Functional Analysis and Its Applications, Vol. 40, No. 3, pp. 188–199, 2006
Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 40, No. 3, pp. 30–43, 2006

The Argument Shift Method and the Gaudin Model*

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Received April 9, 2005

Abstract. We construct a family of maximal commutative subalgebras in the tensor product of n copies of the universal enveloping algebra $U(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$. This family is parameterized by finite sequences μ, $z_1, \dots, z_n$, where $\mu \in \mathfrak{g}^*$ and $z_i \in \mathbb{C}$. The construction presented here generalizes the famous construction of the higher Gaudin Hamiltonians due to Feigin, Frenkel, and Reshetikhin. For $n = 1$, the corresponding commutative subalgebras in the Poisson algebra $S(\mathfrak{g})$ were obtained by Mishchenko and Fomenko with the help of the argument shift method. For commutative algebras of our family, we establish a connection between their representations in the tensor products of finite-dimensional $\mathfrak{g}$-modules and the Gaudin model.

Key words: Gaudin model, argument shift method, Mishchenko–Fomenko subalgebra, affine Kac–Moody algebra, critical level.

1. Introduction

Let $\mathfrak{g}$ be a semisimple complex Lie algebra, and let $U(\mathfrak{g})$ be its universal enveloping algebra. The algebra $U(\mathfrak{g})$ bears the natural filtration by degree with respect to the generators. The associated graded algebra is the symmetric algebra $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$ by the Poincaré–Birkhoff–Witt theorem. The commutator on $U(\mathfrak{g})$ defines the Poisson–Lie bracket on $S(\mathfrak{g})$.

The *argument shift method* gives a way to construct subalgebras in $S(\mathfrak{g})$ commutative with respect to the Poisson–Lie bracket. The method is as follows. Let $ZS(\mathfrak{g}) = S(\mathfrak{g})^{\mathfrak{g}}$ be the center of $S(\mathfrak{g})$ with respect to the Poisson bracket, and let $\mu \in \mathfrak{g}^*$ be a regular semisimple element. Then the subalgebra $A_\mu \subset S(\mathfrak{g})$ generated by the elements $\partial_\mu^n \Phi$ with $\Phi \in ZS(\mathfrak{g})$ (or, equivalently, by the central elements of $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$ shifted by $t\mu$ for all $t \in \mathbb{C}$) is commutative with respect to the Poisson bracket and has maximal possible transcendence degree, equal to $\frac{1}{2}(\dim \mathfrak{g} + \operatorname{rk} \mathfrak{g})$ (see [10]). Moreover, the subalgebras A_μ are maximal subalgebras in $S(\mathfrak{g})$ commutative with respect to the Poisson–Lie bracket [17]. In [19], the subalgebras $A_\mu \subset S(\mathfrak{g})$ were named the *Mishchenko–Fomenko subalgebras*.

In the present paper, we lift the subalgebras $A_\mu \subset S(\mathfrak{g})$ to commutative subalgebras in the universal enveloping algebra $U(\mathfrak{g})$. More precisely, for each semisimple Lie algebra $\mathfrak{g}$ we construct a family of commutative subalgebras $\mathscr{A}_\mu \subset U(\mathfrak{g})$ parameterized by regular semisimple elements $\mu \in \mathfrak{g}^*$, so that $\operatorname{gr} \mathscr{A}_\mu = A_\mu$. For classical Lie algebras $\mathfrak{g}$, this was done (by other methods) by Olshanski and Nazarov (see [14], [11]) and also by Tarasov in the case $\mathfrak{g} = sl_r$ [16].

The construction presented here is a modification of the famous construction of higher Gaudin Hamiltonians (cf. [7], [3]). The Gaudin model was introduced in [8] as a spin model related to the Lie algebra sl_2 and generalized in [9, 13.2.2] to the case of an arbitrary semisimple Lie algebra. The generalized Gaudin model has the following algebraic interpretation. Let V_λ be an irreducible representation of $\mathfrak{g}$ with highest weight λ. For any finite sequence $(\lambda) = \lambda_1, \dots, \lambda_n$ of integral dominant weights, let $V_{(\lambda)} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$. For any $x \in \mathfrak{g}$, consider the operator $x^{(i)} = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1$ (x is in the ith position) acting on the space $V_{(\lambda)}$. Let $\{x_a\}$, $a = 1, \dots, \dim \mathfrak{g}$, be an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form, and let $z_1, \dots, z_n$ be pairwise distinct complex numbers. The Hamiltonians of the Gaudin model are the following commuting operators

*The research was supported by CRDF grant RM1-2543-MO-03 and RFBR grant 05-01-00988-a.

 0016–2663/06/4003–0188

in the space $V_{(\lambda)}$:

$$H_i = \sum_{k\neq i} \sum_{a=1}^{\dim \mathfrak{g}} \frac{x_a^{(i)} x_a^{(k)}}{z_i - z_k}\,. \tag{1}$$

We can treat H_i as elements of $U(\mathfrak{g})^{\otimes n}$. A method for constructing a large commutative subalgebra $\mathscr{A}(z_1,\dots,z_n) \subset U(\mathfrak{g})^{\otimes n}$ containing H_i was suggested in [7]. For $\mathfrak{g} = sl_2$, the algebra $\mathscr{A}(z_1,\dots,z_n)$ is generated by H_i and the central elements of $U(\mathfrak{g})^{\otimes n}$. In other cases, the algebra $\mathscr{A}(z_1,\dots,z_n)$ also has some new generators, known as higher Gaudin Hamiltonians. The construction of $\mathscr{A}(z_1,\dots,z_n)$ uses the deep fact [4] that the completed universal enveloping algebra of the affine Kac–Moody algebra $\hat{\mathfrak{g}}$ at the critical level has a large center $Z(\hat{\mathfrak{g}})$. To any finite sequence $z_1,\dots,z_n$ of pairwise distinct complex numbers, one can naturally assign a homomorphism $Z(\hat{\mathfrak{g}}) \to U(\mathfrak{g})^{\otimes n}$. The image of this homomorphism is $\mathscr{A}(z_1,\dots,z_n)$.

In the present paper, we construct a family of homomorphisms $Z(\hat{\mathfrak{g}}) \to U(\mathfrak{g})^{\otimes n} \otimes S(\mathfrak{g})$ parameterized by finite sequences $z_1,\dots,z_n$ of pairwise distinct complex numbers. For each finite sequence $z_1,\dots,z_n$, the image of the corresponding homomorphism is a commutative subalgebra $\mathscr{A}(z_1,\dots,z_n,\infty) \subset U(\mathfrak{g})^{\otimes n} \otimes S(\mathfrak{g})$. Evaluation at any point $\mu \in \mathfrak{g}^* = \operatorname{Spec} S(\mathfrak{g})$ gives a commutative subalgebra $\mathscr{A}_\mu(z_1,\dots,z_n) \subset U(\mathfrak{g})^{\otimes n}$ depending on $z_1,\dots,z_n$ and $\mu \in \mathfrak{g}^*$. For $n = 1$, we obtain commutative subalgebras $\mathscr{A}_\mu(z_1) = \mathscr{A}_\mu \subset U(\mathfrak{g})$, which do not depend on z_1. We show that $\operatorname{gr} \mathscr{A}_\mu = A_\mu$ for regular semisimple μ, i.e., that the subalgebras $\mathscr{A}_\mu \subset U(\mathfrak{g})$ are lifts of Mishchenko–Fomenko subalgebras. For $\mu = 0$, we have $\mathscr{A}_0(z_1,\dots,z_n) = \mathscr{A}(z_1,\dots,z_n)$; i.e., the subalgebras $\mathscr{A}_0(z_1,\dots,z_n) \subset U(\mathfrak{g})^{\otimes n}$ are generated by (higher) Gaudin Hamiltonians. We show that the subalgebras $\mathscr{A}_\mu(z_1,\dots,z_n)$ for generic $z_1,\dots,z_n$, and μ have maximal possible transcendence degree. These subalgebras contain the following "non-homogeneous Gaudin Hamiltonians":

$$H_i = \sum_{k\neq i} \sum_{a=1}^{\dim \mathfrak{g}} \frac{x_a^{(i)} x_a^{(k)}}{z_i - z_k} + \sum_{a=1}^{\dim \mathfrak{g}} \mu(x_a) x_a^{(i)}.$$

