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On the Technique for Passing to the Limit in Nonlinear Elliptic Equations*

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Abstract. We consider the problem of passing to the limit in a sequence of nonlinear elliptic problems. The "limit" equation is known in advance, but it has a nonclassical structure; namely, it contains the p-Laplacian with variable exponent $p = p(x)$. Such equations typically exhibit a special kind of nonuniqueness, known as the Lavrent'ev effect, and this is what makes passing to the limit nontrivial. Equations involving the $p(x)$-Laplacian occur in many problems of mathematical physics. Some applications are included in the present paper. In particular, we suggest an approach to the solvability analysis of a well-known coupled system in non-Newtonian hydrodynamics ("stationary thermo-rheological viscous flows") without resorting to any smallness conditions.

Key words: $p(x)$-Laplacian, compensated compactness, weak convergence of flows to a flow.

1. Introduction

1. Consider the Dirichlet problem

$$\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \operatorname{div} g, \quad u|_{\partial\Omega} = 0, \qquad g \in (L^\infty(\Omega))^d, \tag{1.1}$$

in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, where the exponent $p(x)$ is measurable and satisfies the condition $1 < \alpha \leqslant p(x) \leqslant \beta < \infty$.

To the variable exponent $p = p(x)$, we assign an Orlicz space. By $L^{p(\cdot)} = L^{p(\cdot)}(\Omega, \mathbb{R}^d)$ we denote the class of all measurable vector functions $f\colon \Omega \to \mathbb{R}^d$ such that

$$\int_\Omega |f(x)|^{p(x)}dx < \infty.$$

This class is a reflective Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0,\ \int_\Omega |\lambda^{-1} f|^p\,dx \leqslant 1\right\}. \tag{1.2}$$

Note that the infimum in (1.2) is attained if $\int_\Omega |f|^p\,dx > 0$. It follows that

$$\|f\|_{L^{p(\cdot)}} = \lambda \iff \int_\Omega |\lambda^{-1} f|^p\,dx = 1. \tag{1.3}$$

We seek the solution of the Dirichlet problem (1.1) in the Sobolev–Orlicz space (see [1], [2])

$$W = W_0^{1,p(\cdot)}(\Omega) = \left\{u \in W_0^{1,1}(\Omega),\ \int_\Omega |\nabla u|^p\,dx < \infty\right\}, \qquad \|u\|_W = \|\nabla u\|_{L^{p(\cdot)}(\Omega,\mathbb{R}^d)}.$$

A function $u \in W_0^{1,p(\cdot)}(\Omega)$ will be called a *weak solution* of the problem if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,dx = \int_\Omega g \cdot \nabla\varphi\,dx \tag{1.4}$$

for every function $\varphi \in C_0^\infty(\Omega)$. This is the usual definition in the sense of distributions. Now what about the uniqueness? If u_1 and u_2 are two solutions, then

$$\int_\Omega [|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2] \cdot \nabla\varphi\,dx = 0.$$

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For constant $p > 1$, the expression in brackets belongs to $(L^{p'}(\Omega))^d$, $p' = p/(p-1)$, and the set $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$. By closure, one can take $\varphi = u_1 - u_2$ and readily obtain $u_1 = u_2$ from the strict monotonicity

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) > 0, \qquad \xi \neq \eta.$$

For variable p, only one point is missing from the argument: C_0^∞ is not dense in $W_0^{1,p(\,\cdot\,)}$, and generally, there is no uniqueness. To single out solutions with some certain properties, one does the following. Let $H = H_0^{1,p(\,\cdot\,)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W = W_0^{1,p(\,\cdot\,)}(\Omega)$. Take an intermediate closed subspace V, $H \subseteq V \subseteq W$, and define a V-solution, i.e., a *variational* solution, as a function $u \in V$ such that identity (1.4) holds for every test function $\varphi \in V$. Then the strong monotonicity argument applies, and there exists a unique V-solution. One can take the solution itself for the test function φ and obtain the energy relation

$$\int_\Omega |\nabla u|^p \, dx = \int_\Omega g \cdot \nabla u \, dx. \tag{1.5}$$

Proposition 1.1. *A weak solution is a variational solution if and only if the energy relation* (1.5) *holds.*

Indeed, for V one can take the least closed subspace containing $C_0^\infty(\Omega)$ and the solution itself. We see that the notion of variational solution can be defined without indicating the intermediate subspace itself.

For $V = H$ or $V = W$, one speaks of H- or W-solutions, respectively.

It is of interest to consider model examples in which H has codimension 1 in W. Then there are no variational solutions other than H- and W-solutions. It is not quite clear a priori whether there exist weak solutions other than these variational solutions. We shall show that the answer is "yes."

Model example (see [3], [4]). Let $d = 2$, let $\Omega = \{|x| < 1\}$ be the unit disk, and let

$$p(x) = \begin{cases} \alpha & \text{if } x_1 x_2 > 0, \\ \beta & \text{if } x_1 x_2 < 0, \ 1 < \alpha < 2 < \beta. \end{cases} \tag{1.6}$$

This example is discussed in detail in Section 6.

If $H = W$, then the exponent p and the Dirichlet problem itself are said to be *regular*. Conditions ensuring regularity are rather subtle; for example, the mere continuity of the exponent p is insufficient. It is known (see [5], [6]) that the exponent p is regular if it has a *logarithmic* modulus of continuity,

$$|p(x) - p(y)| \leqslant \frac{C}{\ln(1/|x-y|)}, \qquad x, y \in \Omega, \ |x - y| \leqslant \frac{1}{4}\,.$$

Nonuniqueness is possible in classical variational and monotone problems, but it is due to the missing strict convexity of the functional or strict monotonicity of the operator, and the solution set is convex and closed. In our problems, nonuniqueness is of completely different nature, for the functionals are formally strictly convex and the operators are strictly monotone. Moreover, as will be shown in the model example (see Section 6), the set of weak solutions is not convex in general. The question as to whether this set is weakly closed in W remains open.

2. For applications, the solutions resulting from a certain "regularization" of the original problem (1.1) are of importance. Consider the regularization

$$\operatorname{div} A_\varepsilon(x, \nabla u_\varepsilon) = \operatorname{div} g, \qquad u_\varepsilon|_{\partial\Omega} = 0, \tag{1.7}$$

$$A_\varepsilon(x, \xi) = |\xi|^{p_\varepsilon(x)-2}\xi + \varepsilon|\xi|^{\beta-2}\xi, \qquad \varepsilon > 0, \tag{1.8}$$

where the exponents p_ε are measurable and obey the condition

$$1 < \alpha \leqslant p_\varepsilon(x) \leqslant \beta < \infty, \qquad p_\varepsilon(x) \to p(x) \ \text{for almost all } x \in \Omega. \tag{1.9}$$

For each $\varepsilon > 0$, the vector function $A_\varepsilon(x,\xi)$ satisfies the classical monotonicity, coercivity, and growth conditions corresponding to the exponent β, and problem (1.7) has a unique solution $u_\varepsilon \in W_0^{1,\beta}(\Omega)$. In particular, the identity

$$\int_\Omega A_\varepsilon(x,\nabla u_\varepsilon)\cdot\nabla\varphi\,dx = \int_\Omega g\cdot\nabla\varphi\,dx \qquad \forall\varphi\in C_0^\infty(\Omega) \tag{1.10}$$

holds, where, by closure, one can take $\varphi = u_\varepsilon$ to obtain the energy relation

$$\int_\Omega (|\nabla u_\varepsilon|^{p_\varepsilon} + \varepsilon|\nabla u_\varepsilon|^\beta)\,dx = \int_\Omega A_\varepsilon(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\,dx = \int_\Omega g\cdot\nabla u_\varepsilon\,dx. \tag{1.11}$$

By the Young inequality, this relation implies that

$$\int_\Omega g\cdot\nabla u_\varepsilon\,dx \leqslant 2^{\alpha'}\int_\Omega |g|^{p'_\varepsilon}\,dx + \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^{p_\varepsilon}\,dx,$$

$$\int_\Omega A_\varepsilon(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\,dx \leqslant 2^{\alpha'+1}\int_\Omega |g|^{p'_\varepsilon}\,dx \leqslant 2^{\alpha'+1}\left(|\Omega| + \int_\Omega |g|^{\alpha'}dx\right). \tag{1.12}$$

We see that the sequence u_ε is bounded in $W_0^{1,\alpha}(\Omega)$. Without loss of generality, we assume that the weak convergence $u_\varepsilon \rightharpoonup u$ takes place in $W_0^{1,\alpha}(\Omega)$. Then it follows from (1.11) that

$$\lim_{\varepsilon\to 0}\int_\Omega A_\varepsilon(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\,dx = \lim_{\varepsilon\to 0}\int_\Omega g\cdot\nabla u_\varepsilon\,dx = \int_\Omega g\cdot\nabla u\,dx, \tag{1.13}$$

$$\int_\Omega |\nabla u|^p\,dx \leqslant \int_\Omega g\cdot\nabla u\,dx, \tag{1.14}$$

because, according to the property of (lower) semicontinuity of convex functionals (for more detail, see Section 3), we have

$$\lim_{\varepsilon\to 0}\int_\Omega A_\varepsilon(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\,dx \geqslant \liminf_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon|^{p_\varepsilon}\,dx \geqslant \int_\Omega |\nabla u|^p\,dx.$$

Thus, the limit function u belongs to the space $W_0^{1,p(\cdot)}(\Omega)$.

