that the potential is real-valued. This fact has long been known for classical potentials (see, e.g., [33]), while for singular distribution potentials, it was proved in [20] and [45]. Note that, for positive λ_k and μ_k, inequalities (2.1) are equivalent to the same inequalities for the square roots of these numbers. Hence conditions (2.1), together with the condition $\mu_1 \geqslant 1/4$ (i.e., the condition $\sigma \in \Gamma^0_B$), are equivalent to the inequalities

$$s_1 \geqslant 0, \qquad s_k - s_{k+1} < 1/2, \qquad k = 1, 2, \dots, \tag{2.2}$$

where the $\{s_k\} = \{s_k(B)\}$ are the regularized spectral data of Borg's problem. The sequence $\{s_k\}_1^\infty$ belongs to l_2, and hence for each real-valued function $\sigma \in L_2$, there exists a number $h = h(\sigma) > 0$ such that

$$s_1 \geqslant 0, \qquad s_k - s_{k+1} \leqslant 1/2 - h, \qquad k = 1, 2, \dots. \tag{2.3}$$

Fix arbitrary numbers $r > 0$ and $h \in (0, 1/2)$. Let $\Omega^\theta_B(r, h)$ be the set of real sequences $\{s_k\}_1^\infty$ satisfying inequalities (2.3) and lying in the closed ball of radius r in l^θ_B, i.e., such that $\|\{s_k\}\|_\theta \leqslant r$. By Ω^θ_B we denote the set of all real sequences $\{s_k\}_1^\infty \in l^\theta_B$ satisfying inequalities (2.2).

Recall that we have associated Borg's problem with the operator $F_B \colon W^\theta_2 \to l^\theta_B$, $F_B(\sigma) = \{s_k\}_1^\infty$, where $\{s_k\}_1^\infty$ is the regularized spectral data of Borg's problem. It follows from the preceding considerations and Theorem 1.3 that F_B maps Γ^θ_B into Ω^θ_B.

Further in this section, *we omit the subscript* B wherever convenient, because we only work with Borg's problem. In particular, the operators F_B, T_B, and Φ_B (Theorem 1.3) will be denoted by F, T, and Φ, respectively. We everywhere write Γ^θ, Ω^θ, and $\Omega^\theta(r, h)$ instead of Γ^θ_B, Ω^θ_B, and $\Omega^\theta_B(r, h)$. However, we retain the symbol l^θ_B for the spaces of regularized spectral data.

Theorem 2.1. *The mapping* $F \colon \Gamma^\theta \to \Omega^\theta$ *is bijective for each* $\theta \geqslant 0$.

Proof. The injectivity of $F \colon \Gamma^\theta \to \Omega^\theta$ was proved in [45, Lemma 6]. The proof of the surjectivity of this mapping is contained in the proof of Lemma 5 in the same paper, but this proof needs a supplement for $\theta \geqslant 1/2$. For $\theta < 1/2$, one has $l^\theta_B = l^\theta_2$, while for $\theta \geqslant 1/2$, l^θ_B additionally contains the $2m$-dimensional subspace $\mathscr{L}^{2m}$ of special sequences, where $m = [\theta/2 + 3/4]$. By analyzing the proof of Lemma 5 in [45], we conclude that to complete this proof one should be able to reconstruct the function σ (or prove its existence) in the case where only the coordinates in the subspace $\mathscr{L}^{2m}$ are varied, while all coordinates in l^θ_2 remain unchanged. The authors cannot see a simple straightforward solution of this problem, not using laborious theorems. Here we prove surjectivity with the use of Theorem 1.3, having in mind that, for the case $\theta \in [0, 1/2)$, this property has already been proved.

We know that the mapping $F \colon \Gamma^\theta \to \Omega^\theta$ is surjective for $\theta \leqslant 1/4$. Let us prove that it is surjective for every $\theta \in (1/4, 1/2]$. Take an arbitrary element $\mathbf{y} \in \Omega^\theta \subset l^\theta_B$, $\theta \in (1/4, 1/2]$. Since the mapping in question is a bijection for $\theta = 1/4$, it follows that there exists a unique function $\sigma \in \Gamma^{1/4}$ such that $F\sigma = \mathbf{y}$. (Here we have taken into account the embedding $l^\theta_B \hookrightarrow l^{1/4}_B$.) By Theorem 1.3 we have $T\sigma = -\Phi\sigma + \mathbf{y} \in l^\theta_B$, since $\mathbf{y} \in l^\theta_B$ and it follows from the condition $\sigma \in W^{1/4}_2$ that $\Phi\sigma \in l^{1/2}_B \hookleftarrow l^\theta_B$. But the linear operator $T \colon W^\theta_2 \to l^\theta_B$ is an isomorphism by Theorem 1.2. Consequently, $\sigma \in W^\theta_{2,\mathbb{R}}$, and so, since $\mathbf{y} \in \Omega^\theta$, we have $\sigma \in \Gamma^\theta$. Thus, we have proved that F is surjective for $\theta \in (1/4, 1/2]$. Now, knowing that the mapping $F \colon \Gamma^\theta \to \Omega^\theta$ is surjective for $\theta \in [0, 1/2]$, we can use the same trick to prove its surjectivity for $\theta \in (1/2, 1]$. By repeating this trick, we show its surjectivity for $\theta \in (1, 2]$ with the use of Theorem 1.3. At the $(k+1)$st step, we obtain surjectivity for $\theta \in (k-1, k]$. Here the number k is arbitrary, and hence the claim holds for all $\theta \geqslant 0$. □

Let $\widehat{\Omega}^\theta_B$ be the set of sequences $\{s_k\}_1^\infty \in l^\theta_B$ for which the numbers $\mu_k = (s_{2k-1} + k - 1/2)^2$ and $\lambda_k = (s_{2k-1} + k - 1/2)^2$ are real and satisfy conditions (2.1).