The main problem in the Gaudin model is the problem of simultaneous diagonalization of (higher) Gaudin Hamiltonians. The bibliography on this problem is enormous (cf. [5], [6], [7], [12]). It follows from the construction in [7] that all elements of $\mathscr{A}(z_1,\dots,z_n) \subset U(\mathfrak{g})^{\otimes n}$ are invariant with respect to the diagonal action of $\mathfrak{g}$, so that it suffices to diagonalize the algebra $\mathscr{A}(z_1,\dots,z_n)$ in the subspace $V_{(\lambda)}^{\text{sing}} \subset V_{(\lambda)}$ of singular vectors with respect to $\operatorname{diag}_n(\mathfrak{g})$ (i.e., with respect to the diagonal action of $\mathfrak{g}$). The standard conjecture says that for generic z_i the algebra $\mathscr{A}(z_1,\dots,z_n)$ has simple spectrum in $V_{(\lambda)}^{\text{sing}}$. This conjecture is proved in [12] for $\mathfrak{g} = sl_r$ and λ_i equal to ω_1 or ω_{r-1} (i.e., for the case in which every V_{λ_i} is the standard representation of sl_r or its dual).

It is also natural to pose the problem of diagonalization of $\mathscr{A}_\mu(z_1,\dots,z_n)$ in the space $V_{(\lambda)}$. We show that the representation of the algebra $\mathscr{A}_\mu(z_1,\dots,z_n)$ in the space $V_{(\lambda)}$ is a limit of the representation of $\mathscr{A}(z_1,\dots,z_{n+1})$ in $[V_{(\lambda)} \otimes M^*_{z_{n+1}\mu}]^{\text{sing}}$ as $z_{n+1} \to \infty$. Here $M^*_{z_{n+1}\mu}$ is the contragredient module of the Verma module with highest weight $z_{n+1}\mu$, and the space $[V_{(\lambda)} \otimes M^*_{z_{n+1}\mu}]^{\text{sing}}$ consists of all singular vectors in $V_{(\lambda)} \otimes M^*_{z_{n+1}\mu}$ with respect to $\operatorname{diag}_{n+1}(\mathfrak{g})$. This means that the representation of $\mathscr{A}_\mu(z_1,\dots,z_n)$ in $V_{(\lambda)}$ is, in a sense, a limit case of the Gaudin model.

We prove the conjecture on the simplicity of the spectrum for the representation of $\mathscr{A}_\mu(z_1,\dots,z_n)$ in the space $V_{(\lambda)}$ for $\mathfrak{g} = sl_r$. The key point of our proof is the fact that the closure of the family $\mathscr{A}_\mu$ contains the Gelfand–Tsetlin subalgebra. (On the level of Poisson algebras, this was proved by Vinberg [19].) Hence, for $\mathfrak{g} = sl_r$, we conclude that the algebra $\mathscr{A}_\mu(z_1,\dots,z_n)$ for generic μ and $z_1,\dots,z_n$ has simple spectrum in $V_{(\lambda)}$ for any $V_{(\lambda)}$. As a consequence, we find that the spectrum of the algebra $\mathscr{A}_0(z_1,\dots,z_n)$ in $V_{(\lambda)}^{\text{sing}}$ is simple for generic z_i and (λ).

The paper is organized as follows. In Secs. 2 and 3, we collect some well-known facts on Mishchenko–Fomenko subalgebras and the center $Z(\hat{\mathfrak{g}})$ at the critical level, respectively. In Sec. 4, we describe the construction of the subalgebras $\mathscr{A}_\mu$ and prove that $\operatorname{gr} \mathscr{A}_\mu = A_\mu$. In Sec. 5, we describe the general construction of the subalgebras $\mathscr{A}_\mu(z_1,\dots,z_n)\subset U(\mathfrak{g})^{\otimes n}$ and prove that these subalgebras have the maximal possible transcendence degree. In Sec. 6, we describe the representation of $\mathscr{A}_\mu(z_1,\dots,z_n)$ in $V_{(\lambda)}$ as a "limit" Gaudin model. Finally, in Sec. 7 we prove the assertions on simplicity of spectrum for $\mathfrak{g}=sl_r$.

I thank B. L. Feigin, E. B. Vinberg, V. V. Shuvalov, A. V. Chervov, and D. V. Talalaev for useful discussions.

2. The Argument Shift Method

The argument shift method is a special case of the following construction (cf. [1]). Let R be a commutative algebra equipped with two compatible Poisson brackets, $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$. (That is, any linear combination of $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ is a Poisson bracket.) Let Z_t be the center of R with respect to $\{\cdot,\cdot\}_1+t\{\cdot,\cdot\}_2$. Let A be the subalgebra of R generated by all Z_t for generic t.

Fact 1 [1, Proposition 4]. *The subalgebra $A\subset R$ is commutative with respect to any Poisson bracket $\{\cdot,\cdot\}_1+t\{\cdot,\cdot\}_2$.*

Proof. Let $a\in Z_{t_1}$ and $b\in Z_{t_2}$, $t_1\neq t_2$. The expression $\{a,b\}_1+t\{a,b\}_2$ is linear in t and, on the other hand, vanishes at two distinct points t_1 and t_2. It follows that $\{a,b\}_1+t\{a,b\}_2=0$ for all t.

Now suppose that $a,b\in Z_{t_0}$. Since t_0 is generic, it follows that there exists a continuous function $a(s)$ such that $a(t_0)=a$ and $a(s)\in Z_s$ for s in a neighborhood of t_0. For any s in a punctured neighborhood of t_0, we have $\{a(s),b\}_1+t\{a(s),b\}_2=0$, and therefore, $\{a,b\}_1+t\{a,b\}_2=0$. □

Corollary 1. *Suppose that $ZS(\mathfrak{g})=S(\mathfrak{g})^{\mathfrak{g}}$ is the center of $S(\mathfrak{g})$ with respect to the Poisson–Lie bracket, and let $\mu\in\mathfrak{g}^*$. Then the algebra $A_\mu\subset S(\mathfrak{g})$ generated by the elements $\partial_\mu^n\Phi$ with $\Phi\in ZS(\mathfrak{g})$ (or, equivalently, by the central elements of $S(\mathfrak{g})=\mathbb{C}[\mathfrak{g}^*]$ shifted by $t\mu$ for all $t\in\mathbb{C}$) is commutative with respect to the Poisson–Lie bracket.*

Proof. Take the Poisson–Lie bracket for $\{\cdot,\cdot\}_1$ and the "frozen argument" bracket for $\{\cdot,\cdot\}_2$. (This means that

$$\{x,y\}_2=\mu([x,y]),\qquad x,y\in\mathfrak{g},$$

for the generators.) Then the algebra Z_t is generated by the central elements of $\mathbb{C}[\mathfrak{g}^*]=S(\mathfrak{g})$ shifted by $t\mu$. □

Since the Lie algebra $\mathfrak{g}$ is semisimple, we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and write $\mu\in\mathfrak{g}$.

Fact 2 [10]. *For a regular semisimple $\mu\in\mathfrak{g}$, the algebra A_μ is a free commutative subalgebra in $S(\mathfrak{g})$ with $\frac{1}{2}(\dim\mathfrak{g}+\operatorname{rk}\mathfrak{g})$ generators. (This means that A_μ is a commutative subalgebra of maximal possible transcendence degree.) One can take the elements $\partial_\mu^n\Phi_k$, $k=1,\dots,\operatorname{rk}\mathfrak{g}$, $n=0,1,\dots,\deg\Phi_k$, where the Φ_k are basis $\mathfrak{g}$-invariants in $S(\mathfrak{g})$, as free generators of A_μ.*

Shuvalov [15] described the closure of the family of subalgebras $A_\mu\subset S(\mathfrak{g})$ under the condition $\mu\in\mathfrak{h}^{\mathrm{reg}}$. (That is, the parameter μ lies in a given Cartan subalgebra.) In particular, the following assertion was proved in [15].

