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High-level dependence in time series models

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Abstract We present several notions of high-level dependence for stochastic processes, which have appeared in the literature. We calculate such measures for discrete and continuous-time models, where we concentrate on time series with heavy-tailed marginals, where extremes are likely to occur in clusters. Such models include linear models and solutions to random recurrence equations; in particular, discrete and continuous-time moving average and (G)ARCH processes. To illustrate our results we present a small simulation study.

Keywords ARCH · COGARCH · Extreme cluster · Extreme dependence measure · Extremal index · Extreme value theory · GARCH · Linear model · Multivariate regular variation · Nonlinear model · Lévy-driven Ornstein–Uhlenbeck process · Random recurrence equation

AMS 2000 Subject Classifications 60G70 · 62G32 · 62M10

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1 Introduction

Throughout we assume that $(X_t)_{t\in I}$ is a strictly stationary stochastic process and its extremes satisfy some analytic regularity condition. The index set I can be discrete or continuous and w.l.o.g. we assume that $0 \in I$. We shall present analytic results on the dependence structure of high level extremes in various time series models. Our motivation is two-fold. Firstly, we review certain known spatial dependence measures in the framework of time series models in discrete and continuous time. We also adapt the notion of extremal index for discrete time models to some useful version for continuous time models. Secondly, we present explicit results for the most prominent time series models, which also prepares the ground for statistical estimation.

Starting point of our investigation is the extremal coefficient, which simply measures for two random variables the probability for joint extremes. We extend this function in the same way as autocovariance and autocorrelation functions extend the covariance and the correlation between two random variables. Concentrating on dependence in the extremes it can be viewed as an analog of a covariance function, but on extreme observations.

Various time series models feature strong clustering in the extremes, a phenomenon, which is captured by the extremal index for time series models in discrete time. This also applies to continuous-time models by introducing discrete time grids on the positive time axes. Since the inverse of the extremal index serves as a measure for the mean cluster size, it is only a crude measure of dependence in extremes. In various models it is possible to calculate also the cluster size distribution. However, the analytic expressions of the cluster size distributions are complicated, have basically no interpretation, and their statistical estimation is mostly not possible. Consequently, we concentrate on the extremal coefficient function (including some multivariate versions) and the extremal index function.

Previous results to describe the extremal behaviour of time series by extreme dependence measures like in Definition 1.1 below have been obtained by Ledford and Tawn (2003) and Gomes et al. (2004), and we extend parts of their results. There also exists a large statistics literature to assess the extremal behavior of multivariate vectors or time series; see e.g. Ledford and Tawn (2003) and Ramos and Ledford (2008) and references therein.

There also exists a vast literature on the extremal index of Definition 1.4 and its estimation. A pathbreaking paper for random recurrence models has been De Haan et al. (1989). We refer in particular to Laurini and Tawn (2003) for recent results on the extremal index and further references, in particular, with respect to statistical estimation.

Independently of our work Davis and Mikosch (2008) introduced what they call the extremogram of multivariate time series in discrete time, which in special cases coincides with our extremal coefficient function defined in Eq. 1.4 below. Their emphasis is, however, on nonparametric estimation of the extremogram. Segers (2007), in his recent paper, on the other hand, studies similar measures for extremes of regularly varying Markov chains and some of our models fall into this framework.

As appropriate regularity condition we require that all finite dimensional distributions of $(X_t)_{t\in I}$ belong to the maximum domain of attraction of some extreme value distribution. In particular, the one-dimensional stationary distribution (represented by X_0) has right endpoint $x_R \le \infty$ and belongs to the maximum domain of attraction of some extreme value distribution G ($X_0 \in \text{MDA}(G)$); i. e. there exist norming sequences $a_n > 0$, $b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} n\mathbb{P}(X_0 > a_n x + b_n) = -\log G(x), \quad x \in \mathbb{R}, \tag{1.1}$$

for some non-degenerate distribution function G, where we define $-\log 0 := \infty$. By the Fisher–Tippett Theorem G has to be an extreme value distribution; i. e. G is either a Frechét, a Weibull or a Gumbel distribution. If we do not specify G, we shall simply write MDA. If all finite dimensional distributions of the strictly stationary stochastic process $(X_t)_{t\in I}$ belong to some maximum domain of attraction, we write $(X_t)_{t\in I} \in \text{MDA}$. Our conditions may not be the most general ones, but under a domain of attraction condition all limits below exist.

We shall formulate extreme dependence measures and related notions for lagged vectors of $(X_t)_{t\in I}$ of arbitrary dimension. For ease of notation, we denote for arbitrary $d \in \mathbb{N}$ by

$$\mathbf{X}_d := (X_{t_1}, \dots, X_{t_d}), \quad \mathrm{t}_1 < \cdots < t_d \text{ in } I,$$

a *generic lagged vector* of $(X_t)_{t\in I}$. Analogous notation will be used for $\overline{\mathbf{X}}_d := (|X_{t_1}|, \dots, |X_{t_d}|)$ and $\mathbf{X}_d^2 := (X_{t_1}^2, \dots, X_{t_d}^2)$.

Definition 1.1 (Extreme dependence measures) Let $(X_t)_{t\in I} \in \text{MDA}$ and $a_n > 0, b_n \in \mathbb{R}$ satisfy Eq. 1.1.

(a) We define the *extreme dependence functions of* $(X_t)_{t\in I}$ for any lagged vector $\mathbf{X}_d$ of $(X_t)_{t\in I}$ and for all $(x_1, \dots, x_d) \in \mathbb{R}^d$ by

$$\underline{\chi}_{(t_1,\dots,t_d)}(x_1, \dots, x_d) := \lim_{n\to\infty} n\mathbb{P}(X_{t_1} > a_n x_1 + b_n, \dots, X_{t_d} > a_n x_d + b_n), \tag{1.2}$$

$$\overline{\chi}_{(t_1,\dots,t_d)}(x_1, \dots, x_d) := \lim_{n\to\infty} n\mathbb{P}(X_{t_1} > a_n x_1 + b_n \text{ or } \dots \text{ or } X_{t_d} > a_n x_d + b_n). \tag{1.3}$$

(b) We also define the *extremal coefficient function* of $(X_t)_{t\in I}$ by

$$\chi(t) := \lim_{x\uparrow x_R} \mathbb{P}(X_t > x \mid X_0 > x) \quad \text{for } t \in I. \tag{1.4}$$

Remark 1.2

(i) The extremal coefficient function satisfies for every $t \in I$

$$\chi(t) = \lim_{n\to\infty} \frac{\mathbb{P}(X_0 > a_n y + b_n\,,\; X_t > a_n y + b_n)}{\mathbb{P}(X_0 > a_n y + b_n)} \tag{1.5}$$

$$= \frac{\underline{\chi}_{(0,t)}(y, y)}{\underline{\chi}_{(0)}(y)} = 2 - \frac{\overline{\chi}_{(0,t)}(y, y)}{\overline{\chi}_{(0)}(y)} \quad \text{for any } y \in \mathbb{R}\,.$$

The right-hand side quotients are indeed independent of y. In the Fréchet case this is a consequence of Lemma 2.4 below.

(ii) Note that the limit relation (Eq. 1.5) implies also

$$\chi(t) = \lim_{x\to\infty} \frac{\mathrm{Cov}(I_{\{X_0>x\}}, I_{\{X_t>x\}})}{\mathbb{P}(X_0 > x)},$$

which presents $\chi(\cdot)$ as covariance function for extremes; see Davis and Mikosch (2008) for more general results in the context of multivariate time series.

(iii) The extremal coefficient function $\theta(\cdot)$ of Schlather and Tawn (2003) is slightly different. They define $\theta(t) = 2 - \chi(t)$ for $t \in I$, and their multivariate extremal coefficient, defined for some index set $A \subset I$ is in our notation

$$\theta_A = \frac{\overline{\chi}_A(y, \ldots, y)}{\overline{\chi}_{(0)}(y)} \quad \text{for any } y \in \mathbb{R},$$

where again the right hand side is independent of y.

The following lemma shows that the definition of the extreme dependence functions are invariant under affine transformations.

Lemma 1.3 *Let* $(X_t)_{t\in I} \in \mathrm{MDA}$ *and* $a_n > 0$, $b_n \in \mathbb{R}$ *satisfy* Eq. 1.1. *Let* $\widetilde{a}_n > 0$, $\widetilde{b}_n \in \mathbb{R}$ *be constants such that for some* $a > 0$ *and* $b \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{\widetilde{a}_n}{a_n} = a \quad \textit{and} \quad \lim_{n\to\infty} \frac{\widetilde{b}_n - b_n}{a_n} = b\,.$$

Consider a lagged vector $\mathbf{X}_d$ *of* $(X_t)_{t\in I}$. *Denote the extreme dependence measures* $\underline{\chi}$ *and* $\overline{\chi}$ *as in Eqs.* 1.2 *and* 1.3, *and define* $\underline{\widetilde{\chi}}$, $\widetilde{\overline{\chi}}$ *in the same way for the constants* $\widetilde{a}_n > 0$, $\widetilde{b}_n \in \mathbb{R}$, *respectively. Then for all* $(x_1, \ldots, x_d) \in \mathbb{R}^d$ *we have*

$$\underline{\widetilde{\chi}}_{(t_1,\ldots,t_d)}(x_1, \ldots, x_d) = \underline{\chi}_{(t_1,\ldots,t_d)}(ax_1 + b, \ldots, ax_d + b),$$
$$\widetilde{\overline{\chi}}_{(t_1,\ldots,t_d)}(x_1, \ldots, x_d) = \overline{\chi}_{(t_1,\ldots,t_d)}(ax_1 + b, \ldots, ax_d + b).$$

Proof Let $\mathbf{F} \in \text{MDA}(\mathbf{G})$ be the distribution function of $\mathbf{X}_d$. Furthermore, let $\mathbf{x} = (x_1, \ldots, x_d)$, $\mathbf{b} = (b, \ldots, b)$, $\widetilde{\mathbf{b}}_n = (\widetilde{b}_n, \ldots, \widetilde{b}_n) \in \mathbb{R}^d$. Then Slutzky's theorem applies and we obtain

$$\begin{aligned}\widetilde{\overline{\chi}}_{(t_1,\ldots,t_d)}(x_1, \ldots, x_d) &= \lim_{n\to\infty} n(1 - \mathbf{F}(\widetilde{a}_n\mathbf{x} + \widetilde{\mathbf{b}}_n)) \\ &= \lim_{n\to\infty} -\log \mathbf{F}^n(\widetilde{a}_n\mathbf{x} + \widetilde{\mathbf{b}}_n) \\ &= -\log \mathbf{G}(a\mathbf{x} + \mathbf{b}) \\ &= \overline{\chi}_{(t_1,\ldots,t_d)}(ax_1 + b, \ldots, ax_d + b), \end{aligned} \tag{1.6}$$

which proves the second equality. The first equality can be obtained by a classical Bonferroni argument and Eq. 1.6. For simplicity we only give the proof for $d = 2$

$$\begin{aligned}\widetilde{\underline{\chi}}_{(t_1,t_2)}(x_1, x_2) &= \lim_{n\to\infty} n\mathbb{P}(X_{t_1} > \widetilde{a}_n x_1 + \widetilde{b}_n, X_{t_2} > \widetilde{a}_n x_2 + \widetilde{b}_n) \\ &= \lim_{n\to\infty} n\mathbb{P}(X_{t_1} > \widetilde{a}_n x_1 + \widetilde{b}_n) + \lim_{n\to\infty} n\mathbb{P}(X_{t_2} > \widetilde{a}_n x_2 + \widetilde{b}_n) \\ &\quad - \lim_{n\to\infty} n\mathbb{P}(X_{t_1} > \widetilde{a}_n x_1 + \widetilde{b}_n \text{ or } X_{t_2} > \widetilde{a}_n x_2 + \widetilde{b}_n) \\ &= -\log G(ax_1 + b) - \log G(ax_2 + b) - \overline{\chi}_{(t_1,t_2)}(ax_1 + b, ax_2 + b) \\ &= \underline{\chi}_{(t_1,t_2)}(ax_1 + b, ax_2 + b).\end{aligned}$$

□

For discrete-time processes the cluster behavior in extremes is often measured by the extremal index; cf. Embrechts et al. (1997), Section 8.1, or Leadbetter et al. (1983), p. 67ff. By dividing the positive real line into blocks of length h this definition can be extended to a function of h, which then applies to discrete and continuous-time processes. For fixed h it serves as a measure for the expected cluster sizes among these blocks. This function has been introduced in Fasen (2004, 2005).

Definition 1.4 (Extremal index, extremal index function) Let $(X_t)_{t\in I}$ be strictly stationary satisfying Eq. 1.1 for $a_n > 0$, $b_n \in \mathbb{R}$.

(a) Let $I = \mathbb{N}_0$. If there exists some $\theta \in (0, 1]$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{i=1,\ldots,n} X_i \le a_n x + b_n\right) = G^\theta(x) \quad \text{for } x \in \mathbb{R},$$

then θ is the extremal index of $(X_t)_{t\in I}$.

(b) Let $I = \mathbb{N}_0$ or $I = [0, \infty)$. For $h > 0$ in I define the sequence

$$M_k(h) := \sup_{(k-1)h \le t \le kh} X_t \quad \text{for } k \in \mathbb{N}.$$

Let $\theta(h)$ be the extremal index of the sequence $(M_k(h))_{k\in\mathbb{N}}$. Then we call the function $\theta : (0, \infty) \to (0, 1]$ extremal index function.