The *main question* is as follows. Is the limit function—an "approximation solution"—a weak solution of the limit Dirichlet problem (1.1)?

This problem can readily be reduced to another one, namely, to the problem of weak convergence of flows to a flow. As is seen from (1.12), the sequence of "flows" $A_\varepsilon(x,\nabla u_\varepsilon)$ is bounded in $(L^{\beta'}(\Omega))^d$. We assume without loss of generality that the weak convergence

$$A_\varepsilon(x,\nabla u_\varepsilon) \rightharpoonup z \quad \text{in } (L^{\beta'}(\Omega))^d \tag{1.15}$$

takes place.

Since the flows $A_\varepsilon(x,\nabla u_\varepsilon)$ depend on the gradients ∇u_ε nonlinearly and the gradients themselves converge only weakly, we see that the relation $z = |\nabla u|^{p-2}\nabla u$ is by no means obvious. If it holds, we say that the flows are *weakly convergent to a flow*.

Proposition 1.2. *If the flows are weakly convergent to a flow, then the limit function is a weak solution of the limit Dirichlet problem* (1.1).

Indeed, the passage to the limit in identity (1.10) gives

$$\int_\Omega z\cdot\nabla\varphi\,dx = \int_\Omega g\cdot\nabla\varphi\,dx, \qquad \varphi\in C_0^\infty(\Omega), \tag{1.16}$$

and, in view of the relation $z = |\nabla u|^{p-2}\nabla u$, this proves the desired assertion.

The choice of the regularization (1.8) depends on the applications (see Section 5). Sometimes, it suffices to use the regularization (1.8) with $p_\varepsilon = p$. In this case, one does not encounter any serious problems concerning the convergence of solutions, and the limit function is the H-solution of problem (1.1) (see Theorem 4.4). The approximation

$$A_\varepsilon(x,\xi) = |\xi|^{p_\varepsilon(x)-2}\xi, \quad p_\varepsilon \text{ are regular for } \varepsilon > 0, \tag{1.17}$$

where, as before, p_ε satisfies condition (1.9), also proves useful. Here we encounter the same problems involving passage to the limit as in the case of the approximation (1.8).

In what follows, unless otherwise stipulated, we deal with the approximation (1.8), and z stands for the vector in (1.15), i.e., a weak limit of flows. Note that actually $z \in L^{p'(\cdot)}(\Omega, \mathbb{R}^d) \subset (L^{\beta'}(\Omega))^d$. This is also a consequence of lower semicontinuity of convex functionals and will be proved in Lemma 3.2. Therefore, by the Young inequality, we have $z \cdot \nabla u \in L^1(\Omega)$.

2. Weak Convergence of Flows to a Flow

1. We impose the constraint

$$\beta < \alpha^* \equiv \frac{\alpha d}{d - \alpha} \quad \text{if } \alpha < d \tag{2.1}$$

on the exponents α and β in (1.9).

Theorem 2.1. *Under condition* (2.1), *the flows are weakly convergent to a flow.*

The proof of this main theorem requires a rather complicated technique, which is developed in what follows and is of interest in itself.

As was mentioned, convex functionals possess the property of semicontinuity. No less important role is played by another type of semicontinuity property related to monotonicity rather than to convexity. In the simplest version, it is as follows.

Proposition 2.2 (a special case of Lemma 3.3)**.** *One has*

$$\liminf_{\varepsilon \to 0} \int_K A_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx \geqslant \int_K z \cdot \nabla u \, dx \tag{2.2}$$

for every measurable set $K \subset \Omega$. *Moreover, if*

$$\lim_{\varepsilon \to 0} \int_K A_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx = \int_K z \cdot \nabla u \, dx, \tag{2.3}$$

then $z(x) = |\nabla u(x)|^{p(x)-2} \nabla u(x)$ *for almost all* $x \in K$.

The choice of the set K will be based on some considerations related to measure theory.

The sequence $A_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon$ is bounded in $L^1(\Omega)$. Passing to a subsequence if necessary, we assume that

$$A_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon dx \rightharpoonup d\mu$$

in the sense of weak convergence of measures, where μ is a finite Borel measure on Ω. By definition, this means that

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi A_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx = \int_\Omega \varphi \, d\mu \qquad \forall \varphi \in C_0(\Omega),$$

where $C_0(\Omega)$ is the set of continuous functions on $\bar{\Omega}$ vanishing on the boundary.

Proposition 2.3. *If the absolutely continuous component of* μ *relative to the Lebesgue measure is equal to* $z \cdot \nabla u \, dx$, *then the flows are weakly convergent to a flow.*

Proof. Let $S \subset \Omega$ denote a set of Lebesgue measure zero on which the singular component μ^s of the measure μ is concentrated, so that $\mu^s(\Omega \backslash S) = 0$. By the properties of weak convergence (see [7]), we have

$$\limsup_{\varepsilon \to 0} \int_K A_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx \leqslant \int_K d\mu = \int_K z \cdot \nabla u \, dx$$

for every closed set $K \subset \Omega \setminus S$. Therefore, by (2.2), relation (2.3) holds, and hence $z|_K = |\nabla u|^{p-2} \nabla u|_K$. Since S is of Lebesgue measure zero, it follows that this suffices for the relation $z(x) = |\nabla u(x)|^{p(x)-2} \nabla u(x)$ to hold for almost all $x \in \Omega$. The proof of the proposition is complete.

We see that the singular component of μ does not play any role in the problem of weak convergence of flows to a flow, but it is required that the absolutely continuous component have the "natural" structure $z \cdot \nabla u\, dx$. This decisive factor is ensured by the following generalization of the Tartar–Murat compensated compactness lemma(see [8], [9], and [10, Chap. 1]).

Lemma 2.4 (on compensated compactness)**.** *Suppose that*

(i) $u_\varepsilon \in W^{1,\beta}(\Omega)$ *and* $u_\varepsilon \rightharpoonup u$ *in* $W^{1,\alpha}(\Omega)$.

(ii) $w_\varepsilon \in (L^{\beta'}(\Omega))^d$, $\operatorname{div} w_\varepsilon = 0$, *and* $w_\varepsilon \rightharpoonup w$ *in* $(L^{\beta'}(\Omega))^d$.

(iii) *The sequence* $w_\varepsilon \cdot \nabla u_\varepsilon$ *is bounded in* $L^1(\Omega)$.

Then, under condition (2.1),

$$d\mu_\varepsilon \equiv w_\varepsilon \cdot \nabla u_\varepsilon\, dx \ \rightharpoonup\ d\mu = w \cdot \nabla u\, dx + d\mu^s, \tag{2.4}$$

where μ^s *is a singular measure* (*relative to the Lebesgue measure*) *on* Ω.

Note that the classical version of this lemma assumes that $\alpha = \beta$. In this case, the limit measure μ is absolutely continuous. For an example with a nontrivial singular component, see the end of Section 4.

Using Lemma 2.4, one can readily prove Theorem 2.1. We set $w_\varepsilon = A_\varepsilon(x, \nabla u_\varepsilon) - g$ and use relation (2.4) with $w = z - g$ to obtain

$$A_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon\, dx \rightharpoonup z \cdot \nabla u\, dx + d\mu^s.$$

Now Proposition 2.3 ensures the desired relation $z = |\nabla u|^{p-2} \nabla u$ as well as the representation of the limit measure in the form

$$d\mu = |\nabla u|^p\, dx + d\mu^s.$$

There are reasons to suspect that condition (2.1) is important for Lemma 2.4 to be true. Thus, without condition (2.1), the main question as to whether the limit function is a weak solution of the limit equation remains open in the general setting.

Lemma 2.4 was proved in [11]. Below we present a simpler proof.