Note that if we add the function $c(x - \pi)$ to the function σ determining the operators L_D and L_{DN}, then these operators turn into $L_D + c$ and $L_{DN} + c$, respectively; i.e., their spectra are shifted by c. We set

$$s_{2k-1}(c)=\sqrt{\mu_k+c}-(k-1/2),\qquad s_{2k}(c)=\sqrt{\lambda_k+c}-k. \tag{2.4}$$

Since $c(x-\pi)\in W_2^\theta$ for all $\theta\geqslant 0$, it follows that $\{s_k(c)\}_1^\infty\in l_B^\theta$ if and only if $\{s_k(0)\}_1^\infty\in l_B^\theta$. Consequently, $\{s_k\}_1^\infty\in\widehat{\Omega}^\theta$ if and only if there exists a $c\geqslant 0$ such that $\{s_k(c)\}_1^\infty\in\Omega^\theta$. Our remarks imply the following claim.

Theorem 2.2. *The mapping $F\colon W_{2,\mathbb{R}}^\theta\to\widehat{\Omega}^\theta$ is a bijection. Two numerical sequences $\{\mu_k\}_1^\infty$ and $\{\lambda_k\}_1^\infty$ are the spectra of L_D and L_{DN} if an only if they satisfy the interlacing conditions* (2.1) *and $\{s_k\}_1^\infty\in l_B^\theta$.*

Marchenko and Ostrovskii ([36], [34]) characterized the spectral data of Borg's problem for positive integer $\theta=1,2,\dots$ in a different form, without using the spaces l_B^θ. One can show that, for these θ, their result combined with Borg's uniqueness theorem is equivalent to Theorem 2.2.

In what follows, we essentially use the analytic properties of the mapping F. We assume that the reader is familiar with the definitions of Fréchet and Gâteaux derivatives of a mapping $F\colon U\to H$, where U is an open subset of E and E and H are separable Hilbert spaces. For complex Hilbert spaces, the Fréchet derivative in the complex sense is naturally defined. A mapping $F\colon U\to H$ is said to be *analytic* if the complex Fréchet derivative of F exists at each point $x\in U$. The Fréchet derivative at x will be denoted by $F'(x)$. The notion of a real-analytic mapping is defined in a natural way; see, e.g., [40]. A mapping $F\colon U\to H$ is said to be *weakly analytic* if the coordinate functions $(F(x),e_k)$, where $\{e_k\}_1^\infty$ is an orthonormal basis in H, are Gâteaux differentiable in the complex sense. The following result of [40] significantly simplifies the verification of analyticity of a mapping.

Proposition 2.3. *If $F\colon U\to H$ is a weakly analytic mapping locally bounded at each point $x\in U$, then F is analytic.*

In what follows, we deal with mappings of closed sets. To avoid additional explanations, we shall say that a mapping $F\colon D\to H$ is analytic on D if there exists an open set U such that $U\supset D$ and $F\colon U\to H$ is analytic.

Theorem 2.4. *If $\theta\geqslant 0$ and $\sigma\in\Gamma^\theta$, then there exists a complex neighborhood $U\in W_2^\theta$ of σ such that the mapping $F\colon U\to l_B^\theta$ is differentiable in the complex sense at all points of this neighborhood. Thus, the mapping $F\colon\Gamma^\theta\to l_B^\theta$ is real-analytic, as well as the mapping $\Phi=F-T\colon\Gamma^\theta\to l_B^\tau$, where $T=T_B$ and τ are defined in Theorem* 1.3. *The derivative at $\sigma\in\Gamma$ is given by*

$$[F'(\sigma)]f=\left\{-\frac{(y_k'(x)y_k(x),\overline{f(x)})}{\rho_k\,(y_k^2(x),1)}\right\}_{k=1}^\infty. \tag{2.5}$$

Here $\rho_{2n-1}=\sqrt{\mu_n}$, $\rho_{2n}=\sqrt{\lambda_n}$, the $y_{2n-1}(x)$ are the eigenfunctions of the operator L_{DN}, the y_{2n} are the eigenfunctions of the operator L_D, and $f\in W_2^\theta$ is a function on which the operator $F'(\sigma)$ acts.