Extreme value analysis is most interesting, when extremes happen in a pronounced way. This is the case in many areas of applications as in insurance and finance, telecommunication and other areas of technical risk. Consequently, models with heavy-tailed marginal distributions are of high interest. Distributions with regularly varying or subexponential tails are most natural in these areas. Heavy tails in linear models originate simply because of a heavy-tailed noise. In nonlinear models, however, regularly varying tails occur, even though the noise sequence may be light-tailed. This applies in particular to solutions of random recurrence equations.

Our paper is organized as follows. In Section 2 we present the extreme dependence measures for discrete time series models with regularly varying marginal distributions and, in particular, for solutions to random recurrence equations. The basic results from Section 2 are used in Section 3 to derive the extreme dependence measures, first, for discrete-time linear models followed by non-linear models as ARCH and GARCH models. We present results for regularly varying linear models, but also for subexponential ones outside of regular variation, which are definitely different in their extreme behaviour. For an illustration of our results we choose an ARCH(1) model, and show the performance of some estimates of the extremal coefficient function in a small simulation study. In Section 4 the results for continuous-time analogues of linear models (with the Lévy-driven Ornstein–Uhlenbeck process as prominent example) as well as for non-linear models like the continuous-time GARCH (COGARCH) process are presented.

Throughout we shall use the following notation. For $a \in \mathbb{R}$ we define $a^+ = \max(0, a)$ and $a^- = \max(0, -a)$. The relation $a(x) \sim b(x)$ as $x \to x_R$ means that the quotient of the left hand side and the right hand side tends to 1 as $x \to x_R$. By $\stackrel{w}{\Longrightarrow}$ we denote weak convergence and by $\stackrel{v}{\Longrightarrow}$ vague convergence of measures. We also denote $\mathbb{R}_+ := (0, \infty)$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. We set $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^d$, $-\boldsymbol{\infty} = (-\infty, \ldots, -\infty) \in \overline{\mathbb{R}}^d$, and $\prod_{j=1}^{0} x_j := 1$. We use the maximum norm on $\mathbb{R}^d$ defined as $|\mathbf{x}| = \max_{i=1,\ldots,d} |x_i|$ for $\mathbf{x} \in \mathbb{R}^d$. We furthermore denote by $\mathbb{S}^{d-1} = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$ the unit sphere in $\mathbb{R}^d$ and by $[-\boldsymbol{\infty}, \mathbf{x}]^c = \overline{\mathbb{R}}^d \setminus [-\boldsymbol{\infty}, \mathbf{x}]$ for $\mathbf{x} \in \mathbb{R}^d$. Finally, the Borel σ-algebra is denoted by $\mathcal{B}$.

2 Basic results for heavy-tailed time series models

2.1 Regularly varying models

Many models considered below have marginal distributions in the maximum domain of attraction of the Fréchet distribution, equivalently, their finite dimensional distributions are regularly varying. We present two equivalent definitions of multivariate regular variation, which we shall both use throughout; see e.g. Resnick (1987), Section 5.4 and Resnick (2007), Chapter 6. For supporting explanations we refer to Mikosch (2004), Section 5.4.2. Further properties and results of multivariate regular variation can be found in Basrak et al. (2002a,b), Basrak and Segers (2009) and some classical references in Bingham et al. (1987).

Definition 2.1 (Multivariate regular variation) A vector $\mathbf{Y}$ in $\mathbb{R}^d$ is *regularly varying with index* κ (we write $\mathbf{Y} \in \mathcal{R}(\kappa)$) for $\kappa > 0$, if one of the following equivalent conditions hold:

(a) There exists a random vector $\mathbf{\Theta}$ with values on the unit sphere $\mathbb{S}^{d-1}$ such that for every $x > 0$

$$\frac{\mathbb{P}(|\mathbf{Y}| > ux, \mathbf{Y}/|\mathbf{Y}| \in \cdot)}{\mathbb{P}(|\mathbf{Y}| > u)} \stackrel{w}{\Longrightarrow} x^{-\kappa}\mathbb{P}(\mathbf{\Theta} \in \cdot) \quad \text{on } \mathcal{B}(\mathbb{S}^{d-1}) \text{ as } u \to \infty. \tag{2.1}$$

The distribution of $\mathbf{\Theta}$ is referred to as the *spectral measure* of $\mathbf{Y}$.

(b) There exists a Radon measure $\rho(\cdot)$ on $\overline{\mathbb{R}}^d\backslash\{\mathbf{0}\}$ with $\rho(E) > 0$ for at least one relatively compact set $E \subseteq \overline{\mathbb{R}}^d\backslash\{\mathbf{0}\}$ and a sequence $a_n \uparrow \infty$ of positive constants such that

$$n\mathbb{P}(a_n^{-1}\mathbf{Y} \in \cdot) \stackrel{v}{\Longrightarrow} \rho(\cdot) \quad \text{on } \mathcal{B}(\overline{\mathbb{R}}^d\backslash\{\mathbf{0}\}) \text{ as } n \to \infty. \tag{2.2}$$

The measure ρ satisfies the homogeneity property $\rho(tA) = t^{-\kappa}\rho(A)$ for all $A \in \mathcal{B}(\overline{\mathbb{R}}^d\backslash\{\mathbf{0}\})$ and $t > 0$.

For ease of notation we introduce the following notation, which is equivalent to saying that $(X_t)_{t\in I} \in$ MDA with Fréchet limit distributions.

Definition 2.2 (Regularly varying stochastic process) Let $(X_t)_{t\in I}$ be a strictly stationary process. If all finite-dimensional distributions of $(X_t)_{t\in I}$ belong to $\mathcal{R}(\kappa)$, we say that $(X_t)_{t\in I}$ is regularly varying with index κ and write $(X_t)_{t\in I} \in \mathcal{R}(\kappa)$.

Remark 2.3

(i) Regular variation of stochastic processes has been defined in Hult and Lindskog (2005). Our definition concerns only the finite-dimensional distributions. Consequently, it is for continuous-time processes weaker than the definition presented in Hult and Lindskog (2005), which requires an additional tightness condition.

(ii) The equivalence of Definitions (a) and (b) above is based on the following transformation to polar coordinates. The map $T : \mathbb{R}^d\backslash\{\mathbf{0}\} \to (0,\infty) \times \mathbb{S}^{d-1}$ defined by $T(\mathbf{x}) = (|\mathbf{x}|, \mathbf{x}/|\mathbf{x}|)$ is a continuous bijection. Furthermore, let $\vartheta_\kappa(dy) = \kappa y^{-\kappa-1}\,dy$ be a measure on $(0,\infty)$. Then with $\mathbb{P}_{\mathbf{\Theta}}$ denoting the distribution of $\mathbf{\Theta}$,

$$\rho \circ T^{-1} = c\vartheta_\kappa \times \mathbb{P}_{\mathbf{\Theta}} \quad \text{on } (0,\infty) \times \mathbb{S}^{d-1},$$

where $c=\lim_{n\to\infty} n\mathbb{P}(|\mathbf{Y}|>a_n)$, see e.g. Resnick (2007), Theorem 6.1. In particular, for all $A\in\mathcal{B}(\mathbb{R}^d\backslash\{\mathbf{0}\})$,

$$\rho(A)=(c\vartheta_\kappa\times\mathbb{P}_{\mathbf{\Theta}})(T(A))=c\int_{\mathbb{S}^{d-1}}\int_0^\infty \mathbf{1}_{\{r\omega\in A\}}\,\kappa r^{-\kappa-1}dr\,\mathbb{P}(\mathbf{\Theta}\in d\omega). \tag{2.3}$$

(iii) On $\mathcal{B}(\mathbb{R}^d\backslash\mathbb{S}^{d-1})$ the right hand side of Eq. 2.3 can be interpreted as the distribution of $Y(\Theta_1,\ldots,\Theta_d)$ times c, where Y and the sequence (Θ_t) are independent, $\mathbb{P}(Y>y)=y^{-\kappa}$ for $y>1$ and $(\Theta_1,\ldots,\Theta_d)$ has distribution $\mathbb{P}(\mathbf{\Theta}\in\cdot)$; see Basrak and Segers (2009) and Segers (2007) for more general results.

Next, we obtain that for regularly varying models the extremal dependence measures are homogeneous.

Lemma 2.4 *Let $(X_t)_{t\in I}\in\mathcal{R}(\kappa)$ for some $\kappa>0$. Assume the tail balance condition $\mathbb{P}(X_0>x)\sim p\mathbb{P}(|X_0|>x)$ as $x\to\infty$ holds for some $p\in(0,1]$, and let $a_n>0$ satisfy*

$$\lim_{n\to\infty} n\mathbb{P}(X_0>a_n)=1.$$

Let $\mathbf{X}_d$ be a lagged vector of $(X_t)_{t\in I}$. Then for all $(x_1,\ldots,x_d)\in\mathbb{R}_+^d$ and $a>0$ the following homogeneity properties hold:

$$\underline{\chi}_{(t_1,\ldots,t_d)}(ax_1,\ldots,ax_d)=a^{-\kappa}\underline{\chi}_{(t_1,\ldots,t_d)}(x_1,\ldots,x_d),$$
$$\overline{\chi}_{(t_1,\ldots,t_d)}(ax_1,\ldots,ax_d)=a^{-\kappa}\overline{\chi}_{(t_1,\ldots,t_d)}(x_1,\ldots,x_d).$$

Proof Let ρ be given as in Definition 2.1 (b). Then

$$\overline{\chi}_{(t_1,\ldots,t_d)}(ax_1,\ldots,ax_d)=\lim_{n\to\infty} n\mathbb{P}(a_n^{-1}\mathbf{X}_d\in[-\infty,a\mathbf{x}]^c)=\rho([-\infty,a\mathbf{x}]^c),$$
$$\underline{\chi}_{(t_1,\ldots,t_d)}(ax_1,\ldots,ax_d)=\lim_{n\to\infty} n\mathbb{P}(a_n^{-1}\mathbf{X}_d\in(a\mathbf{x},\infty])=\rho((a\mathbf{x},\infty]).$$

The result follows then from the homogeneity of ρ. □

The following result presents the extreme dependence measures for a strictly stationary stochastic process with regularly varying finite dimensional distributions. The condition on the spectral measure is very natural; it holds, for instance, for every example of this paper. It has also been shown to hold for some discrete time Markov chains in Segers (2007), who also calculated $\overline{\chi}$ in his Corollary 6.3.

Theorem 2.5 *Let $(X_t)_{t\in I}\in\mathcal{R}(\kappa)$ for some $\kappa>0$ and assume that the tail balance condition $\mathbb{P}(X_0>x)\sim p\mathbb{P}(|X_0|>x)$ as $x\to\infty$ holds for some $p\in(0,1]$.*

Suppose there exists a stochastic process $(W_t)_{t\in I}$ *such that the spectral measure of a lagged vector* $\mathbf{X}_d$ *of* $(X_t)_{t\in I}$ *has the representation*

$$\mathbb{P}(\mathbf{\Theta}_d \in \cdot) = \frac{\mathbb{E}\left(|\mathbf{W}_d|^\kappa\, \mathbf{1}_{\{\mathbf{W}_d/|\mathbf{W}_d|\in\cdot\}}\right)}{\mathbb{E}|\mathbf{W}_d|^\kappa} \quad \text{on } \mathcal{B}(\mathbb{S}^{d-1}), \tag{2.4}$$

where $\mathbf{W}_d = (W_{t_1}, \ldots, W_{t_d})$*. Furthermore, assume that* $a_n > 0$ *are such that*

$$\lim_{n\to\infty} n\mathbb{P}(X_0 > a_n) = 1.$$

Then for all $(x_1, \ldots, x_d) \in \mathbb{R}_+^d$*,*

$$\underline{\chi}_{(t_1,\ldots,t_d)}(x_1, \ldots, x_d) = \frac{\mathbb{E}\left(\min_{i=1,\ldots,d}\{x_i^{-1} W_{t_i}^+\}^\kappa\right)}{\mathbb{E}(W_0^+)^\kappa},$$

$$\overline{\chi}_{(t_1,\ldots,t_d)}(x_1, \ldots, x_d) = \frac{\mathbb{E}\left(\max_{i=1,\ldots,d}\{x_i^{-1} W_{t_i}^+\}^\kappa\right)}{\mathbb{E}(W_0^+)^\kappa},$$

where $W_t^+ = \max(W_t, 0)$*. Furthermore,*

$$\chi(t) = \frac{\mathbb{E}(\min\{W_0^+, W_t^+\}^\kappa)}{\mathbb{E}(W_0^+)^\kappa} \quad \text{for } t \in I.$$

Proof Note that by strict stationarity of $(X_t)_{t\in I}$ we have $\mathbb{E}(W_t^+)^\kappa = \mathbb{E}(W_0^+)^\kappa$ for all $t \in I$. Furthermore,

$$\mathbb{P}(X_0 > x) \sim \frac{\mathbb{E}(W_0^+)^\kappa}{\mathbb{E}|W_0|^\kappa}\mathbb{P}(|X_0| > x) \quad \text{as } x \to \infty,$$

such that $\mathbb{E}(W_0^+)^\kappa > 0$ by the tail balance condition. First, we calculate c of Remark 2.3(ii):

$$c = \lim_{n\to\infty} n\mathbb{P}(|\mathbf{X}_d| > a_n) = \lim_{u\to\infty} \frac{\mathbb{P}(|\mathbf{X}_d| > u)}{\mathbb{P}(X_0 > u)} = \frac{\mathbb{E}|\mathbf{W}_d|^\kappa}{\mathbb{E}(W_0^+)^\kappa}.$$