2. Proof of Lemma 2.4. It follows from conditions (i) and (ii) that

$$\int_\Omega \varphi\, d\mu_\varepsilon = \int_\Omega w_\varepsilon \cdot \nabla u_\varepsilon \varphi\, dx = -\int_\Omega w_\varepsilon u_\varepsilon \cdot \nabla\varphi\, dx$$

for $\varphi \in C_0^\infty(\Omega)$. One can readily pass to the limit on the right-hand side, because $w_\varepsilon \rightharpoonup w$ weakly in $L^{\beta'}(\Omega)$ and $u_\varepsilon \to u$ strongly in $L^\beta(\Omega)$ by condition (2.1) and the Sobolev embedding theorem. As a result,

$$\int_\Omega \varphi\, d\mu = -\int_\Omega wu \cdot \nabla\varphi\, dx = -\int_\Omega w(u - t) \cdot \nabla\varphi\, dx \qquad \forall t \in \mathbb{R}^1. \tag{2.5}$$

Recall the notion of a Lebesgue point. Let $Q_r(x_0) = x_0 + (r/2, r/2)^d$ be the cube with edge length $r > 0$ centered at a point $x_0 \in \Omega$. If $f \in L^\gamma(\Omega)$, $\gamma \geqslant 1$, then

$$\lim_{r\to 0} \frac{1}{r^d} \int_{Q_r(x_0)} |f(x) - f(x_0)|^\gamma\, dx = \lim_{r\to 0} \int_{Q_1(0)} |f(x_0 + ry) - f(x_0)|^\gamma\, dy = 0$$

for almost all $x_0 \in \Omega$; in particular,

$$\lim_{r\to 0} \int_{Q_1(0)} f(x_0 + ry)\varphi(y)\, dy = \lim_{r\to 0} \frac{1}{r^d} \int_{Q_r(x_0)} f(x)\varphi_r(x)\, dx = f(x_0) \int_{Q_1(0)} \varphi(y)\, dy$$

for $\varphi \in C_0^\infty(Q_1(0))$, where $\varphi_r(x) = \varphi(r^{-1}(x - x_0))$. It follows from (2.5) that

$$J \equiv \frac{1}{r^d} \int_{Q_r(x_0)} \varphi_r\, d\mu = -\frac{1}{r^d} \int_{Q_r(x_0)} (u-t)(w-C) \cdot \nabla\varphi_r\, dx - \frac{1}{r^d} \int_{Q_r(x_0)} (u-t)C \cdot \nabla\varphi_r\, dx = J_1 + J_2$$

for constant $C \in \mathbb{R}^d$.

Let x_0 be a Lebesgue point of the functions w and ∇u, and let $C = w(x_0)$, $t = \frac{1}{r^d}\int_{Q_r(x_0)} u\,dx$, and $k_0 = \max(|\varphi| + |\nabla\varphi|)$. Then

$$|J_1| \leqslant \frac{k_0}{r^d}\int_{Q_r(x_0)} |w - C|\left|\frac{u-t}{r}\right| dx \leqslant k_0\left(\frac{1}{r^d}\int_{Q_r(x_0)} |w-C|^{\beta'}\,dx\right)^{1/\beta'}\left(\frac{1}{r^d}\int_{Q_r(x_0)}\left|\frac{u-t}{r}\right|^{\beta} dx\right)^{1/\beta}$$
$$\leqslant k_1\left(\frac{1}{r^d}\int_{Q_r(x_0)} |w-C|^{\beta'}\,dx\right)^{1/\beta'}\left(\frac{1}{r^d}\int_{Q_r(x_0)} |\nabla u|^{\alpha}\,dx\right)^{1/\alpha} \to 0$$

as $r \to 0$. Here we have consecutively used the Hölder inequality, the Poincaré–Sobolev inequality (where condition (2.1) is used), and the properties of Lebesgue points.

Furthermore, by Green's formula and the properties of Lebesgue points, we have

$$J_2 = \frac{1}{r^d}\int_{Q_r(x_0)} C\cdot\nabla u\varphi_r\,dx, \qquad \lim_{r\to 0} J_2 = w(x_0)\cdot\nabla u(x_0)\int_{Q_1(0)}\varphi\,dy.$$

Thus, the relation

$$\lim_{r\to 0} J \equiv \lim_{r\to 0}\frac{1}{r^d}\int_{Q_r(x_0)}\varphi_r\,d\mu = w(x_0)\cdot\nabla u(x_0)\int_{Q_1(0)}\varphi\,dy \tag{2.6}$$

has been proved.

It remains to use the classical theorem on the differentiation of a measure μ with respect to the Lebesgue measure (see [12, Chap. III]). Let us present a suitable statement of this theorem. We define a measure μ_{r,r_0} on the unit cube $Q_1(0)$ by the relation

$$\int_{Q_1(0)}\varphi\,d\mu_{r,x_0} = \frac{1}{r^d}\int_{Q_r(x_0)}\varphi_r\,d\mu \qquad \forall\varphi\in C_0^\infty(Q_1(0)),\ \varphi_r(x) = \varphi(r^{-1}(x-x_0)).$$

Differentiation theorem. *For almost all $x_0 \in \Omega$ (with respect to the Lebesgue measure), the relation*

$$\mu_{r,x_0} \rightharpoonup a(x_0)\,dx \quad \text{as } r \to 0$$

holds, where $a(x)\,dx$ is the absolutely continuous component of the measure μ. In other words,

$$\lim_{r\to 0}\frac{1}{|Q_r|}\int_{Q_r(x_0)}\varphi_r\,d\mu = a(x_0)\int_{Q_1(0)}\varphi\,dy.$$

Now formula (2.6) shows that $a(x) = w(x)\cdot\nabla u(x)$. The proof of the lemma is complete.

3. Lower Semicontinuity Properties

In this section, we do not use Sobolev spaces and present the material at the level of Lebesgue and Orlicz spaces.

1. Convex integrands. Let K be a bounded measurable set in $\mathbb{R}^d$. Consider the class of integrands $f(x,\xi)$ that are convex in $\xi \in \mathbb{R}^d$, measurable in $x \in K$, and satisfy the nonstandard estimate

$$c_1|\xi|^\alpha - \varphi(x) \leqslant f(x,\xi) \leqslant c_2|\xi|^\beta + \varphi(x), \qquad 1 < \alpha \leqslant \beta < \infty,\ c_1, c_2 > 0,\ \varphi \in L^1(K).$$

Let integrands f_ε and f belong to this class, and let the condition

$$\lim_{\varepsilon\to 0} f_\varepsilon(x,\xi) = f(x,\xi) \text{ for almost all } x\in K \text{ and for all } \xi\in\mathbb{R}^d$$

hold. One can readily verify that the similar condition

$$\lim_{\varepsilon\to 0} f_\varepsilon^*(x,\xi) = f^*(x,\xi) \tag{3.1}$$

then holds for the conjugate integrands, where, by definition, $f^*(x,\xi) = \sup_{\eta\in\mathbb{R}^d}\{\xi\cdot\eta - f(x,\eta)\}$.

Lemma 3.1. *If $v_\varepsilon \rightharpoonup v$ in $L^1(K)^d$, then*

$$\liminf_{\varepsilon\to 0}\int_{\bar K} f_\varepsilon(x,v_\varepsilon)\,dx \geqslant \int_K f(x,v)\,dx. \tag{3.2}$$

Moreover, if

$$\lim_{\varepsilon\to 0}\int_K f_\varepsilon(x,v_\varepsilon)\,dx = \int_K f(x,v)\,dx < \infty, \tag{3.3}$$

then the weak convergence

$$f_\varepsilon(x,v_\varepsilon) \rightharpoonup f(x,v) \quad \text{in } L^1(K) \tag{3.4}$$

takes place.

Proof. By the Young inequality, we have

$$\int_K f_\varepsilon(x,v_\varepsilon)\,dx \geqslant \int_K [z\cdot v_\varepsilon - f_\varepsilon^*(x,z)]\,dx \qquad \forall z\in L^\infty(K)^d.$$

By (3.1), it follows that

$$\liminf_{\varepsilon\to 0}\int_K f_\varepsilon(x,v_\varepsilon)\,dx \geqslant \sup_{z\in L^\infty(K)^d}\int_K [z\cdot v - f^*(x,z)]\,dx = \int_K f(x,v)\,dx,$$

where, at the concluding stage, use has been made of the classical theorem on the conjugate functional (see [13, Chap. IX]). Thus, inequality (3.2) has been proved. Assume that relation (3.3) holds. Let $K_0 \subset K$ be an arbitrary measurable subset, and let $K_1 = K\setminus K_0$. By comparing inequalities of the form (3.2) for K_0 and K_1 with the equality for K, we conclude that

$$\lim_{\varepsilon\to 0}\int_{K_0} f_\varepsilon(x,v_\varepsilon)\,dx = \int_{K_0} f(x,v)\,dx. \tag{3.5}$$

By the arbitrariness of K_0, formula (3.5) implies the weak convergence (3.4). The proof of the lemma is complete.

As an example of a sequence of convex integrands, one can take

$$f_\varepsilon(x,\xi) = |\xi|^{p_\varepsilon(x)} + \varepsilon|\xi|^\beta \to f(x,\xi) = |\xi|^{p(x)},$$

where the exponents p_ε satisfy condition (1.9).

Lemma 3.2. *Assume that*
(i) $v_\varepsilon \in (L^1(K))^d$ *and* $\int_K(|v_\varepsilon|^{p_\varepsilon} + \varepsilon|v_\varepsilon|^\beta)\,dx \leqslant C < \infty$.
(ii) $w_\varepsilon \equiv |v_\varepsilon|^{p_\varepsilon-2}v_\varepsilon + \varepsilon|v_\varepsilon|^{\beta-2}v_\varepsilon \rightharpoonup z$ *in* $(L^{\beta'}(K))^d$.
Then $z \in L^{p'(\cdot)}(\Omega,\mathbb{R}^d)$.