Proof. The theorem was proved in [46, Sec. 5]. The proof is based on using Theorem 1.3 and Proposition 2.3 after calculating the derivatives of the coordinates. Here it is important that the denominators in formula (2.5) vanish nowhere for a real-valued function σ. According to [43], the eigenfunctions continuously depend on the primitive σ of the potential, and hence the numbers $(y_k^2(x),1)$ do not vanish in some complex neighborhood. (One should also take into account the asymptotics of y_k as $k\to\infty$).) The theorem remains valid if, instead of the condition $\sigma\in\Gamma^\theta$, one requires that σ be real-valued and that none of the ρ_k be zero. In that case, one has usual analyticity rather than real analyticity. □

Lemma 2.5. *If the functions $y_k(x)$ in Theorem* 2.4 *are normalized by the conditions $y_k^{[1]}(0)=1$, then the system of functions*

$$\varphi_k(x)=\frac{2}{\pi}\,y_k'(x)y_k(x),\qquad k=1,2,\dots, \tag{2.6}$$

is a Riesz basis in $L_2(0,\pi)$*. The system biorthogonal to* $\{\varphi_k(x)\}_1^\infty$ *has the form*

$$\psi_k(x) = \pi\rho_k^{1/2} y_k(x) w_k(x), \tag{2.7}$$

where the function w_k *is the solution of the equation* $-y'' + \sigma' y = \lambda_n y$ *with initial conditions*

$$w_k^{[1]}(\pi) = 0, \qquad w_k(\pi) = \left(\int_0^\pi y_k^2(x)\,dx \cdot y_k^{[1]}(\pi)\right)^{-1}$$

for $k = 2n$ *and of the equation* $-y'' + \sigma' y = \mu_n y$ *with initial conditions*

$$w_k(\pi) = 0, \qquad w_k^{[1]}(\pi) = -\left(\int_0^\pi y_k^2(x)\,dx \cdot y_k(\pi)\right)^{-1}$$

for $k = 2n-1$*.*

Proof. The first claim, that the system $\{\varphi_k(x)\}_1^\infty$ is a Riesz basis, was proved in [45, Lemma 6], where the relations $(\varphi_k(x), \psi_m(x)) = 0$ for $k \neq m$ were proved as well. The relations $(\varphi_k(x), \psi_k(x)) = 1$ can be proved by straightforward computations, which we omit, because the specific form of the functions φ_k and ψ_k will not be used here. □

Theorem 2.6. *Let* $\theta \geqslant 0$*. Then each point* $\mathbf{y}_0 \in \Omega^\theta = F(\Gamma^\theta)$ *has a complex neighborhood* $U(\mathbf{y}_0)$ *in which the inverse mapping* $F^{-1}(\mathbf{y})$ *is defined and has Fréchet derivative in the complex sense. This derivative has the form*

$$(F^{-1})'(\mathbf{y}) = (F')^{-1}(\mathbf{y}) = \sum_{k=1}^\infty s_k \widetilde{\psi}_k(x), \qquad \mathbf{y} = (s_1, s_2, \dots). \tag{2.8}$$

Here $\widetilde{\psi}_k(x) = \gamma_k \psi_k(x)$*, where* $\{\psi_k(x)\}_1^\infty$ *is the biorthogonal system specified in Lemma* 2.5 *and* $\gamma_k = \rho_k \int_0^1 y_k^2(x)\,dx$*.*

Proof. First, suppose that $\theta > 0$. We have

$$F'(\sigma_0) = T + \Phi'(\sigma_0), \qquad \mathbf{y}_0 = F(\sigma_0).$$

The operator $T\colon W_2^\theta \to l_B^\theta$ is an isomorphism by Theorem 1.2, and the operator $\Phi'(\sigma_0)\colon W_2^\theta \to l_B^\tau$ is bounded by Theorem 2.4; hence the operator $\Phi'(\sigma_0)\colon W_2^\theta \to l_B^\theta$ is compact. Consequently, $F'(\sigma_0)$ is a Fredholm operator of index zero, and hence it is invertible if its kernel is trivial. It follows from formulas (2.5) and the completeness of system (2.6) in the space L_2 that the relation $F'(\sigma_0)f = 0$ for $f \in L_2$ implies that $f = 0$. This is all the more true if $f \in W_2^\theta$ for $\theta > 0$. Now (2.8) can be obtained by a straightforward verification. It suffices to verify that $F'(\sigma_0)(F^{-1})'(\mathbf{y}_0) = \mathbf{y}_0$. This readily follows from (2.5) and (2.8) with regard to the mutual biorthogonality of the systems $\{\gamma_k^{-1}\varphi_k\}_1^\infty$ and $\{\gamma_k\psi_k\}_1^\infty$.

Now, let $\theta = 0$. It readily follows from the asymptotic formulas obtained in [44, Theorems 2.6 and 2.7] for the eigenvalues ρ_k^2 and the eigenfunctions y_k that $\gamma_k \asymp 1$, provided that the functions y_k are normalized by the condition $y_k^{[1]}(0) = 1$. Hence it follows from Lemma 2.5 that the system $\{\widetilde{\psi}_k\}_1^\infty$ is a Riesz basis, and the boundedness of the operator $(F')^{-1}(\mathbf{y}_0)$ given by formula (2.8) follows from the definition of Riesz basis. The existence of the inverse operator for every $\theta \geqslant 0$ in a small complex neighborhood of the point $\mathbf{y}_0$ and its complex differentiability follow from the inverse mapping theorem. □

Note that Theorem 2.6 readily implies local estimates of the difference of potentials via the difference of spectral data and vice versa. As was noted in the introduction, various forms of such estimates were proved by various methods in many papers for the classical case $\theta = 1$ ($q \in L_2$). However, these papers deal with the mappings $q \to \{\text{spectral data}\}$, while we study the mapping $\int q(t)\,dt = \sigma \to \{\text{spectral data}\}$, and hence our systems and formulas look differently.

In what follows, we show that, for $\theta > 0$, Theorem 1.3 can be used to obtain a substantially stronger result while avoiding technicalities with function systems.

