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ORIGINAL RESEARCH

On Theory Construction in Physics: Continuity from Classical to Quantum

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Abstract It is well known that the process of *quantization*—constructing a quantum theory out of a classical theory—is not in general a uniquely determined procedure. There are many inequivalent methods that lead to different choices for what to use as our quantum theory. In this paper, I show that by requiring a condition of continuity between classical and quantum physics, we constrain and inform the quantum theories that we end up with.

1 Introduction

The process of *quantization*—constructing a quantum theory from a classical theory for some system—is full of technical and conceptual problems. Philosophers of physics have recently created a stir because there appear to be many inequivalent quantization procedures for theories of great physical interest, including quantum field theory and quantum statistical mechanics. And if there are many inequivalent quantization procedures, how are we supposed to know which to use to construct our quantum theories? The purpose of this paper is to argue that careful attention to the mathematical tools of classical physics can help us narrow the playing field and choose from among the different quantization methods.

Both classical and quantum theories come equipped with topologies on their classes of physical observables. These topologies are used, for example, to represent how an observable can be approximated in a certain limit, with different topologies corresponding to different notions of approximation. One can constrain the construction of quantum theories by requiring that the topology on classical

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observables be preserved in a quantization procedure. Inequivalent quantizations appear in standard approaches to even the quantization of nonrelativistic particle systems, which only require that a particular topology—called the *norm topology*—is preserved. But there are other topologies on the algebras of observables that we can use—in particular *weak topologies*. I will argue that requiring preservation of the weak topologies[1] rules out one source of inequivalent quantizations.[2]

Quantizing a classical theory typically involves two steps: first, one constructs an abstract algebra of observables, and second, one represents those observables as operators on a Hilbert space. Philosophers of physics have focused on the ambiguity inherent in the second step—in quantum field theory and quantum statistical mechanics, there are many inequivalent Hilbert space representations of the abstract algebra of observables (see, e.g., Earman and Fraser 2006; Ruetsche 2003, 2011). But even the first step of this procedure involves some ambiguity (see Ashtekar and Isham 1992; Emch 1997)—there are in fact many different abstract algebras one might choose to represent the observables of the theory, although this has played almost no role in the philosophical discussions. This paper proposes a change in perspective from the standard philosophical literature by focusing on the construction of the abstract algebra rather than inequivalent Hilbert space representations. I claim that this perspective lends some insight to the physical significance of states in quantum theories and their relation to classical physics.

Philosophers of physics consistently use only one algebra of observables in the first step of quantization—a structure known as the *Weyl algebra*. This is for good reason: the Weyl algebra appears to play a central role in the physics literature as a standard tool in the mathematical physicist's toolbox (see, e.g. Petz 1990). In this paper, I will show that the Weyl algebra has a topology that is, in a certain sense, incompatible with the classical theory it derives from. Specifically, I will show that the weak topology on the algebra of observables of a quantum theory can be understood to have physical significance through its origins in the manifestly physically significant notion of pointwise approximation from the weak topology on the classical observables. I will argue that the weak topology on the Weyl algebra fails to carry this important information about approximation because it fails to preserve the notion of pointwise approximation from classical physics, and this is precisely the information that the weak topology is designed to encode. In other words, the weak topology on the Weyl algebra fails to capture the physically significant notion of approximation that philosophers of physics have assumed it represents. Thus, I will argue that topological considerations count against the Weyl algebra. This demonstrates how certain aspects of current theories provide methodological tools for the construction of new physical theories.

[1] As we will see later, I specifically propose the requirement that the weak topology on a quantum algebra of observables preserves the information encoded in the classical topology of pointwise convergence, which can be understood itself as a weak topology.

[2] In particular, this rules out nonregular representations of the Weyl algebra.