Proof. Set $f_\varepsilon(x,\xi) = |\xi|^{p_\varepsilon(x)}/p_\varepsilon(x) + \varepsilon|\xi|^\beta/\beta$. Since $w_\varepsilon = f'_\xi(x,v_\varepsilon)$, it follows that the identity $f_\varepsilon(x,v_\varepsilon) + f_\varepsilon^*(x,w_\varepsilon) \equiv w_\varepsilon\cdot v_\varepsilon$ holds, which implies that

$$\int_K f_\varepsilon^*(x,w_\varepsilon)\,dx \leqslant C.$$

Since $f_\varepsilon^*(x,\xi) \to f^*(x,\xi) = |\xi|^{p'(x)}/p'(x)$ as $\varepsilon\to 0$, we have

$$\int_\Omega \frac{|z|^{p'}}{p'}\,dx \leqslant C$$

by Lemma 3.1, as desired.

2. Monotone operators. Let $A_\varepsilon(x,\xi)$ and $A(x,\xi)$ be vector functions satisfying the Carathéodory condition (i.e., continuous in ξ for almost all $x\in\Omega$ and measurable in x for all $\xi\in\mathbb{R}^d$)

such that

$$(A_\varepsilon(x,\xi)-A_\varepsilon(x,\eta))\cdot(\xi-\eta)\geqslant 0 \quad \text{(monotonicity)},$$

$$A_\varepsilon(x,\xi)\to A(x,\xi) \qquad \forall \xi\in\mathbb{R}^d \text{ for almost all } x\in\Omega, \tag{3.6}$$

$$|A(x,\xi)|\leqslant c_1(|\xi|^{p(x)-1}+1). \tag{3.7}$$

Thus, the growth conditions are imposed only on the limit operator A.

Lemma 3.3. *Suppose that*

$$v_\varepsilon \rightharpoonup v, \quad A_\varepsilon(x,v_\varepsilon)\rightharpoonup z \ \text{ in } \ (L^1(K))^d, \qquad v\in L^{p(\cdot)}(K,\mathbb{R}^d),\ z\in L^{p'(\cdot)}(K,\mathbb{R}^d),$$

and the sequence $A_\varepsilon(x,v_\varepsilon)\cdot v_\varepsilon$ *is bounded in* $L^1(K)$*. Then*

$$\liminf_{\varepsilon\to 0}\int_K A_\varepsilon(x,v_\varepsilon)\cdot v_\varepsilon\,dx \geqslant \int_K z\cdot v\,dx, \tag{3.8}$$

and if equality holds in (3.8)*, i.e., if* $\lim_{\varepsilon\to 0}\int_K A_\varepsilon(x,v_\varepsilon)\cdot v_\varepsilon\,dx=\int_K z\cdot v\,dx$*, then* $z=A(x,v)$ *and*

$$A_\varepsilon(x,v_\varepsilon)\cdot v_\varepsilon \rightharpoonup A(x,v)\cdot v \quad \text{in} \quad L^1(K). \tag{3.9}$$

Proof. Assume that

$$\liminf_{\varepsilon\to 0}\int_K A_\varepsilon(x,v_\varepsilon)\cdot v_\varepsilon\,dx \leqslant \int_K z\cdot v\,dx. \tag{3.10}$$

By monotonicity, we have the inequality

$$\int_K (A_\varepsilon(x,v_\varepsilon)-A_\varepsilon(x,\psi))\cdot(v_\varepsilon-\psi)\,dx\geqslant 0 \qquad \forall\psi\in L^\infty(K)^d,$$

in which we pass to the limit considering each of the four terms separately. The limit of the term $\int_K A_\varepsilon(x,v_\varepsilon)\cdot v_\varepsilon\,dx$ can be estimated from above with the use of inequality (3.10), and the limits of the other three terms can be calculated easily. As a result, we obtain

$$\int_K (z-A(x,\psi))\cdot(v-\psi)\,dx\geqslant 0 \qquad \forall\psi\in L^\infty(K)^d. \tag{3.11}$$

For fixed v and z, the integrand in this inequality has the structure $g(x,\psi(x))$, where $g(x,\xi)$ is a Carathéodory vector function satisfying the estimate

$$|g(x,\xi)|\leqslant c|\xi|^p+a(x), \qquad a\in L^1(K), \tag{3.12}$$

which can readily be verified by using inequality (3.7) and the fact that $v\in L^{p(\cdot)}(K,\mathbb{R}^d)$ and $z\in L^{p'(\cdot)}(K,\mathbb{R}^d)$. It follows that the superposition operator $G(\psi)(x)=g(x,\psi(x))$ acts continuously from $L^{p(\cdot)}(K,\mathbb{R}^d)$ into $L^1(K)$. Indeed, let $\psi_n\to\psi$ in $L^{p(\cdot)}(K,\mathbb{R}^d)$; i.e., $\int_K|\varphi-\varphi_n|^p\,dx\to 0$ by (1.3). As is seen from the estimate $|\psi_n|^p\leqslant 2^\beta(|\psi-\psi_n|^p+|\psi|^p)$, the family $|\psi_n|^p$ is equiintegrable. Therefore, inequality (3.12) implies the equiintegrability of the family $G(\psi_n)$. (For the notion of equiintegrability, see [13, Chap. VII, Sec. 1].) It can be assumed that $\psi_n(x)\to\psi(x)$ almost everywhere on K. By the Lebesgue theorem, we have $G(\psi_n)\to G(\psi)$ in $L^1(K)$, which proves the continuity of the superposition operator.

Since the set $(L^\infty(K))^d$ is dense in $L^{p(\cdot)}(K,\mathbb{R}^d)$, it follows that inequality (3.11) holds for every test function $\psi\in L^{p(\cdot)}(K,\mathbb{R}^d)$. We now use the Minty technique and take $\psi=v\pm th$, $h\in L^{p(\cdot)}(K,\mathbb{R}^d)$, in (3.11) to arrive at the desired relation $z=A(x,v)$. It is also clear that strict inequality is impossible in (3.10). The weak convergence (3.9) can be established by the same argument as in the case of convex integrands. The proof of the lemma is complete.

4. Some Individual Cases of Passage to the Limit

In this section, neither condition (2.1) nor the compensated compactness lemma is used, and our argument is solely based on lower semicontinuity properties. At the same time, some special constraints are imposed on the limit and prelimit exponents etc.

1. We start from the following simplest result.

Theorem 4.1. *If the limit exponent p is regular, then the flows are weakly convergent to a flow.*

Furthermore, an exponent p is said to be *regular outside a set* $S \subset \Omega$ if every function in $W_0^{1,p(\cdot)}(\Omega)$ vanishing in some neighborhood of S belongs to the space $H_0^{1,p(\cdot)}(\Omega)$. For a regular exponent, we have $S = \varnothing$.

Theorem 4.2. *Suppose that the limit exponent is regular outside a closed set S of Lebesgue measure zero. Then the flows are weakly convergent to a flow, and moreover,*

$$A_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \rightharpoonup |\nabla u|^p \quad \text{in } L^1(\Omega \setminus S_\delta), \tag{4.1}$$

where S_δ is an arbitrary δ-neighborhood of S. Here the measure μ^s is concentrated on S.

We shall construct an example in which the sequence $A_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon$ is not weakly compact in $L^1(\Omega)$ and the singular component of the measure μ^s is nontrivial.

Theorem 4.3. *If the energy relation* (1.5) *holds for the limit function, then this function is a variational solution of the Dirichlet problem* (1.1). *Moreover, $\mu^s = 0$.*

Theorem 4.4. *If $p_\varepsilon \geqslant p$, then the limit function is the H-solution of Dirichlet problem* (1.1).

2. Proof of Theorem 4.1. Consider the limit identity (1.16).

Since the exponent p is regular and $z \in L^{p'(\cdot)}(\Omega)$ by Lemma 3.2, it follows that every function $\varphi \in W_0^{1,p(\cdot)}$, in particular, $\varphi = u$, can be taken for a test function. As a result, we obtain

$$\int_\Omega z \cdot \nabla u \, dx = \int_\Omega g \cdot \nabla u \, dx. \tag{4.2}$$

The comparison with (1.13) shows that relation (2.3) with $K = \Omega$ holds, and Theorem 4.1 follows from Proposition 2.2.