Lemma 2.7. *Fix $\theta > 0$. For each $R > 0$, there exist positive numbers $r = r(R)$ and $h = h(R)$ such that*

$$F(\mathscr{B}^\theta_\Gamma(R)) \subset \Omega^\theta(r,h), \quad \text{where } \mathscr{B}^\theta_\Gamma(R) = \Gamma \cap \mathscr{B}^\theta_\mathbb{R}(R).$$

Proof. If $\|\sigma\|_\theta \leqslant R$, $\sigma \in \Gamma^\theta$, then it follows from Theorem 1.3 that $F\sigma = \mathbf{y} \in \Omega^\theta(r)$, where $r = r(R)$ depends on R but is independent of σ. It remains to show that, for all elements $\mathbf{y} = F\sigma$, $\sigma \in \mathscr{B}^\theta_\Gamma(R)$, inequalities (2.3) hold with some $h = h(R) > 0$ depending on R but independent of σ.

Note that there exists a number $N = N(\theta, r)$ such that the inequalities $s_{k+1} - s_k \leqslant 1/4$ hold for all $\mathbf{y} = (s_1, s_2, \dots) \in \Omega^\theta(r)$ and all $k \geqslant N$. (Here $1/4$ can be replaced by any $\varepsilon > 0$.) This readily follows from the definition of the norm in l^θ_B for $\theta > 0$. (See [46, Section 5] for details; the assertion fails for $\theta = 0$.) Now, assume that the assertion of the theorem is not true and there exist elements $\mathbf{y}^n = F\sigma_n$, $\sigma_n \in \mathscr{B}^\theta_\Gamma(R)$, such that $s^n_k - s^n_{k+1} \to 1/2$ as $n \to \infty$ for some fixed k, $1 \leqslant k < N$. (Here the s^n_k are the coordinates of the elements $\mathbf{y}^n$.) The ball in the space W^θ_2 is weakly compact, and hence the sequence $\{\sigma_n\}$ contains a weakly convergent subsequence. Without loss of generality, we assume that the sequence itself weakly converges to a function $\sigma \in W^\theta_{2,\mathbb{R}}$. Since the space W^θ_2 is compactly embedded in L_2, it follows that the sequence σ_n strongly converges to σ in the norm of L_2. Let the index k for which $s^n_k - s^n_{k+1} \to 1/2$ be, say, even, $k = 2p$. Then $\lambda_p(\sigma_n) - \mu_{p+1}(\sigma_n) \to 0$. By [43, Theorem 2], the convergence of the functions σ_n in L_2 implies the convergence of their eigenvalues; i.e., $\lambda_p(\sigma_n) \to \lambda_p(\sigma)$ and $\mu_{p+1}(\sigma_n) \to \mu_{p+1}(\sigma)$. Hence $s^n_k - s^n_{k+1} \to 1/2$ implies that $\lambda_p(\sigma) = \mu_{p+1}(\sigma)$, which is impossible in view of the interlacing condition (2.1). □

Lemma 2.8. *Let $\theta > 0$. The converse of Lemma 2.7 is true: for any numbers r and h, there exists a number $R > 0$ such that*

$$F^{-1}(\Omega^\theta(r,h)) \subset \mathscr{B}^\theta_\Gamma(R).$$

One has the representation $F^{-1} = T^{-1} + \Psi$, $\Psi\colon \Omega^\theta \to W^\tau_2$, where the number τ is defined in Theorem 1.3. The mapping $\Psi\colon \Omega^\theta \to W^\tau_2$ is analytic, and moreover,

$$\|\Psi\mathbf{y}\|_\tau \leqslant C\|\mathbf{y}\|_\theta \quad \text{for all } \mathbf{y} \in \Omega^\theta(r,h), \tag{2.9}$$

where the constant C depends only on r and h.

Proof. If the first claim of the lemma is false, then there exist elements $\mathbf{y}^n \in \Omega^\theta(r,h)$ such that $F^{-1}\mathbf{y}^n = \sigma_n$, $\|\sigma_n\|_\theta \to \infty$. To be definite, we assume that $\theta \in (0,1]$. For $\theta > 1$, the proof does not change; we only need to replace $\theta/2$ by $\theta - 1$ according to Theorem 1.3. Let us extract a subsequence of $\mathbf{y}^n$ weakly convergent in l^θ_B. We assume that the sequence itself weakly converges to some element $\mathbf{y} \in l^\theta_B$. Weak convergence implies coordinatewise convergence. It follows from the definition of the set $\Omega^\theta(h,r)$ and its closedness that $\mathbf{y} \in \Omega^\theta(h,r)$. By Theorem 2.1, there exists a function $\sigma \in \Gamma^\theta$ such that $F\sigma = \mathbf{y}$. The weak convergence $\mathbf{y}^n \rightharpoonup \mathbf{y}$ in l^θ_B implies the strong convergence $\mathbf{y}^n \to \mathbf{y}$ in the norm of $l^{\theta/2}_B$, and it follows from the analyticity (continuity would suffice) of the mapping $F^{-1}\colon \Omega^{\theta/2} \to \Gamma^{\theta/2}$ that $\|\sigma_n - \sigma\|_{\theta/2} \to 0$. By Theorem 1.3, $\|\Phi\sigma_n\|_\theta \leqslant \|\sigma_n\|_{\theta/2} \leqslant C$. Therefore,

$$\|T\sigma_n\|_\theta \leqslant \|\Phi\sigma_n\|_\theta + \|\mathbf{y}^n\|_\theta \leqslant C + C = 2C.$$

(We have again used Theorem 1.3 and the boundedness of a weakly convergent sequence.) Since the operator $T\colon W^\theta_2 \to l^\theta_B$ is an isomorphism, it follows that $\|\sigma_n\|_\theta \leqslant 2C$. This contradiction completes the proof of the first claim of the lemma.