Fact 3. *Suppose that $\mu(t)=\mu_0+t\mu_1+t^2\mu_2+\dots\in\mathfrak{h}^{\mathrm{reg}}$ for generic t. Set $\mathfrak{z}_k=\bigcap_{i=0}^k\mathfrak{z}_{\mathfrak{g}}(\mu_i)$ (where $\mathfrak{z}_{\mathfrak{g}}(\mu_i)$ is the centralizer of μ_i in $\mathfrak{g}$) and $\mathfrak{z}_{-1}=\mathfrak{g}$. Then*

1. *The subalgebra $\lim_{t\to0}A_{\mu(t)}\subset S(\mathfrak{g})$ is generated by all elements of $S(\mathfrak{z}_k)^{\mathfrak{z}_k}$ and their derivatives (of any order) along μ_{k+1} for all k.*
2. *$\lim\limits_{t\to0}A_{\mu(t)}$ is a free commutative algebra. As the free generators one can take some of the derivatives of the generators of $S(\mathfrak{z}_k)^{\mathfrak{z}_k}$ along μ_{k+1}.*

This means, in particular, that the closure of the family A_μ for $\mathfrak{g}=sl_r$ contains the Gelfand–Tsetlin algebra (see [19, 6.1–6.4]). We shall discuss this case in Sec. 7.

The following results were obtained by Tarasov.

Fact 4 [17]. *The subalgebras A_μ and the limit subalgebras $\lim_{t\to0}A_{\mu(t)}$ are maximal commutative subalgebras; i.e., they coincide with their Poisson centralizers in $S(\mathfrak{g})$.*

The *symmetrization map* $\sigma\colon S(\mathfrak{g}) \to U(\mathfrak{g})$ is defined by the following property:

$$\sigma(x^k) = x^k \quad \forall\, x \in \mathfrak{g},\ k = 0, 1, 2, \dots .$$

Fact 5 ([16], [18]). *For $\mathfrak{g} = sl_r$, some systems of generators of A_μ and the limit subalgebras $\lim_{t\to 0} A_{\mu(t)}$ can be lifted to commuting elements of $U(\mathfrak{g})$ by the symmetrization map. This lift of A_μ to the universal enveloping algebra is unique.*

Remark 1. The generators of A_μ and the limit subalgebras $\lim_{t\to 0} A_{\mu(t)}$ to be lifted by the symmetrization are explicitly indicated in [16]. Up to proportionality, these are the elements $\partial_\mu^n \Phi_k$, $k = 1, \dots, r-1$, $n = 0, 1, \dots, \deg \Phi_k$, and their limits, respectively (where $\Phi_k \in S(sl_r)^{sl_r}$ are the coefficients of the characteristic polynomial as functions on sl_r). We only use the fact that this system of generators, up to proportionality, is continuous in the parameter μ.

3. The Center at the Critical Level

Let $\hat{\mathfrak{g}}$ be the affine Kac–Moody algebra corresponding to $\mathfrak{g}$. The Lie algebra $\hat{\mathfrak{g}}$ is a central extension of the formal loop algebra $\mathfrak{g}((t))$ by an element K. The commutator relations are defined as follows:

$$[g_1 \otimes x(t), g_2 \otimes y(t)] = [g_1, g_2] \otimes x(t)y(t) + \kappa_c(g_1, g_2)\operatorname{Res}_{t=0} x(t)dy(t)\cdot K, \tag{2}$$

where κ_c is the invariant inner product on $\mathfrak{g}$ defined by the formula

$$\kappa_c(g_1, g_2) = -\frac{1}{2}\operatorname{Tr}_{\mathfrak{g}} \operatorname{ad}(g_1)\operatorname{ad}(g_2). \tag{3}$$

Set $\hat{\mathfrak{g}}_+ = \mathfrak{g}[[t]] \subset \hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}_- = t^{-1}\mathfrak{g}[t^{-1}] \subset \hat{\mathfrak{g}}$.

Define the completion $\widetilde{U}(\hat{\mathfrak{g}})$ of $U(\hat{\mathfrak{g}})$ as the inverse limit of $U(\hat{\mathfrak{g}})/U(\hat{\mathfrak{g}})(t^n\mathfrak{g}[[t]])$, $n > 0$. The action of $\widetilde{U}(\hat{\mathfrak{g}})$ is well defined on $\hat{\mathfrak{g}}$-modules in the category $\mathscr{O}^0$ (i.e., $\hat{\mathfrak{g}}$-modules on which the Lie subalgebra $\hat{\mathfrak{g}}_+$ acts locally finitely). We set $\widetilde{U}(\hat{\mathfrak{g}})_c = \widetilde{U}(\hat{\mathfrak{g}})/(K-1)$. This algebra acts on $\hat{\mathfrak{g}}$-modules of the *critical level* (i.e., $\hat{\mathfrak{g}}$-modules on which the element K acts as the unity). The name "critical" is explained by the fact that representation theory at this level is most complicated. In particular, the algebra $\widetilde{U}(\hat{\mathfrak{g}})_c$ has a nontrivial center $Z(\hat{\mathfrak{g}})$. The following fact shows that this center is rather large.

Fact 6 [4]. 1. *The natural homomorphism $Z(\hat{\mathfrak{g}}) \to (U(\hat{\mathfrak{g}})/U(\hat{\mathfrak{g}})(\hat{\mathfrak{g}}_+ + \mathbb{C}(K-1)))^{\hat{\mathfrak{g}}_+}$ is surjective.*

2\. *The Poincaré–Birkhoff–Witt filtration on the enveloping algebra yields a filtration on the $\hat{\mathfrak{g}}_+$-module $U(\hat{\mathfrak{g}})/U(\hat{\mathfrak{g}})(\hat{\mathfrak{g}}_+ + \mathbb{C}(K-1))$. We have $\operatorname{gr}(U(\hat{\mathfrak{g}})/U(\hat{\mathfrak{g}})(\hat{\mathfrak{g}}_+ + \mathbb{C}(K-1)))^{\hat{\mathfrak{g}}_+} = (S(\hat{\mathfrak{g}})/S(\hat{\mathfrak{g}})(\hat{\mathfrak{g}}_+ + \mathbb{C}K))^{\hat{\mathfrak{g}}_+}$ with respect to this filtration.*

Now let us give an explicit description of the algebra $(S(\hat{\mathfrak{g}})/S(\hat{\mathfrak{g}})(\hat{\mathfrak{g}}_+ + \mathbb{C}K))^{\hat{\mathfrak{g}}_+}$. Since $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_- \oplus \mathbb{C}K$ as vector spaces, it follows that every element of $U(\hat{\mathfrak{g}})/U(\hat{\mathfrak{g}})(\hat{\mathfrak{g}}_+ + \mathbb{C}(K-1))$ (respectively, $S(\hat{\mathfrak{g}})/S(\hat{\mathfrak{g}})(\hat{\mathfrak{g}}_+ + \mathbb{C}K)$) has a unique representative in $U(\hat{\mathfrak{g}}_-)$ (respectively, in $S(\hat{\mathfrak{g}}_-)$). Thus we obtain the natural embeddings

$$(U(\hat{\mathfrak{g}})/U(\hat{\mathfrak{g}})(\hat{\mathfrak{g}}_+ + \mathbb{C}(K-1)))^{\hat{\mathfrak{g}}_+} \hookrightarrow U(\hat{\mathfrak{g}}_-) \tag{4}$$

and

$$(S(\hat{\mathfrak{g}})/S(\hat{\mathfrak{g}})(\hat{\mathfrak{g}}_+ + \mathbb{C}K))^{\hat{\mathfrak{g}}_+} \hookrightarrow S(\hat{\mathfrak{g}}_-). \tag{5}$$

Let $\mathscr{A} \subset U(\hat{\mathfrak{g}}_-)$ and $A \subset S(\hat{\mathfrak{g}}_-)$ be the images of these embeddings, respectively. Consider the following derivations of the Lie algebra $\hat{\mathfrak{g}}_-$:

$$\partial_t(g \otimes t^m) = mg \otimes t^{m-1} \quad \forall g \in \mathfrak{g},\ m = -1, -2, \dots, \tag{6}$$

$$t\partial_t(g \otimes t^m) = mg \otimes t^m \quad \forall g \in \mathfrak{g},\ m = -1, -2, \dots . \tag{7}$$

The derivations (6) and (7) can be extended to derivations of the associative algebras $S(\hat{\mathfrak{g}}_-)$ and $U(\hat{\mathfrak{g}}_-)$. The derivation (7) defines a grading of these algebras.