2 Preliminaries

2.1 Algebraic Tools

The bounded observables of a physical theory carry the structure of a *C*-algebra*.[3] This means that one may add and multiply observables, and multiply observables by scalars. In addition, a C*-algebra has an operation of involution that is a generalization of complex conjugation. A C*-algebra $\mathfrak{A}$ comes equipped with a norm, which is required to satisfy the C*-identity:

$$||A^*A|| = ||A||^2$$

The norm defines a topology, called the *norm topology*, which is characterized by the following condition for convergence. A net $\{A_i\} \subseteq \mathfrak{A}$ converges to A in the norm topology[4] iff

$$||A_i - A|| \rightarrow 0$$

where the convergence is now in the standard topology on $\mathbb{R}$. The C*-algebra $\mathfrak{A}$ is required to be complete with respect to this topology in the sense that for every Cauchy net $\{A_i\} \subseteq \mathfrak{A}$, i.e. for every net such that

$$||A_i - A_j|| \rightarrow 0$$

as $i, j \rightarrow \infty$, there is an $A \in \mathfrak{A}$ such that $A_i \rightarrow A$ in the norm topology. Standard results in the theory of normed vector spaces tell us that every normed vector space has a unique completion.[5]

Since $\mathfrak{A}$ is a vector space, we can also consider the dual space $\mathfrak{A}^*$ of bounded (i.e. norm continuous) linear functionals $\rho : \mathfrak{A} \rightarrow \mathbb{C}$. States on a C*-algebra $\mathfrak{A}$ are particular elements of the dual space $\mathfrak{A}^*$—namely ones that are positive and normalized.[6] A state is called *pure* if it cannot be written as a convex combination of distinct states. Otherwise, a state is called *mixed*. Pure states represent the possible states of an individual system while mixed states are typically taken to represent some sort of probabilistic combination (whether it be via an ensemble interpretation or mere epistemic uncertainty).

The dual space $\mathfrak{A}^*$ can be used to define an alternative to the norm topology on $\mathfrak{A}$, called the *weak topology*, which is characterized by the following condition for convergence. A net $\{A_i\} \subseteq \mathfrak{A}$ converges in the weak topology to $A \in \mathfrak{A}$ iff for every $\rho \in \mathfrak{A}^*$,

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[3] For more on C*-algebras and W*-algebras, see Kadison and Ringrose (1997), Sakai (1971), and Landsman (1998). For more on algebraic quantum theory, see Haag (1992), Bratteli and Robinson (1987), and Emch (1972). For philosophical introductions, see Halvorson (2006) and Ruetsche (2011).

[4] One could restrict attention here to sequences because the norm topology is second countable, but for the weak topologies considered later, which are not second countable, one must work with arbitrary nets.

[5] A complete normed vector space is called a Banach space. A C*-algebra is thus a Banach algebra whose norm is, in a certain sense, compatible with multiplication and involution.

[6] A linear functional $\rho \in \mathfrak{A}^*$ is *positive* if $\rho(A^*A) \geq 0$ for all $A \in \mathfrak{A}$ and *normalized* if $||\rho|| = 1$.

$$\rho(A_i) \to \rho(A)$$

where the convergence is now in the standard topology on $\mathbb{C}$. The weak topology is the coarsest topology on $\mathfrak{A}$ with respect to which all of the linear functionals in $\mathfrak{A}^*$ are continuous.

A C*-algebra $\mathfrak{A}$ need not be complete with respect to its weak topology; there may be nets $\{A_i\} \subseteq \mathfrak{A}$ that are Cauchy in the sense that

$$\rho(A_i - A_j) \to 0$$

as $i, j \to \infty$ for every $\rho \in \mathfrak{A}^*$ without the net having a limit point $A \in \mathfrak{A}$ such that $A_i \to A$ in the weak topology. However, a C*-algebra can always be completed in its weak topology to form a *W*-algebra*.[7] A W*-algebra is a C*-algebra $\mathfrak{R}$ with a *predual* i.e. a vector space $\mathfrak{R}_*$ such that $(\mathfrak{R}_*)^* = \mathfrak{R}$. We can understand the elements of the predual as canonically embedded in the dual space by $\rho \in \mathfrak{R}_* \mapsto \hat{\rho} \in \mathfrak{R}^*$ with $\hat{\rho}$ defined by

$$\hat{\rho}(A) = A(\rho)$$

for all $A \in \mathfrak{R}$. The elements of the predual $\mathfrak{R}_*$ define a further topology on $\mathfrak{R}$, called the weak* topology, which is characterized by the following condition for convergence. A net $\{A_i\} \subseteq \mathfrak{R}$ converges in the weak* topology to $A \in \mathfrak{R}$ iff for every $\rho \in \mathfrak{R}_*$,

$$A_i(\rho) \to A(\rho)$$

The weak* topology is the coarsest topology that makes every element of the predual continuous when considered as a linear functional on $\mathfrak{R}$. One can show that every W*-algebra is complete in its weak* topology.