Obviously, $\Psi = -T^{-1}\Phi F^{-1}$. Consequently, the mapping $\Psi\colon \Omega^\theta \to W^\tau_2$ is analytic as a composition of analytic mappings. We obtain the estimate $\|\Psi\mathbf{y}\|_\tau \leqslant C$ for all $\mathbf{y} \in \Omega^\theta(r,h)$ from the first claim of the lemma and the uniform boundedness of the mapping $\Phi\colon \Omega^\theta \to W^\tau_2$ on every ball. Since $\Psi(0) = 0$ and Ψ is analytic, we arrive at the estimate (2.9). □

The following claim is very simple, but it is convenient for us to state it separately.

Lemma 2.9. *Assume that X and X_1 are metric spaces, X is complete, and a function $\Phi\colon X \to X_1$ is continuous on X. If the set $U \subset X$ is precompact in X, then $\Phi\colon U \to X_1$ is uniformly continuous and uniformly bounded.*

Proof. Under the assumptions of the lemma, the closure $\overline{U}$ is a compact set in X, and the function $\Phi\colon \overline{U} \to X_1$ is continuous. Hence the claim follows from properties of continuous functions on compact sets. $\square$

Lemma 2.10. *Let $\theta > 0$. For any $R > 0$, one has the estimate*

$$\|F'(\sigma)\|_\theta \leqslant C \quad \text{for all } \sigma \in \mathscr{B}^\theta_\Gamma(R), \tag{2.10}$$

where the constant C depends on R but is independent of σ.

Proof. Without loss of generality, we assume that $\theta \in (0,1]$. If $\theta > 1$, then $\theta/2$ should be replaced by $\theta - 1$ in what follows. Since $F' = \Phi' + T$, it suffices to prove estimate (2.10) with Φ instead of F. By Theorem 2.3, the mapping $\Phi\colon W^{\theta/2} \to l^\theta_B$ is analytic on the closed set $\mathscr{B}^{\theta/2}_\Gamma(R_1)$ for every $R_1 > 0$, and hence the numerical function $\|\Phi'(\sigma)\|_\theta$ is continuous on that set. Since the embedding $W^\theta_2 \hookrightarrow W^{\theta/2}_2$ is continuous, it follows that there exists a number $R_1 = R_1(R,\theta)$ such that $\mathscr{B}^\theta_\Gamma(R) \subset \mathscr{B}^{\theta/2}_\Gamma(R_1)$. Here the first set is compact in the second, and therefore Lemma 2.9 provides estimate (2.10), where F should be replaced by Φ. $\square$

Lemma 2.11. *Let $\theta > 0$. Then, for any $r > 0$ and $h \in (0,1/2)$, the inverse mapping satisfies the estimate*

$$\|(F^{-1})'(\mathbf{y})\| \leqslant C \quad \text{for all } \mathbf{y} \in \Omega^\theta(r,h), \tag{2.11}$$

where the constant C depends on r and h but is independent of $\mathbf{y}$.

Proof. To be definite, consider the case of $\theta \in (0,1]$. The argument is similar to that used in the proof of Lemma 2.10. Fix some numbers $r > 0$ and $h \in (0,1/2)$. By using the continuity of the embedding $l^\theta_B \hookrightarrow l^{\theta/2}_B$, we find a number r_1 such that $\Omega^\theta(r,h) \subset \Omega^{\theta/2}(r_1,h)$. By Lemma 2.8, the mapping $\Psi = -F^{-1}\Phi T^{-1}\colon \Omega^{\theta/2}(r_1,h) \to W^\theta_2$ is analytic. Hence the numerical function $\|\Psi'(\mathbf{y})\|_\theta$ is continuous for $\mathbf{y} \in \Omega^{\theta/2}(r_1,h)$. By using Lemma 2.9 and the compactness of the embedding $\Omega^\theta(r,h) \subset \Omega^{\theta/2}(r_1,h)$, we obtain estimate (2.11) with F^{-1} replaced by Ψ. Since $F^{-1} = T^{-1} + \Psi$, it follows that the estimate is also true for F^{-1}. $\square$

Now we can prove the main result of this section.

Theorem 2.12. *Fix $\theta > 0$. Assume that the sequences $\mathbf{y}$ and $\mathbf{y}_1$ of regularized spectral data lie in $\Omega^\theta_B(r,h)$. Then the preimages $\sigma = F_B^{-1}\mathbf{y}$ and $\sigma_1 = F_B^{-1}\mathbf{y}_1$ lie in $\mathscr{B}^\theta_\Gamma(R)$, and one has the estimates*

$$C_1\|\mathbf{y} - \mathbf{y}_1\|_\theta \leqslant \|\sigma - \sigma_1\|_\theta \leqslant C_2\|\mathbf{y} - \mathbf{y}_1\|_\theta, \tag{2.12}$$

where the number R and the constants C_1 and C_2 depend only on r and h. The number R and the constants C_2 and C_1^{-1} increase as $r \to \infty$ or $h \to 0$. Conversely, if σ and σ_1 lie in the ball $\mathscr{B}^\theta_\mathbb{R}(R)$, then the sequences $\mathbf{y}$ and $\mathbf{y}_1$ of regularized spectral data of these functions lie in $\Omega^\theta(r,h)$, and one has the estimates

$$C_1\|\sigma - \sigma_1\|_\theta \leqslant \|\mathbf{y} - \mathbf{y}_1\|_\theta \leqslant C_2\|\sigma - \sigma_1\|_\theta. \tag{2.13}$$

Here the numbers $r > 0$ and $h \in (0,1/2)$ and the constants C_1 and C_2 depend only on R. The numbers r, h^{-1}, C_2, and C_1^{-1} increase as $R \to \infty$.

