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Persistence of Nonhyperbolic Measures for C^1-Diffeomorphisms*

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ABSTRACT. In the space of diffeomorphisms of an arbitrary closed manifold of dimension $\geqslant 3$, we construct an open set such that each diffeomorphism in this set has an invariant ergodic measure with respect to which one of its Lyapunov exponents is zero. These diffeomorphisms are constructed to have a partially hyperbolic invariant set on which the dynamics is conjugate to a soft skew product with the circle as the fiber. It is the central Lyapunov exponent that proves to be zero in this case, and the construction is based on an analysis of properties of the corresponding skew products.

KEY WORDS: Lyapunov exponent, partial hyperbolicity, dynamical system, skew product.

1. Introduction

By definition, hyperbolic dynamical systems have nonzero Lyapunov exponents. However, they are not generic in the space of all dynamical systems [1]. A wider class is that of nonuniformly hyperbolic systems studied in Pesin's theory [2]. Such systems as well have nonzero Lyapunov exponents and are also globally nongeneric.

In more detail, Pesin's theory deals with diffeomorphisms that have nonzero Lyapunov exponents with respect to some given invariant measure and describes the behavior of trajectories generic with respect to this measure. The invariant measure can be given in advance and be coordinated with the smooth structure, or it can be determined by the dynamical system. These two cases substantially differ from each other.

Bochi [7] discovered that generic area-preserving C^1-diffeomorphisms of two-dimensional manifolds are either Anosov or have zero Lyapunov exponents.

Shub and Wilkinson [10] discovered a surprising property of volume-preserving diffeomorphisms in dimensions greater than two. They found an open set of partially hyperbolic diffeomorphisms of the three-dimensional torus with the following property. Every diffeomorphism belonging to this set admits an invariant fibration of the torus into circles. These fibers continuously (*but not absolutely continuously*) depend on the initial condition. However, almost all (in the sense of the invariant Lebesgue measure) the orbits of each of these maps have nonzero (two positive and one negative) Lyapunov exponents. Paradoxically, the set of full measure consisting of these orbits meets each fiber in a set of measure zero on the circle. Later, Katok observed and Ruelle an Wilkinson [11] proved that the intersection consists of finitely many points.

For volume-preserving partially hyperbolic systems, under certain additional assumptions, Baraviera and Bonatti [9] proved that the zero central Lyapunov exponent for the preserved volume (taken as an invariant measure) can be eliminated by a C^1-small perturbation in the class of such systems. The cited results belong to so-called conservative dynamics.

Our research follows the "nonconservative approach," i.e., the study of systems whose invariant measures are determined by the dynamics and may not be coordinated with the smooth structure. One important problem of nonconservative dynamics was stated in [10]: "Is it true that a generic diffeomorphism has nonzero Lyapunov exponents with respect to all 'good' invariant measures?"

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An invariant measure is said to be *good* if it can be obtained as a partial limit of the time averages of the Lebesgue measure (the Krylov–Bogolyubov procedure).

Until recently, even the simpler question as to whether *a generic diffeomorphism has nonzero Lyapunov exponents with respect to **all** invariant measures* has been open. We provide a negative answer this question. The present paper is a continuation of [18].

Earlier, Gorodetski and Ilyashenko [14]–[17] proved that the presence of a *single* orbit with zero central Lyapunov exponent may be irremovable by small perturbations.

2. Main Results

In the present paper, we prove that, in a sense (see Theorem 1 below), zero Lyapunov exponents are persistent. We follow the strategy suggested by Gorodetski and Ilyashenko. First, the new phenomenon is revealed for step skew products. (We recall the definitions in Sec. 3.) Then the technique is adapted to the case of soft skew products. Finally, the smooth realization technique is used to derive the corresponding result for smooth systems. The first step (the study of Lyapunov exponents for step skew products) was made in [18]. Here we make the second and third steps.

Theorem 1 (main result)**.** *Let M be a closed manifold with* $\dim M \geqslant 3$. *Then there exists a domain $U \subset \mathrm{Diff}^1(M)$ such that every diffeomorphism $f \in U$ has the following property. There exists a locally maximal partially hyperbolic set $\Lambda \subset M$ and a nonatomic ergodic invariant measure μ with* $\operatorname{supp} \mu \subset \Lambda$ *such that one of the Lyapunov exponents of f with respect to μ is zero.*

2.1. Scheme of proof of the main result. In this section, we outline the proof of the main result. As was already mentioned, this theorem is proved by combining the following two theorems. The first of these (rigorously stated in Sec. 4) establishes that zero Lyapunov exponents are generic for soft skew products.

Theorem 2. *The space of Hölder soft skew products over the Smale horseshoe contains an open domain including step skew products arbitrarily close to the identity product and consisting of products each of which has a measure with zero Lyapunov exponent along the fiber.*

The methods used to prove this assertion are similar to those applied in [18] to step skew products. Namely, we use an iterative procedure to construct a sequence of periodic orbits with increasing periods. In this sequence, the subsequent orbits become more and more "similar" to the preceding ones (more precisely, to the preceding orbits passed several times). The Lyapunov exponents corresponding to these orbits tend to zero. As a result, the sequence of measures corresponding to these orbits weakly converges to some ergodic measure whose Lyapunov exponent is zero.

The other theorem, which is due to Gorodetski [16], is rigorously stated in Sec. 5. Roughly speaking, it states that, under certain conditions, the property of a smooth dynamical system to have a (locally maximal partially hyperbolic) invariant set on which the dynamics is a soft skew product is preserved under small perturbations, the soft skew products for the perturbed systems being Hölder and close to the original soft skew product.

The main result can be derived from these two theorems as follows. One considers a smooth realization of the Smale horseshoe and constructs a step skew product over it in the domain guaranteed by Theorem 2. For each diffeomorphism sufficiently close to the one thus constructed, there exists a subset on which the dynamics is conjugate to a Hölder soft skew product close to the original one. Thus, this skew product belongs to the same domain and hence has an invariant measure with zero Lyapunov exponent along the fiber. It follows that one of the Lyapunov exponents for the corresponding invariant measure of the perturbed diffeomorphism is zero.

The technique of smooth realizations used in this argument was developed by Gorodetski and Ilyashenko [14]–[16].

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2.3. Outline of the paper. Section 3 introduces notation and definitions. Theorem 2 is rigorously stated in Sec. 4, where we also describe the idea and outline of the proof. In Sec. 5, we use the smooth realization technique and the theory of partially hyperbolic systems to derive Theorem 1 from Theorem 2. Finally, we prove Theorem 2 in Secs. 6–9.

