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Discrete Nonlinear Hyperbolic Equations. Classification of Integrable Cases*

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Abstract. We consider discrete nonlinear hyperbolic equations on quad-graphs, in particular on $\mathbb{Z}^2$. The fields are associated with the vertices and an equation of the form $Q(x_1, x_2, x_3, x_4) = 0$ relates four vertices of one cell. The integrability of equations is understood as 3D-consistency, which means that it is possible to impose equations of the same type on all faces of a three-dimensional cube so that the resulting system will be consistent. This allows one to extend these equations also to the multidimensional lattices $\mathbb{Z}^N$. We classify integrable equations with complex fields x and polynomials Q multiaffine in all variables. Our method is based on the analysis of singular solutions.

Key words: integrability, quad-graph, multidimensional consistency, zero curvature representation, Bäcklund transformation, Bianchi permutability, Möbius transformation.

1. Introduction

The idea of consistency (or compatibility) is at the core of the theory of integrable systems. It appears already in the very definition of complete integrability of a Hamiltonian flow in the Liouville–Arnold sense, which says that the flow should be included in a complete family of commuting (compatible) Hamiltonian flows [1]. Similarly, it is a characteristic feature of soliton (integrable) partial differential equations that they do not appear separately but are always organized in hierarchies of commuting (compatible) systems. The condition of the existence of a number of commuting systems can be taken as the basis of the *symmetry approach*, which is used to single out integrable systems in some general classes and classify them [18]. Another manifestation of the compatibility idea is the relation between continuous and discrete systems based on the notion of *Bäcklund transformations* and the *Bianchi permutability theorem* [9]. The latter has developed into one of the fundamental principles of discrete differential geometry [12].

Thus, the consistency of discrete equations takes center stage in the integrability theater. We say that

> A d-dimensional discrete equation possesses the *consistency* property if it can be imposed in a consistent way on all d-dimensional sublattices of a $(d+1)$-dimensional lattice.

(A more precise definition will be stated below.) As the above remarks show, the idea that this notion is closely related to integrability is not new. For $d = 1$, it was used as a possible definition of integrability of mappings in [24]. For $d = 2$, a decisive step was made in [10] and independently in [19]: it was shown that the integrability in the usual sense of soliton theory (as the existence of a zero curvature representation) *follows* for two-dimensional systems from the three-dimensional consistency. Thus, the latter property can be viewed as a definition of integrability. It is a criterion that can be checked in a completely algorithmic manner starting with no more information than the equation itself. Moreover, if this criterion gives a positive result, it also delivers the discrete zero curvature representation.

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Basic building blocks of systems on quad-graphs are quad-equations, i.e., equations of the form

$$Q(x_1, x_2, x_3, x_4) = 0 \tag{1}$$

on quadrilaterals, where the field variables $x_i \in \mathbb{CP}^1$ are assigned to the four vertices of a quadrilateral as shown in Figure 1. On $\mathbb{Z}^2$, equations of this type can be treated as discrete analogs of nonlinear hyperbolic equations. Boundary value problems of Goursat type for such systems were studied in [6].

Assumption. In this paper, we assume that Q is a *multiaffine polynomial*, i.e., a polynomial of degree one in each argument. It follows that equation (1) can be solved for each variable, and the solution is a rational function of the other three variables.

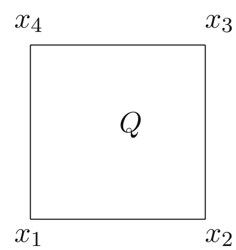

Fig. 1. A quad-equation $Q(x_1, x_2, x_3, x_4) = 0$;
the variables x_i are assigned to the vertices

The general idea of integrability as consistency in this case is shown in Figure 2. We assign six quad-equations to the faces of a coordinate cube. The subscript j corresponds to the shift in the jth coordinate direction. If one starts from arbitrary values x, x_1, x_2, x_3, then the values x_{12}, x_{13}, x_{23} are found from three equations on the left, front, and bottom faces, and the equations on the right, back, and top faces yield, in general, three different values of x_{123}. The system is said to be 3D-consistent if these three values identically coincide for arbitrary initial data x, x_1, x_2, x_3.

In [3], we classified 3D-consistent systems of a particular type. The equations on all faces coincided up to the parameter values associated with the three directions of edges. Moreover, cubic symmetry, as well as a certain additional condition called the tetrahedron property (see below), was imposed. A 3D-consistent system without the tetrahedron property was found in [13]. Later, this system was shown in [22] to be linearizable. In [26], it was shown that the 3D-consistent equations classified in [3] satisfy the integrability test based on the notion of algebraic entropy.

The consistency approach was generalized in various directions. Systems with fields on edges lead to Yang–Baxter maps ([25], [23], [20]). Quadrirational Yang–Baxter maps were classified in [4]. The 4D-consistency of discrete 3D-systems is related to the functional tetrahedron equation studied in [17], [16], [15], and [8].

In the present paper, we classify 3D-consistent multiaffine quad-equations in a more general setting. The faces of the consistency cube can carry a priori different quad-equations. Neither symmetry nor the tetrahedron property are assumed. This leads to a general classification of integrable quad-equations.

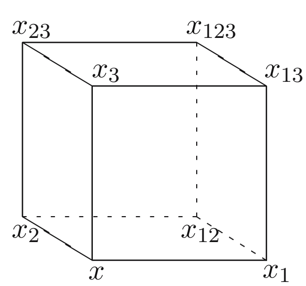

Fig. 2. A 3D consistent system of quad-equations. The equations are associated to the faces of the cube

The outline of our approach is the following.

(a) By applying discriminant-like operators to successively eliminate variables, one can descend from a multiaffine polynomial in four variables, associated to a quadrilateral, to quadratic polynomials in two variables, associated to its edges, and finally to quartic polynomials in one variable, associated to its vertices (Section 2).

(b) By analyzing singular solutions, we prove that the biquadratic polynomials that come to an edge of the cube from the two adjacent faces coincide up to a constant factor (see Section 3). At this point, an additional nondegeneracy assumption is needed. We assume that all biquadratic polynomials do not have factors of the form $x-c$ with constant c. (Examples of equations without this property are presented in Section 7.)

(c) This allows us to associate a quartic polynomial in the respective variable to each vertex of the cube; admissible sets of polynomials are classified modulo Möbius transformations, and each variable is transformed independently (Section 4).

(d) Finally, we reverse the procedure and reconstruct the biquadratic polynomials on the edges of the cube and then the multiaffine equations themselves (Section 6).

2. Multiaffine and Biquadratic Polynomials

Our approach is based on the descent from the faces to the edges and further to the vertices of the cube. In this section, we consider a single face and describe this descent irrespective of 3D-consistency. Let $\mathscr{P}_n^m$ denote the set of polynomials in n variables which are of degree m in each variable. We consider the following action of Möbius transformations on polynomials $f \in \mathscr{P}_n^m$:

$$M[f](x_1,\dots,x_n) = (c_1x_1+d_1)^m\cdots(c_nx_n+d_n)^m f\left(\frac{a_1x_1+b_1}{c_1x_1+d_1},\dots,\frac{a_nx_n+b_n}{c_nx_n+d_n}\right),$$

where $a_id_i - b_ic_i = \Delta_i \neq 0$. The operations

$$\mathscr{P}_4^1 \xrightarrow{\delta_{x_i,x_j}} \mathscr{P}_2^2 \xrightarrow{\delta_{x_k}} \mathscr{P}_1^4, \qquad \delta_{x,y}(Q) = Q_xQ_y - QQ_{xy}, \quad \delta_x(h) = h_x^2 - 2hh_{xx}$$

are covariant with respect to Möbius transformations. (The subscripts x and y denote partial differentiation.) More precisely, if $Q \in \mathscr{P}_4^1$ and $h \in \mathscr{P}_2^2$, then

$$\delta_{x_i,x_j}(M[Q]) = \Delta_i\Delta_j M[\delta_{x_i,x_j}(Q)], \quad \delta_{x_i}(M[h]) = \Delta_i^2 M[\delta_{x_i}(h)]. \tag{2}$$