Then Remark 2.3(ii) results in

$$\begin{aligned}\rho(A) &= \frac{\mathbb{E}|\mathbf{W}_d|^\kappa}{\mathbb{E}(W_0^+)^\kappa} \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}_{\{r\omega\in A\}}\, \kappa r^{-\kappa-1}\, dr\, \mathbb{P}(\mathbf{\Theta}_d \in d\omega) \\ &= \frac{1}{\mathbb{E}(W_0^+)^\kappa} \int_0^\infty \mathbb{E}(|\mathbf{W}_d|^\kappa\, \mathbf{1}_{\{r\mathbf{W}_d/|\mathbf{W}_d|\in A\}})\kappa r^{-\kappa-1}\, dr, \quad A \in \mathcal{B}(\mathbb{R}^d\backslash\{\mathbf{0}\}),\end{aligned}$$

by Fubini's theorem together with assumption 2.4. By Definition 2.1 (b), using Fubini's theorem again we obtain

$$\begin{aligned}
\overline{\chi}_{(t_1,\ldots,t_d)}(x_1,\ldots,x_d) &= \rho(\mathbb{R}^d\setminus(-\infty,\mathbf{x}]) \\
&= \frac{1}{\mathbb{E}(W_0^+)^\kappa}\mathbb{E}\left(|\mathbf{W}_d|^\kappa\int_0^\infty \mathbf{1}_{\{r\mathbf{W}_d/|\mathbf{W}_d|\in\mathbb{R}^d\setminus(-\infty,\mathbf{x}]\}}\,\kappa r^{-\kappa-1}\,dr\right) \\
&= \frac{1}{\mathbb{E}(W_0^+)^\kappa}\mathbb{E}\left(|\mathbf{W}_d|^\kappa\int_{\{r>\min_{i=1,\ldots,d}(|\mathbf{W}_d|/(x_i^{-1}W_{t_i}^+))\}}\kappa r^{-\kappa-1}\,dr\right) \\
&= \frac{1}{\mathbb{E}(W_0^+)^\kappa}\mathbb{E}\left(\max_{i=1,\ldots,d}\{x_i^{-1}W_{t_i}^+\}^\kappa\right).
\end{aligned}$$

In the same way we obtain $\underline{\chi}_{(t_1,\ldots,t_d)}$, and Remark 1.2(i) yields the expression for χ. □

For several examples below we shall show that extreme dependence decreases (when measured by $\chi(\cdot)$), when κ increases and even that for $\kappa\uparrow\infty$ extremal dependence disappears completely. Since one of our goals is the comparison of extremal dependence for linear and non-linear models, this makes it clear that we should compare the extremal dependence in models with the same heavy-tailedness.

2.2 Random recurrence equations

Important examples of time series with multivariate regularly varying marginal distributions are solutions to multivariate random recurrence equations of the form

$$\mathbf{Y}_n = \mathbf{A}_n\mathbf{Y}_{n-1} + \mathbf{B}_n \quad \text{for } n\in\mathbb{N}, \tag{2.5}$$

with an i. i. d. sequence $((\mathbf{A}_n,\mathbf{B}_n))_{n\in\mathbb{N}}$ of $d\times d$ matrices $\mathbf{A}_n$ and d-dimensional random vectors $\mathbf{B}_n\neq 0$ a. s.

For $((\mathbf{A}_n,\mathbf{B}_n))_{n\in\mathbb{N}}$ all with non-negative entries Kesten (1973) presents in his Theorems 3 and 4 natural, non-trivial conditions for the existence of a unique strictly stationary solution $(\mathbf{X}_n)_{n\in\mathbb{N}}$ to the stochastic recurrence Eq. 2.5. In the one-dimensional case results can be reformulated as shown in Goldie (1991). Then the spectral measure can be written down explicitly as well as the extreme dependence measures and the extremal index. The multivariate version 2.5 is more involved and we refer to Theorem 3.1 of Basrak et al. (2002a) for a precise formulation. Parts (a) and (b) of the following result are classic, part (c) gives a representation of the spectral measure in terms of Theorem 2.5.

Proposition 2.6 *Let $(X_t)_{t\in\mathbb{N}_0}$ be a stochastic process defined by $X_t = A_t X_{t-1} + B_t$, where $((A_t, B_t))_{t\in\mathbb{N}}$, (A, B) are i. i. d. and independent of X_0. Assume that $\kappa > 0$ and the following conditions are satisfied:*

(i) $\mathbb{E}|A|^\kappa = 1$.
(ii) *The law of* $\log|A|$, *given* $|A| \neq 0$, *is not concentrated on a lattice* $-\infty \cup r\mathbb{Z}$ *for any* $r > 0$ *and* $-\infty \le \mathbb{E}(\log|A|) < 0$.
(iii) $\mathbb{E}(|A|^\kappa \log^+|A|) < \infty$.
(iv) $\mathbb{E}|B|^\kappa < \infty$.

Then the following results hold:

(a) *The equation $X_\infty \overset{d}{=} AX_\infty + B$, where X_∞ is independent of (A, B), has the solution unique in distribution*

$$X_\infty \overset{d}{=} \sum_{m=1}^{\infty} B_m \prod_{k=1}^{m-1} A_k.$$

If we take $X_0 \overset{d}{=} X_\infty$ then $(X_t)_{t\in\mathbb{N}_0}$ is strictly stationary.

(b) *The tail of X_∞ in (a) satisfies*

$$\mathbb{P}(X_\infty > x) \sim C_+ x^{-\kappa} \quad \text{and} \quad \mathbb{P}(X_\infty < -x) \sim C_- x^{-\kappa} \quad \text{as } x \to \infty,$$

where $C_+ + C_- > 0$ if and only if $\mathbb{P}(B = (1 - A)c) < 1$ for every $c \in \mathbb{R}$. If $A \ge 0$ a.s. then

$$C_+ = \frac{\mathbb{E}\left[((AX_\infty + B)^+)^\kappa - ((AX_\infty)^+)^\kappa\right]}{\kappa\mathbb{E}(|A|^\kappa \log|A|)},$$

$$C_- = \frac{\mathbb{E}\left[((AX_\infty + B)^-)^\kappa - ((AX_\infty)^-)^\kappa\right]}{\kappa\mathbb{E}(|A|^\kappa \log|A|)}.$$

(c) *Assume that $\mathbb{P}(B = (1 - A)c) < 1$ for every $c \in \mathbb{R}$. Define a Bernoulli variable R independent of $(A_t)_{t\in\mathbb{N}}$ by*

$$\mathbb{P}(R = 1) = \lim_{x\to\infty} \frac{\mathbb{P}(X_\infty > x)}{\mathbb{P}(|X_\infty| > x)} \quad \text{and} \quad \mathbb{P}(R = -1) = \lim_{x\to\infty} \frac{\mathbb{P}(X_\infty < -x)}{\mathbb{P}(|X_\infty| > x)}. \tag{2.6}$$

Consider a strictly stationary version of $(X_t)_{t\in\mathbb{N}_0}$. Then $(X_t)_{t\in\mathbb{N}_0} \in \mathcal{R}(\kappa)$. The spectral measure of a lagged vector $\mathbf{X}_d$ of $(X_t)_{t\in\mathbb{N}_0}$ is given by

$$\mathbb{P}(\mathbf{\Theta}_d \in \cdot) = \frac{\mathbb{E}\left(\max_{i=1,\dots,d}\left\{\prod_{j=1}^{t_i}|A_j|^\kappa\right\}\mathbf{1}_{\left\{R\left(\prod_{j=1}^{t_1}A_j,\dots,\prod_{j=1}^{t_d}A_j\right)/\left(\max_{i=1,\dots,d}\prod_{j=1}^{t_i}|A_j|\right)\in\cdot\right\}}\right)}{\mathbb{E}\left(\max_{i=1,\dots,d}\prod_{j=1}^{t_i}|A_j|^\kappa\right)}.$$

Proof Parts (a) and (b) are consequences of Goldie (1991), Theorem 4.1. For a proof of part (c) define the random vectors

$$\mathbf{a} = \left(\prod_{j=1}^{t_1} A_j, \ldots, \prod_{j=1}^{t_d} A_j\right) \quad \text{and} \quad \mathbf{b} = \left(\sum_{i=1}^{t_1} B_i \prod_{j=i+1}^{t_1} A_j, \ldots, \sum_{i=1}^{t_d} B_i \prod_{j=i+1}^{t_d} A_j\right).$$

Then $\mathbf{X}_d = \mathbf{a}X_0 + \mathbf{b}$, so that the multivariate regular variation is inherited from the one-dimensional regular variation property of X_0. We prove first Eq. 2.1 for the random vector $\mathbf{a}X_0$ with $\mathbf{\Theta}_d$ as in (c). If we take $\mathbb{E}|\mathbf{a}|^\kappa < \infty$ and $\mathbb{P}(|X_0| > x) \sim (C_+ + C_-)x^{-\kappa}$ as $x \to \infty$, into account, the multivariate version of Breiman's (1965) classical result in Basrak et al. (2002b), Proposition A.1, guarantees the multivariate regular variation of $\mathbf{a}X_0$. The spectral measure was explicitly calculated in Fasen (2005), Lemma 2.1 as

$$\lim_{u\to\infty} \frac{\mathbb{P}(|\mathbf{a}||X_0| > ux, \mathbf{a}X_0/(|\mathbf{a}||X_0|) \in S)}{\mathbb{P}(|\mathbf{a}X_0| > u)} = x^{-\kappa}\frac{\mathbb{E}(|\mathbf{a}|^\kappa \mathbf{1}_{\{R\mathbf{a}/|\mathbf{a}|\in S\}})}{\mathbb{E}|\mathbf{a}|^\kappa} \tag{2.7}$$

for $x > 0$ and $S \in \mathcal{B}(\mathbb{S}^{d-1})$. Finally, since $\mathbb{E}|\mathbf{b}|^\kappa < \infty$, in particular $\lim_{x\to\infty} x^\kappa \mathbb{P}(|\mathbf{b}| > x) = 0$. By Jessen and Mikosch (2006), Lemma 3.12, we conclude

$$\frac{\mathbb{P}(|\mathbf{X}_d| > ux, \mathbf{X}_d/|\mathbf{X}_d| \in S)}{\mathbb{P}(|\mathbf{X}_d| > u)} \sim \frac{\mathbb{P}(|\mathbf{a}||X_0| > ux, \mathbf{a}X_0/(|\mathbf{a}||X_0|) \in S)}{\mathbb{P}(|\mathbf{a}X_0| > u)} \quad \text{as } u\to\infty. \tag{2.8}$$

Hence, the result follows from Eqs. 2.7–2.8. □

Remark 2.7 Note that Basrak et al. (2002b), Proposition A.1, and Fasen (2005), Lemma 2.1, require the stronger condition $\mathbb{E}|\mathbf{a}|^\beta < \infty$ for some $\beta > \kappa$. Since $|X_0|$ is not only regularly varying, but has a Pareto-like tail, Breiman's result $\mathbb{P}(|\mathbf{a}||X_0| > x) \sim \mathbb{E}|\mathbf{a}|^\kappa \mathbb{P}(|X_0| > x)$ as $x \to \infty$ holds under the weaker condition $\mathbb{E}|\mathbf{a}|^\kappa < \infty$; see Jessen and Mikosch (2006), Lemma 4.2(3). This interesting detail has been communicated to us by the referee.

Parts of the following Proposition can be found under more restrictive assumptions in Gomes et al. (2004). They applied their results to the ARCH(1) model; cf. Section 3.2.1.

Proposition 2.8 *Let the assumptions of Proposition* 2.6 (*c*) *hold and* X_0 *be as in* (*a*). *Furthermore, let* $a_n > 0$ *be such that*

$$\lim_{n\to\infty} n\mathbb{P}(X_0 > a_n) = 1.$$

Then the extreme dependence functions of the strictly stationary process $(X_t)_{t\in\mathbb{N}_0}$ *are given for a lagged vector* $\mathbf{X}_d$ *of* $(X_t)_{t\in\mathbb{N}_0}$ *and for all* $(x_1,\ldots,x_d)\in\mathbb{R}_+^d$ *by*

$$\underline{\chi}_{(t_1,\ldots,t_d)}(x_1,\ldots,x_d)=\frac{\mathbb{E}\Big(\min_{i=1,\ldots,d}\Big\{x_i^{-\kappa}\Big(\big(R\prod_{j=1}^{t_i}A_j\big)^+\Big)^\kappa\Big\}\Big)}{\mathbb{P}(R=1)},\tag{2.9}$$

$$\overline{\chi}_{(t_1,\ldots,t_d)}(x_1,\ldots,x_d)=\frac{\mathbb{E}\Big(\max_{i=1,\ldots,d}\Big\{x_i^{-\kappa}\Big(\big(R\prod_{j=1}^{t_i}A_j\big)^+\Big)^\kappa\Big\}\Big)}{\mathbb{P}(R=1)}.\tag{2.10}$$

In particular,

$$\chi(t)=\mathbb{E}\left(\min\left\{1,\left(\left(\prod_{j=1}^{t}A_j\right)^+\right)^\kappa\right\}\right)\geq 0\quad \textit{for } t\in\mathbb{N}_0.\tag{2.11}$$

In particular, if $\mathbb{P}(A>0)>0$*, then* $\chi(t)>0$ *for* $t\in\mathbb{N}_0$*. Furthermore, the extremal index of* $(X_t)_{t\in\mathbb{N}_0}$ *is*

$$\theta=\mathbb{E}\left(1-\bigvee_{k=1}^{\infty}\left(\left(\prod_{i=1}^{k}A_i\right)^+\right)^\kappa\right)^+\leq 1\tag{2.12}$$

with $\theta<1$ *if* $\mathbb{P}(A>0)>0$.