Of course, more nets converge in the weak* topology on $\mathfrak{R}$ than in the weak topology because it only requires convergence of expectation values on a subspace of the dual space $\mathfrak{R}^*$. Nevertheless, the weak* topology is a natural generalization of the weak topology in the special case where the predual of $\mathfrak{R}$ is itself the dual space of a C*-algebra. In this case, one has a C*-algebra $\mathfrak{A}$, its dual space $\mathfrak{A}^*$, and a W*-algebra $\mathfrak{A}^{**}$ called the *bidual*. The original algebra $\mathfrak{A}$ is canonically embedded in its bidual as above by $A \in \mathfrak{A} \mapsto \hat{A} \in \mathfrak{A}^{**}$, with $\hat{A}$ defined by

$$\hat{A}(\rho) = \rho(A)$$

for all $\rho \in \mathfrak{A}^*$. With respect to this embedding, the W*-algebra $\mathfrak{A}^{**}$ is the completion of $\mathfrak{A}$ in its weak topology, which is the subspace topology of the weak* topology on $\mathfrak{A}^{**}$. The weak* topology on the bidual $\mathfrak{A}^{**}$ corresponds precisely to the extension of the condition of convergence for the weak topology on $\mathfrak{A}$ to the larger algebra $\mathfrak{A}^{**}$. In particular, the weak* topology on $\mathfrak{A}^{**}$ is the coarsest topology on $\mathfrak{A}^{**}$ that makes every linear functional in $\mathfrak{A}^*$ continuous (when considered as a linear functional on $\mathfrak{A}^{**}$ by the canonical embedding above).

[7] See Feintzeig (2016) for more on the completion of a C*-algebra into a W*-algebra.

2.2 Classical and Quantum Algebras

The state space (phase space) of a classical theory is represented by a manifold $\mathcal{M}$, points of which may represent, for example, the positions and momenta of particles.[8] The observables of a classical theory are given by functions $f : \mathcal{M} \rightarrow \mathbb{C}$.[9] Each observable represents a physical quantity, with the value $f(x)$ on any state $x \in \mathcal{M}$ representing the value of the quantity f in the determinate state x. For simplicity and concreteness, let us restrict attention to the case of a single particle moving in one-dimension, with phase space given by $\mathcal{M} = \mathbb{R}^2$ and canonical coordinates (q, p), representing the position q and momentum p of the particle. Everything that follows will also hold more generally for theories with finitely many degrees of freedom, i.e. for which the phase space $\mathcal{M}$ is locally compact and simply connected.

For the observables of the classical theory, one can use the C*-algebra $C_0(\mathcal{M})$, the collection of continuous functions that vanish at infinity, equipped with algebraic operations of pointwise multiplication, addition, and complex conjugation.[10] The norm of the algebra is given by the usual supremum function norm:

$$||f|| = \sup_{x \in \mathcal{M}} |f(x)|$$

Convergence in the norm topology corresponds precisely with the familiar notion of uniform function convergence.

One can provide at least two considerations in favor of using this algebra of observables. First, the pure state space of $C_0(\mathcal{M})$ coincides with the points of the manifold $\mathcal{M}$ (Fell and Doran 1988), which are precisely what we would like to call the pure states of the physical theory. And, as expected, mixed states on $C_0(\mathcal{M})$ correspond to probability measures on $\mathcal{M}$, mapping each observable to the expectation value or average of the function integrated with respect to that measure. Second, $C_c^\infty(\mathcal{M})$, the collection of smooth functions of compact support, is norm dense in $C_0(\mathcal{M})$. This means that anyone willing to admit $C_c^\infty(\mathcal{M})$ as their observables will be able to approximate functions in $C_0(\mathcal{M})$ arbitrarily well in the norm topology. And it seems that even an operationalist who insisted that we only ever observe a finite (compact) region of phase space should admit the elements of $C_c^\infty(\mathcal{M})$ as observables.