Let $i_{-1}\colon S(\mathfrak{g}) \hookrightarrow S(\hat{\mathfrak{g}}_-)$ be the embedding that takes $g \in \mathfrak{g}$ to $g \otimes t^{-1}$. Let Φ_k, $k = 1, \dots, \operatorname{rk}\mathfrak{g}$, be the generators of the algebra $S(\mathfrak{g})^{\mathfrak{g}}$ of invariants.

Fact 7 [2], [6], [13]. *The subalgebra* $A \subset S(\hat{\mathfrak{g}}_-)$ *is freely generated by the elements* $\partial_t^n \overline{S_k}$, $k = 1, \dots, \operatorname{rk}\mathfrak{g}$, $n = 0, 1, 2, \dots,$ *where* $\overline{S_k} = i_{-1}(\Phi_k)$.

It follows from Fact 6 that the generators $\overline{S_k}$ can be lifted to the (commuting) generators of $\mathscr{A}$. This means that the following assertion is true.

Corollary 2. 1. *There exist elements* $S_k \in \mathscr{A}$ *homogeneous with respect to* $t\partial_t$ *and satisfying* $\operatorname{gr} S_k = \overline{S_k}$.

2. $\mathscr{A}$ *is the free commutative algebra generated by* $\partial_t^n S_k$, $k = 1, \dots, \operatorname{rk}\mathfrak{g}$, $n = 0, 1, 2, \dots$.

In subsequent considerations, we use only the existence of the commutative subalgebra $\mathscr{A} \subset U(\hat{\mathfrak{g}}_-)$ and its description in Corollary 2.

Remark 2. No general explicit formulas for the elements S_k are known at the moment. For the quadratic Casimir element Φ_1, the corresponding element $S_1 \in \mathscr{A}$ is obtained from $\overline{S_1} = i_{-1}(\Phi_1)$ by the symmetrization map.

Remark 3. The construction of the higher Gaudin Hamiltonians is as follows. The commutative subalgebra $\mathscr{A}(z_1, \dots, z_n) \subset U(\mathfrak{g})^{\otimes n}$ is the image of the subalgebra $\mathscr{A} \subset U(\hat{\mathfrak{g}}_-)$ under the homomorphism $U(\hat{\mathfrak{g}}_-) \to U(\mathfrak{g})^{\otimes n}$ of specialization at the points $z_1, \dots, z_n$ (see [7], [3]). We discuss this in Section 5.

4. Maximal Commutative Subalgebras in $U(\mathfrak{g})$

For each $z \neq 0$, we have the evaluation homomorphism

$$\phi_z \colon U(\hat{\mathfrak{g}}_-) \to U(\mathfrak{g}), \qquad g \otimes t^m \mapsto z^m g. \tag{8}$$

We also have the homomorphism

$$\phi_\infty \colon U(\hat{\mathfrak{g}}_-) \to S(\mathfrak{g}), \qquad g \otimes t^{-1} \mapsto g, \qquad g \otimes t^m \mapsto 0, \quad m = -2, -3, \dots. \tag{9}$$

Let $\Delta \colon U(\hat{\mathfrak{g}}_-) \hookrightarrow U(\hat{\mathfrak{g}}_-) \otimes U(\hat{\mathfrak{g}}_-)$ be the comultiplication. For each $z \neq 0$, we have the homomorphism

$$\phi_{z,\infty} = (\phi_z \otimes \phi_\infty) \circ \Delta \colon U(\hat{\mathfrak{g}}_-) \to U(\mathfrak{g}) \otimes S(\mathfrak{g}). \tag{10}$$

More explicitly,

$$\phi_{z,\infty}(g \otimes t^m) = z^m g \otimes 1 + \delta_{-1,m} \otimes g.$$

We set

$$\mathscr{A}(z, \infty) = \phi_{z,\infty}(\mathscr{A}) \subset U(\mathfrak{g}) \otimes S(\mathfrak{g}).$$

Proposition 1. *The subalgebra* $\mathscr{A}(z, \infty)$ *is generated by the coefficients of the principal parts of the Laurent series for the functions* $S_k(w) = \phi_{w-z,\infty}(S_k)$ *around* z *and by the values of these functions at* ∞.

Proof. Indeed, $\mathscr{A}(z, \infty)$ is generated by the elements $\phi_{z,\infty}(\partial_t^n S_k)$. These elements are the Taylor coefficients of $S_k(w) = \phi_{w-z,\infty}(S_k)$ around $w = 0$. Since $S_k(w)$ has a unique pole at z, it follows that the Taylor coefficients of $S_k(w)$ around $w = 0$ are linear expressions in the coefficients of the principal part of the Laurent series for the same function around z and its value at ∞, and vice versa. □

Corollary 3. *The subalgebra* $\mathscr{A}(z, \infty) \subset U(\mathfrak{g}) \otimes S(\mathfrak{g})$ *is independent of* z.

Proof. Indeed, the Laurent coefficients of the functions $S_k(w) = \phi_{w-z,\infty}(S_k)$ around the point z and the values of these functions at ∞ are independent of z. □

Every $\mu \in \mathfrak{g}^*$ defines the homomorphism $S(\mathfrak{g}) \to \mathbb{C}$ of "specialization at the point μ". We denote this homomorphism also by μ. Consider the following family of commutative subalgebras of $U(\mathfrak{g})$ parameterized by $\mu \in \mathfrak{g}^*$:

$$\mathscr{A}_\mu := (\operatorname{id} \otimes \mu)(\mathscr{A}(z, \infty)) \subset U(\mathfrak{g}). \tag{11}$$

Proposition 2. *All elements of the subalgebra* $\mathscr{A}_\mu \subset U(\mathfrak{g})$ *are* $\mathfrak{z}_{\mathfrak{g}}(\mu)$*-invariant* (*where* $\mathfrak{z}_{\mathfrak{g}}(\mu)$ *is the centralizer of* μ *in* $\mathfrak{g}$).

Proof. Indeed, $\mathscr{A}(z,\infty) \subset [U(\mathfrak{g}) \otimes S(\mathfrak{g})]^{\Delta(\mathfrak{g})}$, and the homomorphism μ is $\mathfrak{z}_{\mathfrak{g}}(\mu)$-equivariant. Therefore, $\mathscr{A}_\mu \subset U(\mathfrak{g})^{\mathfrak{z}_{\mathfrak{g}}(\mu)}$. □

Now let us prove that the subalgebras $\mathscr{A}_\mu \subset U(\mathfrak{g})$ give a quantization of the Mishchenko–Fomenko subalgebras in $S(\mathfrak{g})$ obtained by the argument shift method.

Theorem 1. $\operatorname{gr} \mathscr{A}_\mu = A_\mu$ *for regular semisimple* $\mu \in \mathfrak{g}^*$.