Further, we make an extensive use of *relative invariants* of polynomials under Möbius transformations. For quartic polynomials $r \in \mathscr{P}_1^4$, these relative invariants are well known and can be defined as the coefficients of the Weierstrass normal form $r = 4x^3 - g_2x - g_3$. For a given polynomial $r(x) = r_4x^4 + r_3x^3 + r_2x^2 + r_1x + r_0$, they are specified by the formulas (e.g., see [28])

$$\begin{aligned} g_2(r,x) &= \frac{1}{48}(2rr^{IV} - 2r'r''' + (r'')^2) = \frac{1}{12}(12r_0r_4 - 3r_1r_3 + r_2^2),\\ g_3(r,x) &= \frac{1}{3456}(12rr''r^{IV} - 9(r')^2r^{IV} - 6r(r''')^2 + 6r'r''r''' - 2(r'')^3)\\ &= \frac{1}{432}(72r_0r_2r_4 - 27r_1^2r_4 + 9r_1r_2r_3 - 27r_0r_3^2 - 2r_2^3). \end{aligned}$$

Under the Möbius change of $x = x_1$, these quantities are just multiplied by constant factors,

$$g_k(M[r],x) = \Delta_1^{2k}g_k(r,x), \quad k = 2,3.$$

For biquadratic polynomials $h \in \mathscr{P}_2^2$,

$$h(x,y) = h_{22}x^2y^2 + h_{21}x^2y + h_{20}x^2 + h_{12}xy^2 + h_{11}xy + h_{10}x + h_{02}y^2 + h_{01}y + h_{00}, \tag{3}$$

the relative invariants are

$$\begin{aligned} i_2(h,x,y) &= 2hh_{xxyy} - 2h_x h_{xyy} - 2h_y h_{xxy} + 2h_{xx}h_{yy} + h_{xy}^2 \\ &= 8h_{00}h_{22} - 4h_{01}h_{21} - 4h_{10}h_{12} + 8h_{02}h_{20} + h_{11}^2, \\ i_3(h,x,y) &= \frac{1}{4}\det\begin{pmatrix} h & h_x & h_{xx} \\ h_y & h_{xy} & h_{xxy} \\ h_{yy} & h_{xyy} & h_{xxyy} \end{pmatrix} = \det\begin{pmatrix} h_{22} & h_{21} & h_{20} \\ h_{12} & h_{11} & h_{10} \\ h_{02} & h_{01} & h_{00} \end{pmatrix}. \end{aligned}$$

Note that i_3 can also be defined by the formula

$$-4i_3(h,x,y) = \delta_{x,y}(\delta_{x,y}(h))/h.$$

The transformation law associated with the Möbius change of $x = x_1$ and $y = x_2$ has the form

$$i_k(M[h],x,y) = \Delta_1^k \Delta_2^k i_k(h,x,y), \qquad k = 2,3.$$

The following properties of the operations $\delta_{x,y}$ and δ_x can be proved by straightforward computations.

Lemma 1. *The identities*

$$\delta_{x_3}(\delta_{x_1,x_2}(Q)) = \delta_{x_2}(\delta_{x_1,x_3}(Q)), \tag{4}$$

$$i_k(\delta_{x_1,x_2}(Q), x_3, x_4) = i_k(\delta_{x_3,x_4}(Q), x_1, x_2), \qquad k = 2,3, \tag{5}$$

hold for any multiaffine polynomial $Q(x_1,x_2,x_3,x_4)$. The identity

$$g_k(\delta_{x_1}(h), x_2) = g_k(\delta_{x_2}(h), x_1), \qquad k = 2,3, \tag{6}$$

holds for any biquadratic polynomial $h(x_1,x_2) \in \mathscr{P}_2^2$.

Let $h^{ij} = h^{ji} = \delta_{x_k,x_l}(Q)$, where $\{i,j,k,l\} = \{1,2,3,4\}$. Then Lemma 1 implies the commutativity of the diagram

$$\begin{array}{ccccc} r_4(x_4) & \xleftarrow{\delta_{x_3}} & h^{34}(x_3,x_4) & \xrightarrow{\delta_{x_4}} & r_3(x_3) \\ {\scriptstyle\delta_{x_1}}\uparrow & & \uparrow{\scriptstyle\delta_{x_1,x_2}} & & \uparrow{\scriptstyle\delta_{x_2}} \\ h^{14}(x_1,x_4) & \xleftarrow{\delta_{x_2,x_3}} & Q(x_1,x_2,x_3,x_4) & \xrightarrow{\delta_{x_1,x_4}} & h^{23}(x_2,x_3) \\ {\scriptstyle\delta_{x_4}}\downarrow & & \downarrow{\scriptstyle\delta_{x_3,x_4}} & & \downarrow{\scriptstyle\delta_{x_3}} \\ r_1(x_1) & \xleftarrow{\delta_{x_2}} & h^{12}(x_1,x_2) & \xrightarrow{\delta_{x_1}} & r_2(x_2) \end{array} \tag{7}$$

Moreover, the biquadratic polynomials on the opposite edges have the same invariants i_2 and i_3, and all four quartic polynomials r_i have the same invariants g_2 and g_3. This diagram can be completed by the polynomials h^{13} and h^{24} corresponding to the diagonals (so that the graph of the tetrahedron appears), but we will not need them. The introduced polynomials also satisfy a number of other interesting relations.

Lemma 2. *For any multiaffine polynomial $Q(x_1,x_2,x_3,x_4) \in \mathscr{P}_4^1$, the following identities hold (where $h^{ij}(x_i,x_j) = \delta_{x_k,x_l}(Q) \in \mathscr{P}_2^2$):*

$$4i_3(h^{12},x_1,x_2)h^{14} = \det\begin{pmatrix} h^{12} & h^{12}_{x_1} & \ell \\ h^{12}_{x_2} & h^{12}_{x_1x_2} & \ell_{x_2} \\ h^{12}_{x_2x_2} & h^{12}_{x_1x_2x_2} & \ell_{x_2x_2} \end{pmatrix}, \tag{8}$$

where $\ell = h^{23}_{x_3x_3}h^{34} - h^{23}_{x_3}h^{34}_{x_3} + h^{23}h^{34}_{x_3x_3}$;

$$h^{12}h^{34} - h^{14}h^{23} = PQ, \qquad P = \det\begin{pmatrix} Q & Q_{x_1} & Q_{x_3} \\ Q_{x_2} & Q_{x_1x_2} & Q_{x_2x_3} \\ Q_{x_4} & Q_{x_1x_4} & Q_{x_3x_4} \end{pmatrix} \in \mathscr{P}_4^1, \tag{9}$$

$$\frac{2Q_{x_1}}{Q}=\frac{h^{12}_{x_1}h^{34}-h^{14}_{x_1}h^{23}+h^{23}h^{34}_{x_3}-h^{23}_{x_3}h^{34}}{h^{12}h^{34}-h^{14}h^{23}}\,. \tag{10}$$

Identity (8) shows that h^{14} can be expressed via three other biquadratic polynomials (provided that $i_3(h^{12})\neq 0$). Identity (9) defines Q as one of the factors in a simple expression built from h^{ij}. Finally, differentiating (10) with respect to x_2 or x_4 leads to a relation of the form $Q^2=F[h^{12},h^{23},h^{34},h^{14}]$, where F is a rational expression in h^{ij} and their derivatives. Therefore, if the biquadratic polynomials on three edges (out of four) is known, then Q can be found explicitly. Of course, it is seen from Lemma 2 that not any set of three biquadratic polynomials can be obtained as h^{ij} from some $Q\in\mathscr{P}^1_4$.