To quantize a classical theory, one looks for a norm continuous surjective linear ‘quantization map’ $\mathcal{Q}$ from the observables of the classical theory into a (non-commutative) C*-algebra $\mathfrak{A}$. Norm continuity of $\mathcal{Q}$ ensures that the norm topology on the C*-algebra carries the same physical information about ‘global’ approximation that the classical norm topology carries. Specifically, norm continuity of $\mathcal{Q}$ guarantees that whenever a net of classical observables $\{f_i\} \subseteq C_0(\mathcal{M})$ converges uniformly to $f \in C_0(\mathcal{M})$, it follows that the quantized observables $\{\mathcal{Q}(f_i)\} \subseteq \mathfrak{A}$ converge in norm to $\mathcal{Q}(f) \in \mathfrak{A}$. Surjectivity of $\mathcal{Q}$ ensures that each quantum

[8] See Landsman (1998, 2006) for a description of classical and quantum theories in the algebraic framework and a detailed investigation of quantization.

[9] We include complex-valued functions for generality. One typically restricts to the self-adjoint functions, which are real-valued, for describing observable quantities.

[10] Of course there are other possible choices for the algebra of classical observables that one may use if one wanted to admit, e.g., unbounded observables.

observable can be given physical significance by tracing its roots back to a classical quantity we already understand. One can always guarantee that $\mathcal{Q}$ is surjective by restricting attention to the subalgebra that is the range of $\mathcal{Q}$, which constitutes precisely the collection of quantities whose origin traces back to classical physics.

The non-commutative quantum algebra $\mathfrak{A}$ is supposed to be the result of implementing some form of the canonical commutation relations on our classical observables (see, e.g. Petz 1990), thereby converting them into quantum observables. The canonical commutation relations are given by[11]

$$[\mathcal{Q}(f), \mathcal{Q}(g)] = i\mathcal{Q}(\{f, g\})$$

for all $f, g \in C^{\infty}(\mathcal{M})$, where the classical Poisson bracket $\{\cdot,\cdot\} : C^{\infty}(\mathcal{M}) \times C^{\infty}(\mathcal{M}) \to C^{\infty}(\mathcal{M})$ is defined by

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$$

We can see that the canonical commutation relations transform information encoded in the classical Poisson bracket into a constraint on the algebraic relations among the quantum observables.[12]

We will put the canonical commutation relations in a slightly different form for the convenience of applying them to a specific collection of bounded observables (see also Petz 1990; Halvorson 2004; Ruetsche 2011). After all, since both $\mathcal{Q}(p)$ and $\mathcal{Q}(q)$ are unbounded, neither can belong to a C*-algebra. As such, one typically considers the exponentiated observables $U_a = e^{ia\mathcal{Q}(q)}$ and $V_b = e^{ib\mathcal{Q}(p)}$ (for $a, b \in \mathbb{R}$), which are bounded. The canonical commutation relations for q and p can be put in the (formally equivalent) Weyl form

$$U_a V_b = e^{-iab} V_b U_a$$

One then searches for a noncommutative C*-algebra $\mathfrak{A}$ to represent the quantum observables with the requirement that $\mathfrak{A}$ contain all the operators U_a, V_b satisfying the Weyl form of the canonical commutation relations. One choice, used often in the physics literature and which has now become standard in philosophy of physics, is the *Weyl algebra*, which is the smallest C*-algebra $\mathcal{W}$ containing all of the operators U_a, V_b.[13] In other words, $\mathcal{W}$ is generated from the collection of all finite linear combinations of U_a, V_b by imposing the Weyl commutation relations and then completing the resultant algebra in the norm topology.[14] As is well known, one can generalize this construction of the Weyl algebra to an arbitrary classical phase space carrying the structure of a symplectic vector space.

[11] Throughout this paper we work in units such that $\hbar = 1$.

[12] This definition captures as a special case the usual commutation relations

$$[\mathcal{Q}(q), \mathcal{Q}(p)] = i$$

for the quantized position observable $\mathcal{Q}(q)$ and the quantized momentum observable $\mathcal{Q}(p)$.

[13] The Weyl algebra can also be uniquely characterized through its Hilbert space representations (see Clifton and Halvorson 2001).

[14] The Weyl operators admit a unique maximal C*-algebra norm (Manuceau et al. 1974).