Proof. Let E be the $\mathfrak{g}$-invariant derivation of $U(\hat{\mathfrak{g}}_-) \otimes S(\mathfrak{g})$ acting as follows on the generators:

$$E((g \otimes x(t)) \otimes 1) = 1 \otimes g \operatorname{Res}_{t=0} x(t)\, dt, \quad E(1 \otimes g) = 0 \qquad \forall\, g \in \mathfrak{g}. \tag{12}$$

In other words,

$$(g \otimes t^{-m}) \otimes 1 \mapsto \delta_{-1,m} \otimes g, \quad 1 \otimes g \mapsto 0 \qquad \forall\, g \in \mathfrak{g}.$$

Lemma 1. *The subalgebra* $\mathscr{A}(z,\infty) \subset U(\mathfrak{g}) \otimes S(\mathfrak{g})$ *is generated by the elements*

$$(\phi_z \otimes \mathrm{id})(E^j(S_k \otimes 1)) \in U(\mathfrak{g}) \otimes S^j(\mathfrak{g}).$$

Proof. Note that

$$(\mathrm{id} \otimes \phi_\infty \circ \Delta)(\partial_t^n S_k) = (\exp E)(\partial_t^n S_k) \in U(\hat{\mathfrak{g}}_-) \otimes S(\mathfrak{g}).$$

Since the elements S_k are homogeneous with respect to $t\partial_t$, it follows that the elements $(\phi_z \otimes \mathrm{id})(E^j(S_k \otimes 1))$ are the Laurent coefficients of the function $S_k(w) = \phi_{w-z,\infty}(S_k) = \phi_{w-z}((\exp E)(S_k \otimes 1))$ at the point $w = z$. Now it remains to use Proposition 1. □

Now let e be a $\mathfrak{g}$-invariant derivation of $S(\mathfrak{g}) \otimes S(\mathfrak{g})$ acting on the generators by the formula

$$e(g \otimes 1) = 1 \otimes g, \qquad e(1 \otimes g) = 0. \tag{13}$$

Clearly, $(\mathrm{id} \otimes \mu) \circ e^j(f \otimes 1) = \partial_\mu^j f$ for each $f \in S(\mathfrak{g})$.

Note that

$$\operatorname{gr}(\phi_z \otimes \mathrm{id})(E^j(S_k \otimes 1)) = z^{(-\deg \Phi_k + j)} e^j(\Phi_k \otimes 1) \in S(\mathfrak{g}) \otimes S^j(\mathfrak{g}),$$

since $\operatorname{gr} S_k = i_{-1}(\Phi_k)$. Hence

$$\operatorname{gr}(\mathrm{id} \otimes \mu) \circ (\phi_z \otimes \mathrm{id})(E^j(S_k \otimes 1)) = z^{(-\deg \Phi_k + j)} \partial_\mu^j(\Phi_k).$$

Since the elements $\partial_\mu^j(\Phi_k)$ generate A_μ, we have $\operatorname{gr} \mathscr{A}_\mu \supset A_\mu$. The elements $\partial_\mu^j(\Phi_k)$ are algebraically independent by Fact 2, and the lemma says that the elements $(\mathrm{id} \otimes \mu) \circ (\phi_z \otimes \mathrm{id})(E^j(S_k \otimes 1))$ generate $\mathscr{A}_\mu$. Thus $\operatorname{gr} \mathscr{A}_\mu = A_\mu$. □

5. Commutative Subalgebras in $U(\mathfrak{g})^{\otimes n}$

Now let us generalize our construction. Let $U(\mathfrak{g})^{\otimes n}$ be the tensor product of n copies of $U(\mathfrak{g})$. We denote the subspace $1 \otimes \cdots \otimes 1 \otimes \mathfrak{g} \otimes 1 \otimes \cdots \otimes 1 \subset U(\mathfrak{g})^{\otimes n}$, where $\mathfrak{g}$ stands at the ith place, by $\mathfrak{g}^{(i)}$. Accordingly, we set

$$u^{(i)} = 1 \otimes \cdots \otimes 1 \otimes u \otimes 1 \otimes \cdots \otimes 1 \in U(\mathfrak{g})^{\otimes n} \tag{14}$$

for each $u \in U(\mathfrak{g})$.

Let $\mathrm{diag}_n \colon U(\hat{\mathfrak{g}}_-) \hookrightarrow U(\hat{\mathfrak{g}}_-)^{\otimes n}$ be the diagonal embedding. For any finite sequence of pairwise distinct complex numbers z_i, $i = 1, \dots, n$, we have the homomorphism

$$\phi_{z_1,\dots,z_n,\infty} = (\phi_{z_1} \otimes \cdots \otimes \phi_{z_n} \otimes \phi_\infty) \circ \mathrm{diag}_{n+1} \colon U(\hat{\mathfrak{g}}_-) \to U(\mathfrak{g})^{\otimes n} \otimes S(\mathfrak{g}). \tag{15}$$

More explicitly,

$$\phi_{z_1,\dots,z_n,\infty}(g \otimes t^m) = \sum_{i=1}^{n} z_i^m g^{(i)} \otimes 1 + \delta_{-1,m} \otimes g.$$

Set

$$\mathscr{A}(z_1, \dots, z_n, \infty) = \phi_{z_1,\dots,z_n,\infty}(\mathscr{A}) \subset U(\mathfrak{g})^{\otimes n} \otimes S(\mathfrak{g}).$$

The following assertion can be proved in the same way as Proposition 1 and Corollary 2.

Proposition 3. 1. *The subalgebras* $\mathscr{A}(z_1,\dots,z_n,\infty)$ *are generated by the coefficients of the principal parts of the Laurent series of the functions*

$$S_k(w;z_1,\dots,z_n)=\phi_{w-z_1,\dots,w-z_n,\infty}(S_k)$$

at the points $z_1,\dots,z_n$ *and by the values of these functions at* ∞.

2. *The subalgebras* $\mathscr{A}(z_1,\dots,z_n,\infty)$ *are stable under simultaneous affine transformations* $z_i\mapsto az_i+b$ *of the parameters.*

3. *All elements of* $\mathscr{A}(z_1,\dots,z_n,\infty)$ *are invariant with respect to the diagonal action of* $\mathfrak{g}$.

Consider the following family of commutative subalgebras in $U(\mathfrak{g})^{\otimes n}$ parameterized by $z_1,\dots,z_n\in\mathbb{C}$ and $\mu\in\mathfrak{g}^*$:

$$\mathscr{A}_\mu(z_1,\dots,z_n):=(\mathrm{id}\otimes\mu)(\mathscr{A}(z_1,\dots,z_n,\infty))\subset U(\mathfrak{g})^{\otimes n}. \tag{16}$$

We obtain the following assertion as an immediate corollary of Proposition 3.

Proposition 4. 1. *The subalgebras* $\mathscr{A}_\mu(z_1,\dots,z_n)$ *are stable under simultaneous translations* $z_i\mapsto z_i+b$ *of the parameters.*

2. *All elements of* $\mathscr{A}_\mu(z_1,\dots,z_n)$ *are invariant with respect to the diagonal action of* $\mathfrak{z}_{\mathfrak{g}}(\mu)$.

Remark 4. The subalgebra $\mathscr{A}_0(z_1,\dots,z_n)\subset U(\mathfrak{g})^{\otimes n}$ can be obtained as the image of the subalgebra $\mathscr{A}\subset U(\hat{\mathfrak{g}}_-)$ under the homomorphism

$$\phi_{z_1,\dots,z_n}=(\phi_{z_1}\otimes\dots\otimes\phi_{z_n})\circ\mathrm{diag}_n\colon U(\hat{\mathfrak{g}}_-)\to U(\mathfrak{g})^{\otimes n}.$$

These subalgebras are just the subalgebras $\mathscr{A}(z_1,\dots,z_n)\subset U(\mathfrak{g})^{\otimes n}$ of higher Gaudin Hamiltonians introduced in [7] (see also [3]). The quadratic Gaudin Hamiltonians (1) are linear combinations of the elements $\phi_{z_1,\dots,z_n}(\partial_t^n\overline{S_1})$, $n=0,1,2,\dots$.

We shall write $\mathscr{A}(z_1,\dots,z_n)$ instead of $\mathscr{A}_0(z_1,\dots,z_n)$.