3. Notation and Definitions

Let Σ^N be the space of two-sided infinite sequences over the alphabet $\{0, \dots, N-1\}$, and let $\sigma\colon \Sigma^N \to \Sigma^N$ be the Bernoulli shift. We equip Σ^N with the metric $d_{\Sigma^N}(\omega, \omega') = 2^{-\min\{|n| : \omega_n \neq \omega'_n\}}$ and set $M = \Sigma^N \times S^1$.

Let $g_i\colon S^1 \to S^1$, $i = 0, \dots, N-1$, be diffeomorphisms of the circle. Consider the map

$$G\colon M \to M, \qquad (\omega, x) \to (\sigma\omega, g_{\omega_0}(x)). \tag{1}$$

Note that the action of G on the fiber over a point ω of the base depends only on the element ω_0 rather than on the entire sequence ω. We refer to such skew products as *step* skew products.

Now let $f_\omega\colon S^1 \to S^1$, $\omega \in \Sigma^N$, be a family of diffeomorphisms of the circle. Consider the map

$$F\colon M \to M, \qquad (\omega, x) \to (\sigma\omega, f_\omega(x)). \tag{2}$$

We refer to such skew products as *soft* skew products, since the map on the fiber depends on the entire word ω and is not determined by any finite part of ω.

Definition 1. The soft skew product F given by (2) is said to be (C, α)-*Hölder* if

$$\forall \omega, \omega' \in \Sigma^N, \quad d_{C^0}(f_\omega, f_{\omega'}) < C \cdot d_{\Sigma^N}(\omega, \omega')^\alpha. \tag{3}$$

Definition 2. An (L, C, α)-*system* is a (C, α)-Hölder skew product such that the maps f_ω depend on ω continuously in the Diff^1-norm (in particular, the skew product is smooth along the fibers) and the estimate

$$\forall \omega \in \Sigma^N, \quad \max_{x \in S^1} \max(f'_\omega(x), (f_\omega^{-1})'(x)) < L$$

holds for the maximum dilatation rate.

We equip the space of (L, C, α)-systems with the metric

$$d(F, \tilde{F}) = \sup_{\omega \in \Sigma^n} d_{\mathrm{Diff}^1}(f_\omega, \tilde{f}_\omega)$$

of the uniform Diff^1-distance.

Let us introduce the notation

$$\bar{f}_m[\omega] = f_{\sigma^{m-1}\omega} \circ \dots \circ f_{\sigma\omega} \circ f_\omega, , \quad \bar{f}_{-m}[\omega] = f^{-1}_{\sigma^{-m}\omega} \circ \dots \circ f^{-1}_{\sigma^{-1}\omega}, \quad \bar{f}_0[\omega] = \mathrm{id}\,.$$

Definition 3. Let F be a skew product of the form (2). The *Lyapunov exponent along the fiber* at a point (ω, x) is the following function (defined at the points where the limit exists):

$$\lambda^c(\omega, x) := \lim_{n \to \infty} \frac{1}{n} \ln |D\bar{f}_n[\omega](x)|.$$

If F has an ergodic probability measure ν, then there exists a set of full ν-measure such that the Lyapunov exponent along the fiber is defined for each point of this set and is independent of the point. Thus, the function $\lambda^c(\omega, x)$ is constant on a set of full ν-measure, and one can speak of the Lyapunov exponent along the fiber with respect to the measure ν. (This exponent will be denoted by $\lambda^c(\nu)$.)

4. Main Result for Soft Skew Products

Theorem 2. *Let the number of symbols satisfy the inequality $N \geqslant 5$. Then for any $L > 1$, $C > 0$, and $\alpha \in (0,1)$ the space of (L, C, α)-systems contains an open domain in which every system has a nonatomic ergodic invariant measure with zero Lyapunov exponent along the fiber. This domain contains step skew products arbitrarily close to the identity* (*along the fiber*) *map.*

To prove Theorem 2, we shall construct the desired measure as a limit of measures uniformly distributed on periodic orbits whose Lyapunov exponents tend to zero and which, in a sense, "resemble" one another.

The idea is implemented as follows. In Sec. 6, following Gorodetski and Ilyashenko (see [15] and [16]), we study the divergence of orbits of soft skew products. Section 7 describes a construction that, given a periodic orbit, produces another periodic orbit that has a larger period and a smaller absolute value of the Lyapunov exponent and sufficiently "resembles" the original orbit. In Sec. 8, we inductively construct the desired sequence of orbits, prove that the limit measure is ergodic and nonatomic, and show that, as a consequence of ergodicity, the Lyapunov exponent along the fiber for the limit measure is the limit of Lyapunov exponents. Finally, in Sec. 9 we construct diffeomorphisms satisfying the conditions in Sec. 6.

5. Smooth Realization and Proof of Theorem 1

Consider the Smale horseshoe realized as a map of pairwise disjoint rectangles. Namely, let $B = [0,1] \times [0,1]$ be the unit rectangle on the coordinate plane. We divide it into eleven equal vertical rectangles C_k, $k = 0, \ldots, 10$, and set $D_i := C_{2i+1}$. Let us divide the same rectangle into eleven equal horizontal rectangles C'_k and set $D'_i := C'_{2i+1}$.

Let $D = \bigcup_{i=0}^{4} D_i$ and $D' = \bigcup_{i=0}^{4} D'_i$. The map $T\colon D \to D'$ acts on each of the rectangles D_i by linear contraction along the vertical and linear dilatation along the horizontal and takes it to the respective rectangle D'_i.

It is well known that T has an invariant set Λ homeomorphic to Σ^5 with $T|_\Lambda$ being conjugate to the Bernoulli shift $\sigma\colon \Sigma^5 \to \Sigma^5$. We denote the conjugating homeomorphism by $\Phi_0\colon \Lambda \to \Sigma^5$.

Let diffeomorphisms $\{g_i\}_{i=0,\ldots,4}$ of the circle be given. Then we can construct a *smooth realization of the step skew product* (1). Namely, set

$$\mathscr{F}\colon D \times S^1 \to D' \times S^1, \quad \mathscr{F}(z, x) = (T(z), g_j(x)) \text{ for } z \in D_j.$$

It is easily seen that $\mathscr{F}|_{\Lambda\times S^1}$ is conjugate to the map (1). The set $\Lambda \times S^1$ is partially hyperbolic for $\mathscr{F}$ (provided that the g_i are sufficiently close to the identity map), and its central fibers are fibers of the projection $\Lambda \times S^1 \to \Lambda$ onto the first factor $\Lambda \simeq \Sigma^5$.