Before we continue, I would like to make some brief remarks about how this paper fits into the existing philosophical literature on algebraic methods in quantum theories. As stated in the introduction, much of that literature focuses on the problem of inequivalent Hilbert space representations of the quantum algebra of observables (see, e.g., Ruetsche 2011). Two interpretive positions have arisen in that literature:[15] the *Hilbert Space Conservative* grants physical significance to only one irreducible representation of the algebra while the *Algebraic Imperialist* grants physical significance to the algebra itself. First, this paper can be seen as supporting the Algebraic Imperialist's position by demonstrating how a certain perspective on algebraic methods helps us to understand theory construction. Second, one might argue that in precisely the same way that inequivalent Hilbert space representations pose a problem for the Hilbert Space Conservative, inequivalent observable algebras pose a problem for the Algebraic Imperialist because we have to choose just one of these algebras in an ad hoc fashion. This paper shows that while the question of which algebra to use is obviously an important question for the Algebraic Imperialist, she can indeed bring appropriate and non-ad hoc considerations to bear on this question. In particular, I will show that the Imperialist has tools from classical physics that she can use to constrain, inform, or perhaps even justify her choice of an algebra of observables to use for some given purpose.

3 Significance of the Weak Topologies

Just as the norm topology on the C*-algebra of classical observables $C_0(\mathcal{M})$ corresponds to a familiar notion of 'global' or uniform approximation, so too does the weak topology on $C_0(\mathcal{M})$ correspond to a familiar notion of approximation. The weak topology on $C_0(\mathcal{M})$ is equivalent to the topology of pointwise convergence of nets with bounded norm. More precisely, a net $\{f_i\} \subseteq C_0(\mathcal{M})$ converges to $f \in C_0(\mathcal{M})$ in the weak topology iff (a) for all $x \in \mathcal{M}, f_i(x) \to f(x)$ in $\mathbb{C}$ and (b) the net of real numbers $\{||f_i||\}$ is bounded (Reed and Simon 1980, p. 1112). The weak topology provides a natural way of capturing this pointwise notion of approximation within the abstract algebraic framework.[16]

Completing our C*-algebra $C_0(\mathcal{M})$ in the weak topology allows us to include as observables of our theory functions that can be pointwise approximated by our original observables. The bidual of the classical algebra of observables is $C_0(\mathcal{M})^{**} = B(\mathcal{M})$, the collection of bounded Borel functions on $\mathcal{M}$, where a

[15] These are not the only interpretive positions available (see Ruetsche 2011). Albeit extreme views, they are illustrative for the purposes of this paper.

[16] One gets this notion of pointwise convergence immediately for abelian C*-algebras of all continuous functions on a compact Hausdorff space (see Kadison and Ringrose 1997, p. 270). However, when we consider functions on a locally compact (not necessarily compact) manifold $\mathcal{M}$, the correspondence of weak convergence and pointwise convergence on $\mathcal{M}$ only holds for the algebra $C_0(\mathcal{M})$, rather than the algebra of all continuous functions, $C(\mathcal{M})$. The reason is that it is only for $C_0(\mathcal{M})$ that the pure states correspond to precisely the points in $\mathcal{M}$. Other algebras of bounded continuous functions allow for 'states at infinity'—pure states that cannot be represented as points of $\mathcal{M}$—which forces the weak topology to diverge from the topology of pointwise convergence on $\mathcal{M}$. Another way to state the main issue of this paper is as the question of whether these 'states at infinity' are physically significant.

Borel function is a measurable function in the Borel σ-algebra generated from the topology on $\mathcal{M}$. This is just a restatement of the familiar fact that while uniform limits of continuous functions must always be continuous, pointwise limits of continuous functions may be discontinuous, as long as they remain measurable. It immediately follows that the weak* topology on $B(\mathcal{M})$ is also precisely the topology of pointwise convergence of nets with bounded norm. This provides further reason for using the algebras $C_0(\mathcal{M})$ and $B(\mathcal{M})$—their naturally defined topologies correspond with familiar topologies on classes of functions that represent notions of approximation that are obviously physically significant.