The biquadratic polynomials h^{ij} for a given $Q\in\mathscr{P}^1_4$ are closely related to singular solutions of the multiaffine equation

$$Q(x_1,x_2,x_3,x_4)=0. \tag{11}$$

The polynomial $Q\in\mathscr{P}^1_4$ is assumed to be irreducible. (In particular, $Q_{x_i}\not\equiv 0$; otherwise, the polynomial Q should be treated as reducible, since the change of variable $x_i\mapsto 1/x_i$ takes it to x_iQ). Obviously, equation (11) can be solved for any variable: let $Q=p(x_j,x_k,x_l)x_i+q(x_j,x_k,x_l)$; then $x_i=-q/p$ for generic x_j,x_k, and x_l. However, x_i is not defined if the point (x_j,x_k,x_l) lies on the curve

$$S_i:\quad p(x_j,x_k,x_l)=q(x_j,x_k,x_l)=0,\quad Q\equiv px_i+q, \tag{12}$$

in $(\mathbb{CP}^1)^3$. The projection of this curve onto the coordinate plane (j,k) is exactly the biquadratic $h^{jk}=pq_{x_l}-p_{x_l}q=0$.

Definition 1. A solution (x_1,x_2,x_3,x_4) of equation (11) is said to be *singular* with respect to x_i if it also satisfies the equation $Q_{x_i}(x_1,x_2,x_3,x_4)=0$. The curve S_i is called the *singular curve* for x_i.

Lemma 3. *If a solution (x_1,x_2,x_3,x_4) of equation (11) is singular with respect to x_i, then $h^{jk}=h^{jl}=h^{kl}=0$. Conversely, if $h^{jk}=0$ for some solution, then this solution is singular with respect to either x_i or x_l.*

Proof. Since $h^{jk}=Q_{x_i}Q_{x_l}-QQ_{x_i,x_l}$, it follows that the equations $h^{jk}=0$ and $Q_{x_i}Q_{x_l}=0$ are equivalent for the solutions of the equation $Q=0$. □

We use the following notion of nondegeneracy for biquadratic polynomials.

Definition 2. A biquadratic polynomial $h(x,y)\in\mathscr{P}^2_2$ is said to be *nondegenerate* if no polynomial in its equivalence class with respect to Möbius transformations is divisible by a factor of the form $x-c$ or $y-c$ (with $c=\text{const}$).

According to this definition, a nondegenerate polynomial $h(x,y)\in\mathscr{P}^2_2$ is either irreducible or has the form $(\alpha_1xy+\beta_1x+\gamma_1y+\delta_1)(\alpha_2xy+\beta_2x+\gamma_2y+\delta_2)$ with $\alpha_i\delta_i\neq\beta_i\gamma_i$. In both cases, the equation $h=0$ defines y as a two-valued function of x, and vice versa. On the other hand, for example, the polynomial $h(x,y)=x-y^2$ (treated as an element of $\mathscr{P}^2_2$) is, according to Definition 2, a degenerate biquadratic, since the inversion $x\mapsto 1/x$ takes it to $x(1-xy^2)$.

The following notion plays a fundamental role in our studies.

Definition 3. A multiaffine function $Q\in\mathscr{P}^1_4$ is said to be *of type* Q if all four of its accompanying biquadratics $h^{jk}\in\mathscr{P}^2_2$ are nondegenerate and *of type* H otherwise.

3. 3D-Consistency and Biquadratic Curves

Consider the system of equations

$$\begin{aligned}
A(x,x_1,x_2,x_{12})&=0, & \bar A(x_3,x_{13},x_{23},x_{123})&=0,\\
B(x,x_1,x_3,x_{13})&=0, & \bar B(x_2,x_{12},x_{23},x_{123})&=0,\\
C(x,x_2,x_3,x_{23})&=0, & \bar C(x_1,x_{12},x_{13},x_{123})&=0
\end{aligned} \tag{13}$$

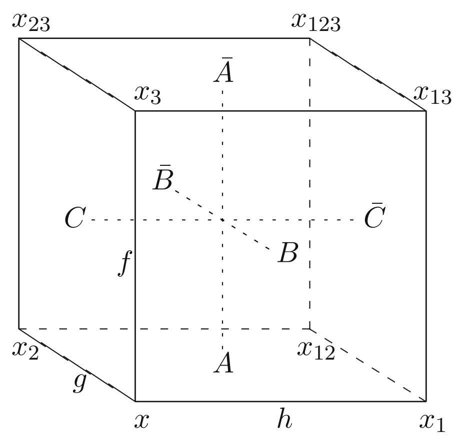

Fig. 3. A 3D-consistent system of quad-equations. The equations are associated with the faces of the cube: A and $\bar{A}$, with the bottom and top ones; B and $\bar{B}$, with the front and back ones; and C and $\bar{C}$, with the left and right ones

on a cube; see Figure 3. The functions $A, \dots, \bar{C}$ are multiaffine (belong to $\mathscr{P}_4^1$) and are not a priori supposed to be related to each other in any way. We will use the notation $A^{ij} = \delta_{x_k,x_l} A$ for the accompanying biquadratic polynomials.

Theorem 1. *Let all six functions $A, \dots, \bar{C}$ be of type Q, and let equations* (13) *be 3D-consistent. Then*

(i) *For any edge of the cube, the two biquadratic polynomials corresponding to this edge* (*coming from the two faces sharing this edge*) *coincide up to a constant factor.*

(ii) *The product of these factors around any vertex is equal to -1; for example,*

$$A^{0,1}B^{0,3}C^{0,2} + A^{0,2}B^{0,1}C^{0,3} = 0. \tag{14}$$

(iii) *System* (13) *possesses the tetrahedron property $\partial x_{123}/\partial x = 0$.*

Proof. The elimination of x_{12}, x_{13}, and x_{23} leads to equations

$$F(\overset{2}{x}, \overset{1}{x}_1, \overset{1}{x}_2, \overset{3}{x}_3, \overset{1}{x}_{123}) = \bar{A}_{x_{13},x_{23}}BC - \bar{A}_{x_{23}}B_{x_{13}}C - \bar{A}_{x_{13}}BC_{x_{23}} + \bar{A}B_{x_{13}}C_{x_{23}} = 0,$$

$$G(\overset{2}{x}, \overset{1}{x}_1, \overset{3}{x}_2, \overset{1}{x}_3, \overset{1}{x}_{123}) = \bar{B}_{x_{12},x_{23}}AC - \bar{B}_{x_{23}}A_{x_{12}}C - \bar{B}_{x_{12}}AC_{x_{23}} + \bar{B}A_{x_{12}}C_{x_{23}} = 0,$$

$$H(\overset{2}{x}, \overset{3}{x}_1, \overset{1}{x}_2, \overset{1}{x}_3, \overset{1}{x}_{123}) = \bar{C}_{x_{12},x_{13}}AB - \bar{C}_{x_{13}}A_{x_{12}}B - \bar{C}_{x_{12}}AB_{x_{13}} + \bar{C}A_{x_{12}}B_{x_{13}} = 0.$$

Here the numbers over the arguments of F, G, and H indicate the degrees of the right-hand side in the respective variables. (The degree is understood in the projective sense, as in the example at the end of the previous section.) Owing to 3D-consistency, the expressions for x_{123} as functions of x, x_1, x_2, x_3, found from these equations, coincide. Therefore, the following factorizations hold:

$$F = f(x, \overset{2}{x}_3)K, \quad G = g(x, \overset{2}{x}_2)K, \quad H = h(x, \overset{2}{x}_1)K, \quad K = K(x, \overset{1}{x}_1, \overset{1}{x}_2, \overset{1}{x}_3, \overset{1}{x}_{123}), \tag{15}$$

where f, g, and h are some polynomials of degree 2 in the second argument. The degree in x remains to be determined.