Theorem 3 (Gorodetski [16])**.** *Let $T : D \to D'$ be a C^{r+1}-smooth* ($0 \leqslant r \leqslant \infty$) *map that is hyperbolic with locally maximal set $\Lambda = \bigcap_{n\in\mathbb{Z}} T^n(D)$, and let M be a closed manifold. Consider the map $\mathscr{F}_0 = T \times \mathrm{id}_M : D \times M \to D' \times M$. There exists a C^{r+1}-neighborhood V of $\mathscr{F}_0$ such that the following assertions hold for each diffeomorphism $\mathscr{B} \in V$:*

1. *There exists an invariant subset $\Delta_{\mathscr{B}}$ homeomorphic to $\Lambda \times M$.*
2. *The projection $\Phi\colon (\Delta_{\mathscr{B}}, \mathscr{B}) \to (\Lambda, T)$ is a semi-conjugation.*
3. *The fibers $\Phi^{-1}(z)$ are C^{r+1}-smooth manifolds for all $z \in \Lambda$.*
4. *The dependence of the fibers $\Phi^{-1}(z)$ on the point $z \in \Lambda$ is Hölder continuous in the C^r-norm. Moreover, the related Hölder exponent and constant can be chosen to be the same for all diffeomorphisms in V.*

In addition, we need the following assertion, which is a special case of Theorem 6.8 in [13].

Theorem 4 (Hirsch, Pugh, and Shub)**.** *Under the assumptions of Theorem* 3, *the central fibers depend on the point in $\Delta_{\mathscr{B}}$ and on the diffeomorphism $\mathscr{B}$ continuously in the C^{r+1}-topology.*

Consider a map $\mathscr{B}\colon D \times S^1 \to U(D') \times S^1$ that is C^1-close to $T \times \mathrm{id}_{S^1}$. By Theorem 3, it has a locally maximal partially hyperbolic set close to $\Lambda \times S^1$ and homeomorphic to this product;

moreover, the dynamics on the set of central fibers is conjugate to the map $T\colon \Lambda \to \Lambda$. Next, by Theorem 4, the central fibers (circles) into which this set is divided are C^1-close to the central fibers of the unperturbed map. The map $\mathscr{B}$ permutes the fibers of the invariant set and hence defines an *induced map* on the set of these fibers. It follows that if $\mathscr{B}$ is sufficiently close to $\mathscr{F}_0$, then the map

$$G_{\mathscr{B}}\colon \Delta_{\mathscr{B}} \to \Sigma^5 \times S^1, \qquad G_{\mathscr{B}}(z,x) = (\Phi_0 \circ \Phi(z,x), x),$$

is a homeomorphism C^1-smooth along the central fibers. Note that $G_{\mathscr{B}}$ preserves the coordinate along S^1 and conjugates the induced map on the set of central fibers with the Bernoulli shift; the coordinates "along the base" (z and ω, respectively) are Hölder functions of each other (uniformly with respect to x). Let $V_2 \subset V$ be a neighborhood of $\mathscr{F}_0$ in which $G_{\mathscr{B}}$ is a homeomorphism.

Let us write out the restriction $\mathscr{B}|_{\Delta_{\mathscr{B}}}$ in these coordinates. We set

$$F_{\mathscr{B}}\colon \Sigma^5 \times S^1 \to \Sigma^5 \times S^1, \qquad F_{\mathscr{B}} = G_{\mathscr{B}} \circ \mathscr{B} \circ G_{\mathscr{B}}^{-1}.$$

We obtain a soft Hölder skew product that is C^1-smooth along the fibers and whose maps depend on the base point C^0-Hölder continuously. Moreover, for $\mathscr{B} \in V_2$ this soft skew product C^1-continuously depends on $\mathscr{B}$; in particular, $F_{\mathscr{B}}$ is close to the identity map.

We refer to $\mathscr{F}_{\mathscr{B}}$ as the *rectification* of $\mathscr{B}$. By Theorem 3, there exists a C^1-neighborhood $V_3 \subset V_2$ of the map $\mathscr{F}_0 = T \times \mathrm{id}_{S^1}$ and constants C and α such that the rectification of any of the maps $\mathscr{B} \in V_3$ is a (C,α)-system. Take an $L_0 > 1$ such that $L_0 2^{-\alpha} < 1$.

By Theorem 2, the intersection of the domain given by that theorem with the domain V_3 contains a step skew product that is an (L_0, C, α)-system. Consider its realization by the map $\mathscr{B}_0$. Note that the rectification of a C^1-small perturbation $\mathscr{B}$ of B_0 is an (L_0, C, α)-system as well and is close to the original system. Indeed, the Hölder constant and exponent are preserved by virtue of the choice of the neighborhood V_3, and the strict inequality on the derivatives is preserved under a C^1-small perturbation.

By the construction of $\mathscr{B}_0$, this rectification satisfies the conclusion of Theorem 2, i.e., has a zero Lyapunov exponent along the fiber (since the rectification still belongs to the same domain). But then one of the Lyapunov exponents (namely, the central Lyapunov exponent) of the corresponding measure for the original map $\mathscr{B}$ is zero as well. We have proved that for each map $\mathscr{B}$ sufficiently C^1-close to $\mathscr{B}_0$ there exists an ergodic invariant measure with one of the Lyapunov exponents equal to zero.

Finally, one can readily see that the map $\mathscr{B}_0$ itself can be extended to a diffeomorphism of an arbitrary three-dimensional manifold. (Indeed, it is easy to extend it to a diffeomorphism, identical on the boundary, of the solid torus $[-1,2] \times [-1,2] \times S^1$, and the solid torus can be embedded in an arbitrary three-dimensional manifold.) Since close diffeomorphisms have close restrictions to $D \times S^1$, this proves Theorem 1.

6. Choice of a Domain in the Space of Soft Skew Products and Divergence of Trajectories

Let a be an arbitrary finite word in the symbols $0,\dots,4$. By $\{\dots|a\dots\}$ we denote an arbitrary infinite sequence $\omega \in \Sigma^5$ in which a occurs starting from the zeroth position. In a similar way, we introduce the notation $\{\dots a|\dots\}$ and $\{\dots a|b\dots\}$. In what follows, we use the Greek letter ω to denote infinite words and the Latin letter w to denote finite words. Finally, for a given finite word w (with the zeroth position indicated), by (w) we denote the (appropriately positioned) infinite periodic sequence with period w.