One might object that not all measurable functions in $B(\mathcal{M})$ are physically significant in the classical theory because this algebra includes many discontinuous functions. To alleviate this worry, notice that $B(\mathcal{M})$ is generated by its projections, the characteristic functions of measurable sets in $\mathcal{M}$. The usual interpretation of measurable sets is that they correspond to propositions, or 'yes-no' questions, that have manifest physical or empirical significance (e.g. do the position and momentum of the particle fall in this range of values?). To the extent that this interpretation of measurable sets is appropriate, the characteristic functions of such sets have manifest physical significance as well for representing the very same propositions. The entire algebra $B(\mathcal{M})$ is then generated by composing the characteristic functions with the usual algebraic relations, which gives an effective procedure for measuring any observable in $B(\mathcal{M})$: first measure the projection observables, then compose their values via the well-defined algebraic relations. Hence, every observable in $B(\mathcal{M})$ carries at least a derived physical significance.[17]

One might think that the weak topology on a quantum algebra also carries physical significance by encoding a notion of approximation by expectation values. After all, the very definition of the weak topology encodes a notion of pointwise approximation of expectation values of states. But this will not do—given a non-commutative C*-algebra $\mathfrak{A}$ satisfying some form of the canonical commutation relations, we do not know a priori that all states on $\mathfrak{A}$, and hence all expectation values, are physically significant. There may be pathological states on $\mathfrak{A}$ that we want to rule out for the purposes of physical approximation. If there are such unphysical states, then the weak topology provides a notion of approximation with respect to these unphysical states, and so does not carry the physical interpretation we have assumed. To be sure that one has an appropriate physically significant quantum state space, one can use a quantization map to connnect the quantum and classical algebras. The following proposition shows that a quantization map $\mathcal{Q}$ can be used to characterize the physical significance of the weak topology of a quantum algebra of observables.

Proposition 1 *Suppose* $\mathcal{Q} : C_0(\mathcal{M}) \to \mathfrak{A}$ *is a norm continuous surjective linear mapping onto a C*-algebra* $\mathfrak{A}$. *Then the weak topology on* $\mathfrak{A}$ *is the coarsest topology that makes continuous every linear functional* $\rho : \mathfrak{A} \to \mathbb{C}$ *whose composition with* $\mathcal{Q}$, $\rho \circ \mathcal{Q} : C_0(\mathcal{M}) \to \mathbb{C}$ *is continuous in the topology of pointwise convergence on* $C_0(\mathcal{M})$.

[17] On the other hand, if we started with a C*-algebra other than $C_0(\mathcal{M})$, then completing in the weak topology would give rise to functions that cannot even be understood as measurable functions on $\mathcal{M}$ because of their values 'at infinity' (cf. footnote 15).

Proof By the definition of the weak topology, it suffices to show that if a linear functional $\rho : \mathfrak{A} \rightarrow \mathbb{C}$ has a norm continuous composition with $\mathcal{Q}$, $\rho \circ \mathcal{Q} : C_0(\mathcal{M}) \rightarrow \mathbb{C}$, then ρ is norm continuous on $\mathfrak{A}$. (The converse is trivial.)

Let O be an open set in $\mathbb{C}$. Since $\rho \circ \mathcal{Q}$ is continuous in the norm topology on $C_0(\mathcal{M})$, $(\rho \circ \mathcal{Q})^{-1}[O]$ is open in the norm topology on $C_0(\mathcal{M})$. Since $\mathcal{Q}$ is a norm continuous surjective mapping between Banach spaces, by the open mapping theorem (Reed and Simon 1980, Thm. III.10, p. 82) we know

$$\mathcal{Q}[(\rho \circ \mathcal{Q})^{-1}[O]] = \mathcal{Q}[\mathcal{Q}^{-1}[\rho^{-1}[O]]] = \rho^{-1}[O]$$

is open in the norm topology on $\mathfrak{A}$. Hence ρ is continuous in the norm topology on $\mathfrak{A}$. □

This shows that when a quantum algebra is constructed by a quantization map $\mathcal{Q}$, then the weak topology on the quantum algebra captures a natural notion of approximation that respects the structure of the classical theory it derives from. Recall that the definition of the weak topology ensured that it was the coarsest topology that made certain maps continuous that ought to be continuous. The above proposition shows that the maps that are made continuous are precisely the ones that behave well with respect to the classical algebra and the quantization map.