Let the initial data x, x_1, and x_2 be free variables, and let x_3 be chosen so as to satisfy the equation $f(x, x_3) = 0$. Then $F \equiv 0$, and thus the system $B = C = \bar{A} = 0$ does not determine the value of x_{123}. Moreover, the equation $B = 0$ can be solved for x_{13}, since otherwise the initial data would be constrained by equation $B^{0,1}(x, x_1) = 0$. Likewise, the equation $C = 0$ is solvable for x_{23}. Therefore, the uncertainty appears from the singularity of the equation $\bar{A} = 0$ with respect to x_{123}. Hence, the relation $\bar{A}^{3,13}(x_3, x_{13}) = 0$ holds. By virtue of the assumption of the theorem, x_{13} is a (two-valued) function of x_3 and does not depend on x_1. This means that the equation $B = 0$ is singular with respect to x_1, and therefore, $B^{0,3}(x, x_3) = 0$. Likewise, $C^{0,3}(x, x_3) = 0$.

Thus, we have proved that if $x_3 = \varphi(x)$ is a zero of the polynomial f, then it is also a zero of the polynomials $B^{0,3}$ and $C^{0,3}$. If one of these three polynomials is irreducible, then this already implies that they coincide up to a constant factor. If the polynomials are reducible, this may be wrong, since it is possible that $f = a^2$, $B^{0,3} = ab$, and $C^{0,3} = ac$, where a, b, and c are multiaffine in x and x_3. In any case, we have $\deg_x f = 2$, and this is sufficient to complete the proof.

Indeed, this implies that $\deg_x K = 0$, that is, the tetrahedron property holds. In turn, this implies the relation (14), as was shown in [3]. Recall this computation: let us rewrite system (13) in the form

$$x_{12} = a(x, x_1, x_2), \quad x_{13} = b(x, x_1, x_3), \quad x_{23} = c(x, x_2, x_3),$$
$$x_{123} = d(x_1, x_2, x_3) = \bar{a}(x_3, x_{13}, x_{23}) = \bar{b}(x_2, x_{12}, x_{23}) = \bar{c}(x_1, x_{12}, x_{13})$$

and find, by differentiation,

$$\begin{aligned} d_{x_1} &= \bar{a}_{x_{13}} b_{x_1}, & d_{x_2} &= \bar{a}_{x_{23}} c_{x_2}, & 0 &= \bar{a}_{x_{13}} b_x + \bar{a}_{x_{23}} c_x, \\ d_{x_1} &= \bar{b}_{x_{12}} a_{x_1}, & d_{x_3} &= \bar{b}_{x_{23}} c_{x_3}, & 0 &= \bar{b}_{x_{12}} a_x + \bar{b}_{x_{23}} c_x, \\ d_{x_2} &= \bar{c}_{x_{12}} a_{x_2}, & d_{x_3} &= \bar{c}_{x_{13}} b_{x_3}, & 0 &= \bar{c}_{x_{12}} a_x + \bar{c}_{x_{13}} b_x. \end{aligned}$$

These equations readily imply the relation

$$a_{x_2} b_{x_1} c_{x_3} + a_{x_1} b_{x_3} c_{x_2} = 0,$$

which is equivalent to (14) by virtue of the identity $a_{x_2}/a_{x_1} = A^{0,1}/A^{0,2}$. The variables in equation (14) separate: $B^{0,3}/C^{0,3} = -A^{0,2}/C^{0,2} \cdot B^{0,1}/A^{0,1}$, so that $B^{0,3}/C^{0,3}$ can only depend on x. In view of the assumption of the theorem, this ratio is constant. □

There exist 3D-consistent systems whose equations are not of type Q. The assertions of Theorem 1 may or may not hold for such systems, as the following examples show.

Example 1. The simplest 3D-consistent equation is the linear equation

$$x + x_i + x_j + x_{ij} = 0.$$

In this case, all biquadratic polynomials are equal to 1, so that assertion (i) is satisfied and assertion (ii) is not. Since (ii) is a consequence of the tetrahedron property (iii), it follows that the latter cannot hold either. Indeed,

$$x_{123} = 2x + x_1 + x_2 + x_3.$$

The factor f in this example is also equal to 1, but this is a coincidence, destroyed by Möbius changes of variables. Indeed, in this case $\deg_x K = 1$, and after the inversion $x_I \to 1/x_I$ of all variables we arrive at $f = xx_3^2$, while $B^{0,3}$ turns into $x^2 x_3^2$.

Example 2. The Hietarinta equation [13]

$$(x - e^{(j)})(x_{ij} - o^{(j)})(x_i - o^{(i)})(x_j - e^{(i)}) - (x - e^{(i)})(x_{ij} - o^{(i)})(x_i - e^{(j)})(x_j - o^{(j)}) = 0 \qquad (16)$$

is 3D-consistent, but assertion (i) does not hold:

$$B^{0,3} = (e^{(3)} - o^{(1)})(o^{(1)} - o^{(3)})(x - e^{(3)})(x - e^{(1)})(x_3 - e^{(1)})(x_3 - o^{(3)}),$$
$$C^{0,3} = (e^{(3)} - o^{(2)})(o^{(2)} - o^{(3)})(x - e^{(3)})(x - e^{(2)})(x_3 - e^{(2)})(x_3 - o^{(3)}).$$

The factor f is proportional to $(x - e^{(3)})(x_3 - e^{(1)})(x_3 - e^{(2)})$. Consequently, $\deg_x K = 1$, and the tetrahedron property does not hold.

Example 3. Probably the best known example of a 3D-consistent system is given by the discrete potential KdV equation

$$(x - x_{ij})(x_i - x_j) + \alpha^{(i)} - \alpha^{(j)} = 0. \qquad (17)$$

In this case, all assertions of the theorem hold, in spite of the degeneracy of the biquadratics:

$$B^{0,3} = \alpha^{(1)} - \alpha^{(3)}, \quad C^{0,3} = \alpha^{(2)} - \alpha^{(3)}, \quad f = 1.$$

(Recall that the degree is understood in the projective sense. Under the inversion, these polynomials turn into $x^2x_3^2$.)

Example 4. Equation (Q_1)

$$Q(x, x_1, x_2, x_{12}; \alpha^{(1)}, \alpha^{(2)}; \delta) = \alpha^{(1)}(x - x_2)(x_1 - x_{12}) - \alpha^{(2)}(x - x_1)(x_2 - x_{12}) + \delta\alpha^{(1)}\alpha^{(2)}(\alpha^{(1)} - \alpha^{(2)}) = 0$$

is consistent not only with its own copies (see [3] and Theorem 4 below), but also with linear equations. Namely, the system formed by the equations

$$Q(x, x_1, x_{12}, x_2; \alpha^{(1)}, \alpha^{(2)}; \delta) = 0, \quad x_{13} - x_3 = x_1 - x, \quad x_{23} - x_3 = x_2 - x$$

and their copies on the opposite faces is 3D-consistent. In this case, the edge (x, x_3) carries the polynomials

$$B^{0,3} = C^{0,3} = -1, \quad f = 1.$$

However, in contrast to the previous example, the tetrahedron property is not valid, and $\deg_x K = 2$. This means that the polynomial f is not biquadratic and its image under inversion is x_3^2. Moreover, the biquadratic polynomials corresponding to the edge (x, x_1) do not coincide,

$$A^{0,1} = \alpha^{(2)}(\alpha^{(1)} - \alpha^{(2)})\big((x_1 - x)^2 - \delta(\alpha^{(1)})^2\big), \quad B^{0,1} = -1, \quad h = 1.$$

We see in this example that it is possible that some of the biquadratic polynomials satisfy the assumptions of the theorem and the others do not.