Proof. Note that the set $\Omega^\theta(r,h)$ is convex. For differentiable functions on convex sets, one has the following analog of Lagrange's theorem (see, e.g., [5, Corollary 12.2.8]):

$$\|\sigma - \sigma_1\| \leqslant \sup_{0<t<1} \|(F^{-1})'(t\mathbf{y} + (1-t)\mathbf{y}_1)\| \cdot \|\mathbf{y} - \mathbf{y}_1\|.$$

Thus, Lemma 2.11 implies the upper bound in inequality (2.12). The upper bounds in (2.13) can be obtained in a similar way from Lemma 2.10. Now the lower bounds in (2.12) and (2.13) follow from the upper bounds and Lemmas 2.7 and 2.8. $\square$

The sets $\mathscr{B}^\theta_\Gamma(R)$ in Theorem 2.12 can be replaced by the usual balls $\mathscr{B}^\theta_\mathbb{R}(R)$, but then the regularized spectral data should be defined by formula (2.4), where c is a constant such that $c \geqslant -\mu_1(\sigma) - 1/4$ for all $\sigma \in \mathscr{B}^\theta_\mathbb{R}(R)$. Theorem 3.1 guarantees the existence of such a constant depending only on R. This follows from the fact that if one adds the function $c(x-\pi)$ to σ, then the spectra of L_D and L_{DN} are shifted by c, and the difference of σ, $\sigma_1 \in \mathscr{B}^\theta_{\mathbb{R}}(R)$ coincides with that of $\sigma + c(x-\pi)$, $\sigma_1 + c(x-\pi) \in \mathscr{B}^\theta_\Gamma(R)$.

3. Problem of Reconstruction of the Operator L_D from Its Spectral Function. Characterization of Spectral Data and Uniform A Priori Estimates

The general scheme of proof of similar results for the problem of reconstructing the operator L_D from its spectral function remains the same, although the proofs of lemmas with similar statements are different. In the course of exposition, we state two lemmas (Lemmas 3.1 and 3.6) whose proofs are of technical character. Owing to space limitations, we only outline these proofs and refer the reader to our electronic preprint [48] for details and detailed computations.

In what follows, it is convenient to work not with the space $W_2^\theta \ominus \{1\}$ but with the quotient space $W_2^\theta/\{1\}$, assuming that all functions in W_2^θ are defined up to an additive constant. The inner product of functions $f, g \in W_2^\theta/\{1\}$ is assumed to be defined by the formula $(f,g)_\theta = (f_0, g_0)_\theta$, where $f_0, g_0 \in W_2^\theta \ominus \{1\}$. Let Γ_D^θ be the set of real functions $\sigma \in W_2^\theta/\{1\}$ such that $\lambda_1(\sigma) \geqslant 1/2$, and let $\mathscr{B}^\theta_\Gamma(R)$ be the intersection of Γ_D^θ with the closed ball $\mathscr{B}^\theta_{\mathbb{R}}(R)$. If $\sigma \in \Gamma_D^\theta$, then the eigenvalues of L_D satisfy the conditions $1/2 \leqslant \lambda_1 < \lambda_2 < \dots$. For the regularized spectral data, these inequalities are equivalent to

$$s_2 \geqslant 0, \qquad s_{2k} - s_{2k+2} < 1, \qquad k = 1, 2, \dots. \tag{3.1}$$

The nonnegativity of all normalizing constants is equivalent to the conditions

$$s_{2k-1} > -\pi/2, \qquad k = 1, 2, \dots. \tag{3.2}$$

The sequence $\{s_k\}_1^\infty$ lies in l_2, and hence for each real function $\sigma \in \Gamma_D^\theta$, there exists a number $h = h(\sigma) > 0$ such that

$$s_2 \geqslant 0, \qquad s_{2k} - s_{2k+2} \leqslant 1 - h, \qquad s_{2k-1} \geqslant -\pi/2 + h, \qquad k = 1, 2, \dots. \tag{3.3}$$

Fix arbitrary numbers $r > 0$ and $h \in (0,1)$. Let $\Omega_D^\theta(r,h)$ be the set or real sequences $\{s_k\}_1^\infty$ which satisfy inequalities (3.3) and lie in the closed ball of radius r in l_D^θ; i.e., $\|\{s_k\}\|_\theta \leqslant r$. By Ω_D^θ we denote the set of all real sequences $\{s_k\}_1^\infty \in l_D^\theta$ satisfying inequalities (3.1) and (3.2). We deal only with the mapping F_D in what follows and *omit the subscript D wherever convenient*. Instead of Γ_D^θ, Ω_D^θ, and $\Omega_D^\theta(r,h)$ we always write Γ^θ, Ω^θ, and $\Omega^\theta(r,h)$, respectively.