Proof The results (Eqs. 2.9–2.11) follow by Theorem 2.5 and Proposition 2.6. The value of the extremal index (Eq. 2.12) was calculated in De Haan et al. (1989) for A, B positive, but it is possible to extend their result to general A, B by an application of Theorem 2.7 in Davis and Hsing (1995). □

3 Time series models in discrete time

For the following examples we calculate the extremal measures for $(|X_t|)_{t\in I}$ such that we have a measure for dependence in extremes on positive and negative levels. A separation into positive and negative extremes is notationally involved.

Note that $\Theta_{\mathbf{d}}$, $\mathbf{W}_d$ and the functions $\underline{\chi}$, $\overline{\chi}$ and χ correspond in this section to $(|X_t|)_{t\in I}$.

3.1 Linear models

In this section we investigate the extremal behavior of a strictly stationary infinite moving average (MA) process

$$X_t=\sum_{n=-\infty}^{\infty}c_{t-n}Z_n\qquad \text{for } t\in\mathbb{N}_0\,,\tag{3.1}$$

where $(Z_t)_{t\in\mathbb{Z}}$ is an i. i. d. sequence. We further assume the tail balance condition

$$\lim_{x\to\infty}\frac{\mathbb{P}(Z_0>x)}{\mathbb{P}(|Z_0|>x)}=p \quad \text{and} \quad \lim_{x\to\infty}\frac{\mathbb{P}(-Z_0>x)}{\mathbb{P}(|Z_0|>x)}=1-p \tag{3.2}$$

for some $p\in[0,1]$. Let $c_{\max}=\max_{i\in\mathbb{Z}}|c_i|$. More details on linear models in the context of extreme value theory can be found in the monographs of Embrechts et al. (1997), Section 5.5, Leadbetter et al. (1983), Section 3.8, and Resnick (1987), Section 4.5.

3.1.1 Linear models with regularly varying tail

The next Lemma is due to Davis and Resnick (1985) and Hult and Samorodnitsky (2008).

Lemma 3.1 *Let $(X_t)_{t\in\mathbb{N}_0}$ be the MA process given in Eq. 3.1 which satisfies the tail balance condition (Eq. 3.2). Further, we assume that for $\kappa>0$ and some slowly varying function ℓ*

$$\mathbb{P}(|Z_0|>x)=\ell(x)x^{-\kappa} \qquad \text{for } x\geq 0,$$

and that one of the following conditions is satisfied:

(i) $\sum_{n=-\infty}^{\infty}|c_n|^\delta<\infty$ *for some* $\delta<\min\{1,\kappa\}$.
(ii) $\sum_{n=-\infty}^{\infty}|c_n|^\delta<\infty$ *for some* $\delta<\kappa$, $\delta\leq 2$, $\kappa>1$ *and* $\mathbb{E}(Z_1)=0$.

Then the following results hold:

(a) *There exists a strictly stationary version of the MA process.*
(b) *The stationary distribution given by X_∞ is regularly varying with index κ such that*

$$\mathbb{P}(X_\infty>x)\sim\left[p\sum_{n=-\infty}^{\infty}(c_n^+)^\kappa+(1-p)\sum_{n=-\infty}^{\infty}(c_n^-)^\kappa\right]\mathbb{P}(|Z_0|>x) \qquad \text{as } x\to\infty.$$

(c) *Let $(X_t)_{t\in\mathbb{N}_0}$ be a strictly stationary version of the MA process. Then $(|X_t|)_{t\in\mathbb{N}_0}\in\mathcal{R}(\kappa)$ with discrete spectral measure given for a lagged vector $\overline{\mathbf{X}}_d$ of $(|X_t|)_{t\in\mathbb{N}_0}$ by*

$$\mathbb{P}(\mathbf{\Theta}_d\in\cdot)=\frac{\mathbb{E}\left(|\mathbf{W}_d|^\kappa\,\mathbf{1}_{\{\mathbf{W}_d/|\mathbf{W}_d|\in\cdot\}}\right)}{\mathbb{E}|\mathbf{W}_d|^\kappa}, \tag{3.3}$$

where for $n\in\mathbb{Z}$, $\max_{i=1,\dots,d}|c_{t_i-n}|\neq 0$, and

$$m^+(n)=\#\{j\in\mathbb{Z}:(|c_{t_1-j}|,\dots,|c_{t_d-j}|)=(|c_{t_1-n}|,\dots,|c_{t_d-n}|)\},$$

setting $\widetilde{c}=\sum_{k=-\infty}^{\infty}\max_{i=1,\dots,d}|c_{t_i-k}|^\kappa$, we have

$$\mathbb{P}(\mathbf{W}_d=(|c_{t_1-n}|,\dots,|c_{t_d-n}|))=\frac{m^+(n)}{\widetilde{c}}\max_{i=1,\dots,d}|c_{t_i-n}|^\kappa.$$

Remark 3.2 The spectral measure given in Eq. 3.3 has the alternative representation

$$\mathbb{P}(\boldsymbol{\Theta}_d \in \cdot) = \frac{1}{\widetilde{c}} \sum_{n=-\infty}^{\infty} \max_{i=1,\ldots,d} |c_{t_i-n}|^\kappa \, \mathbf{1}_{\{(|c_{t_1-n}|,\ldots,|c_{t_d-n}|)/\max_{i=1,\ldots,d}|c_{t_i-n}| \in \cdot\}} \,.$$

Theorem 3.3 *Let the strictly stationary MA process $(X_t)_{t\in\mathbb{N}_0}$ satisfy the assumptions of Lemma 3.1. Furthermore, let $a_n > 0$ be such that*

$$\lim_{n\to\infty} n\mathbb{P}(|X_\infty| > a_n) = 1.$$

Then the extreme dependence functions of $(|X_t|)_{t\in\mathbb{N}_0}$ are given for a lagged vector $\overline{\mathbf{X}}_d$ of $(|X_t|)_{t\in\mathbb{N}_0}$ and for all $(x_1, \ldots, x_d) \in \mathbb{R}^d_+$ by

$$\underline{\chi}_{(t_1,\ldots,t_d)}(x_1,\ldots,x_d) = \frac{\sum_{n=-\infty}^{\infty} \min_{i=1,\ldots,d} \{x_i^{-1}|c_{t_i-n}|\}^\kappa}{\sum_{n=-\infty}^{\infty} |c_n|^\kappa},$$

$$\overline{\chi}_{(t_1,\ldots,t_d)}(x_1,\ldots,x_d) = \frac{\sum_{n=-\infty}^{\infty} \max_{i=1,\ldots,d} \{x_i^{-1}|c_{t_i-n}|\}^\kappa}{\sum_{n=-\infty}^{\infty} |c_n|^\kappa}.$$

In particular,

$$\chi(t) = \frac{\sum_{n=-\infty}^{\infty} \min\{|c_n|, |c_{t-n}|\}^\kappa}{\sum_{n=-\infty}^{\infty} |c_n|^\kappa} \qquad \textit{for } t \in \mathbb{N}_0.$$

Furthermore, the extremal index of $(|X_t|)_{t\in\mathbb{N}_0}$ is $\theta = (c_{max})^\kappa \Big/ \sum_{n=-\infty}^{\infty} |c_n|^\kappa$.

Proof Theorem 2.5 and Lemma 3.1 (c) lead to the extreme dependence functions. If condition (i) of Lemma 3.1 holds, the extremal index follows from Theorem 3.2 in Davis and Resnick (1985). Under condition (ii) (also condition (i)), the extremal index of $(|X_t|)_{t\in\mathbb{N}_0}$ can be calculated as in Fasen (2005) for continuous-time moving average processes. □

Example 3.4 (AR(1) process) Let $(X_t)_{t\in\mathbb{N}_0}$ be a strictly stationary AR(1) process with moving average representation

$$X_t = \sum_{k=0}^{\infty} \alpha^k Z_{t-k} = \sum_{k=-\infty}^{t} \alpha^{t-k} Z_k\,, \quad t \in \mathbb{N}_0, \tag{3.4}$$

for some $0 < \alpha < 1$ and $(Z_k)_{k\in\mathbb{Z}}$ be an i. i. d. sequence satisfying the assumptions of Lemma 3.1 with tail index $\kappa > 0$. Then

$$\chi(t) = \alpha^{\kappa t} \quad \text{for } t \in \mathbb{N}_0 \quad \text{and} \quad \theta = 1 - \alpha^{\kappa}.$$

Thus, $\chi(\cdot)$ decreases exponentially fast. Furthermore, if the tail index κ of the noise variable $|Z_0|$ increases (recall this is also the tail index of the stationary distribution), the dependence in the extremes becomes weaker. Moreover, for $\kappa \to \infty$ we obviously have $\chi(t) \to 0$ for all $t \in \mathbb{N}$ and $\theta \to 1$.

3.1.2 Linear models with tails in $\mathcal{S} \cap MDA(\Lambda)$

Let $(X_t)_{t\in\mathbb{N}_0}$ be the MA process given in Eq. 3.1 with tail balance condition (Eq. 3.2) and $|Z_0|$ *subexponential* ($|Z_0| \in \mathcal{S}$), i. e. if F denotes the distribution function of $|Z_0|$ then $F(x) < 1$ for every $x \in \mathbb{R}$ and the following conditions hold:

(i) For all $y \in \mathbb{R}$ locally uniformly $\lim_{x\to\infty} \overline{F}(x+y)/\overline{F}(x) = 1$.
(ii) $\lim_{x\to\infty} \overline{F^{2*}}(x)/\overline{F}(x)$ exists and is finite.

Typical examples for subexponential distribution functions are those with regularly varying tails, heavy-tailed Weibull and lognormal distributions. In this section we restrict our attention to $|Z_0| \in \mathcal{S} \cap \text{MDA}(\Lambda)$ which excludes regularly varying distribution functions. Tails in this class are lighter tailed than polynomial. Surveys of the class of subexponential distributions with support on $\mathbb{R}_+$ provide Goldie and Klüppelberg (1998) or Fasen and Klüppelberg (2008), see also Embrechts et al. (1997), Appendix A3. We assume $c^+ \geq c^-$ and define

$$m^+ := \#\{i : c_i = c^+\} \quad \text{and} \quad m^- := \#\{i : c_i = -c^+\}.$$

The following Lemma presented here is due to Davis and Resnick (1988).

Lemma 3.5 *Let $(X_t)_{t\in\mathbb{N}_0}$ be a MA process given in* Eq. 3.1 *which satisfies the tail balance condition* (Eq. 3.2)*. Furthermore, we assume that $|Z_0| \in \mathcal{S} \cap \text{MDA}(\Lambda)$ and $\sum_{n=-\infty}^{\infty} |c_n|^{\delta} < \infty$ for some $0 < \delta < 1$. Then the following results hold:*

(a) *There exists a strictly stationary version of the MA process.*
(b) *The stationary distribution given by X_∞ belongs to $\mathcal{S} \cap \text{MDA}(\Lambda)$ and*

$$\mathbb{P}(X_\infty > x) \sim (pm^+ + (1-p)m^-)\mathbb{P}(c^+|Z_0| > x) \quad \text{as } x \to \infty.$$

Theorem 3.6 *Let the strictly stationary MA process $(X_t)_{t\in\mathbb{N}_0}$ satisfy the assumptions of Lemma 3.2. Furthermore, let $a_n > 0$, $b_n \in \mathbb{R}$ be such that*

$$\lim_{n\to\infty} n\mathbb{P}(|X_\infty| > a_n x + b_n) = \mathrm{e}^{-x} \quad \text{for } x \in \mathbb{R}.$$

Suppose $m_+ = 1$ *and* $m_- = 0$. *Then the extreme dependence functions are given for a lagged vector* $\overline{\mathbf{X}}_d$ *of* $(|X_t|)_{t\in\mathbb{N}_0}$ *and for all* $(x_1, \ldots, x_d) \in \mathbb{R}^d$ *by*

$$\underline{\chi}_{(t_1,\ldots,t_d)}(x_1, \ldots, x_d) = 0 \quad \textit{and} \quad \overline{\chi}_{(t_1,\ldots,t_d)}(x_1, \ldots, x_d) = \sum_{i=1}^{d} \mathrm{e}^{-x_i}.$$

In particular,

$$\chi(t) = 0 \quad \textit{for } t \in \mathbb{N} \quad \textit{and} \quad \theta = 1.$$

Proof The proof of the representation of the dependence measures is an application of Fasen (2006), Lemma 11, and

$$\mathbb{P}(|X_\infty| > x) \sim \mathbb{P}(X_\infty > x) \quad \text{as } x \to \infty.$$

The extremal index of $(X_t)_{t\in\mathbb{N}_0}$ (and hence also $(|X_t|)_{t\in\mathbb{N}_0}$) was calculated in Davis and Resnick (1988). □

Example 3.7 (AR(1) process, continuation of Example 3.4) For the model in Eq. 3.4 with $\alpha = 0.7$ we consider two different regimes. Once we take $(Z_k)_{k\in\mathbb{Z}}$ as an i.i.d. Pareto distributed sequence ($F(x) = 1 - x^{-1/2}$ for $x \geq 1$), which falls in the framework of Example 3.4. Then we take $(Z_k)_{k\in\mathbb{Z}}$ as an i.i.d. Weibull distributed sequence with shape parameter less than 1 ($F(x) = 1 - \exp(-x^{0.9})$ for $x \geq 0$), which belongs to $\mathcal{S} \cap \mathrm{MDA}(\Lambda)$. This model has extreme dependence functions as in Theorem 3.6. Figure 1 compares the extremal coefficient functions of both models. The functions are estimated from a sample path of length 10,000. The estimation is based on data above a threshold, which is chosen as the empirical 0.5% quantile of the data. The estimated extremal coefficient function of the AR(1) process with Pareto noise follows nicely the theoretical one, which decreases exponentially fast with rate $0.5\log(0.7)$. For the Weibull noise the estimate at lag 1 is still positive, but for higher order lags the empirical estimate is near 0.