4. Classification of Biquadratic Polynomials

Diagram (7) suggests an algorithm for the classification of multiaffine equations $Q = 0$ modulo Möbius transformations. The first step is to use Möbius transformations to bring the polynomials $r_i(x_i)$ associated with the vertices of the quadrilateral into canonical form. According to formulas (2),

$$\delta_{x_l}(\delta_{x_j,x_k}(M[Q])) = \Delta_j^2\Delta_k^2\Delta_l^2 M[\delta_{x_l}(\delta_{x_j,x_k}(Q))] = \frac{C}{\Delta_i^2}M[r_i],$$

where $C = \Delta_1^2\Delta_2^2\Delta_3^2\Delta_4^2$. Since the polynomial Q is defined up to an arbitrary factor, we can assume that Möbius changes of variables in the equation $Q = 0$ induce the transformations

$$r_i \mapsto \frac{1}{\Delta_i^2}M[r_i]$$

of the polynomials r_i. This allows us to bring each r_i into one of the following six forms:

$$r = (x^2 - 1)(k^2x^2 - 1), \quad r = x^2 - 1, \quad r = x^2, \quad r = x, \quad r = 1, \quad r = 0,$$

according to the six possibilities for the root distribution of r: four simple roots, two simple roots and one double root, two pairs of double roots, one simple root and one triple root, one quadruple root, or, finally, r vanishes identically. Note that in the first canonical form it is always assumed that $k \neq 0, \pm 1$, so that the second and third forms are not considered as special cases of the first one.

Not every pair of such polynomials is admissible as a pair of polynomials at two adjacent vertices, since the relative invariants of the polynomials in such a pair must coincide according to (6). We identify all admissible pairs and then solve the problem of reconstructing the biquadratic polynomial (3) from the pair

$$\delta_y(h) = h_y^2 - 2hh_{yy} = r_1(x), \qquad \delta_x(h) = h_x^2 - 2hh_{xx} = r_2(y) \tag{18}$$

of its discriminants; this is equivalent to a system of ten (bilinear) equations for the nine unknown coefficients of the polynomial h.

Theorem 2. *Biquadratic polynomials with the pair $(r_1(x), r_2(y))$ of discriminants in canonical form exist for the following pairs, up to the permutation of x and y:*

	$(y^2-1)(k^2y^2-1)$	y^2-1	y^2	y	1	0
$(x^2-1)(k^2x^2-1)$	+					
x^2-1		+	+			
x^2			+			
x				+	+	
1					+	+
0						+

These polynomials h and their relative invariants i_2 and i_3 are given in the following list:

$(r(x), r(y)),\ r(x) = (x^2-1)(k^2x^2-1)$:

$$h = \frac{1}{2\alpha}(k^2\alpha^2x^2y^2 + 2Axy - x^2 - y^2 + \alpha^2), \quad A^2 = r(\alpha), \tag{19}$$
$$i_2 = 3(k^2\alpha^2 + \alpha^{-2}) - k^2 - 1, \quad 4i_3 = A(k^2\alpha - \alpha^{-3});$$

$$(x^2-\delta, y^2-\delta): \quad h = \frac{\alpha}{1-\alpha^2}(x^2+y^2) - \frac{1+\alpha^2}{1-\alpha^2}xy + \frac{\delta(1-\alpha^2)}{4\alpha}, \tag{20}$$
$$i_2 = \frac{1+10\alpha^2+\alpha^4}{(1-\alpha^2)^2}, \quad i_3 = \frac{\alpha^2(1+\alpha^2)}{(1-\alpha^2)^3};$$

$$(x,y): \quad h = \frac{1}{4\alpha}(x-y)^2 - \frac{\alpha}{2}(x+y) + \frac{\alpha^3}{4}, \quad i_2 = \frac{3}{4\alpha^2},\ i_3 = \frac{1}{32\alpha^3}; \tag{21}$$

$$(x^2,y^2): \quad h = \lambda x^2 + \mu xy + \nu y^2, \quad \mu^2 - 4\lambda\nu = 1,\ i_2 = 1 + 12\lambda\nu,\ i_3 = -\lambda\mu\nu; \tag{22}$$

$$h = \lambda x^2y^2 + \mu xy + \nu, \quad \mu^2 - 4\lambda\nu = 1,\ i_2 = 1 + 12\lambda\nu,\ i_3 = \lambda\mu\nu; \tag{23}$$

$$(1,1): \quad h = \lambda(x \pm y)^2 + \mu(x \pm y) + \nu, \quad \mu^2 - 4\lambda\nu = 1,\ i_2 = 12\lambda^2,\ i_3 = \mp 2\lambda^3; \tag{24}$$

$$(0,0): \quad h = (\varkappa xy + \lambda x + \mu y + \nu)^2, \quad i_2 = 12(\varkappa\nu - \lambda\mu)^2,\ i_3 = 2(\varkappa\nu - \lambda\mu)^3; \tag{25}$$

$$(x^2-1, y^2): \quad h = \alpha y^2 \pm xy + \frac{1}{4\alpha}, \quad i_2 = 1,\ i_3 = 0; \tag{26}$$

$$(x,1): \quad h = \pm\frac{1}{4}(y-\alpha)^2 \mp x, \quad i_2 = 0,\ i_3 = 0; \tag{27}$$

$$(1,0): \quad h = \lambda y^2 + \mu y + \nu, \quad \mu^2 - 4\lambda\nu = 1,\ i_2 = 0,\ i_3 = 0. \tag{28}$$

Proof. The list is obtained by a straightforward solution of system (18) for various canonical pairs (r_1, r_2). The exhaustion of cases is shortened if we notice that $g_2^3 \neq 27g_3^2$ in one case only and that the relative invariants for the polynomial $r_1 = ax^2 + bx + c$ are $12g_2 = a^2$ and $216g_3 = -a^3$, so that the second polynomial must be of the form $r_2 = ay^2 + \tilde{b}y + \tilde{c}$. The solution for the pair $(x, 0)$ turns out to be empty. □

5. Classification of Multiaffine Equations of Type *Q*

It is important to note that after bringing the polynomials $r_i(x_i)$ into canonical forms one still has some freedom. Namely, one can use Möbius transformations that do not change the form of r to further simplify the biquadratics h and the multiaffine equation Q. In particular, the list in Theorem 2 is slightly more detailed than the list of biquadratics modulo Möbius transformations.

Indeed, the polynomial (22) turns into (23) under the inversion of x; the change $x \mapsto -x$ allows to fix the signs in the polynomials (24) and (26); in case (27), the sign is fixed by the change $x \mapsto -x$, $y \mapsto iy$; the polynomials (25) and (28) admit further simplification.

However, a transformation of any one of the four variables for a quadrilateral affects the biquadratic polynomials on the two edges adjacent to the correspondent vertex, and therefore it cannot a priori be guaranteed that all four biquadratics can be brought to some definite form simultaneously. For example, if the polynomial $r_i = x_i^2$ corresponds to each vertex, then the polynomials corresponding to the edges may have the form (22) or (23). We do not know a priori that these polynomials can be always brought into the same form (even with different coefficients). Actually, this is possible, as the proof of the following theorem shows.

The next step is the reconstruction of the multiaffine polynomials from the biquadratic ones. Since our goal is only the classification of systems of type Q equations, we will not solve this problem in full generality. We leave aside cases (26), (27), and (28), since the corresponding biquadratics are degenerate. For the same reason, we impose additional restrictions on the parameter values: $\lambda\nu \neq 0$ in cases (22) and (23), $\lambda \neq 0$ in case (24), and $\varkappa\nu - \lambda\mu \neq 0$ in case (25).