Definition 4. Let a soft skew product F be an (L, C, α)-system in the sense of Definition 2. We say that it exhibits

- The *dilatation property* if there exist $\nu > 1$ and $\delta_1 > 0$ such that for an arbitrary interval $I \subset S^1$ with $|I| < \delta_1$ one has

$$\exists j_1 \in \{0,\dots,4\} : \forall \omega = \{\dots|j_1\dots\}\, \forall x \in I, \quad (Df_\omega)(x) > \nu. \tag{4}$$

- The *inverse dilatation property* if there exist $\nu > 1$ and $\delta_1 > 0$ such that for an arbitrary interval $I \subset S^1$ with $|I| < \delta_1$ one has

$$\exists j_2 \in \{0, \dots, 4\} : \forall \omega = \{\dots j_2| \dots\} \, \forall x \in I, \quad (Df^{-1}_{\sigma^{-1}\omega})(x) > \nu. \tag{5}$$

- A δ_2*-rotation* if

$$d_{C^0}(f_\omega, R_{\delta_2}) < \frac{\delta_2^2}{40} \tag{6}$$

for each sequence $\omega = \{\dots |0 \dots\}$; here R_{δ_2} stands for the rotation by the angle δ_2.
- A *weakly attracting orbit* if there exists an attracting periodic orbit X whose Lyapunov exponent $\lambda(X) < 0$ along the fiber satisfies

$$\lambda(X) + \ln \nu > 0. \tag{7}$$

- γ*-predictability of trajectories*, $\gamma > 0$, if

$$\operatorname{diam}\{\bar{f}_m[\omega](x) \mid \omega = \{\dots w^* \dots\}\} < \gamma, \tag{8}$$

$$\operatorname{diam}\{\bar{f}_{-m}[\omega](x) \mid \omega = \{\dots w^* \dots\}\} < \gamma \tag{9}$$

for each $x \in S^1$, each positive integer m, and each finite word $w^* = \{w_{-m} \dots w_{-1}|w_0 \dots w_{m-1}\}$.

Finally, we say that the system is *controllable* if it possesses all of the properties mentioned above and the constants in these properties can be chosen to satisfy the following *constant compatibility condition*:

$$\gamma < \delta_2/40, \qquad \delta_1 > 3\delta_2, \tag{10a}$$

and

$$\alpha > \log_2 L. \tag{10b}$$

Clearly, all these properties except for predictability of trajectories can be satisfied simultaneously for an appropriately chosen step skew product as well as for all soft systems sufficiently close to it. The following lemma, which is due to Gorodetski, derives predictability of trajectories from the condition on the Hölder exponent and the dilatation rate.

Lemma 1 [15, Lemma 3.1]**.** *Let there be given constants L, C, and α such that condition* (10b) *is satisfied. Then there exists a $K = K(L, C, \alpha)$ such that*

$$d_{C_0}(\bar{f}_{\pm m}[\omega], \bar{f}_{\pm m}[\omega']) \leqslant \gamma := K\delta^\beta, \tag{11}$$

where $\beta = 1 - (\ln L / \ln 2^\alpha)$, for any (L, C, α)-system δ-close to a step system.

7. Main Lemma on Periodic Orbits

Definition 5. Let X be a periodic orbit of F with period P, and let $\varepsilon > 0$. A point y is said to be (ε, P)*-good* for X if there exists a point $x \in X$ such that

$$\forall l = 0, 1 \dots P - 1, \qquad d(F^l(x), F^l(y)) < \varepsilon.$$

Lemma 2 (Main Lemma)**.** *Let F be a controllable skew product of the form* (2)*. Let X be an arbitrary periodic orbit of F with period P and fiber multiplier θ, $0 < \theta < 1$, and suppose that the fiber Lyapunov exponent $\lambda := \ln \theta / P$ of X satisfies the inequality*

$$\lambda + \ln \nu > 0.$$

Then for each $\varepsilon > 0$ there exists a periodic orbit Y of F with period $P' > 2P$ and fiber Lyapunov exponent $\lambda' < 0$ such that

1. $|\lambda'| < C|\lambda|$, *where $C = C(F)$, $0 < C < 1$, is a global constant depending only on F.*
2. $\lambda' + \ln \nu > 0$.
3. *There exists a subset $\widetilde{Y} \subset Y$ and a projection $\pi \colon \widetilde{Y} \to X$ such that*

(a) *All points of $\widetilde{Y}$ are (ε, P)-good with respect to X, and one can take $x = \pi(y)$ in Definition* 5.

(b) *The proprotion* $\varkappa := \#\widetilde{Y}/\#Y$ *of points where the projection* π *is defined admits the lower bound*

$$\varkappa \geqslant 1 - \frac{3|\lambda|}{\ln L}.$$

(c) *The number of elements in the preimage* $\pi^{-1}(x)$ *is the same for all* $x \in X$.

A periodic orbit of a skew product is specified by an initial point (ω, x), $\omega \in \Sigma^N$, $x \in S^1$, where $\omega = (w)$ is a periodic sequence with period $w = \{\ | w_0 \dots w_{P-1}\}$ and

$$\sigma^P \omega = \omega, \qquad \bar{f}_P[\omega](x) = x.$$

The idea of the proof is to fix an interval J on the central fiber and find a series of words $w'(k)$ in the base whose length increases with increasing positive integer k. The corresponding maps of the skew product contract J into itself, thus guaranteeing the existence of a fixed point attractive along the fiber, and their derivative on J is bounded uniformly with respect to k. By increasing k, one can ensure a negative Lyapunov exponent arbitrarily close to zero.

When constructing the new orbit Y, we incidentally ensure the closeness of points of the new and old orbits (properties 3(a)–3(c)). This permits us to obtain the ergodicity of the limit measure as a consequence of the Birkhoff–Khinchin theorem.

Proof of Lemma 2. Let X be a periodic orbit with initial point $(\omega, x) = ((w), x)$, where w is the period of the sequence ω. We shall construct a new periodic orbit Y with initial point $(\omega', y) = ((w'), y)$, where

$$w' = \{R_1 w^{n_1} w^{r'(k)} w^k | w^k w^{n_1} R_2 R_3\}. \tag{12}$$

Here k, $r'(k)$, and n_1 are large positive integers to be chosen below, w, w', and R_i are words of finite length, w^m is the mth power of the word w, the vertical bar in the period marks the place corresponding to the origin in ω', and finally, x and y are points on the circle.