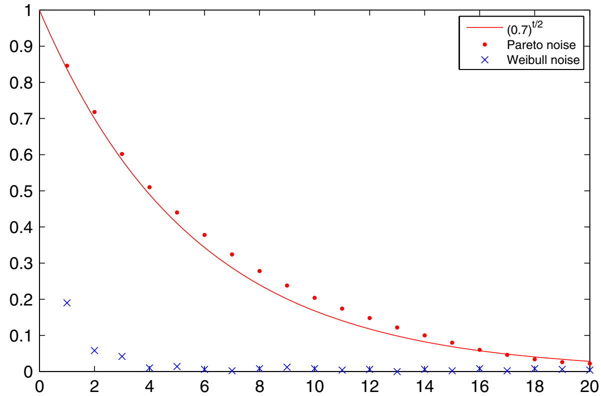

Fig. 1 Empirical estimates of the extremal coefficient function for an AR(1) process with Pareto and Weibull noise.

It can be debated, and indeed has, whether the influence of the marginal distributions (regular variation versus lighter-tailed) blurs the interpretation of estimated extremal dependence. Ramos and Ledford (2008) argue that it is more reasonable to separate the influence of the marginals and the extreme dependence by standardizing the marginals first to unit Fréchet distributions. Only afterwards they investigate bivariate joint tail distributions. The spectral measure in case of unit Fréchet marginals is then given in Eq. 3.3 with $\kappa = 1$. There are two points to be aware of, when standardizing marginals. Firstly, some statistical uncertainty will be introduced into the model by estimating the parameters of transformation. Secondly, transformation of the data involves the whole model. For instance, a linear exponential model will transform into a product model, resulting in a change of the dependence structure as well.

3.2 Non-linear models

3.2.1 ARCH(1) and GARCH(1,1) processes

The ARCH(1) process $(X_t)_{t\in\mathbb{N}_0}$ is defined as

$$X_t = (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2} Z_t \qquad \text{for } t \in \mathbb{N}\,, \tag{3.5}$$

where X_0 is independent of the i. i. d. sequence $(Z_t)_{t\in\mathbb{N}_0}$ and $\alpha_0, \alpha_1 > 0$. As a generalization, the GARCH(1,1) process $(X_t)_{t\in\mathbb{N}_0}$ is defined by

$$X_t = \sigma_t Z_t \qquad \text{for } t \in \mathbb{N}_0, \tag{3.6}$$

where $(Z_t)_{t\in\mathbb{N}_0}$ is an i. i. d. sequence independent of σ_0 and the volatility process $(\sigma_t^2)_{t\in\mathbb{N}_0}$ is the solution of the stochastic recurrence equation

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta\sigma_{t-1}^2 = \alpha_0 + (\alpha_1 Z_{t-1}^2 + \beta)\sigma_{t-1}^2 \qquad \text{for } t \in \mathbb{N},$$

where $\alpha_0, \alpha_1 > 0$ and $0 < \beta < 1$. Thus, setting $B_t = \alpha_0$ and $A_t = \alpha_1 Z_{t-1}^2 + \beta$, we see that $\sigma_t^2 = A_t\sigma_{t-1}^2 + B_t$. Indeed, if we define

$$\mathbf{Y}_t = \begin{pmatrix} X_t^2 \\ \sigma_t^2 \end{pmatrix}, \quad \mathbf{A}_t = \begin{pmatrix} \alpha_1 Z_t^2 & \beta Z_t^2 \\ \alpha_1 & \beta \end{pmatrix} \quad \text{and} \quad \mathbf{B}_t = \begin{pmatrix} \alpha_0 Z_t^2 \\ \alpha_0 \end{pmatrix},$$

then $\mathbf{Y}_t = \mathbf{A}_t\mathbf{Y}_{t-1} + \mathbf{B}_t$ for $t \in \mathbb{N}$. Setting $\beta = 0$ we obtain again the ARCH(1) process and its volatility process.

This model can be considered as the solution to a multivariate stochastic recurrence equation and hence, general results by Kesten (1973) and Bougerol and Picard (1992b) can be applied. These general results can, however, for this model be considerably reduced; cf. Nelson (1990) and Bougerol and Picard (1992a). The multivariate regular variation of this model was derived by Basrak et al. (2002b), and Mikosch and Stărică (2000). For a survey on GARCH processes and their properties see Mikosch (2004).

Lemma 3.8 *Let $(X_t)_{t\in\mathbb{N}_0}$ be the ARCH(1) process given in Eq. 3.5 or the GARCH(1,1) process given in Eq. 3.6, respectively. Suppose Z_0 has a positive* *density on $\mathbb{R}$, and either $\mathbb{E}|Z_0|^h < \infty$ for all $0 < h < h_0$ and $\mathbb{E}|Z_0|^{h_0} = \infty$ for some finite $h_0 > 0$, or $\mathbb{E}|Z_0|^h < \infty$ for all $h > 0$. Furthermore, we assume that*

$$\mathbb{E}(\log(\alpha_1 Z_0^2 + \beta)) < 0.$$

Then the following results hold:

(a) *There exists a $\kappa > 0$ such that*

$$\mathbb{E}(\alpha_1 Z_0^2 + \beta)^{\kappa/2} = 1. \tag{3.7}$$

(b) *There exists a strictly stationary version of the bivariate process $(X_t, \sigma_t)_{t \in \mathbb{N}_0}$.*

(c) *The stationary distributions given by $|X_\infty|$ and σ_∞ of $(|X_t|)_{t \in \mathbb{N}_0}$ and $(\sigma_t)_{t \in \mathbb{N}_0}$, respectively, are regularly varying with index κ such that*

$$\mathbb{P}(\sigma_\infty > x) \sim C x^{-\kappa} \quad \text{and} \quad \mathbb{P}(|X_\infty| > x) \sim C\, \mathbb{E}|Z_0|^\kappa x^{-\kappa} \quad \text{as } x \to \infty,$$

where

$$C = \frac{\mathbb{E}[(\alpha_0 + (\alpha_1 Z_0^2 + \beta)\sigma_\infty^2)^{\kappa/2} - ((\alpha_1 Z_0^2 + \beta)\sigma_\infty^2)^{\kappa/2}]}{[(\kappa/2)\mathbb{E}((\alpha_1 Z_0^2 + \beta)^{\kappa/2} \log(\alpha_1 Z_0^2 + \beta))]}.$$

(d) *Let $(X_t^2, \sigma_t^2)_{t \in \mathbb{N}_0}$ be a strictly stationary version of the ARCH(1) or the GARCH(1,1) process, respectively, and its volatility process. Then $(X_t^2, \sigma_t^2)_{t \in \mathbb{N}_0} \in \mathcal{R}(\kappa/2)$ with spectral measure given for a lagged vector $(X_{t_1}^2, \sigma_{t_1}^2, \dots, X_{t_d}^2, \sigma_{t_d}^2)$ by*

$$\mathbb{P}(\boldsymbol{\Theta}_d \in \cdot) = \frac{\mathbb{E}\left(|\mathbf{W}_d|^{\kappa/2} \mathbf{1}_{\{\mathbf{W}_d/|\mathbf{W}_d| \in \cdot\}}\right)}{\mathbb{E}|\mathbf{W}_d|^{\kappa/2}},$$

where $\mathbf{W}_d = \left(\prod_{i=1}^{t_1}(\alpha_1 Z_{i-1}^2 + \beta)(Z_{t_1}^2, 1), \dots, \prod_{i=1}^{t_d}(\alpha_1 Z_{i-1}^2 + \beta)(Z_{t_d}^2, 1)\right)$.

Theorem 3.9 *Let $(X_t)_{t \in \mathbb{N}_0}$ be a strictly stationary ARCH(1) or GARCH(1,1) process, respectively, satisfying the assumptions of Lemma 3.8. Furthermore, let $a_n > 0$ be such that*

$$\lim_{n \to \infty} n \mathbb{P}(|X_\infty| > a_n) = 1.$$

Then the extreme dependence functions of $(|X_t|)_{t \in \mathbb{N}_0}$ are given for a lagged vector $\overline{\mathbf{X}}_d$ of $(|X_t|)_{t \in \mathbb{N}_0}$ and for all $(x_1, \dots, x_d) \in \mathbb{R}_+^d$ by

$$\underline{\chi}_{(t_1,\dots,t_d)}(x_1, \dots, x_d) = \frac{\mathbb{E}\left(\min_{i=1,\dots,d}\left\{x_i^{-\kappa}|Z_{t_i}|^\kappa \prod_{j=1}^{t_i}(\alpha_1 Z_{j-1}^2 + \beta)^{\kappa/2}\right\}\right)}{\mathbb{E}|Z_0|^\kappa},$$

$$\overline{\chi}_{(t_1,\dots,t_d)}(x_1, \dots, x_d) = \frac{\mathbb{E}\left(\max_{i=1,\dots,d}\left\{x_i^{-\kappa}|Z_{t_i}|^\kappa \prod_{j=1}^{t_i}(\alpha_1 Z_{j-1}^2 + \beta)^{\kappa/2}\right\}\right)}{\mathbb{E}|Z_0|^\kappa}.$$

In particular,

$$\chi(t) = \frac{\mathbb{E}\left(\min\left\{|Z_0|^\kappa, |Z_t|^\kappa \prod_{i=1}^{t}(\alpha_1 Z_{i-1}^2 + \beta)^{\kappa/2}\right\}\right)}{\mathbb{E}|Z_0|^\kappa} \quad \text{for } t \in \mathbb{N}_0. \tag{3.8}$$

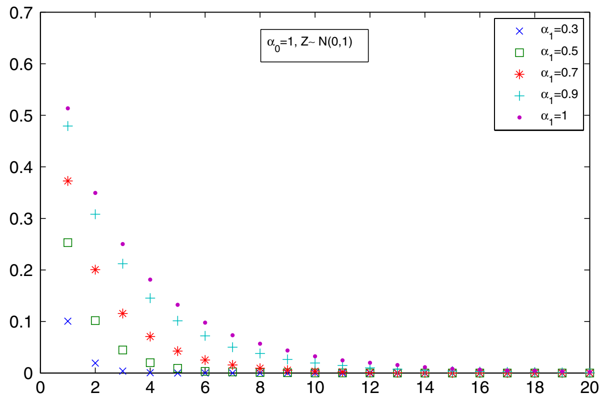

Fig. 2 Extremal coefficient function $\chi(\cdot)$ of different ARCH(1) processes: Monte Carlo simulation of Eq. 3.8 with parameters α_0 and α_1 as above and κ calculated from Eq. 3.7.