Proposition 5. *The subalgebras* $\mathscr{A}_\mu(z_1,\dots,z_n)$ *contain the "nonhomogeneous Gaudin Hamiltonians"*

$$H_i=\sum_{k\neq i}\sum_{a=1}^{\dim\mathfrak{g}}\frac{x_a^{(i)}x_a^{(k)}}{z_i-z_k}+\sum_{a=1}^{\dim\mathfrak{g}}\mu(x_a)x_a^{(i)}.$$

Proof. Since the element $S_1\in\mathscr{A}$ is the symmetrization of $\overline{S_1}=i_{-1}(\Phi_1)$, it follows that H_i is the coefficient of $1/(z-z_i)$ in the expansion of $S_1(w;z_1,\dots,z_n)=\phi_{w-z_1,\dots,w-z_n,\infty}(S_1)$ at the point $w=z_i$. Now it remains to apply Proposition 3. $\square$

The algebra $U(\mathfrak{g})^{\otimes n}\otimes S(\mathfrak{g})^{\otimes m}$ has an increasing filtration by finite-dimensional spaces, $U(\mathfrak{g})^{\otimes n}\otimes S(\mathfrak{g})^{\otimes m}=\bigcup_{k=0}^\infty(U(\mathfrak{g})^{\otimes n}\otimes S(\mathfrak{g})^{\otimes m})_{(k)}$ (by degree with respect to the generators). We define the limit $\lim_{s\to\infty}B(s)$ for any one-parameter family of subalgebras $B(s)\subset U(\mathfrak{g})^{\otimes n}\otimes S(\mathfrak{g})^{\otimes m}$ as

$$\bigcup_{k=0}^\infty\lim_{s\to\infty}B(s)\cap(U(\mathfrak{g})^{\otimes n}\otimes S(\mathfrak{g})^{\otimes m})_{(k)}.$$

It is clear that the limit of a family of *commutative* subalgebras is a commutative subalgebra. It is also clear that the passage to the limit commutes with homomorphisms of filtered algebras (in particular, with the projection onto any factor and with finite-dimensional representations).

Theorem 2. $\lim_{s\to\infty}\mathscr{A}_\mu(sz_1,\dots,sz_n)=\mathscr{A}_\mu^{(1)}\otimes\dots\otimes\mathscr{A}_\mu^{(n)}\subset U(\mathfrak{g})^{\otimes n}$ *for regular semisimple* $\mu\in\mathfrak{g}^*$.

Proof. We use the following lemma.

Lemma 2. $\lim_{z\to\infty}\phi_z=\varepsilon$, *where* $\varepsilon\colon U(\hat{\mathfrak{g}}_-)\to\mathbb{C}\cdot 1\subset U(\mathfrak{g})$ *is the counit.*

Proof. It suffices to verify this on the generators. We have

$$\lim_{z\to\infty}\phi_z(g\otimes t^m)=\lim_{z\to\infty}z^mg=0\quad\forall\,g\in\mathfrak{g},\ m=-1,-2,\dots. \qquad\square$$

Now let us choose the generators of $\mathscr{A}(sz_1,\dots,sz_n,\infty)$ as in Proposition 3. The coefficients of the Laurent expansion of $S_k(w;sz_1,\dots,sz_n)$ at any point sz_i are equal to the Laurent coefficients of $S_k(w+sz_i;sz_1,\dots,sz_n)$ at the point 0. On the other hand,

$$\lim_{s\to\infty} S_k(w+sz_i;sz_1,\dots,sz_n) = \lim_{s\to\infty} \phi_{w-s(z_1-z_i),\dots,w,\dots,w-s(z_n-z_i),\infty}(S_k)$$
$$= (\varepsilon\otimes\dots\otimes\varepsilon\otimes\phi_w\otimes\varepsilon\otimes\dots\otimes\varepsilon\otimes\phi_\infty)\circ \operatorname{diag}_{n+1}(S_k) = S_k^{(i)}(w;0)$$

by Lemma 2. This means that the generators of $\mathscr{A}(sz_1,\dots,sz_n,\infty)$ give the generators of $\mathscr{A}(z_1,\infty)^{(1)}\cdots\mathscr{A}(z_n,\infty)^{(n)}$ in the limit. Hence

$$\lim_{s\to\infty}\mathscr{A}(sz_1,\dots,sz_n,\infty)\supset\mathscr{A}(z_1,\infty)^{(1)}\cdots\mathscr{A}(z_n,\infty)^{(n)},$$

and therefore,

$$\lim_{s\to\infty}\mathscr{A}_\mu(sz_1,\dots,sz_n)\supset\mathscr{A}_\mu^{(1)}\otimes\dots\otimes\mathscr{A}_\mu^{(n)}.$$

By Fact 4, the subalgebra $\mathscr{A}_\mu^{(1)}\otimes\dots\otimes\mathscr{A}_\mu^{(n)}\subset U(\mathfrak{g})^{\otimes n}$ coincides with its own centralizer. Thus $\lim_{s\to\infty}\mathscr{A}_\mu(sz_1,\dots,sz_n)=\mathscr{A}_\mu^{(1)}\otimes\dots\otimes\mathscr{A}_\mu^{(n)}$. □

Corollary 4. *For generic parameter values, the commutative subalgebra* $\mathscr{A}_\mu(z_1,\dots,z_n)\subset U(\mathfrak{g})^{\otimes n}$ *has the maximal possible transcendence degree* (*which is equal to* $\frac{n}{2}(\dim\mathfrak{g}+\operatorname{rk}\mathfrak{g})$).

Proof. Indeed, for generic μ the subalgebra $\mathscr{A}_\mu^{(1)}\otimes\dots\otimes\mathscr{A}_\mu^{(n)}\subset U(\mathfrak{g})^{\otimes n}$ has the maximal possible transcendence degree owing to Fact 2. Since these subalgebras are contained in the closure of the family $\mathscr{A}_\mu(z_1,\dots,z_n)$, it follows that for generic parameter values the subalgebra $\mathscr{A}_\mu(z_1,\dots,z_n)$ has the maximal possible transcendence degree as well. □

Consider the one-parameter family $U(\mathfrak{g})_t$ of associative algebras generated by $\mathfrak{g}$ with the defining relations

$$xy-yx=t[x,y]\quad\forall\, x,y\in\mathfrak{g}.\tag{17}$$

For each $t\neq0$, the map $\mathfrak{g}\to\mathfrak{g}$, $x\mapsto t^{-1}x$, induces an associative algebra homomorphism

$$\psi_t\colon U(\mathfrak{g})\tilde{\to}U(\mathfrak{g})_t.\tag{18}$$

For $t=0$, we have $U(\mathfrak{g})_0=S(\mathfrak{g})$.

Consider the commutative subalgebra

$$(\mathrm{id}^{\otimes n}\otimes\psi_{z^{-1}})(\mathscr{A}(z_1,\dots,z_n,z))\subset U(\mathfrak{g})^{\otimes n}\otimes U(\mathfrak{g})_{z^{-1}}.$$

Passing to the limit as $z\to\infty$, we obtain a commutative subalgebra in $U(\mathfrak{g})^{\otimes n}\otimes S(\mathfrak{g})$.

Theorem 3.

$$\lim_{z\to\infty}(\mathrm{id}^{\otimes n}\otimes\psi_{z^{-1}})(\mathscr{A}(z_1,\dots,z_n,z))=\mathscr{A}(z_1,\dots,z_n,\infty)\subset U(\mathfrak{g})^{\otimes n}\otimes S(\mathfrak{g}).$$

Proof. We use the following lemma.

Lemma 3. $\lim_{z\to\infty}\psi_{z^{-1}}\circ\phi_z=\phi_\infty$.