The words w^k in (12) ensure simultaneous unlimited growth in the length of w' and strong contraction of some interval J to a tiniest size. The word R_2 dilates the image of J to J_2, using fewer letters. (The specific efficiency, i.e., the logarithm of the derivative per letter, is at least $\ln \nu$ compared with $|\lambda|$, which is close to zero, for the letters of the orbit X.) The word R_1 dilates the preimage of J to J_1 under inverse iterations, thus creating a "funnel"; entering the (large) "opening" J_1 of this funnel guarantees entering J after a number of iterations. The word R_3 maps the interval J_2 into J_1, thus closing the chain of embeddings, so that the resulting periodic word maps J into itself over the period. The words w^{n_1} and $w^{r'(k)}$ play a technical role; they permit to control the "error" along the fiber.

Fix an $\varepsilon > 0$. By the Diff^1-continuous dependence of f_ω on the parameter in the base, for any θ^+ and θ^-, $0 < \theta^- < \theta < \theta^+ < 1$, there exists a number $n_1(\theta^\pm) \in \mathbb{N}$ and an interval $J(n_1, \theta^\pm) \subset S^1$, $J \ni x$, $L^P|J| < \varepsilon$, $|J| < \delta_1$, such that

$$\bar{f}_P[\omega^*](J) \subset J, \tag{13}$$

$$D\bar{f}_P[\omega^*]|_J \in (\theta^-, \theta^+) \tag{14}$$

for each sequence $\omega^* = \{\dots w^{n_1} | w^{n_1} \dots\}$ and

$$\operatorname{diam}\{\bar{f}_l[\omega^*](y) \mid \omega^* = \{\dots w^{n_1} | w^{n_1} \dots\}\} < \varepsilon \tag{15}$$

for all l, $0 \leqslant l \leqslant P-1$, and $y \in J$. The upper bound on the length of J readily implies the inequality

$$\rho(\bar{f}_l[\omega](x), \bar{f}_l[\omega](y)) < \varepsilon \tag{16}$$

for any pair of points $x, y \in J$, each integer $l = 0, \dots, P-1$, and fixed ω.

Take θ^- and θ^+ sufficiently close to θ (the specific conditions on these will be indicated later) and fix $n_1(\theta^\pm)$ and $J(n_1, \theta^\pm)$.

Naturally, all constants and intervals to be constructed below depend on the choice of J, n_1, and $\theta^\pm$. In what follows, we do not emphasize this dependence.

Proposition 1 (the "funnel"). *There exists a finite word R_1 and an interval J_1 of length $|J_1| \geqslant \delta_1 - 4\gamma$ such that*

$$\bar{f}_{-(|R_1|+n_1P)}[\omega](J) \supset J_1 \tag{17}$$

for all $\omega = \{\dots R_1 w^{n_1}|w^{n_1+\lceil |R_1|/P\rceil}\dots\}$.

Proof. We construct the word R_1 by induction, starting from the empty word and writing out the letters from right to left. Suppose that the last l letters have already been constructed. Let $R_1(l)$ be the word formed by these letters. Consider some word ω of the form $\{\dots R_1(l)w^{n_1}|w^{n_1+\lceil l/P\rceil}\dots\}$ and find the preimage of the interval $I_l = \bar{f}_{-(l+n_1P)}[\omega](J)$.

If the length of I_l exceeds $\delta_1 - 2\gamma$, then, by applying the predictability property (9) for trajectories to the ends of J, we see that $\bar{f}_{-(l+n_1P)}[\omega](J) \supset J_1$ for *any other* word ω of the same form, where the interval J_1 is obtained from I_l by retreating *inside* by γ from each end. The desired interval J_1 and word $R_1 = R_1(l)$ have thus been constructed.

If the length of I_l is less than $\delta_1 - 2\gamma$, then the same predictability property for trajectories implies that

$$\bar{f}_{-(l+n_1P)}[\omega](J) \subset I_l^* := U_\gamma(I_l)$$

for any other word ω of the same form. Since $|I_l^*| < \delta_1$, it follows from the inverse dilatation property that there exists a letter $a_{l+1} \in \{0,\dots,4\}$ such that all maps corresponding to it dilate at least by a factor of ν at each point of I_l^* when taking the preimage. And we append just this letter at the beginning, $R_1(l+1) = a_{l+1}R_l$.

It can easily be seen that this process terminates in at most $C = \lceil \log_\nu(\delta_2 L^{n_1P}/|J|)\rceil$ steps, since the preimage decreases at most by a factor of L at each of the first n_1P steps and increases at least by a factor of ν at each of the subsequent steps. $\square$

Note that the embedding (17) implies the embedding

$$\bar{f}_{|R_1|+n_1P}[\omega](J_1) \subset J \tag{18}$$

for any sequence $\omega = \{\dots|R_1 w^{2n_1+\lceil l/P\rceil}\dots\}$.

Proposition 2 ("rotation"). *For each interval J_2 of length $|J_2| < \delta_2$, there exists j, $0 \leqslant j \leqslant M := \lceil 1/\delta_2\rceil$, such that*

$$\bar{f}_j[\omega](J_2) \subset J_1 \tag{19}$$

for each word of the form $\omega = \{\dots|0^j\dots\}$. (We set $R_3 := 0^j$.)

Proof. First, note that if the maps $f_1,\dots,f_j$ are $\bar{\delta}$-close in the C^0-metric to the rotation R_{δ_2}, then their composition is $j\bar{\delta}$-close to the rotation $R_{j\delta_2}$,

$$\begin{aligned} d(f_1\circ\dots\circ f_j, R_{\delta_2}^j) &\leqslant d(f_1\circ\dots\circ f_j, R_{\delta_2}\circ f_2\circ\dots\circ f_j) \\ &\quad + d(R_{\delta_2}\circ f_2\circ\dots\circ f_j, R_{\delta_2}^2\circ f_3\circ\dots\circ f_j) + \dots + d(R_{\delta_2}^{j-1}\circ f_j, R_{\delta_2}^j) \\ &\leqslant d(f_1, R_{\delta_2}) + \dots + d(f_j, R_{\delta_2}) \leqslant j\bar{\delta}. \end{aligned}$$

A simple geometric argument shows that there exists a j, $0 \leqslant j \leqslant M$, such that the rotation by the angle $j\delta_2$ moves J_2 to the inside of J_1 in such a way that the length of each of the two complementary intervals is not less than $\delta_2/2$. (Recall that $\delta_1 > 3\delta_2$.)

Then, by the preceding,

$$\bar{f}_j[\omega](J_2) \subset U_{j\cdot\delta_2^2/40}(R_{\delta_2}^j(J_2)) \subset U_{\delta_2/5}(R_{j\delta_2}(J_2)) \subset J_1$$

for each word $\omega = \{\dots|0^j\dots\}$. $\square$

Fix C and M according to Propositions 1 and 2. Set

$$\delta = \min\left(\frac{1}{L^{2n_1(J,\theta^\pm)P+C(J)+M}}, \frac{\delta_2 - 2\gamma}{L^{n_1P}|J|}\right). \tag{20}$$

Let $k > \max(C(J), M)$ be an arbitrary large number. (In what follows, we subject k to additional lower bounds.)