3. Proof of Theorem 4.2. Let

$$\rho \in C^\infty(\overline{\Omega}), \quad \rho \geqslant 0, \quad \rho = 0 \text{ in a neighborhood of } S.$$

Proposition 4.5. *Suppose that*

(i) *$v_\varepsilon \in W_0^{1,\beta}(\Omega)$ and $v_\varepsilon \rightharpoonup v$ in $W_0^{1,\alpha}(\Omega)$.*

(ii) *The sequence v_ε is compact in $L^\beta(\Omega)$.*

Then

$$\lim_{\varepsilon \to 0} \int_\Omega \rho A_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla v_\varepsilon \, dx = \int_\Omega \rho z \cdot \nabla v \, dx.$$

Proof. It follows from identity (1.10) that

$$J_\varepsilon \equiv \int_\Omega A_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla v_\varepsilon \rho \, dx = -\int_\Omega A_\varepsilon(x, \nabla u_\varepsilon) v_\varepsilon \cdot \nabla \rho \, dx + \int_\Omega g \cdot \nabla(v_\varepsilon \rho) \, dx,$$

$$\lim_{\varepsilon \to 0} J_\varepsilon = -\int_\Omega z v \cdot \nabla \rho \, dx + \int_\Omega g \cdot \nabla(v\rho) \, dx, \tag{4.3}$$

where the convergence $A_\varepsilon(x, \nabla u_\varepsilon) v_\varepsilon \rightharpoonup zv$ in $(L^1(\Omega))^d$ and the convergence $v_\varepsilon \rho \rightharpoonup v\rho$ in $W_0^{1,\alpha}(\Omega)$, which follow from (ii), have been used. Since $v\rho \in H_0^{1,p(\cdot)}(\Omega)$ by the assumption of the theorem and the choice of ρ, it follows that the right-hand side in (4.2) is equal to $\int_\Omega \rho z \cdot \nabla v \, dx$, which completes the proof of the proposition.

It is now tempting to take $v_\varepsilon = u_\varepsilon$, but condition (ii) does not hold for u_ε in general (since condition (2.1) is not assumed). For this reason, for v_ε we take the cutoff function

$$u_\varepsilon^{(N)} = \begin{cases} u_\varepsilon(x) & \text{for } |u(x)| \leqslant N, \\ \pm N & \text{for } |u(x)| > N. \end{cases}$$

Then for fixed N we have

$$\lim_{\varepsilon\to 0} \int_\Omega \rho A_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon^{(N)}\, dx = \int_\Omega \rho z \cdot \nabla u^{(N)}\, dx. \tag{4.4}$$

In what follows (see Lemma 4.6), we indicate a relation permitting one to remove the cutoff from (4.4). Then the inequality

$$\limsup_{\varepsilon\to 0} \int_{\Omega\cap B} A_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon\, dx \leqslant \int_{\Omega\cap B} z \cdot \nabla u\, dx,$$

where B is a ball and $\overline{B} \cap S = \varnothing$, can readily be obtained. Now all assertions of Theorem 4.2 follow from Proposition 2.2.

Lemma 4.6. *One has*

$$0 \leqslant A_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon^{(N)} \leqslant A_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon,$$

$$\lim_{N\to\infty} \int_\Omega A_\varepsilon(x, \nabla u_\varepsilon) \cdot (\nabla u_\varepsilon - \nabla u_\varepsilon^{(N)})\, dx = 0 \quad \textit{uniformly with respect to } \varepsilon.$$

Proof. Set $T_N^\varepsilon = \{x \in \Omega,\ |u_\varepsilon(x)| > N\}$. We have

$$|T_N^\varepsilon| \leqslant N^{-1} \|u_\varepsilon\|_{L^1(\Omega)} \leqslant cN^{-1}.$$

Take $\varphi = u_\varepsilon$ and $\varphi = u_\varepsilon^{(N)}$ in (1.10). Then

$$\begin{aligned} \int_\Omega A_\varepsilon(x, \nabla u_\varepsilon)(\nabla u_\varepsilon - \nabla u_\varepsilon^{(N)})\, dx &= \int_{T_N^\varepsilon} g \cdot \nabla u_\varepsilon\, dx \\ &\leqslant \biggl(\int_\Omega |\nabla u_\varepsilon|^\alpha\, dx\biggr)^{1/\alpha} \biggl(\int_{T_N^\varepsilon} |g|^{\alpha'}\, dx\biggr)^{1/\alpha'} \leqslant c_1 \biggl(\int_{T_N^\varepsilon} |g|^{\alpha'}\, dx\biggr)^{1/\alpha'}, \end{aligned}$$

and it suffices to use the absolute integrability of the integrable function $|g|^{\alpha'}$. The proof of the lemma and hence of Theorem 4.2 is complete.

4. Proof of Theorem 4.3. It follows from (1.13) and (1.5) that

$$\lim_{\varepsilon\to 0} \int_\Omega A_\varepsilon(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon\, dx = \int_\Omega |\nabla u|^p\, dx.$$

Since the functional $f(x, \xi) = |\xi|^{p(x)}$ is strictly convex, by the Reshetnyak theorem (see [14]) we obtain the strong convergence $u_\varepsilon \to u$ in $W_0^{1,\alpha}(\Omega)$. Therefore, we can assume that $\nabla u_\varepsilon(x) \to \nabla u(x)$ for almost all $x \in \Omega$. Then $A_\varepsilon(x, \nabla u_\varepsilon(x)) \to |\nabla u(x)|^{p(x)-2} \nabla u(x)$ almost everywhere. Consequently, $A_\varepsilon(x, \nabla u_\varepsilon) \rightharpoonup |\nabla u|^{p-2} \nabla u$ in $L^{\beta'}(\Omega)$, as desired.

We omit the proof of Theorem 4.4, because a similar assertion (Lemma 4.8) is proved below.

5. So far, the approximation (1.8) has been studied. Now let us consider the approximation (1.17) and study which solutions of the original problem (1.1) are attainable, i.e., can be obtained with the use of the approximation (1.17). (In this connection, see [15] and [16].)

Lemma 4.7. *If $p_\varepsilon \leqslant p$, then the limit function is the W-solution.*

Proof. By closure, identity (1.10) holds for every function $\varphi \in H_0^{1,p_\varepsilon(\cdot)}(\Omega) \equiv W_0^{1,p_\varepsilon(\cdot)}(\Omega)$, in particular, for $\varphi \in W_0^{1,p(\cdot)}(\Omega) = W$. Here the regularity of the exponents p_ε (see (1.17)) and the inequality $p_\varepsilon \leqslant p$ have been used. Furthermore, the flows $A_\varepsilon(x, \nabla u_\varepsilon)$ turn out to be bounded in $L^{p'(\cdot)}(\Omega, \mathbb{R}^d)$, and hence $A_\varepsilon(x, \nabla u_\varepsilon) \rightharpoonup z$ in $L^{p'(\cdot)}(\Omega, \mathbb{R}^d)$. It follows that

$$\int_\Omega z \cdot \nabla \varphi \, dx = \int_\Omega g \cdot \nabla \varphi \, dx \qquad \forall \varphi \in W; \tag{4.5}$$

in particular, relation (4.2) holds. The comparison with (1.13) shows that relation (2.3) holds with $K = \Omega$. It follows that $z = |\nabla u|^{p-2} \nabla u$, and identity (4.5) itself means that u is the W-solution. The proof of the lemma is complete.

Lemma 4.8. *If $p_\varepsilon \geqslant p$, then the limit function u is the H-solution.*

Proof. By the Minty lemma (see [17, Chap. III]), we have

$$\int_\Omega [A_\varepsilon(x, \nabla \varphi) - g] \cdot (\nabla \varphi - \nabla u_\varepsilon) \, dx \geqslant 0 \qquad \forall \varphi \in C_0^\infty(\Omega),$$

and the passage to the limit leads to the inequality

$$\int_\Omega [|\nabla \varphi|^{p-2} \nabla \varphi - g] \cdot (\nabla \varphi - \nabla u) \, dx \geqslant 0 \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{4.6}$$

Since the exponents p_ε are regular and $u_\varepsilon \in H_0^{1,p_\varepsilon(\cdot)}(\Omega) \subset H_0^{1,p(\cdot)}(\Omega)$, we have $u \in H_0^{1,p(\cdot)}(\Omega)$. Relation (4.6) holds automatically for $\varphi \in H_0^{1,p(\cdot)}(\Omega)$, and therefore (by the Minty lemma), it means that u is the H-solution. The proof of the lemma is complete.

These lemmas imply the following assertion.

Lemma 4.9. *If the H- and W-solutions do not coincide, then there exists a continuum of approximation solutions.*

Proof. Let $\bar{p}_\varepsilon$ and $\hat{p}_\varepsilon$ be approximations leading to the W- and the H-solution, respectively. Take $p_\varepsilon(x, t) = \bar{p}_\varepsilon(x) t + \hat{p}_\varepsilon(x)(1-t)$, $0 \leqslant t \leqslant 1$, and denote the corresponding solution by $u_n(x, t)$. If u_1 is the W-solution, u_2 is the H-solution, and $u_1 \neq u_2$, then there exists a linear functional $l \in (L^1(\Omega))^* = L^\infty(\Omega)$ such that $l(u_1) \neq l(u_2)$. Consider the approximation $p_\varepsilon(x) = p_\varepsilon(x, t_\varepsilon)$, where t_ε is determined by the condition

$$\lim_{\varepsilon \to 0} l(u_\varepsilon(\cdot, t_\varepsilon)) = \frac{l(u_1) + l(u_2)}{2}.$$

Then $u \neq u_1$ and $u \neq u_2$ for $u = \lim_{\varepsilon \to 0} u(\cdot, t_\varepsilon)$. The proof of the lemma is complete.