Proof. It suffices to check this on the generators. We have

$$\psi_{z^{-1}}\circ\phi_z(g\otimes t^m)=z\cdot z^m g\in U(\mathfrak{g})_{z^{-1}}\quad\forall\, g\in\mathfrak{g},\ m=-1,-2,\dots.$$

Hence

$$\lim_{z\to\infty}\psi_{z^{-1}}\circ\phi_z(g\otimes t^m)=\delta_{-1,m}g=\psi_\infty(g\otimes t^m)\in S(\mathfrak{g}).$$ □

Using Lemma 3, we obtain

$$\lim_{z\to\infty}(\mathrm{id}^{\otimes n}\otimes\psi_{z^{-1}})(\mathscr{A}(z_1,\dots,z_n,z))=\lim_{z\to\infty}(\phi_{z_1}\otimes\dots\otimes\phi_{z_n}\otimes(\psi_{z^{-1}}\circ\phi_z))\circ\operatorname{diag}_{n+1}(\mathscr{A})$$
$$=\lim_{z\to\infty}(\phi_{z_1}\otimes\dots\otimes\phi_{z_n}\otimes\phi_\infty)\circ\operatorname{diag}_{n+1}(\mathscr{A})$$
$$=\mathscr{A}(z_1,\dots,z_n,\infty)\subset U(\mathfrak{g})^{\otimes n}\otimes S(\mathfrak{g}).$$ □

6. The "Limit" Gaudin Model

Let V_λ be a finite-dimensional irreducible $\mathfrak{g}$-module of highest weight λ.

Consider the $U(\mathfrak{g})^{\otimes n}$-module

$$V_{(\lambda)} := V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}. \tag{19}$$

The subalgebra $\mathscr{A}(z_1, \dots, z_n) \subset U(\mathfrak{g})^{\otimes n}$ consists of $\mathrm{diag}_n(\mathfrak{g})$-invariant elements and hence acts on the space $V_{(\lambda)}^{\mathrm{sing}} \subset V_{(\lambda)}$ of singular vectors with respect to $\mathrm{diag}_n(\mathfrak{g})$. This representation of $\mathscr{A}(z_1, \dots, z_n)$ is known as the (n-point) *Gaudin model*.

We claim that for semisimple $\mu \in \mathfrak{g}^*$ the representation of the subalgebra $\mathscr{A}_\mu(z_1, \dots, z_n) \subset U(\mathfrak{g})^{\otimes n}$ in the space $V_{(\lambda)}$ is a limit case of the $(n+1)$-point Gaudin model.

Let M_χ^* be the contragredient module of the Verma module with highest weight χ. This module can be constructed as follows. Let Δ_+ be the set of positive roots of $\mathfrak{g}$. Then $M_\chi^* = \mathbb{C}[x_\alpha]_{\alpha \in \Delta_+}$ (the generators x_α have (multi-)degree α), and the elements of $\mathfrak{g}$ act by the following formulas:

1. The elements e_α, $\alpha \in \Delta_+$, of the subalgebra $\mathfrak{n}_+$ act as

$$\frac{\partial}{\partial x_\alpha} + \sum_{\beta > \alpha} P_\beta^\alpha \frac{\partial}{\partial x_\beta},$$

where P_β^α is a certain polynomial of degree $\beta - \alpha$.

2. The elements $h \in \mathfrak{h}$ act as

$$\chi(h) - \sum_{\beta \in \Delta_+} \beta(h) x_\beta \frac{\partial}{\partial x_\beta}.$$

(3) The generators $e_{-\alpha_i}$ (where α_i are the simple roots) of the subalgebra $\mathfrak{n}_-$ act as

$$\chi(h_{\alpha_i}) x_{\alpha_i} + \sum_{\beta \in \Delta_+} Q_\beta^{\alpha_i} \frac{\partial}{\partial x_\beta},$$

where $Q_\beta^{\alpha_i}$ is a polynomial of degree $\beta + \alpha_i$.

Consider the $U(\mathfrak{g})^{\otimes n} \otimes U(\mathfrak{g})$-module $V_{(\lambda)} \otimes M_{z\mu}^*$. We identify the vector space $M_{z\mu}^*$ with $\mathbb{C}[x_\alpha]$ and rescale the generators by setting $y_\alpha = z^{ht(\alpha)} x_\alpha$, where $ht(\alpha)$ stands for the height of a root α. The formulas for the action of the Lie algebra $\mathfrak{g}$ on $M_{z\mu}^* = \mathbb{C}[y_\alpha]$ now look as follows:

$$e_\alpha = z^{ht(\alpha)} \frac{\partial}{\partial y_\alpha} + \sum_{\beta > \alpha} z^{ht(\alpha)} P_\beta^\alpha \frac{\partial}{\partial y_\beta},$$

$$h = z\mu(h) - \sum_{\beta \in \Delta_+} \beta(h) y_\beta \frac{\partial}{\partial y_\beta},$$

$$e_{-\alpha_i} = \mu(h_{\alpha_i}) y_{\alpha_i} + z^{-1} \sum_{\beta \in \Delta_+} Q_\beta^{\alpha_i} \frac{\partial}{\partial y_\beta}.$$

Thus we can assume that the basis of $V_{(\lambda)} \otimes M_{z\mu}^* = V_{(\lambda)} \otimes \mathbb{C}[y_\alpha]$ is independent of z and that the operators in $U(\mathfrak{g})^{\otimes n} \otimes U(\mathfrak{g})$ do depend on z. Now the subspace $[V_{(\lambda)} \otimes M_{z\mu}^*]^{\mathrm{sing}} \subset V_{(\lambda)} \otimes \mathbb{C}[y_\alpha]$ of singular vectors depends on z as well. Furthermore, the space $V_{(\lambda)} \otimes M_{z\mu}^* = V_{(\lambda)} \otimes \mathbb{C}[y_\alpha]$ is graded by the weights of the diagonal action of $\mathfrak{g}$, where the homogeneous components are independent of z and have finite dimensions. The subspace $[V_{(\lambda)} \otimes M_{z\mu}^*]^{\mathrm{sing}} \subset V_{(\lambda)} \otimes \mathbb{C}[y_\alpha]$ is contained in a finite sum of homogeneous components, and hence the limit $\lim_{z\to\infty} [V_{(\lambda)} \otimes M_{z\mu}^*]^{\mathrm{sing}} \subset V_{(\lambda)} \otimes \mathbb{C}[y_\alpha]$ is well defined. Moreover, the limit of the image of $\mathscr{A}(z_1, \dots, z_n, z)$ in $\mathrm{End}([V_{(\lambda)} \otimes M_{z\mu}^*]^{\mathrm{sing}})$ as $z \to \infty$ is a commutative subalgebra in $\mathrm{End}(\lim_{z\to\infty} [V_{(\lambda)} \otimes M_{z\mu}^*]^{\mathrm{sing}})$.

Theorem 4. *As $z \to \infty$,*

1. *The limit of $[V_{(\lambda)} \otimes M_{z\mu}^*]^{\mathrm{sing}} \subset V_{(\lambda)} \otimes \mathbb{C}[y_\alpha]$ is $V_{(\lambda)} \otimes 1$.*

(2) *The limit of the image of* $\mathscr{A}(z_1,\dots,z_n,z)$ *in* $\operatorname{End}([V_{(\lambda)}\otimes M^*_{z\mu}]^{\text{sing}})$ *contains the image of* $\mathscr{A}_\mu(z_1,\dots,z_n)$ *in* $\operatorname{End}(V_{(\lambda)}\otimes 1)=\operatorname{End}(V_{(\lambda)})$.

Proof. Let us prove the first assertion. The subspace $[V_{(\lambda)}\otimes M^*_{z\mu}]^{\text{sing}}\subset V_{(\lambda)}\otimes M^*_{z\mu}$ is the intersection of kernels of the operators $\operatorname{diag}_{n+1}(e_\alpha)=\sum_{i=1}^{n+1}e_\alpha^{(i)}$, $\alpha\in\Delta_+$. Clearly,

$$\lim_{z\to\infty} z^{-ht(\alpha)}\operatorname{diag}_{n+1}(e_\alpha)=1^{\otimes n}\otimes\left(\frac{\partial}{\partial y_\alpha}+\sum_{\beta>\alpha}P^\alpha_\beta\frac{\partial}{\partial y_\beta}\right).$$

Hence

$$\lim_{z\to\infty}[V_{(\lambda)}\otimes M^*_{z\mu}]^{\text{sing}}\subset\bigcap_{\alpha\in\Delta_+}\operatorname{Ker}1^{\otimes n}\otimes\left(\frac{\partial}{\partial y_\alpha}+\sum_{\beta>\alpha}P^\alpha_\beta\frac{\partial}{\partial y_\beta}\right)=V_{(\lambda)}\otimes 1.$$

Since $\dim[V_{(\lambda)}\otimes M^*_{z\mu}]^{\text{sing}}\geqslant\dim V_{(\lambda)}$, we conclude that $\lim_{z\to\infty}[V_{(\lambda)}\otimes M^*_{z\mu}]^{\text{sing}}=V_{(\lambda)}\otimes 1$.