Furthermore, the extremal index of $(|X_t|)_{t\in\mathbb{N}_0}$ *is*

$$\theta = \frac{\mathbb{E}\left(|Z_0|^\kappa - \bigvee_{m=1}^{\infty} |Z_m|^\kappa \prod_{i=1}^{m}(\alpha_1 Z_{i-1}^2 + \beta)^{\kappa/2}\right)^+}{\mathbb{E}|Z_0|^\kappa}.$$

Proof By Lemma 3.8 the lagged vector $\mathbf{X}_d^2$ has spectral measure

$$\mathbb{P}(\boldsymbol{\Theta}_d \in \cdot) = \frac{\mathbb{E}\left(|\mathbf{W}_d|^{\kappa/2}\,\mathbf{1}_{\{\mathbf{W}_d/|\mathbf{W}_d|\in\cdot\}}\right)}{\mathbb{E}|\mathbf{W}_d|^{\kappa/2}},$$

where $\mathbf{W}_d = \left(\prod_{i=1}^{t_1}(\alpha_1 Z_{i-1}^2 + \beta) Z_{t_1}^2, \ldots, \prod_{i=1}^{t_d}(\alpha_1 Z_{i-1}^2 + \beta) Z_{t_d}^2\right)$. The result follows then from Theorem 2.5. The extremal index of $(|X_t|)_{t\in\mathbb{N}_0}$ is given in Mikosch and Stărică (2000). □

Example 3.10 (ARCH(1) process: the extremal coefficient function) We see in Fig. 2 the extremal coefficient function $\chi(\cdot)$ for ARCH(1) processes with standard normal noise and parameters $\alpha_0 = 1$ and $\alpha_1 = 0.3, 0.5, 0.7, 0.9, 1.0$, which correspond to tail indices $\kappa = \kappa(\alpha_1) = 8.36, 4.74, 3.18, 3.3, 2.0$, respectively. The values of κ were computed from Eq 3.7, which has for a normal noise an analytic representation; cf. Embrechts et al. (1997), Table 8.4.8. The values for $\chi(\cdot)$ are the results of a Monte Carlo simulation based on Eq. 3.8 with 100,000 standard normal random numbers. For every choice of α_1 the

Table 1 Estimation of the extremal coefficient function for an ARCH(1) process with parameters $\alpha_0 = 1, \alpha_1 = 0.5$ and standard normal noise

k	1	2	3	4	5	6	7	8	9	10
$\chi(k)$	0.251	0.099	0.042	0.018	0.008	0.003	0.002	0.001	0.001	0.000
$\chi(k)_{\text{emp}}$	0.314	0.152	0.098	0.070	0.052	0.050	0.044	0.054	0.062	0.042
$\chi(k)_{\text{POT}}$	0.183	0.110	0.051	0.021	0.002	0.001	0.000	0.002	0.000	0.000
$\chi(k)_{\text{Block}}$	0.249	0.044	0.011	0.004	0.000	0.001	0.000	0.001	0.001	0.001

The first line shows the values from the Monte Carlo estimation

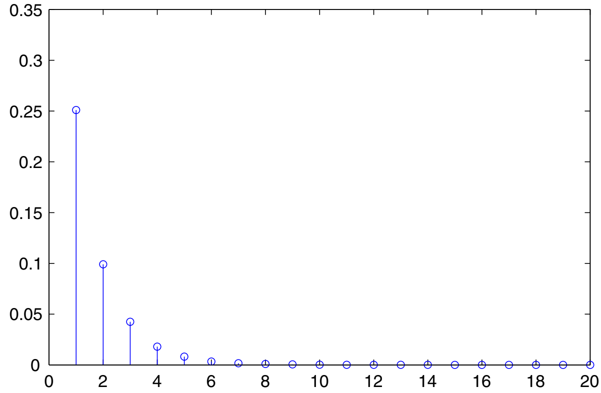

Fig. 3 Monte Carlo simulation of $\chi(\cdot)$ as explained in Example 3.10.

function $\chi(\cdot)$ decreases exponentially in t. Obviously, the dependence in the extremes decreases with time. We also see that for fixed t the function $\chi(t) = \chi(t; \alpha_1)$ is an increasing function in α_1, hence, $\chi(t; \kappa)$ decreases, if κ increases. This suggests, not surprisingly, that for heavier tailed ARCH(1) processes the dependence in the extremes is higher than for lighter tailed ARCH(1) processes.

Example 3.11 (ARCH(1): estimating the extremal coefficient function) In Table 1 and Figs. 3, 4, 5, and 6 we present the estimation of the extremal coefficient function $\chi(\cdot)$ of a simulated ARCH(1) process with parameters $\alpha_0 = 1$, $\alpha_1 = 0.5$ and standard normal noise. All estimates are based on a sample path of length 10,000. For comparison, Fig. 3 depicts again $\chi(\cdot)$ calculated as explained in Example 3.10. In Figs. 4, 5, and 6 we estimate $\chi(\cdot)$ by three different methods. In Fig. 4 the estimator is computed via the empirical conditional tail distribution function of $\min(X_0, X_t)$ given X_0. By definition of $\chi(\cdot)$, estimates have to be based on large values of the process

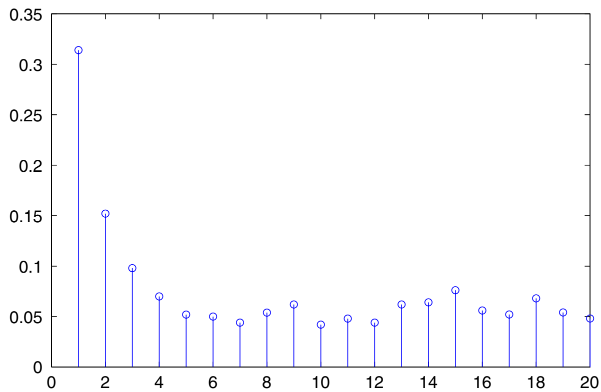

Fig. 4 Empirical estimator $\chi(\cdot)_{\text{emp}}$ of Eq. 1.4.

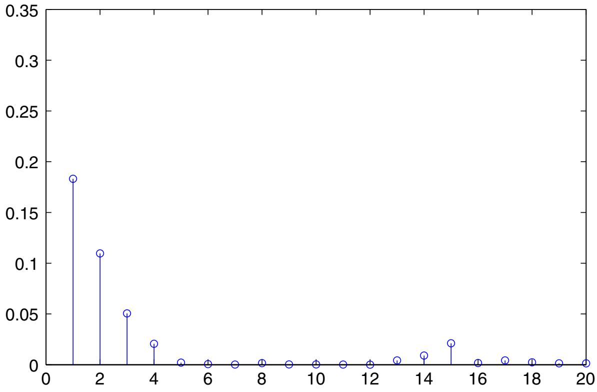

Fig. 5 POT estimator $\chi(\cdot)_{\text{POT}}$ of Eq. 1.4 based on high-level exceedances.

only, and we take the largest 500 values. Better results are to be expected invoking methods from extreme value theory. The last two estimators, $\chi(k)_{\text{POT}}$ and $\chi(k)_{\text{Block}}$, respectively, apply the POT-method and the block-method; see Embrechts et al. (1997), Chapter 6. For the POT method we approximated the distribution of 500 exceedances by a generalized Pareto distribution. The block method is based on a block size of 30 and approximates the distribution of the block maxima by a generalized extreme value distribution. Table 1 shows the corresponding values of the plots. Improved estimators for the extremal coefficient function for either multivariate models or time series models are presented in Ledford and Tawn (2003), Schlather and Tawn (2003), and Naveau et al. (2008).

Example 3.12 (GARCH(1,1)) Figure 7 shows the extremal coefficient function $\chi(\cdot)$ of different GARCH(1,1) processes with standard normal noise and parameters $\beta = 0.7$, $\alpha_0 = 0.01$ and $\alpha_1 = 0.09, 0.14, 0.19, 0.24, 0.29$. From Eq. 3.7 we computed the values for κ by Monte-Carlo simulations based on

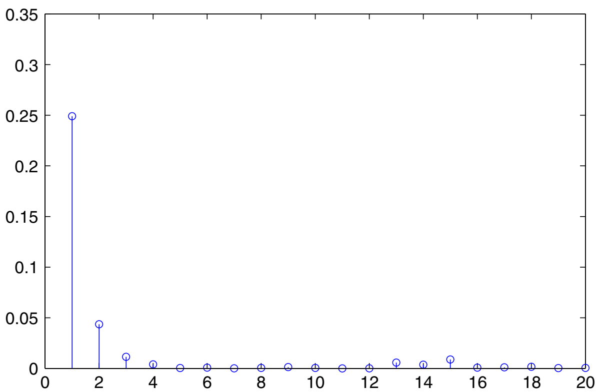

Fig. 6 Block-maxima estimator $\chi(k)_{\text{Block}}$ of Eq. 1.4 based on block maxima of block size 30.

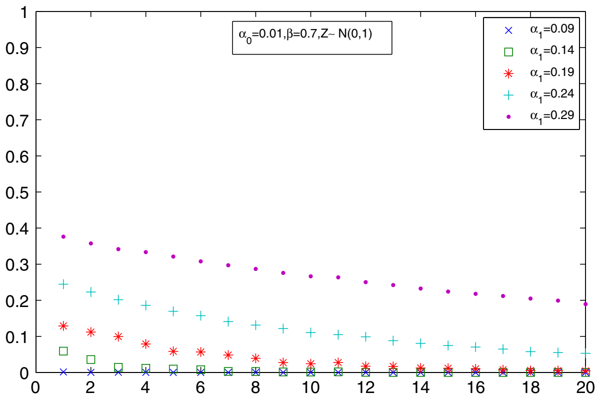

Fig. 7 Extremal coefficient function $\chi(\cdot)$ of different GARCH(1, 1) processes: Monte Carlo simulation of Eq. 3.8 with parameters as above and κ found from a Monte Carlo simulation of Eq. 3.7.

100,000 standard normal random numbers, which gave $\kappa = 18.4, 10.5, 6.5, 4, 2.5$, respectively. The values of κ show that the tails of the GARCH(1,1) process become heavier with increasing α_1. Similar interpretations as for the extremal coefficient function of the ARCH(1) process in Example 3.10 are possible.

3.2.2 The AR(1) process with ARCH(1) errors

In this section we study the AR(1) process with ARCH(1) errors defined by

$$X_t = \lambda X_{t-1} + \sqrt{\alpha_0 + \alpha_1 X_{t-1}^2}\, Z_t \quad \text{for } t \in \mathbb{N}, \tag{3.9}$$

where $\lambda \in \mathbb{R}$, $\alpha_0, \alpha_1 > 0$ and $(Z_t)_{t \in \mathbb{N}_0}$ is an i. i. d. sequence independent of X_0. This model was investigated in Borkovec (2001) and Borkovec and Klüppelberg (2001) by analytic methods; see also the review paper Klüppelberg (2004). Note that the model is Markovian, but it is not easy to prove regular variation of its stationary distribution. Parts (a)–(c) of the following lemma state sufficient conditions to ensure this.

Lemma 3.13 *Let $(X_t)_{t \in \mathbb{N}_0}$ be the AR(1) process with ARCH(1) errors given in Eq. 3.9. We assume that Z_1 is symmetric with continuous density f, which has full support on $\mathbb{R}$, and that $\mathbb{E}(Z_1^2) < \infty$. Furthermore, we assume that f satisfies the following technical conditions:*

(i) *$f(x) \geq f(x')$ for every $0 \leq x < x'$.*
(ii) *The lower and upper Matuszewska indices of $\overline{F}$ are equal, i. e.*

$$-\infty \leq \gamma := \lim_{v \to \infty} \frac{\log \limsup_{x \to \infty} \overline{F}(vx)/\overline{F}(x)}{\log v}$$
$$= \lim_{v \to \infty} \frac{\log \liminf_{x \to \infty} \overline{F}(vx)/\overline{F}(x)}{\log v} \leq 0.$$

(iii) *If* $\gamma = -\infty$ *then, for all* $\delta > 0$*, there exist constants* $q \in (0, 1)$ *and* $x_0 > 0$ *such that for all* $x > x_0$ *and* $t > x^q$*,*

$$f\left(\frac{x \pm \lambda t}{\sqrt{\alpha_1 t^2}}\right) \geq (1 - \delta)\, f\left(\frac{x \pm \lambda t}{\sqrt{\alpha_0 + \alpha_1 t^2}}\right). \tag{3.10}$$

If $\gamma > -\infty$ *then for all* $\delta > 0$ *there exist constants* $x_0 > 0$ *and* $T > 0$ *such that for all* $x > x_0$ *and* $t > T$ *the inequality 3.10 holds.*

(iv) $\mathbb{E}(\log|\lambda + \sqrt{\alpha_1} Z_0|) < 0$.

Then the following results hold:

(a) *There exists a* $\kappa > 0$ *such that*

$$\mathbb{E}(|\lambda + \sqrt{\alpha_1} Z_0|^\kappa) = 1.$$

(b) *There exists a strictly stationary version of the AR(1) process with ARCH(1) errors.*

(c) *The stationary distribution* X_∞ *is regularly varying with index* κ *such that*

$$\mathbb{P}(X_\infty > x) \sim C x^{-\kappa} \qquad \text{as } x \to \infty,$$

where

$$C = \frac{1}{2\kappa} \frac{\mathbb{E}\left(\left[|\lambda| X_\infty| + \sqrt{\alpha_0 + \alpha_1 X_\infty^2} Z_0|\right]^\kappa - \left[|\lambda + \sqrt{\alpha_1}|X_\infty||\right]^\kappa\right)}{\mathbb{E}(|\lambda + \sqrt{\alpha_1} Z_0|^\kappa \log|\lambda + \sqrt{\alpha_1} Z_0|)}.$$

(d) *Let* $(X_t)_{t\in\mathbb{N}_0}$ *be a strictly stationary version of the* $AR(1)$ *process with* $ARCH(1)$ *errors. Then* $(X_t^2)_{t\in\mathbb{N}_0} \in \mathcal{R}(\kappa/2)$ *with spectral measure given for a lagged vector* $\mathbf{X}_d^2$ *of* $(X_t^2)_{t\in\mathbb{N}_0}$ *by*

$$\mathbb{P}(\mathbf{\Theta}_d \in \cdot) = \frac{\mathbb{E}\left(|\mathbf{W}_d|^{\kappa/2}\, \mathbf{1}_{\{\mathbf{W}_d/|\mathbf{W}_d| \in \cdot\}}\right)}{\mathbb{E}|\mathbf{W}_d|^{\kappa/2}},$$

where $\mathbf{W}_d = \left(\prod_{i=1}^{t_1}(\lambda + \sqrt{\alpha_1} Z_i)^2, \ldots, \prod_{i=1}^{t_d}(\lambda + \sqrt{\alpha_1} Z_i)^2\right)$.