6. Consider the model example and an approximation $p_\varepsilon \to p$ that gives a weak solution u other than the H- and W-solutions. In this example, the exponent (see (1.6)) is regular outside the origin, and by Theorem 4.2 we have

$$|\nabla u_\varepsilon|^{p_\varepsilon} \rightharpoonup |\nabla u|^p \quad \text{in } L^1(\Omega \setminus B_\delta), \quad B_\delta = \{|x| < \delta\}. \tag{4.7}$$

Here there is no convergence $|\nabla u_\varepsilon|^{p_\varepsilon} \rightharpoonup |\nabla u|^p$ in $L^1(\Omega)$. Indeed, this convergence would mean (see (1.13)) that

$$\int_\Omega |\nabla u|^p \, dx = \lim_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon|^{p_\varepsilon} dx = \int_\Omega g \cdot \nabla u \, dx;$$

i.e., u would be a variational solution, which is impossible, since the H- and W-solutions are the only variational solutions. Thus,

$$\lim_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon|^{p_\varepsilon} \, dx > \int_\Omega |\nabla u|^p \, dx.$$

This, together with (4.7), implies that the sequence $|\nabla u_\varepsilon|^{p_\varepsilon}$ is not weakly compact in $L^1(\Omega)$ and that the limit measure μ admits the representation $d\mu = |\nabla u|^p \, dx + d\mu^s$, where μ^s is a nontrivial measure concentrated at the point $x = 0$.

5. Some Applications

1. We start from the well-known thermistor problem. Consider the system

$$\begin{cases} \operatorname{div}(|\nabla u|^{\sigma(\theta)-2}\nabla u) = \operatorname{div} g, & u|_{\partial\Omega} = 0, \\ -\Delta\theta = |\nabla u|^{\sigma(\theta)}, & \theta|_{\partial\Omega} = 0, \end{cases} \tag{5.1}$$

where $\sigma(\theta)$ is a measurable function on $[0,\infty)$ satisfying the condition

$$1 < \alpha \leqslant \sigma(\theta) \leqslant \beta \quad \text{and} \quad \beta < \alpha^* \quad \text{for } \alpha < d. \tag{5.2}$$

System (5.1) provides a simultaneous description of the electric field (u is the electric potential) and temperature θ. Such systems, and especially systems of similar structure in hydromechanics, have been subject of much attention (see [18]–[22]). Earlier, existence theorems were proved only under some smallness conditions, e.g., in the case of a sufficiently small Lipschitz constant for the function σ. For a survey of this kind of results, see [24]. The above-developed technique for passing to the limit allows one to dispose of any smallness requirements. Let us introduce the regularized system

$$\begin{cases} \operatorname{div} A_\varepsilon(x,\nabla u_\varepsilon) = \operatorname{div} g, & u_\varepsilon|_{\partial\Omega} = 0, \\ -\Delta\theta_\varepsilon = A_\varepsilon(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon, & \theta_\varepsilon|_{\partial\Omega} = 0, \end{cases} \tag{5.3}$$

where $A_\varepsilon(x,\nabla u_\varepsilon) = |\nabla u_\varepsilon|^{p_\varepsilon(x)-2}\nabla u_\varepsilon + \varepsilon|\nabla u_\varepsilon|^{\beta-2}\nabla u_\varepsilon$ and $p_\varepsilon(x) = \sigma(\theta_\varepsilon(x))$.

By the estimate (1.12), the right-hand side $f \equiv A_\varepsilon(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon$ of the second equation in (5.3) is bounded in $L^1(\Omega)$. By the L^1-theory for the Laplacian (see [24]),

$$\|\theta_\varepsilon\|_{W_0^{1,\gamma}(\Omega)} \leqslant c(\gamma)\|f\|_{L^1(\Omega)} \qquad \forall\gamma \in \left[1, \frac{d}{d-1}\right). \tag{5.4}$$

We assume without loss of generality that $u_\varepsilon \rightharpoonup u$ in $W_0^{1,\alpha}$, $\theta_\varepsilon \rightharpoonup \theta$ in $W_0^{1,1}$, and $p_\varepsilon(x) \to p(x) = \sigma(\theta(x))$ almost everywhere. Then, by Theorem 2.1, the limit function u is a weak solution of the first equation in (5.1). Let us pass to the limit in the second equation in system (5.3). To this end, we rewrite the equation in the form

$$-\operatorname{div}\nabla\theta_\varepsilon = \operatorname{div}[(A_\varepsilon(x,\nabla u_\varepsilon) - g)u_\varepsilon] + g\cdot\nabla u_\varepsilon$$

and use Theorem 2.1. Since the flows are weakly convergent to a flow in $L^{\beta'}(\Omega)$ and (by virtue of the condition $\beta < \alpha^*$) $u_\varepsilon \to u$ *strongly* in $L^\beta(\Omega)$, we have

$$-\operatorname{div}\nabla\theta = \operatorname{div}[(|\nabla u|^{\sigma(\theta)-2}\nabla u - g)u] + g\cdot\nabla u$$

in the sense of distributions. Note that the passage to the limit can also be performed right in the second equation of system (5.3) (without the above-indicated transformation), but then there arises a measure μ on the right-hand side of the equation,

$$-\operatorname{div}\nabla\theta = \mu = |\nabla u|^{\sigma(\theta)} + \mu^s.$$

This equation with measure is understood in the sense of distributions as well.

Theorem 5.1. *There exist functions*

$$\theta \in W_0^{1,\gamma}(\Omega) \qquad \forall\,\gamma \in [1, d/(d-1)),$$
$$u \in W_0^{1,p(\,\cdot\,)}(\Omega), \qquad p(x) = \sigma(\theta(x)),$$

such that

$$\operatorname{div}(|\nabla u|^{\sigma(\theta)-2}\nabla u) = \operatorname{div} g,$$
$$-\Delta\theta = \operatorname{div}[(|\nabla u|^{\sigma(\theta)-2} - g)u] + g\cdot\nabla u = |\nabla u|^{\sigma(\theta)} + \mu^s$$

in the sense of distributions. Furthermore,

$$\int_\Omega |\nabla u|^{\sigma(\theta)}\,dx \leqslant \int_\Omega g\cdot\nabla u\,dx. \tag{5.5}$$

If equality holds in (5.5), *then* $\mu^s = 0$.

2. Consider the well-known coupled system in hydromechanics,

$$\begin{cases} \operatorname{div}[(a(\theta)|Du|^{\sigma(\theta)-2}Du - u \otimes u)] = \operatorname{div} g + \nabla\pi, \\ \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0, \\ -\Delta\theta + u \cdot \nabla\theta = a(\theta)|Du|^{\sigma(\theta)}, \quad \theta|_{\partial\Omega} = 0. \end{cases} \tag{5.6}$$

Here u is the velocity vector, Du is the symmetric part of the gradient ∇u, π is pressure, θ is temperature, $g = \{g_{ij}\}$ is a symmetric matrix, $g_{ij} \in L^\infty(\Omega)$, and the function $a(\theta)$ is measurable and satisfies the condition $0 < a_1 \leqslant a(\theta) \leqslant a_2 < \infty$, where a_1 and a_2 are some constants. To be definite, let $d = 2, 3$. We assume that $\sigma(\theta)$ satisfies condition (5.2) with

$$\alpha > \frac{3d}{d+2}. \tag{5.7}$$

This assumption is related to the presence of the inertial term $\operatorname{div}(u \otimes u)$.

Let us introduce the regularized system

$$\begin{cases} \operatorname{div}[A_\varepsilon(x, Du_\varepsilon) - u_\varepsilon \otimes u_\varepsilon] = \operatorname{div} g + \nabla\pi_\varepsilon \\ \operatorname{div} u_\varepsilon = 0, \quad u_\varepsilon|_{\partial\Omega} = 0 \\ -\Delta\theta_\varepsilon + u_\varepsilon \cdot \nabla\theta_\varepsilon = A_\varepsilon(x, Du_\varepsilon) \cdot Du_\varepsilon, \quad \theta_\varepsilon|_{\partial\Omega} = 0, \end{cases} \tag{5.8}$$

where $A_\varepsilon(x, Du_\varepsilon) = a(\theta_\varepsilon)|Du_\varepsilon|^{p_\varepsilon(x)-2}Du_\varepsilon + \varepsilon|Du_\varepsilon|^{\beta-2}Du_\varepsilon$, $\varepsilon > 0$, and $p_\varepsilon(x) = \sigma(\theta_\varepsilon(x))$. For fixed $\varepsilon > 0$, the solvability of this system is well known ([19], [22]). The energy relation

$$\int_\Omega A_\varepsilon(x, Du_\varepsilon) \cdot Du_\varepsilon \, dx = \int_\Omega g \cdot Du_\varepsilon \, dx$$

implies an estimate of the form (1.12) with ∇u_ε replaced by Du_ε. It follows from this estimate and from the Korn inequality that the sequence u_ε is bounded in $W_0^{1,\alpha}(\Omega)$ and that the sequence of “flows” $A_\varepsilon(x, Du_\varepsilon)$ is bounded in $L^{\beta'}(\Omega)$. It is important that, as follows from condition (5.7) and embedding theorems, the sequence $u_\varepsilon \otimes u_\varepsilon$ is bounded in $L^{\alpha'}$. It is also necessary to study the sequence of pressures π_ε. We assume without loss of generality that $\int_\Omega \pi_\varepsilon \, dx = 0$. Consider the “total flow”

$$w_\varepsilon = A_\varepsilon(x, Du_\varepsilon) - u_\varepsilon \otimes u_\varepsilon - g - \pi_\varepsilon I, \qquad \text{where } I \text{ is the identity matrix.}$$