Take an $r(k)$ such that

$$\frac{\delta}{L} \leqslant (\theta^+)^k L^{r(k)} < \delta. \tag{21}$$

Set $r'(k) := [r(k)/P] + 1$.

Proposition 3 ("dilatation"). *There exists a word $R_2 = R_2(k)$ of length $|R_2| = r(k)$ and an interval $J_2 = J_2(k)$ of length not exceeding δ_2 such that*

$$D\bar{f}_{S+r(k)}[\omega]|_J > (\theta^-)^k L^{-n_1 P} \nu^{r(k)}, \tag{22}$$

$$\bar{f}_{S+r(k)}[\omega](J) \subset J_2 \tag{23}$$

for each sequence $\omega = \{\dots w^{n_1+r'(k)+k} | w^{k+n_1} R_2 \dots\}$, where $S := (k+n_1)P$.

Proof. Just as in Proposition 1, we write out the word R_2 by induction, starting from the empty word. Assuming that the first l letters (which form a word that we denote by $R_2^*(l)$) have already be written out, consider the $(l+S)$th image

$$I_l = \bar{f}_{l+S}[\omega](J)$$

of the interval J for some word $\omega = \{\dots w^{n_1+r'(k)+k} | w^{k+n_1} R_2 \dots\}$. Note that the length of this image does not exceed

$$|J| \cdot (\theta^+)^k \cdot L^{n_1 P + l} \leqslant (|J| L^{n_1 P}) \cdot (\theta^+)^k \cdot L^{r(k)} \leqslant \delta_2 - 2\gamma.$$

By the predictability property (8) for trajectories, it follows that the lth image of J under the action of any other word ω of the same form is contained in the corresponding γ-neighborhood $I_l^* := U_\gamma(I_l)$; moreover, $|I_l^*| \leqslant 2\gamma + |I_l| \leqslant \delta_2$. By the dilatation property (4), there exists a letter b_{l+1} such that all maps corresponding to it dilate at least by a factor of ν on I_l^*. We append this letter at the end of the part already constructed, $R_2^*(l+1) = R_2^*(l) b_{l+1}$. On having written out $r(k)$ letters, we terminate the process and set $R_2 = R_2^*(r(k))$. Then it follows from the preceding that

$$\forall \omega = \{\dots w^{n_1+r'(k)+k} | w^{k+n_1} R_2 \dots\}, \quad \bar{f}_{S+r(k)}[\omega](J) \subset J_2 := I_{r(k)}^*.$$

On the other hand, since the derivative of each of the maps used on the part R_2 is not less than ν on the corresponding interval, we obtain

$$D\bar{f}_{S+r(k)}[\omega]|_J > (\theta^-)^k L^{-n_1 P} \nu^{r(k)}. \qquad \square$$

Take and fix R_1 and J_1 according to Proposition 1 and R_2 and J_2 according to Proposition 3. Take R_3 according to Proposition 2.

Consider the periodic sequence ω' specified by the word

$$w' = \{R_1 w^{n_1+r'(k)+k} | w^{k+n_1} R_2 R_3\}.$$

It readily follows from the chain of embeddings (23), (19), (18) and (13) (the last of them is applied $k + r'(k)$ times) that

$$\bar{f}_{|w'|}[\omega'](J) \subset J.$$

Now (20) and (21) imply the estimate

$$(D\bar{f}_{|w'|}[\omega'])|_J \leqslant (\theta^+)^{2k+r'(k)} L^{2n_1 P + r(k) + M + C(J)} < 1;$$

hence the map $\bar{f}_{|w'|}[\omega']$ has an attractive fixed point y on J. We have constructed the orbit $Y = (\omega', y)$.

Proposition 4. *For sufficiently large k, the Lyapunov exponent of Y possesses properties* 1 *and* 2 *in Lemma* 2.

Proof. Recall that so far we have not imposed any requirements on $\theta^{\pm}$ in the construction of Y. These requirements will be specified in the proof of this proposition.

Let us find a lower bound for the derivative of $\bar{f}_{|w'|}[\omega']$ on J:

$$(D\bar{f}_{|w'|}[\omega'])|_J \geqslant (\theta^-)^{2k+r'(k)}\nu^{r(k)}L^{-(2n_1P+M+C(J))}.$$

By (21), there exists a constant $C_1 = C_1(J, \theta^{\pm}, n_1)$, independent of k, such that

$$r(k) > \frac{1}{\ln L}(-k\ln\theta^+) + C_1.$$

Hence there exist constants C_2 and C_3, independent of k either, such that

$$\begin{aligned}\ln\|D\bar{f}_{|w'|}[\omega']\| &\geqslant (2k + \frac{r(k)}{P})\ln\theta^- + r(k)\ln\nu + C_2\\ &> 2k\ln\theta^- - \frac{k\ln\theta^+\ln\theta^-}{P\ln L} + \frac{C_1\ln\theta^-}{P\ln L} - \frac{k\ln\theta^+\ln\nu}{\ln L} + C_3,\end{aligned}$$

which implies the estimate

$$\lambda' := \frac{\ln\|D\bar{f}_{|w'|}[\omega'](y)\|}{|w'|} > \frac{\ln\|D\bar{f}_{|w'|}[\omega'](y)\|}{2kP} = h(\alpha) + O\Big(\frac{1}{k}\Big)$$

for the Lyapunov exponent of Y, where

$$h(\alpha) := \lambda\bigg((1-\alpha) - \frac{(1-\alpha^2)\ln\theta}{2P\ln L} - \frac{(1+\alpha)\ln\nu}{2\ln L}\bigg), \qquad \theta^{\pm} := \theta^{1\pm\alpha},\ \alpha > 0.$$

The function $h(\alpha)$ is continuous at zero; furthermore, the inequalities

$$0 < q := \frac{\lambda + \ln\nu}{2\ln L} < 1$$

imply the estimate

$$h(0) = \lambda\bigg(1 - \frac{\ln\theta/P + \ln\nu}{2\ln L}\bigg) > (1-q)\lambda.$$

We fix a sufficiently small $\alpha < 1$ (and, simultaneously, $\theta^{\pm}$) such that $h(\alpha) > (1-q/2)\lambda$. Then the estimate

$$\lambda' > (1-q/3)\lambda$$

for sufficiently large k proves property 1 in Lemma 2.