Theorem 3. *Any multiaffine equation of type Q is equivalent, up to Möbius transformations, to one of the equations in the following list:*

$$\begin{aligned}&\operatorname{sn}(\alpha)\operatorname{sn}(\beta)\operatorname{sn}(\alpha+\beta)(k^2x_1x_2x_3x_4+1) - \operatorname{sn}(\alpha)(x_1x_2+x_3x_4)\\ &\qquad - \operatorname{sn}(\beta)(x_1x_4+x_2x_3) + \operatorname{sn}(\alpha+\beta)(x_1x_3+x_2x_4) = 0,\end{aligned} \tag{29}$$

$$\begin{aligned}&(\alpha-\alpha^{-1})(x_1x_2+x_3x_4) + (\beta-\beta^{-1})(x_1x_4+x_2x_3) - (\alpha\beta-\alpha^{-1}\beta^{-1})(x_1x_3+x_2x_4)\\ &\qquad + \frac{\delta}{4}(\alpha-\alpha^{-1})(\beta-\beta^{-1})(\alpha\beta-\alpha^{-1}\beta^{-1}) = 0,\end{aligned} \tag{30}$$

$$\begin{aligned}&\alpha(x_1-x_4)(x_2-x_3) + \beta(x_1-x_2)(x_4-x_3)\\ &\qquad - \alpha\beta(\alpha+\beta)(x_1+x_2+x_3+x_4) + \alpha\beta(\alpha+\beta)(\alpha^2+\alpha\beta+\beta^2) = 0,\end{aligned} \tag{31}$$

$$\alpha(x_1-x_4)(x_2-x_3) + \beta(x_1-x_2)(x_4-x_3) - \delta\alpha\beta(\alpha+\beta) = 0. \tag{32}$$

Proof. Let the polynomials h^{12}, h^{23}, h^{34}, and h^{14} be of the form (19) with parameters (α, A), (β, B), $(\tilde\alpha, \tilde A)$ and $(\tilde\beta, \tilde B)$, respectively, lying on the elliptic curve $A^2 = r(\alpha)$. The relative invariants i_2 and i_3 of h^{12} and h^{34} must coincide by virtue of (5), and one can readily verify that this condition allows only the following possible values for $(\tilde\alpha, \tilde A)$:

$$(\alpha, A), \quad (-\alpha, -A), \quad \frac{1}{k\alpha^2}(\alpha, -A), \quad \frac{1}{k\alpha^2}(-\alpha, A);$$

similar values are possible for $(\tilde\beta, \tilde B)$. At first glance, it seems as if we have to examine 16 quadruples of h^{ij}, but actually the situation is much better. Indeed, according to (2), a Möbius change of variables in the equation $Q = 0$ yields

$$\delta_{x_k,x_l}(M[Q]) = \Delta_k\Delta_l M[\delta_{x_k,x_l}(Q)] = \frac{C}{\Delta_i\Delta_j}M[h^{ij}],$$

where $C = \Delta_1\Delta_2\Delta_3\Delta_4$. Since Q is only defined up to a multiplicative constant, we can assume that a Möbius change of variables induces the transformations

$$h^{ij} \mapsto \frac{1}{\Delta_i\Delta_j}M[h^{ij}]$$

of the biquadratic polynomials h^{ij}. In particular, if

$$h^{34} = h(x_3, x_4, -\alpha, -A) \quad \text{or} \quad h^{34} = h\Big(x_3, x_4, \frac{1}{k\alpha}, -\frac{A}{k\alpha^2}\Big),$$

then the respective Möbius transformation $x_3 \mapsto -x_3$ or $x_3 \mapsto 1/(kx_3)$ will reduce h^{34} to the form

$$-h(-x_3, x_4; -\alpha, -A), \quad \text{respectively,} \quad -kx_3^2 h\Big(\frac{1}{kx_3}, x_4; \frac{1}{k\alpha}, -\frac{A}{k\alpha^2}\Big), \tag{33}$$

which coincides with $h(x_3, x_4, \alpha, A)$ owing to symmetries of the polynomial (19). Thus, performing a suitable Möbius transformation of the variable x_3 (which does not affect the polynomial $r(x_3)$), we can assume without loss of generality that $(\tilde\alpha, \tilde A) = (\alpha, A)$. After that, the polynomial h^{14} is uniquely found according to formula (8), and it turns out that the relation $(\tilde\beta, \tilde B) = (\beta, B)$ is satisfied automatically. Thus, a change of one variable allows one to achieve the equality of the parameters corresponding to the opposite edges of the square. A straightforward computation using formula (10) yields the equation

$$\alpha\beta\gamma(k^2x_1x_2x_3x_4+1)+\alpha(x_1x_2+x_3x_4)+\beta(x_1x_4+x_2x_3)+\gamma(x_1x_3+x_2x_4)=0,$$

where $\gamma=(\alpha B+\beta A)/(k^2\alpha^2\beta^2-1)$, and the final change $\alpha\to\operatorname{sn}(\alpha)$, $A\to\operatorname{sn}'(\alpha)$ and similarly for β brings it to the form (29).

In the other cases, suitable Möbius changes of the variables x_2, x_3, and x_4 also allow us to bring the polynomials into the form $h^{12}=h(x_1,x_2,\alpha)$, $h^{23}=h(x_2,x_3,\beta)$, $h^{34}=h(x_3,x_4,\alpha)$. Moreover, a straightforward computation using formula (8) proves that, in addition, $h^{14}=h(x_1,x_4,\beta)$. Then the answer is found with the use of (10).

In more detail, the polynomials (20) give rise to equation (30). In this case, equations (5) imply that the parameters α of the polynomials h^{12} and h^{34} differ at most in sign. This is compensated for by the change of variables $x_3\to -x_3$, which is possible in view of the symmetry $h(x,y,\alpha)=-h(-x,y,-\alpha)$.

In cases (22), (23), appropriate scalings and, if necessary, inversions of the variables x_2, x_3, and x_4 allow us to bring h^{12}, h^{23}, and h^{34} into the form (20) without the constant term; thus, we arrive at the same case for $\delta=0$.

The polynomial (21) corresponds to equation (31). This is the simplest case, since the parameters are already fixed by condition (5).

In case (24), appropriate shifts and, if necessary, changes of sign of the variables x_2, x_3, and x_4 allow us to bring h^{12}, h^{23}, and h^{34} into the form $2h(x,y,\alpha)=\alpha^{-1}(x-y)^2-\delta\alpha$ with $\delta=1$. Likewise, in case (25) an appropriate Möbius transform of general form brings h^{12}, h^{23}, and h^{34} into the same form with $\delta=0$. In both cases, the invariants are $i_2=3\alpha^{-2}$ and $4i_3=\alpha^{-3}$, and therefore, the parameters of h^{12} and h^{34} coincide and no further changes are necessary. The resulting equation is (32). □

6. Classification of 3D-Consistent Systems of Type *Q*

Theorem 1 provides very strong necessary conditions for 3D-consistency for the case in which all equations are of type Q. This allows us to classify such systems in this section. At this final step, we have to arrange the obtained equations around the cube and choose the parameters in such a way that condition (14) be satisfied. This condition can result in a change of sign or an inversion of one of the parameters.

In the following theorem, we return to the notation of the variables and parameters corresponding to shifts on the lattice. The ordering of the equations corresponds to the preceding theorem, and we label these equations as in [3].

Theorem 4. *Each* 3*D-consistent system* (13) *of type* Q *is*, *up to Möbius transformations*, *one of the systems in the following list*:

$$\begin{aligned}&\operatorname{sn}(\alpha^{(i)})\operatorname{sn}(\alpha^{(j)})\operatorname{sn}(\alpha^{(i)}-\alpha^{(j)})(k^2xx_ix_jx_{ij}+1)+\operatorname{sn}(\alpha^{(i)})(xx_i+x_jx_{ij})\\&\qquad-\operatorname{sn}(\alpha^{(j)})(xx_j+x_ix_{ij})-\operatorname{sn}(\alpha^{(i)}-\alpha^{(j)})(xx_{ij}+x_ix_j)=0,\end{aligned}\qquad (Q_4)$$

$$\begin{aligned}&\Big(\alpha^{(i)}-\frac{1}{\alpha^{(i)}}\Big)(xx_i+x_jx_{ij})-\Big(\alpha^{(j)}-\frac{1}{\alpha^{(j)}}\Big)(xx_j+x_ix_{ij})-\Big(\frac{\alpha^{(i)}}{\alpha^{(j)}}-\frac{\alpha^{(j)}}{\alpha^{(i)}}\Big)(xx_{ij}+x_ix_j)\\&\qquad-\frac{\delta}{4}\Big(\alpha^{(i)}-\frac{1}{\alpha^{(i)}}\Big)\Big(\alpha^{(j)}-\frac{1}{\alpha^{(j)}}\Big)\Big(\frac{\alpha^{(i)}}{\alpha^{(j)}}-\frac{\alpha^{(j)}}{\alpha^{(i)}}\Big)=0,\end{aligned}\qquad (Q_3)$$

$$\begin{aligned}&\alpha^{(i)}(x-x_j)(x_i-x_{ij})-\alpha^{(j)}(x-x_i)(x_j-x_{ij})+\alpha^{(i)}\alpha^{(j)}(\alpha^{(i)}-\alpha^{(j)})(x+x_i+x_j+x_{ij})\\&\qquad-\alpha^{(i)}\alpha^{(j)}(\alpha^{(i)}-\alpha^{(j)})((\alpha^{(i)})^2-\alpha^{(i)}\alpha^{(j)}+(\alpha^{(j)})^2)=0,\end{aligned}\qquad (Q_2)$$

$$\alpha^{(i)}(x-x_j)(x_i-x_{ij})-\alpha^{(j)}(x-x_i)(x_j-x_{ij})+\delta\alpha^{(i)}\alpha^{(j)}(\alpha^{(i)}-\alpha^{(j)})=0.\qquad (Q_1)$$