We use a well-known result due to Bogovskii [26]: *if* $h \in L^\beta(\Omega)$ *and* $\int_\Omega h \, dx = 0$, *then there exists a vector* $\varphi \in W_0^{1,\beta}(\Omega)$ *such that* $\operatorname{div} \varphi = h$ *and* $\|\nabla\varphi\|_{L^\beta(\Omega)} \leqslant c_0\|h\|_{L^\beta(\Omega)}$. Now it follows from the relation $\operatorname{div} w_\varepsilon = 0$ that

$$\int_\Omega \pi_\varepsilon h \, dx = \int_\Omega \pi_\varepsilon I \cdot \nabla\varphi \, dx = \int_\Omega (\pi_\varepsilon I - w_\varepsilon) \cdot \nabla\varphi \, dx \leqslant c_1\|h\|_{L^\beta},$$

since $\|w_\varepsilon - \pi_\varepsilon\|_{L^{\beta'}} \leqslant c_2$. This implies the inequality $\|\pi_\varepsilon\|_{L^{\beta'}} \leqslant c_1$, i.e., the boundedness of the total flow w_ε in $L^{\beta'}(\Omega)$. By the relation $\pi_\varepsilon I \cdot \nabla u_\varepsilon = 0$ and the above-mentioned boundedness of the sequence $u_\varepsilon \otimes u_\varepsilon$ in $L^{\alpha'}(\Omega)$, the sequence $w_\varepsilon \cdot \nabla u_\varepsilon$ is bounded in $L^1(\Omega)$. (Thus, we have condition (iii) in Lemma 2.4.) The right-hand side $f = A_\varepsilon(x, Du_\varepsilon) \cdot Du_\varepsilon$ of the last relation in (5.8) is bounded in L^1. As will be shown later, the estimate (5.4) remains valid. Therefore, we assume without loss of generality that the following convergence relations hold:

$$\theta_\varepsilon \rightharpoonup \theta \text{ in } W_0^{1,\gamma}, \quad p_\varepsilon(x) = \sigma(\theta_\varepsilon(x)) \to p(x) = \sigma(\theta(x)) \text{ almost everywhere,}$$
$$u_\varepsilon \rightharpoonup u \text{ in } W_0^{1,\alpha}, \quad u_\varepsilon \otimes u_\varepsilon \to u \otimes u \text{ in } L^{\alpha'},$$
$$A_\varepsilon(x, Du_\varepsilon) \rightharpoonup z, \quad \pi_\varepsilon \rightharpoonup \pi \text{ in } L^{\beta'}.$$

Here $u \in W_0^{1,p(\cdot)}$ and $z \in L^{p'(\cdot)}$. By Lemma 2.4,

$$w_\varepsilon \cdot \nabla u_\varepsilon \, dx \rightharpoonup (z - u \otimes u - g - \pi) \cdot \nabla u \, dx + d\mu^s,$$

and therefore,

$$A_\varepsilon(x, Du_\varepsilon) \cdot \nabla u_\varepsilon = A_\varepsilon(x, Du_\varepsilon) \cdot Du_\varepsilon \rightharpoonup z \cdot Du + d\mu^s.$$

For an arbitrary closed set $K \subset \Omega \setminus S$, where S is the support of the measure μ^s, we have

$$\limsup_{\varepsilon \to 0} \int_K A_\varepsilon(x, Du_\varepsilon) \cdot Du_\varepsilon \, dx \leqslant \int_K z \cdot Du \, dx,$$

and by Lemma 3.3 we conclude that $z = a(\theta)|Du|^{p-2}Du$. This allows us to pass to the limit in the first equation in (5.8). Next, we write

$$-\operatorname{div} \nabla\theta_\varepsilon + u_\varepsilon \cdot \nabla\theta_\varepsilon = \operatorname{div}[(A_\varepsilon(x, Du_\varepsilon) - g)u_\varepsilon] + g \cdot Du_\varepsilon.$$

On the left-hand side, the convergence $u_\varepsilon \cdot \nabla\theta_\varepsilon \rightharpoonup u \cdot \nabla\theta$ in $L^1(\Omega)$ takes place. Indeed, using conditions (5.7) for $d = 2, 3$, we see that $u_\varepsilon \to u$ in $L^{(\alpha^*+d)/2}$ (because $\alpha^* > d$) and $\nabla\theta_\varepsilon \rightharpoonup \nabla\theta$ in $L^\gamma(\Omega)$, $\gamma = (\frac{\alpha^*+d}{2})' < \frac{d}{d-1}$. The passage to the limit on the right-hand side is based on the same argument as in the termistor problem considered above.

It remains to justify estimate (5.4) for the solution of the second equation in (5.8).

The sequence of solenoidal vectors u_ε is bounded in L^{α^*}, and $\alpha^* > d$ by condition (5.7). Therefore, the representation $u_\varepsilon = \operatorname{div} G_\varepsilon$ holds, where $G_\varepsilon \in W^{1,\alpha^*}(\Omega)$, $\|G_\varepsilon\|_{L^\infty} \leqslant M$, is a skew-symmetric matrix. By condition (5.7), our equation can be rewritten in the divergence form

$$-\Delta\theta_\varepsilon + u_\varepsilon \cdot \nabla\theta_\varepsilon = -\operatorname{div}((I - G_\varepsilon)\nabla\theta_\varepsilon) = f,$$

and the estimate (5.4) follows from the results of the L^1-theory [25].

Theorem 5.2. *There exist functions*

$$\theta \in W_0^{1,\gamma}(\Omega) \qquad \forall \gamma \in [1, d/(d-1)),$$
$$u \in (W_0^{1,p(\cdot)}(\Omega))^d, \quad p = \sigma(\theta), \ \operatorname{div} u = 0, \quad \textit{and} \quad \pi \in L^{\beta'}(\Omega)$$

such that

$$\begin{cases} \operatorname{div}[a(\theta)|Du|^{\sigma(\theta)-2}Du - u \otimes u] = \operatorname{div} g + \nabla\pi, \\ -\Delta\theta + u \cdot \nabla\theta = \operatorname{div}[(a(\theta)|Du|^{\sigma(\theta)-2}Du - g)u] + g \cdot Du = a(\theta)|Du|^{\sigma(\theta)} + \mu^s \end{cases}$$

in the sense of distributions, where μ^s is a singular measure in Ω. Furthermore,

$$\int_\Omega a(\theta)|Du|^{\sigma(\theta)} \, dx \leqslant \int_\Omega g \cdot Du \, dx. \tag{5.9}$$

If equality holds in (5.9), *then* $\mu^s = 0$.

6. Model Example

1. Let $d = 2$, let $\Omega = \{|x| < 1\}$ be the unit disk, and let the exponent p be given by relation (1.6). The coordinate axes divide the disk Ω into four sectors Ω_1, Ω_2, Ω_3, and Ω_4 in accordance with the natural numbering of quarter-planes. Consider the function

$$\psi_0(x) = (1 - r^2)\psi(x), \quad \psi(x) = \begin{cases} 1 & \text{for } x \in \Omega_1, \\ \sin\theta & \text{for } x \in \Omega_2, \\ 0 & \text{for } x \in \Omega_3, \\ \cos\theta & \text{for } x \in \Omega_4, \end{cases} \tag{6.1}$$

where r, θ are polar coordinates. Note that $|\nabla\psi_0| \leqslant 2$ on $\Omega_1 \cup \Omega_3$ and $|\nabla\psi_0| \leqslant 2r^{-1}$ on $\Omega_2 \cup \Omega_4$. Therefore, $\int_\Omega |\nabla\psi_0|^{p(x)} \, dx < \infty$; i.e., $\psi_0 \in W_0^{1,p(\cdot)}$. Let us verify that $\psi_0 \notin H_0^{1,p(\cdot)}(\Omega)$. Assuming the contrary, we find a sequence $u_\varepsilon \in C_0^\infty(\Omega)$ such that

$$\int_\Omega |\nabla u_\varepsilon|^p \, dx \leqslant C < \infty, \qquad \nabla u_\varepsilon \to \nabla\psi_0 \ \text{in} \ L^\alpha(\Omega).$$

It follows that the family u_ε is bounded in $W^{1,\beta}(\Omega_1)$ and $W^{1,\beta}(\Omega_3)$. We have $\beta > 2$, and therefore, by the Sobolev embedding theorem, this family is uniformly Hölder in the closed domain $\overline{\Omega}_1 \cup \overline{\Omega}_3$. Therefore, the function ψ_0 itself must be continuous in the closed domain $\overline{\Omega}_1 \cup \overline{\Omega}_3$ as well, which is obviously untrue.