Proof Parts (a)–(c) were proven in Borkovec and Klüppelberg (2001), Theorem 3 and Theorem 8. It remains to show (d). First, note that $(X_t^2)_{t\in\mathbb{N}_0}$ satisfies the stochastic recurrence equation

$$\begin{aligned} X_t^2 &= \left(\lambda X_{t-1} + \sqrt{\alpha_0 + \alpha_1 X_{t-1}^2} Z_t\right)^2 \\ &= (\lambda + \sqrt{\alpha_1} Z_t)^2 X_{t-1}^2 + 2\lambda X_{t-1} Z_t \left(\sqrt{\alpha_0 + \alpha_1 X_{t-1}^2} - \sqrt{\alpha_1} X_{t-1}\right) + \alpha_0 Z_t^2 \\ &=: A_t X_{t-1}^2 + B_t, \end{aligned}$$

where

$$A_t = (\lambda + \sqrt{\alpha_1} Z_t)^2 \quad \text{and} \quad B_t = 2\lambda X_{t-1} Z_t \left(\sqrt{\alpha_0 + \alpha_1 X_{t-1}^2} - \sqrt{\alpha_1} X_{t-1}\right) + \alpha_0 Z_t^2.$$

Further, we obtain recursively

$$X_t = \prod_{j=1}^{t} A_j X_0 + \sum_{m=1}^{t} B_m \prod_{j=m+1}^{t} A_j.$$

Since

$$|B_t| \leq 2\lambda\sqrt{\alpha_0}|X_{t-1}||Z_t| + \alpha_0 Z_t^2,$$

we have also $\mathbb{E}|B_t|^{\kappa/2} < \infty$. We are now in the same situation as in Proposition 2.6, whose proof does neither require the independence of $(A_t)_{t\in\mathbb{N}}$ and $(B_t)_{t\in\mathbb{N}}$ nor the independence of X_{t-1} and B_t. Following the proof of Proposition 2.6 step by step we conclude that every lagged vector $\mathbf{X}_d^2$ is multivariate regularly varying with index $\kappa/2$ and spectral measure as given above. □

Note, that the normal, Student's and Laplace distribution satisfy the technical conditions of the above Lemma.

Theorem 3.14 *Let the strictly stationary* $AR(1)$ *process with* $ARCH(1)$ *errors* $(X_t)_{t\in\mathbb{N}_0}$ *satisfy the assumptions of Lemma 3.13. Furthermore, let* $a_n > 0$ *be a sequence of constants such that*

$$\lim_{n\to\infty} n\mathbb{P}(|X_\infty| > a_n) = 1.$$

Then the extreme dependence functions of $(|X_t|)_{t\in\mathbb{N}_0}$ *are for every lagged vector* $\overline{\mathbf{X}}_d$ *of* $(|X_t|)_{t\in\mathbb{N}_0}$ *and for all* $(x_1,\ldots,x_d) \in \mathbb{R}_+^d$ *given by*

$$\underline{\chi}_{(t_1,\ldots,t_d)}(x_1,\ldots,x_d) = \mathbb{E}\left(\min_{i=1,\ldots,d}\{x_i^{-\kappa}\prod_{j=1}^{t_i}|\lambda+\sqrt{\alpha_1}Z_j|^\kappa\}\right),$$

$$\overline{\chi}_{(t_1,\ldots,t_d)}(x_1,\ldots,x_d) = \mathbb{E}\left(\max_{i=1,\ldots,d}\{x_i^{-\kappa}\prod_{j=1}^{t_i}|\lambda+\sqrt{\alpha_1}Z_j|^\kappa\}\right).$$

In particular,

$$\chi(t) = \mathbb{E}\left(\min\left\{1, \prod_{j=1}^{t}|\lambda+\sqrt{\alpha_1}Z_j|^\kappa\right\}\right) \quad \textit{for } t \in \mathbb{N}_0.$$

Furthermore, the extremal index of $(|X_t|)_{t\in\mathbb{N}_0}$ *is*

$$\theta = \mathbb{E}\left(1 - \bigvee_{m=1}^{\infty}\prod_{i=1}^{m}|\lambda+\sqrt{\alpha_1}Z_i|^\kappa\right)^+.$$

Proof We obtain the extreme dependence functions of an AR(1) process with ARCH(1) errors by Lemma 3.13 (d) and Theorem 2.5. The extremal index of

$(X_t^2)_{t\in\mathbb{N}_0}$ is presented in Borkovec (2000), Theorem 3.1. Hence, we obtain the index of $(|X_t|)_{t\in\mathbb{N}_0}$. □

4 Time series models in continuous-time

Introducing discrete time grids in the time axes we calculate the extreme dependence functions and also the extremal index function for continuous time models. We shall see that the dependence structure of extremal events in continuous-time models lead to analogous results as for their discrete-time counterparts. In this section we assume throughout that the underlying probability space is complete and that there exists a separable version of $(X_t)_{t\geq 0}$. We further assume that $\mathbb{P}(\sup_{0\leq t\leq 1}|X_t|<\infty)=1$, a condition, which is for some kernel functions and driving Lévy processes automatically satisfied. Recall again that regular variation of a stochastic process is defined as regular variation of all finite-dimensional distributions.

4.1 Linear models in continuous-time

A continuous-time moving average (MA) process has the representation

$$X_t=\int_{-\infty}^{\infty} f(t-s)\,dL_s \qquad \text{for } t\geq 0\,, \tag{4.1}$$

where $f:\mathbb{R}\to\mathbb{R}$ is a deterministic measurable function and $(L_t)_{t\in\mathbb{R}}$ is a Lévy process. For background on Lévy processes we refer to the excellent monograph by Sato (1999). We also assume that f is bounded with $f_{\max}=\sup_{t\in\mathbb{R}}|f(t)|<\infty$ and the following tail balance condition

$$\lim_{x\to\infty}\frac{\mathbb{P}(L_1>x)}{\mathbb{P}(|L_1|>x)}=p \quad \text{and} \quad \lim_{x\to\infty}\frac{\mathbb{P}(-L_1>x)}{\mathbb{P}(|L_1|>x)}=1-p \tag{4.2}$$

hold for some $p\in[0,1]$. More details on continuous-time linear models can be found in Fasen (2005, 2006, 2009b).

4.1.1 Linear models with regularly varying tails

Lemma 4.1 *Let $(X_t)_{t\geq 0}$ be the continuous-time MA process given in Eq. 4.1 which satisfies the tail balance condition (Eq. 4.2). Furthermore, we assume that for $\kappa>0$ and some slowly varying function ℓ*

$$\mathbb{P}(|L_1|>x)=\ell(x)x^{-\kappa} \qquad \text{for } x\geq 0,$$

and one of the following conditions is satisfied:

(i) $\int_{-\infty}^{\infty}|f(t)|^{\delta}\,dt<\infty$ *for some* $\delta<\min\{1,\kappa\}$.
(ii) $\int_{-\infty}^{\infty}|f(t)|^{\delta}\,dt<\infty$ *for some* $\delta<\kappa$, $\delta\leq 2$, $\kappa>1$, *and* $\mathbb{E}(L_1)=0$.

Then the following results hold:

(a) *There exists a strictly stationary version of the MA process.*
(b) *The stationary distribution given by X_∞ is regularly varying with index κ such that*

$$\mathbb{P}(X_\infty > x) \sim \left(p \int_{-\infty}^{\infty} (f(s)^+)^\kappa \, ds + (1-p) \int_{-\infty}^{\infty} (f(s)^-)^\kappa \, ds\right) \times \mathbb{P}(|L_1| > x) \quad as \quad x \to \infty.$$

(c) *Let $(X_t)_{t\geq 0}$ be a strictly stationary version of the MA process. Then $(|X_t|)_{t\geq 0} \in \mathcal{R}(\kappa)$ with spectral measure given for a lagged vector $\overline{\mathbf{X}}_d$ of $(|X_t|)_{t\geq 0}$ by*

$$\mathbb{P}(\mathbf{\Theta}_d \in \cdot) = \frac{1}{\widetilde{c}} \int_{-\infty}^{\infty} \max_{i=1,\dots,d} |f(t_i - s)|^\kappa \, \mathbf{1}_{\{(|f(t_1-s)|,\dots,|f(t_d-s)|)/\max_{i=1,\dots,d}|f(t_i-s)| \in \cdot\}} \, ds$$

with $\widetilde{c} = \int_{-\infty}^{\infty} \max_{i=1,\dots,d} |f(t_i - s)|^\kappa \, ds$.

Remark 4.2 Let $f \geq 0$ be strictly decreasing. Then the spectral measure has the alternative representation

$$\mathbb{P}(\mathbf{\Theta}_d \in \cdot) = \frac{\mathbb{E}\left(|\mathbf{W}_d|^\kappa \, \mathbf{1}_{\{\mathbf{W}_d/|\mathbf{W}_d| \in \cdot\}}\right)}{\mathbb{E}|\mathbf{W}_d|^\kappa},$$

where for $s \in \mathbb{R}$ and $\max_{i=1,\dots,d} |f(t_i - s)| \neq 0$, $\mathbf{W}_d$ has Lebesgue density

$$\mathbb{P}(\mathbf{W}_d \in d(|f(t_1 - s)|, \dots, |f(t_d - s)|)) = \frac{1}{\widetilde{c}} \max_{i=1,\dots,d} |f(t_i - s)|^\kappa \, ds.$$

A similar argument as in Theorem 2.5 and Lemma 4.1 (c) leads to the following result.

Theorem 4.3 *Let the strictly stationary MA process $(X_t)_{t\geq 0}$ satisfy the assumptions of Lemma 4.1. Furthermore, let $a_n > 0$ satisfy*

$$\lim_{n\to\infty} n\mathbb{P}(|X_\infty| > a_n) = 1.$$

Then the extreme dependence functions of $(|X_t|)_{t\geq 0}$ are for a lagged vector $\overline{\mathbf{X}}_d$ of $(|X_t|)_{t\geq 0}$ and for all $(x_1, \dots, x_d) \in \mathbb{R}_+^d$ given by

$$\underline{\chi}_{(t_1,\dots,t_d)}(x_1, \dots, x_d) = \frac{\int_{-\infty}^{\infty} \min_{i=1,\dots,d} \{x_i^{-1} |f(t_i - s)|\}^\kappa \, ds}{\int_{-\infty}^{\infty} |f(s)|^\kappa \, ds},$$

$$\overline{\chi}_{(t_1,\dots,t_d)}(x_1, \dots, x_d) = \frac{\int_{-\infty}^{\infty} \max_{i=1,\dots,d} \{x_i^{-1} |f(t_i - s)|\}^\kappa \, ds}{\int_{-\infty}^{\infty} |f(s)|^\kappa \, ds}.$$

In particular, for $t > 0$,

$$\chi(t) = \frac{\int_{-\infty}^{\infty} \min\{|f(-s)|, |f(t-s)|\}^{\kappa}\, ds}{\int_{-\infty}^{\infty} |f(s)|^{\kappa}\, ds}.$$

Furthermore, the extremal index function of $(|X_t|)_{t\geq 0}$ is

$$\theta(h) = h \frac{f_{max}^{\kappa}}{\int_{-\infty}^{\infty} \sup_{0\leq t\leq h} |f(t-s)|^{\kappa}\, ds} \quad \textit{for } h > 0.$$

Example 4.4 (Ornstein–Uhlenbeck process) The strictly stationary OU process is for $\lambda > 0$ defined as

$$X_t = \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)}\, dL_s \qquad \text{for } t \geq 0\,.$$

Assume that $L_1 \in \mathcal{R}(\kappa)$ for $\kappa > 0$ and satisfies Eq. 4.2. Then

$$\chi(t) = \mathrm{e}^{-\kappa\lambda t} \qquad \text{for } t \geq 0,$$

and $(X_t)_{t\geq 0}$ has extreme dependence functions given for a lagged vector $\mathbf{X}_d$ and for all $x \in \mathbb{R}_+$ by

$$\underline{\chi}_{(t_1,\ldots,t_d)}(x,\ldots,x) = x^{-\kappa}\mathrm{e}^{-\kappa\lambda(t_d-t_1)}\,,$$

$$\overline{\chi}_{(t_1,\ldots,t_d)}(x,\ldots,x) = x^{-\kappa}\Big(d\kappa\lambda - \sum_{i=2}^{d} \mathrm{e}^{-\kappa\lambda(t_i-t_{i-1})}\Big).$$

As in the AR(1) model $\chi(\cdot)$ decreases exponentially with rate $\kappa\lambda$. Consequently, the extremal dependence function increases when the tail of L_1 becomes heavier, also when the parameter λ becomes smaller. Furthermore, the extremal index function is given by

$$\theta(h) = \frac{h\kappa\lambda}{h\kappa\lambda + 1} \qquad \text{for } h > 0,$$

which reflects that the cluster probability increases, when κ or λ decreases. As for the discrete time AR(1) process, for $\kappa \to \infty$ it is obvious that $\chi(t) \to 0$ and $\theta(h) \to 1$.

4.1.2 Linear models with tails in $\mathcal{S} \cap MDA(\Lambda)$

The conclusions of this section can be found in a more general context in Fasen (2004, 2006).