Proof. First of all, note that equations of different types (29)–(32) cannot be consistent with each other, since the corresponding singular curves are different. In particular, the parameters k^2

in case (29) and δ in cases (30) and (32) must be the same on all faces of the cube. Moreover, each equation in the list possesses the square symmetry, that is, is invariant with respect to the changes $(x_1 \leftrightarrow x_2,\ x_3 \leftrightarrow x_4)$ and $(x_1 \leftrightarrow x_3,\ \alpha \leftrightarrow \beta)$.

Therefore, the equations on all faces may differ only in the values of α and β. Consider the equations corresponding to three faces meeting in one vertex, say, x:

$$Q(x, x_1, x_2, x_{12}, \alpha, \tilde{\beta}) = 0, \quad Q(x, x_2, x_3, x_{23}, \beta, \tilde{\gamma}) = 0, \quad Q(x, x_3, x_1, x_{13}, \gamma, \tilde{\alpha}) = 0.$$

Let

$$\delta_{x_2,x_{12}} Q(x, x_1, x_2, x_{12}, \alpha, \tilde{\beta}) = \kappa(\alpha, \tilde{\beta}) h(x, x_1, \alpha).$$

Then, owing to the symmetry,

$$\delta_{x_1,x_{12}} Q(x, x_1, x_2, x_{12}, \alpha, \tilde{\beta}) = \kappa(\tilde{\beta}, \alpha) h(x, x_2, \tilde{\beta}),$$

and according to Theorem 1, the parameters should be related as follows:

$$\frac{h(x, x_1, \alpha)}{h(x, x_1, \tilde{\alpha})} = m(\alpha, \tilde{\alpha}), \quad \frac{h(x, x_2, \beta)}{h(x, x_2, \tilde{\beta})} = m(\beta, \tilde{\beta}), \quad \frac{h(x, x_3, \gamma)}{h(x, x_3, \tilde{\gamma})} = m(\gamma, \tilde{\gamma}),$$

$$\frac{\kappa(\alpha, \tilde{\beta})\kappa(\beta, \tilde{\gamma})\kappa(\gamma, \tilde{\alpha})}{\kappa(\tilde{\beta}, \alpha)\kappa(\tilde{\gamma}, \beta)\kappa(\tilde{\alpha}, \gamma)} m(\alpha, \tilde{\alpha}) m(\beta, \tilde{\beta}) m(\gamma, \tilde{\gamma}) = -1.$$

In case (29), a straightforward computation proves that $\kappa(\alpha, \beta) = 2 \operatorname{sn}(\alpha) \operatorname{sn}(\beta) \operatorname{sn}(\alpha + \beta)$ and

$$h(x, y, \alpha) = \frac{1}{2 \operatorname{sn}(\alpha)} (k^2 \operatorname{sn}^2(\alpha) x^2 y^2 + 2 \operatorname{sn}'(\alpha) xy - x^2 - y^2 + \operatorname{sn}^2(\alpha));$$

therefore, $\tilde{\alpha}$ can take the values $\pm\alpha$, and the same is true for β and γ. Obviously, up to renumbering, two cases are possible:

$$\tilde{\alpha} = -\alpha, \quad \tilde{\beta} = -\beta, \quad \tilde{\gamma} = -\gamma \qquad \text{or} \qquad \tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta, \quad \tilde{\gamma} = -\gamma.$$

Moreover, this is actually only one case, since we can make the change $(\alpha, \tilde{\beta}) \to (-\alpha, -\tilde{\beta})$, which keeps the equation $Q(x, x_1, x_2, x_{12}, \alpha, \tilde{\beta}) = 0$ invariant, as one can readily see from (29). It is not difficult to verify that we can always adjust the signs on the whole cube as in system (Q_4).

Now consider case (30). Here

$$\kappa(\alpha, \beta) = -\frac{(1 - \alpha^2\beta^2)(1 - \alpha^2)(1 - \beta^2)}{\alpha^2\beta^2},$$

$$h(x, y, \alpha) = \frac{\alpha}{1 - \alpha^2}(x^2 + y^2) - \frac{1 + \alpha^2}{1 - \alpha^2} xy + \frac{(1 - \alpha^2)\delta}{4\alpha},$$

and $\tilde{\alpha} = \alpha$ or $\tilde{\alpha} = 1/\alpha$. Taking into account the invariance of equation (30) with respect to the simultaneous inversion of α and β, we can set, without loss of generality,

$$\tilde{\alpha} = 1/\alpha, \quad \tilde{\beta} = 1/\beta, \quad \tilde{\gamma} = 1/\gamma,$$

which results in system (Q_3). In cases (31) and (32), we accordingly have

$$\kappa(\alpha, \beta) = -4\alpha\beta(\alpha + \beta), \quad h(x, y, \alpha) = \frac{1}{4\alpha}(x - y)^2 - \frac{\alpha}{2}(x + y) + \frac{\alpha^3}{4},$$

$$\kappa(\alpha, \beta) = -2\alpha\beta(\alpha + \beta), \quad h(x, y, \alpha) = \frac{1}{2\alpha}(x - y)^2 - \frac{\alpha\delta}{2},$$

and we can set $\tilde{\alpha} = -\alpha$, $\tilde{\beta} = -\beta$, and $\tilde{\gamma} = -\gamma$ exactly as before. This leads to systems (Q_2) and (Q_1). □

The master equation (Q_4) in the list was first derived in [2] and further studied in [5]. A Lax representation for (Q_4) was found in [19] with the help of the method based on the three-dimensional consistency. Equations (Q_1) and ($Q_3|_{\delta=0}$) go back to [21]. Equations (Q_2) and ($Q_3|_{\delta=1}$) first appeared explicitly in [3].

7. Examples of Type H Systems

In contrast to type Q systems, systems of type H can be viewed as "degenerate." Their classification seems to be a rather tedious task. Presently, we cannot suggest any effective procedure to solve this problem. On the other hand, the examples given in Section 3 demonstrate that this class should not be just neglected as "pathological." Indeed, the discrete KdV example (17) suggests that in some cases the degeneracy of the biquadratics is just an unessential coincidence that does not spoil the integrability properties of an equation. Here we consider some more examples of this kind, corresponding to cases (22) and (23) with $\lambda\mu = 0$, (24) with $\lambda = 0$, and (25) with $\varkappa\nu - \lambda\mu = 0$, which were excluded in the previous section. It turns out that if we apply the same algorithm in these cases (in spite of the fact that there is no justification for this) then the list H from our previous paper [3] will be reproduced:

$$\alpha^{(i)}(xx_i + x_jx_{ij}) - \alpha^{(j)}(xx_j + x_ix_{ij}) + \delta((\alpha^{(i)})^2 - (\alpha^{(j)})^2) = 0, \tag{H_3}$$

$$(x - x_{ij})(x_i - x_j) + (\alpha^{(j)} - \alpha^{(i)})(x + x_i + x_j + x_{ij}) + (\alpha^{(j)})^2 - (\alpha^{(i)})^2 = 0, \tag{H_2}$$

$$(x - x_{ij})(x_i - x_j) + \alpha^{(j)} - \alpha^{(i)} = 0. \tag{H_1}$$

One can directly verify that all assertions of Theorem 1 remain valid for these equations, in spite of the degeneracy of the biquadratics.