The inequality $\lambda' > \lambda$ implies property 2. $\square$

It remains to justify property 3 in Lemma 2. We define the set $\widetilde{Y}$ and the projection π as follows. Let $K = K(\varepsilon, w)$ be the minimum positive integer such that $2^{-KP} \leqslant \varepsilon$. For $k > K$, set

$$\widetilde{Y} = \{F^j(\omega', x') \mid -(k-K)P \leqslant j < (k-K-1)P\}.$$

Define the projection $\pi\colon \widetilde{Y} \to X$ by the formula

$$\pi(F^j(\omega', x')) = F^{\rho}(\omega, x),$$

where ρ is the remainder of the division of j by P. Obviously, the number of points in the preimage $\pi^{-1}(\tilde{\omega}, \tilde{x})$ is independent of $(\tilde{\omega}, \tilde{x})$ and is equal to $2k - 2K - 1$. Hence assertion 3(c) of the lemma holds.

Proposition 5. *All points of $\widetilde{Y}$ are $(3\varepsilon, P)$-good for X.*

Proof. Let us estimate the distance along the base. By the choice of K, for each $\tilde{y} \in \widetilde{Y}$ the distance between the Σ^N-coordinates of the points $F^l(\tilde{y}) \in \widetilde{Y}$ and $\pi(F^l(\tilde{y})) \in X$ does not exceed ε for all $l = 0, \ldots, P-1$.

Let us estimate the distance along the fiber. By construction, after iterations on the part $w^{k+n_1}R_2R_3R_1w^{n_1}$, the point x' enters the interval J. The subsequent iterations map J into itself by virtue of the choice of n_1 and relation (13). Hence the distance between $F^j(\omega', x')$ and $F^j(\omega, x) =$

$F^\rho(\omega, x)$ is the distance between the ρth iterations of some points of J under the action of some (distinct!) words that have the form $\{\dots w^{n_1}|w^{n_1}\dots\}$, where ρ is the remainder of the division of j by P.

Consider iterations of the point corresponding to the new orbit by the word ω. (The coordinate is taken on the base of the old orbit.) By virtue of inequality (15), the change in the coordinate along the fiber in ρ iterations does not exceed ε. But now both points are iterated by the word of the old orbit, and the number of iterations does not exceed P; consequently, by inequality (16), the distance between them does not exceed ε. Thus, the distance between $F^j(\omega', x')$ and $F^j(\omega, x)$ along the fiber does not exceed 2ε.

It follows that the first P iterations of the points $\tilde{y} \in \widetilde{Y}$ and $\pi(\tilde{y})$ diverge in $\Sigma^N \times S^1$ by a distance less than 3ε. This proves assertion 3(a) of Lemma 2. □

Let us estimate the proportion of points of Y that are not good for X. There exists a constant C_4, independent of k, such that

$$1 - \frac{\#\widetilde{Y}}{\#Y} = \frac{2n_1P + r'(k)P + r(k) + M + C(J) + (2K+1)P}{|w'|} \leqslant \frac{2r(k) + C_4}{2kP}\,.$$

By (21), there exists a constant C_5, independent of k, such that

$$r(k) < \frac{1}{\ln L}(-k\ln\theta^+) + C_5 < \frac{1}{\ln L}(-2k\ln\theta) + C_5$$

and hence

$$1 - \frac{\#\widetilde{Y}}{\#Y} < -\frac{2\lambda}{\ln L} + O\left(\frac{1}{k}\right).$$

This implies assertion 3(b) in Lemma 2 for sufficiently large k. The proof of Lemma 2 is complete. □

8. Sequence of Periodic Orbits, Ergodicity, and Zero Lyapunov Exponents

In this section, we give a lemma on the zero Lyapunov exponent and prove Theorem 2.

Lemma 3. *For an arbitrary controllable system, there exists an ergodic invariant measure with zero Lyapunov exponent along the fiber.*

Proof. Using Lemma 2, we can construct a sequence, starting from a weakly attractive orbit, of periodic orbits attractive along the fiber (see (7)). The Lyapunov exponents for these orbits tend to zero exponentially, and each subsequent orbit spends most of the time near the preceding orbit.

Consider the sequence of atomic measures uniformly distributed on these orbits. It can be derived from the condition that the orbits "resemble" one another (see assertion 3 in Lemma 2) with the use of the Birkhoff–Khinchin ergodic theorem that each limit point of this sequence is an ergodic invariant measure; one can also readily verify that the limit measure is nonatomic. Since the space of measures on $\Sigma^5 \times S^1$ is weakly* compact, it follows that this sequence has a convergent subsequence, whose limit, by virtue of the preceding, is an ergodic invariant measure. This limit measure is precisely the desired measure.

Indeed, the Lyapunov exponent for an ergodic invariant measure can be expressed as the integral of a continuous function, namely, of the derivative of the map along the fiber, over this measure. Hence the Lyapunov exponent of the limit measure is the limit of the Lyapunov exponents, i.e., zero.

This argument is carried out rigorously in [18] (see Lemmas 1 and 2) for the case of step systems and can be transferred to the soft case word for word. Hence we do not fully reproduce it here. □

Proof of Theorem 2. By Lemma 4 (see Sec. 9), in the space of (L, C, α)-systems there exists an open domain that contains step systems arbitrarily close to the identity (along the fiber) and has the property that each map in this domain is controllable. On the other hand, by Lemma 3, for each controllable system there exists an ergodic measure with zero Lyapunov exponent along the fiber.

Thus, each map in the domain thus constructed possesses an ergodic measure with zero Lyapunov exponent along the fiber. The proof of Theorem 2 is complete. □

9. Construction of Domains of Controllable Systems

In this section, for any $L > 1$, $C > 0$, and $\alpha \in (0, 1)$ we construct diffeomorphisms $g_0, g_1, \dots, g_4$ of the circle that are arbitrarily close to the identity diffeomorphisms and possess the property that all (L, C, α)-systems close to the corresponding step product are controllable. This construction completes the proof of Theorem 2 and hence of the main result, Theorem 1.

Let there be given some constants L, C, and α. Take an $L_1 < L$ such that $1 < L_1 < 2^\alpha$. All diffeomorphisms g_i to be constructed below will satisfy the condition

$$\max_{x \in S^1} \max(g_i'(x), (g_i^{-1})'(x)) < L_1$$

on the derivatives.