Lemma 4.5 *Let $(X_t)_{t\geq 0}$ be a continuous-time MA process given in Eq. 4.1. Furthermore, we assume that $|L_1| \in \mathcal{S} \cap \mathrm{MDA}(\Lambda)$ and one of the following conditions is satisfied:*

(i) $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$.
(ii) $\int_{-\infty}^{\infty} |f(t)|^2\, dt < \infty$ *and* $\mathbb{E}(L_1) = 0$.

Then there exists a strictly stationary version of the MA process, which is infinitely divisible.

The proof of the following Theorem is similar to the proof of Theorem 3.6 for the discrete-time case.

Theorem 4.6 *Let the strictly stationary MA process $(X_t)_{t\geq 0}$ satisfy the assumptions of Lemma 4.5. Furthermore, let $a_n > 0$, $b_n \in \mathbb{R}$ satisfy*

$$\lim_{n\to\infty} n\mathbb{P}(|X_\infty| > a_n x + b_n) = \mathrm{e}^{-x} \qquad \text{for } x \in \mathbb{R}.$$

Suppose that $f(t) = 0$ for $t \leq 0$, $f(t) < f(0)$ for $t > 0$, and that f is non-increasing on $[0,\infty)$. Then the extreme dependence functions of $(|X_t|)_{t\geq 0}$ are given for a lagged vector $\overline{\mathbf{X}}_d$ of $(|X_t|)_{t\geq 0}$ and for all $(x_1,\ldots,x_d) \in \mathbb{R}^d$ by

$$\underline{\chi}_{(t_1,\ldots,t_d)}(x_1,\ldots,x_d) = 0 \quad \text{and} \quad \overline{\chi}_{(t_1,\ldots,t_d)}(x_1,\ldots,x_d) = \sum_{i=1}^{d} \mathrm{e}^{-x_i}.$$

In particular, $\chi(t) = 0$ for $t > 0$. Furthermore, the extremal index function of $(|X_t|)_{t\geq 0}$ is given by $\theta(h) = 1$ for $h > 0$.

In the framework of the above Theorem 4.6 we have

$$\mathbb{P}(|X_\infty| > x) \sim o(\mathbb{P}(|L_1| > x)) \qquad \text{as } x \to \infty.$$

A typical example for a process satisfying the assumptions of Theorem 4.6 is the OU-process of Example 4.4 with $|L_1|$ in $\mathcal{S} \cap \mathrm{MDA}(\Lambda)$.

4.2 Continuous-time GARCH models

Let $(L_t)_{t\geq 0}$ be a Lévy process and define the auxiliary càdlàg process $(R_t)_{t\geq 0}$ by

$$R_t = \eta t - \sum_{0<s\leq t} \log(1 + \varphi(\Delta L_s)^2), \qquad t \geq 0,$$

for $\eta, \varphi > 0$. The auxiliary process $(R_t)_{t\geq 0}$ itself is a spectrally negative Lévy process of bounded variation. Then with $\beta > 0$ and σ_0^2 independent of $(L_t)_{t\geq 0}$, the volatility process $(\sigma_t^2)_{t\geq 0}$ is defined as

$$\sigma_t^2 = \left(\beta \int_0^t \mathrm{e}^{R_{s-}} ds + \sigma_0^2\right) \mathrm{e}^{-R_t} \qquad \text{for } t \geq 0. \tag{4.1}$$

The integrated continuous-time GARCH(1, 1) (COGARCH(1, 1)) process $(X_t)_{t\geq 0}$ is a càdlàg process satisfying

$$X_t = \int_0^t \sigma_{s-}\, dL_s \quad \text{for } t > 0 \quad \text{and} \quad X_0 = 0, \tag{4.2}$$

where $\sigma_t := \sqrt{\sigma_t^2}$. In a financial context the logarithmic returns over time periods of length $r > 0$ are then modeled by

$$X_t^{(r)} = X_{t+r} - X_t \quad \text{for } t \geq 0. \tag{4.3}$$

The next Lemma is based on Klüppelberg et al. (2004, 2006) and Fasen (2009a).

Lemma 4.7 *Let $(\sigma_t^2)_{t\geq 0}$ be the volatility process of the COGARCH(1,1) process given in Eq. 4.1. Furthermore, assume that there exists some $\kappa > 0$ such that*

$$\mathbb{E}|L_1|^\kappa \log^+ |L_1| < \infty \quad \text{and} \quad \mathbb{E}(\mathrm{e}^{-R_1\kappa/2}) = 1.$$

Then the following results hold:

(a) *There exists a strictly stationary version of $(\sigma_t^2)_{t\geq 0}$.*
(b) *The stationary distribution of the volatility process given by σ_∞^2 is regularly varying with index $\kappa/2$ such that for some $C > 0$,*

$$\mathbb{P}(\sigma_\infty^2 > x) \sim Cx^{-\kappa/2} \quad \text{as } x \to \infty.$$

(c) *Let $(\sigma_t^2)_{t\geq 0}$ be a strictly stationary version of the volatility process. Then $(\sigma_t^2)_{t\geq 0} \in \mathcal{R}(\kappa/2)$ with spectral measure given for a lagged vector $(\sigma_{t_1}^2, \ldots, \sigma_{t_d}^2)$ by*

$$\mathbb{P}(\boldsymbol{\Theta}_d \in \cdot) = \frac{\mathbb{E}\left(|\mathbf{W}_d|^{\kappa/2}\, \mathbf{1}_{\{\mathbf{W}_d/|\mathbf{W}_d| \in \cdot\}}\right)}{\mathbb{E}|\mathbf{W}_d|^{\kappa/2}},$$

where $\mathbf{W}_d = \left(\mathrm{e}^{-R_{t_1}}, \ldots, \mathrm{e}^{-R_{t_d}}\right)$.

Remark 4.8 The condition $\mathbb{E}(\mathrm{e}^{-R_1\kappa/2}) = 1$ can be expressed in terms of the Lévy measure ν of L. Let $\Psi(s) = \log \mathbb{E}(\mathrm{e}^{-sR_1})$, then

$$\Psi(s) = -s\eta + \int_{\mathbb{R}} ((1 + \varphi y^2)^s - 1)\, \nu(dy) \tag{4.4}$$

and κ is the solution to $\Psi(2s) = 0$.

By Theorem 2.5, Lemma 4.7 (c) and Fasen (2009a), Theorem 4.3, the following result holds.

Theorem 4.9 *Let the strictly stationary volatility process $(\sigma_t^2)_{t\geq 0}$ satisfy the assumptions of Lemma 4.7. Furthermore, let $a_n > 0$ satisfy*

$$\lim_{n\to\infty} n\mathbb{P}(\sigma_\infty > a_n) = 1.$$

Then the extreme dependence functions of $(\sigma_t)_{t\geq 0}$ are given for a lagged vector $(\sigma_{t_1}, \ldots, \sigma_{t_d})$ of $(\sigma_t)_{t\geq 0}$ and for all $(x_1, \ldots, x_d) \in \mathbb{R}_+^d$ by

$$\underline{\chi}_{(t_1,\ldots,t_d)}(x_1, \ldots, x_d) = \mathbb{E}\left(\min_{i=1,\ldots,d} \{x_i^{-\kappa} \mathrm{e}^{-R_{t_i}\kappa/2}\}\right),$$

$$\overline{\chi}_{(t_1,\ldots,t_d)}(x_1, \ldots, x_d) = \mathbb{E}\left(\max_{i=1,\ldots,d} \{x_i^{-\kappa} \mathrm{e}^{-R_{t_i}\kappa/2}\}\right).$$

In particular,

$$\chi(t) = \mathbb{E}\left(\min\{1, \mathrm{e}^{-R_t\kappa/2}\}\right) \quad \text{for } t \geq 0.$$

Furthermore, the extremal index function of $(\sigma_t)_{t\geq 0}$ *is*

$$\theta(h) = \frac{\mathbb{E}\left(\sup_{0\leq t\leq h} \mathrm{e}^{-R_t\kappa/2} - \sup_{t\geq h} \mathrm{e}^{-R_t\kappa/2}\right)^+}{\mathbb{E}\left(\sup_{0\leq t\leq h} \mathrm{e}^{-R_t\kappa/2}\right)} \quad \text{for } h > 0.$$

Lemma 4.10 *Let* $(X_t)_{t\geq 0}$ *be the COGARCH(1,1) process given in Eq. 4.2. Suppose* $(L_t)_{t\geq 0}$ *is of finite variation and* $(-L_t)_{t\geq 0}$ *is not a subordinator. Furthermore, we assume there exist* $\kappa > 0$ *and* $\delta > 0$ *such that*

$$\mathbb{E}|L_1|^{2\kappa+\delta} < \infty \quad \text{and} \quad \mathbb{E}(\mathrm{e}^{-R_1\kappa/2}) = 1.$$

Then the following results hold for $r > 0$*:*

(a) *There exists a strictly stationary version of* $(X_{tr}^{(r)})_{t\in\mathbb{N}_0}$.
(b) *The stationary distribution given by* $X_\infty^{(r)}$ *satisfies for some* $C > 0$,

$$\mathbb{P}(X_\infty^{(r)} > x) \sim Cx^{-\kappa} \quad \text{as } x \to \infty.$$

(c) *Let* $(X_{tr}^{(r)})_{t\in\mathbb{N}_0}$ *be a strictly stationary version of the increments of the COGARCH(1,1) process. Then* $((X_{tr}^{(r)})^2)_{t\in\mathbb{N}_0} \in \mathcal{R}(\kappa)$ *with spectral measure given for a lagged vector* $((X_{t_1 r}^{(r)})^2, \ldots, (X_{t_d r}^{(r)})^2)$ *by*

$$\mathbb{P}(\boldsymbol{\Theta}_d \in \cdot) = \frac{\mathbb{E}\left(|\mathbf{W}_d|^{\kappa/2}\, \mathbf{1}_{\{\mathbf{W}_d/|\mathbf{W}_d|\in\cdot\}}\right)}{\mathbb{E}|\mathbf{W}_d|^{\kappa/2}},$$

where

$$\mathbf{W}_d = \left(\left(\int_{t_1 r}^{(t_1+1)r} \mathrm{e}^{-R_s/2}\, dL_s\right)^2, \ldots, \left(\int_{t_d r}^{(t_d+1)r} \mathrm{e}^{-R_s/2}\, dL_s\right)^2\right).$$

By Theorem 2.5, Lemma 4.8 (c) and Fasen (2009a), Theorem 4.3, the following result holds.

Theorem 4.11 *Let the strictly stationary COGARCH(1,1) process* $(X_t)_{t\geq 0}$ *satisfy the assumptions of Lemma* 4.10*. Furthermore, let* $a_n > 0$ *satisfy*

$$\lim_{n\to\infty} n\mathbb{P}(|X_\infty^{(r)}| > a_n) = 1.$$

Then the extreme dependence functions of $(|X_{tr}^{(r)}|)_{t\in\mathbb{N}_0}$ *are given for a lagged vector* $(|X_{t_1 r}^{(r)}|, \ldots, |X_{t_d r}^{(r)}|)$ *of* $(|X_{tr}^{(r)}|)_{t\in\mathbb{N}_0}$ *and for all* $(x_1, \ldots, x_d) \in \mathbb{R}_+^d$ *by*

$$\underline{\chi}_{(t_1,\ldots,t_d)}(x_1, \ldots, x_d) = \frac{\mathbb{E}\left(\min_{i=1,\ldots,d}\left\{x_i^{-\kappa}\left|\int_{t_i r}^{(t_i+1)r} \mathrm{e}^{-R_s/2}\, dL_s\right|^\kappa\right\}\right)}{\mathbb{E}\left|\int_0^r \mathrm{e}^{-R_s/2}\, dL_s\right|^\kappa},$$

$$\overline{\chi}_{(t_1,\ldots,t_d)}(x_1, \ldots, x_d) = \frac{\mathbb{E}\left(\max_{i=1,\ldots,d}\left\{x_i^{-\kappa}\left|\int_{t_i r}^{(t_i+1)r} \mathrm{e}^{-R_s/2}\, dL_s\right|^\kappa\right\}\right)}{\mathbb{E}\left|\int_0^r \mathrm{e}^{-R_s/2}\, dL_s\right|^\kappa}.$$

In particular,

$$\chi(t) = \frac{\mathbb{E}\min\left\{\left|\int_0^r \mathrm{e}^{-R_s/2}\,dL_s\right|^\kappa, \left|\int_{tr}^{(t+1)r} \mathrm{e}^{-R_s/2}\,dL_s\right|^\kappa\right\}}{\mathbb{E}\left|\int_0^r \mathrm{e}^{-R_s/2}\,dL_s\right|^\kappa} \quad \textit{for } t \in \mathbb{N}_0.$$

Furthermore, the extremal index of $(|X_{tr}^{(r)}|)_{t\in\mathbb{N}_0}$ *is*

$$\theta = \frac{\mathbb{E}\left(\left|\int_0^r \mathrm{e}^{-R_s/2}\,dL_s\right|^\kappa - \bigvee_{k=1}^\infty \left|\int_{kr}^{(k+1)r} \mathrm{e}^{-R_s/2}\,dL_s\right|^\kappa\right)}{\mathbb{E}\left|\int_0^r \mathrm{e}^{-R_s/2}\,dL_s\right|^\kappa}.$$

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