Considering the asymmetric cases (26), (27), and (28) with different polynomials associated with different vertices, one finds that the following cases are possible, up to permutations:

$$(x_1^2 - 1, x_2^2, x_3^2, x_4^2), \quad (x_1^2 - 1, x_2^2 - 1, x_3^2, x_4^2), \quad (x_1^2 - 1, x_2^2 - 1, x_3^2 - 1, x_4^2),$$
$$(x_1, 1, 1, 1), \quad (x_1, x_2, 1, 1), \quad (x_1, x_2, x_3, 1), \quad (1, 0, 0, 0), \quad (1, 1, 0, 0), \quad (1, 1, 1, 0).$$

(There is clearly no distinction between edges and diagonals when we are dealing with a single equation.) A straightforward verification shows that the cases like $\begin{pmatrix} r_2(x_4) & r_1(x_3) \\ r_1(x_1) & r_2(x_2) \end{pmatrix}$ are realizable and lead to the following list of 3D-consistent equations:

$$\alpha(x_1x_2 + x_3x_4) - \beta(x_1x_4 + x_2x_3) + (\alpha^2 - \beta^2)\Big(\delta + \frac{\varepsilon x_2x_4}{\alpha\beta}\Big) = 0, \tag{H_3^ε}$$

$$\begin{aligned}(x_1 - x_3)(x_2 - x_4) + (\beta - \alpha)(x_1 + x_2 + x_3 + x_4) + \beta^2 - \alpha^2 \\ + \varepsilon(\beta - \alpha)(2x_2 + \alpha + \beta)(2x_4 + \alpha + \beta) + \varepsilon(\beta - \alpha)^3 = 0,\end{aligned} \tag{H_2^ε}$$

$$(x_1 - x_3)(x_2 - x_4) + (\beta - \alpha)(1 + \varepsilon x_2x_4) = 0. \tag{H_1^ε}$$

This list can be viewed as a deformation of the list H, which corresponds to the case $\varepsilon = 0$. However, we use the notation with cyclic indices rather than shifts, since owing to lack of symmetry the arrangement of the equations on the faces of a cube requires a more explicit description (see below). Note that in (H_1^ε) the polynomial $1+\varepsilon x_2x_4$ can be replaced by the polynomial $\varkappa x_2x_4+\mu(x_2+x_4)+\nu$ with arbitrary coefficients. The corresponding biquadratic polynomials and their discriminants are given in the following table (up to multiplication by a suitable constant, $Q \to \mu(\alpha, \beta)Q$):

	$h(x_1, x_2)$	$r_1(x_1)$	$r_2(x_2)$
(H_3^ε)	$x_1x_2 + \varepsilon\alpha^{-1}x_2^2 + \delta\alpha$	$x_1^2 - 4\delta\varepsilon$	x_2^2
(H_2^ε)	$x_1 + x_2 + \alpha + 2\varepsilon(x_2 + \alpha)^2$	$1 - 8\varepsilon x_1$	1
(H_1^ε)	$1 + \varepsilon x_2^2$	-4ε	0

Each of these equations possesses the rhombic symmetry

$$Q(x_1, x_2, x_3, x_4, \alpha, \beta) = -Q(x_3, x_2, x_1, x_4, \beta, \alpha) = -Q(x_1, x_4, x_3, x_2, \beta, \alpha)$$

but not the square symmetry, since the vertices x_1 and x_2 correspond to polynomials with zeroes of different multiplicities. The equation is 3D-consistent on the black-white lattice $i + j + k \pmod 2$. That is, each face must carry a copy of the equation in such a way that the parameters on opposite edges coincide and the vertices $x, x_{12}, x_{13}, x_{23}$ are of the same type (here we again switch to the notation where the indices denote shifts, as in Figure 3):

$$Q(x, x_i, x_{ij}, x_j, \alpha^{(i)}, \alpha^{(j)}) = 0, \qquad Q(x_{ik}, x_k, x_{jk}, x_{123}, \alpha^{(i)}, \alpha^{(j)}) = 0, \qquad \{i, j, k\} = \{1, 2, 3\}.$$

Obviously, the equations on opposite faces of the cube do not coincide, but the equation may nevertheless be extended to the entire lattice $\mathbb{Z}^3$. The tetrahedron property is satisfied.

Finally, note that it is also possible to combine equations with square and trapezoidal symmetry. Consider equation (Q_1) again. Let one pair of opposite faces carry the equations

$$Q_1(x, x_1, x_{12}, x_2; \alpha^{(1)}, \alpha^{(2)})_{\delta=1} = 0, \qquad Q_1(x_3, x_{13}, x_{123}, x_{23}; \alpha^{(1)}, \alpha^{(2)})_{\delta=0} = 0,$$

and let the other two pairs carry the equations

$$Q(x, x_i, x_{i,3}, x_3, \alpha^{(i)}, \varepsilon) = 0, \qquad Q(x_j, x_{ij}, x_{123}, x_{j,3}, \alpha^{(i)}, \varepsilon) = 0, \qquad \{i, j\} = \{1, 2\},$$

where the polynomial

$$Q(x_1, x_2, x_3, x_4, \gamma, \varepsilon) = (x_1 - x_2)(x_3 - x_4) + \gamma(\varepsilon^{-1} - \varepsilon x_3 x_4)$$

actually coincides with (H_1^ε) up to the permutation of x_2 and x_3. This awkward structure is 3D-consistent and, surprisingly, satisfies the tetrahedron property. It can be also extended to the lattice $\mathbb{Z}^3$.

8. Concluding Remarks

Noncommutative analogs of some of the equations in the Q-list are known. In particular, a quantum version of $(Q_1|_{\delta=0})$ appeared in [27]. In [11], the consistency approach was extended to the noncommutative context, where the fields take values in an arbitrary associative algebra. The definition of three-dimensional consistency remains the same in this case; however, the assumption of the multiaffine property is replaced by the requirement that the equation can be brought to the linear form $px = q$ with respect to any variable x. These two properties are not equivalent in the noncommutative case, as is seen from the following examples. The first one was found in [11], and the other two were discovered by V. V. Sokolov and V. E. Adler (unpublished):

$$\alpha^{(1)}(x - x_2)(x_2 - x_{12})^{-1} = \alpha^{(2)}(x - x_1)(x_1 - x_{12})^{-1}, \qquad (\widehat{Q}_1|_{\delta=0})$$

$$\alpha^{(1)}(x_1 - x_{12} + \alpha^{(2)})(x - x_1 - \alpha^{(1)})^{-1} = \alpha^{(2)}(x_2 - x_{12} + \alpha^{(2)})(x - x_2 - \alpha^{(2)})^{-1}, \qquad (\widehat{Q}_1|_{\delta=1})$$

$$(1 - (\alpha^{(1)})^2)(x_1 - \alpha^{(2)} x_{12})(\alpha^{(1)} x - x_1)^{-1} = (1 - (\alpha^{(2)})^2)(x_2 - \alpha^{(1)} x_{12})(\alpha^{(2)} x - x_2)^{-1}. \qquad (\widehat{Q}_3|_{\delta=0})$$

The existence of noncommutative analogs of (Q_2), $(Q_3|_{\delta=1})$ and (Q_4) remains an open question. Although the analysis of singular solutions can still be useful as a general principle, our technique is based on the algebraic properties of multiaffine and biquadratic polynomials and therefore does not apply to this problem.

More general quantum systems with consistency property were found recently in [8], [7].

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