To prove the counterparts of Theorems 2.1 and 2.2, we need the following important result, which gives an explicit description of the preimage of F_D under the variation of only one coordinate in the space l_D^θ. Similar formulas for the problem of reconstruction from one spectrum can be found in the book [40]. However, the proof of our result is different.

Lemma 3.1. *Let $\{\lambda_k\}$ and $\{\alpha_k\}$ be the eigenvalues and the normalizing constants of the operator L_D with real-valued function $\sigma \in W_2^\theta \in \Gamma^\theta$, $\theta \geqslant 0$. Then, for any $n \geqslant 1$ and $t \in (\lambda_{n-1} - \lambda_n, \lambda_{n+1} - \lambda_n)$, there exists a function $\sigma(x,t) \in W_2^\theta$ such that the corresponding operator $L_D = L_D(\sigma)$ has spectrum $\{\lambda_k + t\delta_{kn}\}_1^\infty$ (here δ_{kn} is the Kronecker delta) and normalizing constants $\{\alpha_k\}$. Moreover, for any $n \geqslant 1$ and $t \in (-\alpha_n, +\infty)$, there exists a function $\sigma(x,t) \in W_2^\theta$ such that the operator L_D constructed from this function has spectrum $\{\lambda_k\}_1^\infty$ and normalizing constants $\{\alpha_k + t\delta_{kn}\}_1^\infty$.*

Proof. The potential $\sigma(x,t)$ can be written out in explicit form. In the first case, where the eigenvalue λ_n varies and the normalizing constants and all other eigenvalues remain unchanged, we set

$$\sigma_n(x,t) = \sigma(x) - 2\frac{d}{dx}\ln G(x,t), \tag{3.4}$$

$$G(x,t) = \left(1 + \alpha_n^{-1}\int_0^x y^2(\xi, \lambda_n + t)\,d\xi\right)\left(1 - \alpha_n^{-1}\int_0^x y^2(\xi,\lambda_n)\,d\xi\right) + \left(\alpha_n^{-1}\int_0^x y(\xi,\lambda_n + t)y(\xi,\lambda_n)\,d\xi\right)^2. \tag{3.5}$$

Here $y(x,\lambda)$ is the solution of the equation $-y''+\sigma' y=\lambda y$ with initial conditions $y(0,\lambda)=0$ and $y^{[1]}(0,\lambda)=\sqrt{\lambda}$. In the second case, where only one normalizing constant α_n varies, we set

$$\sigma_n(x,t)=\sigma(x)-2\frac{d}{dx}\ln G(x,t), \quad \text{where } G(x,t)=1+((\alpha_n+t)^{-1}-\alpha_n^{-1})\int_0^x y^2(\xi,\lambda_n)\,d\xi. \quad (3.6)$$

To obtain these formulas, we write out the Gelfand–Levitan–Marchenko equation in the form obtained by Hryniv and Mykytyuk [19] for distribution potentials. If one seeks solutions of this equation satisfying the assumptions of the lemma in the form of a linear combination of two functions (cf. [29, Chap. 2, Sec. 7]), then one obtains a system of two linear equations, which can be solved explicitly. Details can be found in [48]. □

Lemma 3.2. *The mapping $F_D\colon \Gamma^\theta \to \Omega_D^\theta$ is surjective for any $\theta \geqslant 0$.*

Proof. First, let us prove the lemma in the case $\theta<1/2$, where the space l_D^θ coincides with l_2^θ. We use a trick of [40]. By Theorems 1.1 and 1.2, the Fréchet derivative of F_D at $\sigma=0$ coincides with the operator T_D, which is an isomorphism. Hence, for each sufficiently small $\varepsilon>0$, there exists a $\delta>0$ such that the image of the ball $\|\sigma\|_\theta<\delta$ under F_D covers the ball $\|s\|_\theta<\varepsilon$. For $\theta<1/2$, the space l_D^θ coincides with l_2^θ. For given $\mathbf{s}=\{s_k\}\in\Omega^\theta$, consider the sequence

$$\mathbf{s}^n=\{0,0,\dots,0,s_n,s_{n+1},\dots\},$$

where n is chosen in such a way that $\|\mathbf{s}^n\|_\theta<\varepsilon$. There exists a unique function $\sigma_n\in W_2^\theta$ whose image $F(\sigma_n)$ coincides with $\mathbf{s}^n$. By applying Lemma 3.1 $n-1$ times, we construct a function $\sigma\in\Gamma^\theta\subset W_{2,\mathbb{R}}^\theta$ such that $F\sigma=\mathbf{s}$. This means that the range of F contains Ω^θ. Now the proof for $\theta\geqslant 1/2$ can be completed by the trick used in the proof of Theorem 2.1. □

Lemma 3.3. *The mapping $F\colon \Gamma^\theta\to\Omega^\theta$ is injective for every $\theta\geqslant 0$.*

Proof. The injectivity of this mapping for $\theta=0$ (and hence for all $\theta\geqslant 0$) was proved by Hryniv and Mykytyuk [19]. Note that injectivity also follows from Lemma 3.6 below. (To prove this, one should reproduce the argument used to prove Lemma 6 in the authors' paper [45].) □

Let $\widehat{\Omega}^\theta$ be the set of sequences $\{s_k\}_{k=1}^\infty\in l_D^\theta$ for which the numbers $\lambda_k=(s_k+k)^2$ are real. By reproducing the argument carried out before the proof of Theorem 2.2, we obtain an analog of Theorem 2.2 from Lemmas 3.2 and 3.3.