Let $W \subset \mathrm{Diff}^1(S^1)$ be an arbitrarily small neighborhood of the identity map of the circle. We represent the circle in the form $S^1 = \mathbb{R}/\mathbb{Z}$; let $g_1 \colon S^1 \to S^1$ be a Morse–Smale diffeomorphism with attractor $p = 0$ and repeller $q = 1/2$ such that $g_1'(x) < 1$ for $x \in [-1/5, 1/5]$, $g_1'(x) > 1$ for $x \in [3/10, 7/10]$, and $g_1 \in W$. Set $g_2(x) := g_1(x + 1/3)$ and $g_3(x) := g_1(x - 1/3)$.

Next, set

$$\delta_1 := \frac{1}{15}, \qquad \nu_0 := \min\Big(\big(\max_{x \in [-1/5 \dots 1/5]} g_1'(x)\big)^{-1}, \min_{x \in [3/10 \dots 7/10]} g_1'(x)\Big). \tag{24}$$

Then for each interval I of length less than δ_1 one of the maps g_i $(i = 1, 2, 3)$ dilates it at each point with derivative not less than ν_0. The same is true for the inverse maps.

Take a δ_2, $0 < \delta_2 < \delta_1/3$, such that the map $g_0 := R_{\delta_2}$ (the rotation of the circle by the angle δ_2) belongs to W.

Let $g_4 \in W$ be a Morse–Smale diffeomorphism with attractive periodic orbit X_0 whose Lyapunov exponent satisfies the inequality

$$\lambda(X_0) + \frac{\ln \nu_0}{2} > 0.$$

Lemma 4. *In the space of (L, C, α)-systems, all systems in a sufficiently small neighborhood of the step system corresponding to the maps $g_0, \dots, g_4$ constructed above are controllable.*

Proof. All desired conditions except for predictability of trajectories are preserved under small perturbations and, by construction, hold for the step system itself. The predictability of trajectories follows from condition (10b) and Lemma 1. For the first part (10a) of the compatibility condition for constants to hold, it suffices to require that the distance δ from the step system to the perturbed system satisfy the inequality $K\delta^\beta < \delta_2/40$.

All conditions comprising the controllability condition hold in a sufficiently small neighborhood of the step system constructed above. □

References

[1] R. Abraham and S. Smale, "Nongenericity of Ω-stability," in: Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, 5–8.

[2] Ya. B. Pesin, "Characteristic Lyapunov exponents and smooth ergodic theory," Uspekhi Mat. Nauk, **32**:4(196) (1977), 55–112; English transl.: Russian Math. Surveys, **32**:4 (1977), 55–114.

[3] H. Furstenberg and H. Kesten, "Products of random matrices," Ann. Math. Statist., **31** (1960), 457–469.

[4] C. Bonatti, X. Gomez-Mont, and M. Viana, "Généricité d'exposants de Lyapunov non-nuls pour des produits déterministes de matrices," Ann. Inst. H. Poincaré, Anal. Non Linéaire, **20**:4 (2003), 579–624.

[5] L. Arnold, N. D. Cong, and V. I. Oseledets, “Jordan normal form for linear cocycles,” Random Oper. Stochastic Equations, **7**:4 (1999), 303–358.
[6] L. Arnold and N. D. Cong, “Linear cocycles with simple spectrum are dense in L^∞,” Ergodic Theory Dynam. Systems, **19**:6 (1999), 1389–1404.
[7] J. Bochi, “Genericity of zero Lyapunov exponents,” Ergodic Theory Dynam. Systems, **22**:6 (2002), 1667–1696.
[8] J. Bochi and M. Viana, “Uniform (projective) hyperbolicity or no hyperbolicity: a dichotomy for generic conservative maps,” Ann. Inst. H. Poincaré, Anal. Non Linéaire, **19**:1 (2002), 113–123.
[9] A. Baraviera and C. Bonatti, “Removing zero Lyapunov exponents,” Ergodic Theory Dynam. Systems, **23** (2003), 1655–1670.
[10] M. Shub and A. Wilkinson, “Pathological foliations and removable zero exponents,” Invent. Math., **139** (2000), 495–508.
[11] D. Ruelle and A. Wilkinson, “Absolutely singular dynamical foliations,” Comm. Math. Phys., **219**:3 (2001), 481–487.
[12] D. Dolgopyat and Y. Pesin, “Every compact manifold carries a completely hyperbolic diffeomorphism,” Ergodic Theory Dynam. Systems, **22**:2 (2002), 409–435.
[13] M. W. Hirsch, C. C. Pugh, and M. Shub, Invariant Manifolds, Lecture Notes in Math., vol. 583, Springer-Verlag, Berlin–New York, 1977.
[14] A. S. Gorodetski and Yu. S. Ilyashenko, “Certain new robust properties of invariant sets and attractors of dynamical systems,” Funkts. Anal. Prilozhen., **33**:2 (1999), 16–30; English transl.: Functional Anal. Appl., **33**:2 (1999), 95–105.
[15] A. S. Gorodetski and Yu. S. Ilyashenko, “Some properties of skew products over the horseshoe and solenoid,” Trudy Mat. Inst. Steklova, **231** (2000), 90–112; English transl.: Proc. Steklov Inst. Math., **231** (2000), 96–118.
[16] A. S. Gorodetski, Minimal Attractors and Partially Hyperbolic Invariant Sets of Dynamical Systems, PhD thesis, Moscow State University, 2001.
[17] A. S. Gorodetski, “The regularity of central leaves of partially hyperbolic sets and its applications,” Izv. Ross. Akad. Nauk Ser. Mat., **70**:6 (2006), 19–44; English transl.: Russian Acad. Sci. Izv. Math., **70**:6 (2006), 1093–1116.
[18] A. S. Gorodetski, Yu. S. Ilyashenko, V. A. Kleptsyn, and M. B. Nalsky, “Nonremovable zero Lyapunov exponents,” Funkts. Anal. Prilozhen., **39**:1 (2005), 27–38; English transl.: Functional Anal. Appl., **39**:1 (2005), 21–30.
[19] Yu. Ilyashenko and W. Li, Nonlocal Bifurcations, Amer. Math. Soc., Providence, R.I., 1998.
[20] A. Katok and B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge Univ. Press, Cambridge, 1995.
[21] A. B. Katok and A. M. Stepin, “Approximation of ergodic dynamical systems by periodic transformations,” Dokl. Akad. Nauk SSSR, **171** (1966), 1268–1271; English transl.: Soviet Math. Dokl., **7** (1966), 1638–1641.
[22] A. B. Katok and A. M. Stepin, “Approximations in ergodic theory,” Uspekhi Mat. Nauk, **22(137)**:5 (1967), 81–106; English transl.: Russian Math. Surveys, **22**:5 (1967), 77–102.

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