Now let us prove the second assertion. The module $V_{(\lambda)}\otimes M^*_{z\mu}=V_{(\lambda)}\otimes\mathbb{C}[y_\alpha]$ can be regarded as $U(\mathfrak{g})^{\otimes n}\otimes U(\mathfrak{g})_{z^{-1}}$-module with highest weight $(\lambda_1,\dots,\lambda_n,\mu)$. Using the formulas for the action of the Lie algebra $\mathfrak{g}$ on $\mathbb{C}[y_\alpha]$, we see that

$$\lim_{z\to\infty}1\otimes\dots\otimes 1\otimes e_{-\alpha_i}=\lim_{z\to\infty}z^{-1}1\otimes\dots\otimes 1\otimes\psi_{z^{-1}}(e_{-\alpha_i})=0$$

for any simple root $\alpha_i\in\Delta_+$.

Therefore, the subspace $V_{(\lambda)}\otimes 1\subset V_{(\lambda)}\otimes\mathbb{C}[y_\alpha]$ is invariant under the action of $\lim_{z\to\infty}U(\mathfrak{g})^{\otimes n}\otimes U(\mathfrak{g})_{z^{-1}}=U(\mathfrak{g})^{\otimes n}\otimes S(\mathfrak{g})$. Moreover, the algebra $1\otimes\dots\otimes 1\otimes S(\mathfrak{g})$ acts on this space via the character μ. By Theorem 3,

$$\lim_{z\to\infty}(\mathrm{id}^{\otimes n}\otimes\psi_{z^{-1}})(\mathscr{A}(z_1,\dots,z_n,z))=\mathscr{A}(z_1,\dots,z_n,\infty)\subset U(\mathfrak{g})^{\otimes n}\otimes S(\mathfrak{g}).$$

This means that the limit of the image of $\mathscr{A}(z_1,\dots,z_n,z)$ in $\operatorname{End}([V_{(\lambda)}\otimes M^*_{z\mu}]^{\text{sing}})$ contains the image of the algebra $(\mathrm{id}\otimes\mu)(\mathscr{A}(z_1,\dots,z_n,\infty))=\mathscr{A}_\mu(z_1,\dots,z_n)$ in $\operatorname{End}(V_{(\lambda)}\otimes 1)$. □

7. The Case of sl_r

In this section, we set $\mathfrak{g}=sl_r$.

Lemma 4. *For* $\mathfrak{g}=sl_r$ *and* $\mu(t)=E_{11}+tE_{22}+\dots+t^{n-1}E_{nn}$, *the limit subalgebra* $\lim_{t\to 0}\mathscr{A}_{\mu(t)}$ *is the Gelfand–Tsetlin subalgebra in* $U(sl_r)$.

Proof. It follows from Shuvalov's results (Fact 3) that the associated graded algebra $\lim_{t\to 0}A_{\mu(t)}\subset S(\mathfrak{g})$ is the Gelfand–Tsetlin subalgebra in $S(\mathfrak{g})$. Indeed, in this case $\mathfrak{z}_k$ is the Lie algebra $sl_{r-k-1}\oplus\mathbb{C}^{k+1}$, consisting of all matrices $A\in sl_r$ satisfying

$$A_{ij}=A_{ji}=0,\qquad i=1,\dots,k+1,\ j=1,\dots,r,\ i\neq j.$$

The subalgebra of $S(sl_r)$ generated by $S(\mathfrak{z}_k)^{\mathfrak{z}_k}$ for all k is the Gelfand–Tsetlin subalgebra.

For any μ, the generators of $\mathscr{A}_\mu$ are the images of those of A_μ under the symmetrization map (Fact 5). Therefore, the generators of $\lim_{t\to 0}\mathscr{A}_{\mu(t)}\subset U(\mathfrak{g})$ are the images of the generators of $\lim_{t\to 0}A_{\mu(t)}\subset S(\mathfrak{g})$ under the symmetrization map as well.

The uniqueness of the lift (Fact 5) implies that $\lim_{t\to 0}\mathscr{A}_{\mu(t)}$ is the subalgebra in $U(sl_r)$ generated by all elements of $ZU(\mathfrak{z}_k)$ for all k, i.e., the Gelfand–Tsetlin subalgebra in $U(sl_r)$. □

Theorem 5. *For any finite sequence* (λ) *of dominant integer weight, the algebra* $\mathscr{A}_\mu(z_1,\dots,z_n)$ *with generic* μ *and* $z_1,\dots,z_n$ *has simple spectrum in* $V_{(\lambda)}$.

Proof. 1. The Gelfand–Tsetlin subalgebra in $U(sl_r)$ has simple spectrum in V_λ for any λ; this is a well-known classical result.

2. Since the Gelfand–Tsetlin subalgebra is a limit of $\mathscr{A}_\mu$, it follows that for generic μ the algebra $\mathscr{A}_\mu$ has simple spectrum in V_λ as well.

3. This means that for generic μ the subalgebra $\mathscr{A}_\mu(z_1)^{(1)}\otimes\dots\otimes\mathscr{A}_\mu(z_n)^{(n)}$ has simple spectrum in $V_{(\lambda)}$. Since the subalgebra $\mathscr{A}_\mu(z_1)^{(1)}\otimes\dots\otimes\mathscr{A}_\mu(z_n)^{(n)}$ belongs to the closure of the family of

subalgebras $\mathscr{A}_\mu(z_1, \dots, z_n)$, it follows that for generic μ and z_i the algebra $\mathscr{A}_\mu(z_1, \dots, z_n)$ has simple spectrum in $V_{(\lambda)}$ as well. □

Corollary 5. *There exists a subset $W \subset \Lambda_+ \times \dots \times \Lambda_+$, which is Zariski dense in $\mathfrak{h}^*$ (where Λ_+ is the set of integral dominant weights), such that for any $(\lambda) = (\lambda_1, \dots \lambda_n) \in W$ the Gaudin subalgebra $\mathscr{A}(z_1, \dots, z_n)$ with generic $z_1, \dots, z_n$ has simple spectrum in $V_{(\lambda)}^{\mathrm{sing}}$.*

Proof. For given $\lambda_1, \dots \lambda_{n-1}$, the condition of nonsimplicity of the spectrum of $\mathscr{A}(z_1, \dots, z_n)$ in the space $[V_{\lambda_1} \otimes \dots \otimes V_{\lambda_{n-1}} \otimes M_{\lambda_n}^*]^{\mathrm{sing}}$ is an algebraic condition on $\lambda_n \in \mathfrak{h}^*$ for any $z_1, \dots, z_n$. By Theorems 4 and 5, this condition is not always satisfied. This means that the set of $\lambda_n \in \Lambda_+$ such that the spectrum of the algebra $\mathscr{A}(z_1, \dots, z_n)$ in $[V_{\lambda_1} \otimes \dots \otimes V_{\lambda_{n-1}} \otimes M_{\lambda_n}^*]^{\mathrm{sing}}$ is simple for generic $z_1, \dots, z_n$ is Zariski dense in $\mathfrak{h}^*$ for any finite sequence $\lambda_1, \dots \lambda_{n-1}$. Since $V_{\lambda_n} \subset M_{\lambda_n}^*$, it follows that the spectrum of the algebra $\mathscr{A}(z_1, \dots, z_n)$ in the space $V_{(\lambda)}^{\mathrm{sing}}$ is simple for any of these finite sequences $\lambda_1, \dots, \lambda_n$. □

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