Thus, we have proved that the exponent (1.6) is not regular in Ω. This is a long-known result (see [3], [4]). By contrast, we also mention the well-known fact stated in the lemma below.

Lemma 6.1. *The exponent* (1.6) *is regular in the annulus* $\{0 < \delta < |x| < 1\}$

2. As was mentioned above, the function $u \in W_0^{1,p(\,\cdot\,)}(\Omega)$ is continuous in each of the closed domains $\overline{\Omega}_1$ and $\overline{\Omega}_3$, and one can speak of the limit values of u at $x = 0$ from the first and third quadrants.

Lemma 6.2. *A function* $u \in W_0^{1,p(\,\cdot\,)}(\Omega)$ *belongs to the subspace* $H_0^{1,p(\,\cdot\,)}(\Omega)$ *if and only if the above-mentioned limit values coincide.*

Proof. We can assume that the common limit value is zero. Take the cutoff function

$$\eta \in C^\infty(\mathbb{R}^2), \quad \eta \equiv 0 \ \text{ for } \ |x| \leqslant 1/2, \quad \eta \equiv 1 \ \text{ for } \ |x| \geqslant 1,$$

and set $u_\varepsilon(x) = \eta_\varepsilon(x)u(x)$ and $\eta_\varepsilon(x) = \eta(\varepsilon^{-1}x)$. Note that $u_\varepsilon \in H_0$ by Lemma 6.1 and it remains to prove the weak convergence

$$\nabla u_\varepsilon = \eta_\varepsilon \nabla u + u \nabla \eta_\varepsilon \rightharpoonup \nabla u \quad \text{in } \ L^{p(\,\cdot\,)}, \tag{6.2}$$

since the space $L^{p(\,\cdot\,)}$ is reflexive (see [1]) and, by the Mazur lemma (see [13, Chap. I, Sec. 1]), there exists a suitable approximation in norm. In turn, since $\nabla u_\varepsilon(x) \to \nabla u(x)$ almost everywhere, to prove the weak convergence (6.2), it suffices to establish the boundedness of the family ∇u_ε in $L^{p(\,\cdot\,)}$. The boundedness of the first term $\eta_\varepsilon \nabla u$ is obvious, and therefore everything is now reduced to the boundedness of $u \nabla \eta_\varepsilon$. By the embedding theorem, we have

$$|u(x)| \leqslant c\,|x|^{(1-2/\beta)} \quad \text{on } \ \Omega_1 \cap \Omega_3,$$

and the gradient $\nabla \eta_\varepsilon$ is concentrated in the disk $B_\varepsilon = \{|x| < \varepsilon\}$. It follows that

$$\int_{\Omega_1 \cup \Omega_3} |\nabla \eta_\varepsilon|^\beta |u|^\beta \, dx \leqslant c_1 \varepsilon^{-\beta} \varepsilon^2 \varepsilon^{\beta(1-2/\beta)} = c_1.$$

Furthermore, by the embedding theorem, $u \in L^{2\alpha/(2-\alpha)}(\Omega_2 \cup \Omega_4)$, and by the Hölder inequality,

$$\int_{\Omega_2 \cup \Omega_4} |\nabla \eta_\varepsilon|^\alpha |u|^\alpha \, dx \leqslant c_2 \varepsilon^{-\alpha} \int_{B_\varepsilon} |u|^\alpha \, dx \leqslant c_2 \varepsilon^{-\alpha} \biggl(\int_{B_\varepsilon} |u|^{2\alpha/(2-\alpha)} \, dx \biggr)^{(2-\alpha)/2} |B_\varepsilon|^{\alpha/2} \to 0.$$

Thus, the proof of the lemma is complete.

Theorem 6.3. *The space* $H_0^{1,p(\cdot)}(\Omega)$ *has codimension* 1 *in* $W_0^{1,p(\cdot)}(\Omega)$.

Proof. Let u be an arbitrary function in $W_0^{1,p(\,\cdot\,)}(\Omega)$, let c_1 and c_3 be the limit values from the sectors Ω_1 and Ω_3, respectively, and let $\lambda = c_1 - c_3$. Then, by Lemma 6.2, we have $u - \lambda \psi_0 \in H_0^{1,p(\,\cdot\,)}(\Omega)$, as desired.

3. Let us indicate a nontrivial functional vanishing on $H_0^{1,p(\,\cdot\,)}$. Set

$$w(x) = w(x_1, x_2) = \psi(-x_2, x_1), \qquad b(x) = \biggl\{ -\frac{\partial w}{\partial x_2}, \frac{\partial w}{\partial x_1} \biggr\}. \tag{6.3}$$

By construction (see also (6.1)), the vector b belongs to the Orlicz space $L^{p'(\,\cdot\,)}(\Omega)$. Furthermore, b is solenoidal; namely, $\operatorname{div} b = 0$ in the sense of distributions, $\int_\Omega b \cdot \nabla \varphi \, dx = 0$ for every $\varphi \in C_0^\infty(\Omega)$. Indeed, since $w \in W^{1,1}(\Omega)$, we have

$$\int_\Omega \biggl(-\frac{\partial w}{\partial x_2} \frac{\partial \varphi}{\partial x_1} + \frac{\partial w}{\partial x_1} \frac{\partial \varphi}{\partial x_2} \biggr) dx = \int_\Omega w \biggl(\frac{\partial^2 \varphi}{\partial x_2 \partial x_1} - \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} \biggr) dx = 0.$$

It follows that the functional

$$l(\varphi)=\int_\Omega b\cdot\nabla\varphi\,dx$$

is continuous on $W_0^{1,p(\,\cdot\,)}(\Omega)$ and is annihilated by $H_0^{1,p(\,\cdot\,)}(\Omega)$. Next, by construction,

$$b(x)\cdot\nabla\psi(x)\equiv 0\quad\text{almost everywhere on }\ \Omega. \tag{6.4}$$

Let us verify that

$$l(\psi_0)=\int_\Omega b\cdot\nabla\psi_0\,dx=-1. \tag{6.5}$$

By (6.4), $b\cdot\nabla\psi_0=-b\cdot\nabla u$, where $u=r^2\psi\in W^{1,\infty}(\Omega)$. Therefore, is suffices to use the proposition below.

Proposition 6.4. *If $u\in W^{1,\infty}(\Omega)$ and $u|_{\partial\Omega}=\psi$, then $\int_\Omega b\cdot\nabla u\,dx=1$.*

Proof. If ν is the unit outward normal to $\partial\Omega=\{|x|=1\}$, then

$$b\cdot\nu=-\frac{\partial w}{\partial x_2}\cos\theta+\frac{\partial w}{\partial x_1}\sin\theta=-\frac{\partial w}{\partial\theta},$$

$$\int_\Omega\nabla u\cdot b\,dx=\int_{\partial\Omega}\psi b\cdot\nu\,ds=-\int_0^{\pi/2}\frac{\partial w}{\partial\theta}\,d\theta=\int_0^{\pi/2}\sin\theta\,d\theta=1,$$

as desired.

4. Let us show that the set of weak solutions is nonconvex. Consider the Dirichlet problem (1.1) with $g=b$, where b is the vector defined in (6.3). Since the vector b is solenoidal, it follows that the H-solution is zero. Relation (6.5) shows that the W-solution is nonzero. We denote it by u. Let us verify that tu is not a weak solution for small $t>0$. Assuming the contrary, we obtain

$$\int_\Omega|t\nabla u|^{p-2}t\nabla u\cdot\nabla\varphi\,dx=\int_\Omega b\cdot\nabla\varphi\,dx=0\qquad\forall\varphi\in C_0^\infty(\Omega),$$

$$t^{\beta-2}\int_{\Omega_1\cup\Omega_3}|\nabla u|^{\beta-2}\nabla u\cdot\nabla\varphi\,dx+t^{\alpha-2}\int_{\Omega_2\cup\Omega_4}|\nabla u|^{\alpha-2}\nabla u\cdot\nabla\varphi\,dx=0,$$

$$\int_{\Omega_2\cup\Omega_4}|\nabla u|^{\alpha-2}\nabla u\cdot\nabla\varphi\,dx=0, \tag{6.6}$$

where the passage to the limit as $t\to 0$ has been performed. By continuity, since $u\in W_0^{1,p(\,\cdot\,)}(\Omega)\subset W_0^{1,\alpha}(\Omega)$, relation (6.6) holds for every $\varphi\in W_0^{1,\alpha}(\Omega)$. It follows that $u=0$ on $\Omega_2\cup\Omega_4$. However, this means that the limit values of u from the sectors Ω_1 and Ω_3 are zero as well, i.e., that $u\in H_0^{1,p(\,\cdot\,)}$, which is impossible.

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