Theorem 3.4. *For any $\theta\geqslant 0$, the mapping $F_D\colon W_{2,\mathbb{R}}^\theta/\{1\}\to\widehat{\Omega}^\theta$ is a bijection. In particular, the numbers $\{\lambda_k\}_1^\infty$ and $\{\alpha_k\}_1^\infty$ are the eigenvalues and the normalizing constants of an operator L_D generated by some function $\sigma\in W_{2,\mathbb{R}}^\theta$ if and only if the sequence $\{\lambda_k\}$ is strictly monotone, the numbers $\{\alpha_k\}$ are positive, and $\{s_k\}_1^\infty\in l_D^\theta$.*

This theorem also implies that the mapping $F_D\colon\Gamma^\theta\to\Omega^\theta$ is a bijection. An analog of Theorem 3.4 for positive integer $\theta=1,2,\dots$ was stated in a different language by Freiling and Yurko [11].

The analyticity of and an explicit expression for the Fréchet derivative are provided by the following theorem.

Theorem 3.5. *Let $\theta\geqslant 0$ and $\sigma\in\Gamma^\theta$. Then there exists a complex neighborhood $U\in W_2^\theta$ of σ such that the mapping $F\colon U\to l_D^\theta$ is real-analytic. In this neighborhood, the mapping $\Phi_D=F_D-T_D\colon U\to l_D^\tau$, where τ is defined in Theorem 1.3, is real-analytic as well. The derivative at a point $\sigma\in U$ is given by the formula*

$$F_D'(\sigma)f=\{(\varphi_k(x),\overline{f(x)})\}_{k=1}^\infty, \quad (3.7)$$

where

$$\varphi_{2k-1}(x)=2\alpha_k\lambda_k\frac{d}{d\lambda}(z(x,\lambda)z'(x,\lambda))\Big|_{\lambda=\lambda_k}, \quad \varphi_{2k}(x)=-\frac{y_k'(x)y_k(x)}{\alpha_k\sqrt{\lambda_k}}, \quad k=1,2,\dots. \quad (3.8)$$

Here $f\in W_2^\theta$ is a function on which the operator $F_D'(\sigma)\colon W_2^\theta\to l_D^\theta$ acts, the $y_n=y(x,\lambda_n)$ are the eigenfunctions of L_D normalized by the conditions $y^{[1]}(0,\lambda_n)=\sqrt{\lambda_n}$, and $z(x,\lambda)$ is the solution of the equation $-y''+\sigma'(x)y=\lambda y$ with initial condition $z(\pi,\lambda)=0$ normalized by the *condition* $\int_0^\pi z^2(x,\lambda)\,dx = 1/\lambda$. *The assertion on (usual) analyticity remains valid if one replaces the condition* $\sigma \in \Gamma^\theta$ *by the condition* $\sigma \in W^\theta_{2,\mathbb{R}}$ *and requires that zero be not an eigenvalue of* L_D.

Proof. The local differentiability of F_D was proved in [47, Sec. 6]. The same paper gives explicit formulas for the Fréchet derivative, but they are less convenient than (3.8). The passage from the old formulas to the new ones requires some technical effort; see [48]. □

Lemma 3.6. *The function system* $\{\varphi_k\}_1^\infty$ *defined in* (3.8) *is a Riesz basis in the space* $L_2(0,\pi)/\{1\}$. *The biorthogonal system has the form*

$$\psi_{2k-1}(x) = \frac{2}{\alpha_k^2}\, y_k^2(x), \quad \psi_{2k}(x) = -\frac{2\sqrt{\lambda_k}}{\alpha_k}\frac{d}{d\lambda}(y^2(x,\lambda))\Big|_{\lambda=\lambda_k}, \qquad k=1,2,\dots, \tag{3.9}$$

and hence is a Riesz basis as well.

Proof. For $k \neq n$, the relations $(\varphi_k(x), \psi_n(x)) = \delta_{kn}$ can be verified in the same way as in [47, Lemma 6]. For $k = n$, the verification is more complicated; see [48]. □

The proofs of the following two theorems reproduce those of Theorems 2.6 and 2.11, respectively, verbatim.

Theorem 3.7. *Let* $\theta \geqslant 0$. *Every point* $\mathbf{y}_0 \in \Omega^\theta = F_D(\Gamma^\theta)$ *has a complex neighborhood* $U(\mathbf{y}_0)$ *where the inverse mapping* $F_D^{-1}(\mathbf{y})$ *is defined and where this mapping has Fréchet derivative in the complex sense. This derivative has the form*

$$(F_D^{-1})'(\mathbf{y}) = (F_D')^{-1}(\mathbf{y}) = \sum_{k=1}^\infty s_k\psi_k(x), \qquad \mathbf{y} = (s_1, s_2, \dots).$$

Here $\{\psi_k(x)\}_1^\infty$ *is the system biorthogonal to* (3.8).

Theorem 3.8. *The assertion of Theorem* 1.12 *remains valid if the mapping* $F = F_B$ *and the set* $\Omega^\theta(r,h) = \Omega^\theta_B(r,h)$ *in this theorem are replaced by* F_D *and* $\Omega^\theta_D(r,h)$, *respectively.*

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