Of course, the collection of functionals whose composition with $\mathcal{Q}$ is continuous turns out to coincide with the collection of all norm continuous linear functionals on $\mathfrak{A}$, which is why we end up providing simply another characterization of the weak topology. But note that it is absolutely crucial that the quantum algebra be a quantization of $C_0(\mathcal{M})$ in order for its weak topology to gain physical significance from the notion of pointwise approximation of the classical theory. If we set things up differently, say by letting the classical algebra to be quantized (the domain of the quantization map $\mathcal{Q}$) be some other algebra of functions besides $C_0(\mathcal{M})$, then we would allow for many states whose composition with $\mathcal{Q}$ is not continuous in the topology of pointwise convergence. In other words, as we will see in the next section, if one tries to quantize an algebra other than $C_0(\mathcal{M})$, one is liable to produce a theory with unphysical states, whose weak topology does not capture the proper notion of pointwise approximation (cf. footnotes 15 and 16).

It follows immediately from the above proposition that we can also characterize the physical significance of the weak* topology of a quantum algebra of observables by reference to the classical theory and quantization map.

Corollary 1 *Suppose* $\mathcal{Q} : C_0(\mathcal{M}) \rightarrow \mathfrak{A}$ *is a norm continuous surjective linear mapping onto a C*-algebra* $\mathfrak{A}$. *Then the weak* topology on* $\mathfrak{A}^{**}$ *is the coarsest topology* τ *such that for every linear functional* $\rho : \mathfrak{A} \rightarrow \mathbb{C}$ *whose composition with* $\mathcal{Q}$, $\rho \circ \mathcal{Q} : C_0(\mathcal{M}) \rightarrow \mathbb{C}$ *is continuous in the topology of pointwise convergence on* $C_0(\mathcal{M})$, *the map* $\hat{\rho} : \mathfrak{A}^{**} \rightarrow \mathbb{C}$ *defined by* $\hat{\rho}(A) = A(\rho)$ *for all* $A \in \mathfrak{A}$ *is continuous with respect to* τ.

This shows that the quantum algebras of observables $\mathfrak{A}$ and $\mathfrak{A}^{**}$ should be understood as analogous to $C_0(\mathcal{M})$ and $B(\mathcal{M})$. $\mathfrak{A}$ should be understood as a restricted subclass of bounded observables, and $\mathfrak{A}^{**}$ should be understood as a larger class of bounded observables that can be approximated by those in $\mathfrak{A}$ in a topology analogous to the topology of pointwise convergence from our classical theory. The problem that I point to in the next section is that one needs to be careful to ensure this correspondence holds—if one makes an unpropitious choice of the quantum algebra of observables, then one cannot understand the weak topology of that algebra as having physical significance, or equivalently, one cannot (at least in the absence of further argument) understand all states of the algebra as having physical significance.

Given this physical interpretation of the weak topologies, one can do real theoretical work by imposing continuity constraints on $\mathcal{Q}$. Namely, one would like to be able to extend the quantization map $\mathcal{Q}$ from $C_0(\mathcal{M})$ to its weak completion $B(\mathcal{M})$. This is important because the projection operators associated with the position and momentum observables belong to $B(\mathcal{M})$, but not $C_0(\mathcal{M})$. So in order to quantize the propositions associated with position and momentum measurements, one must extend the quantization map to these observables. It follows from Kadison and Ringrose Cor. 1.2.3 (1997, p. 15) that if $\mathcal{Q}$ is continuous in the weak topology on $C_0(\mathcal{M})$ and the weak topology on $\mathfrak{A}$, then $\mathcal{Q}$ has a unique extension $\tilde{\mathcal{Q}} : B(\mathcal{M}) \rightarrow \mathfrak{A}^{**}$ that is continuous in the weak* topology on $B(\mathcal{M})$ and the weak* topology on $\mathfrak{A}^{**}$. Of course, this extension is unique *only if* one imposes the continuity condition; otherwise one could extend $\mathcal{Q}$ in any which way. As a precondition for imposing the continuity condition, we must already require that $\mathcal{Q}$ is continuous in the weak topologies on $C_0(\mathcal{M})$ and $\mathfrak{A}$. Thus, it is desirable to use a quantization map that is weakly continuous. The weak topologies have real import for physics by constraining which maps $\mathcal{Q}$ we use and how we extend them to larger algebras.

4 Against the Weyl Algebra

In this section, we show that if the Weyl algebra $\mathcal{W}$ is used as the quantum algebra of observables $\mathfrak{A}$, the quantization map $\mathcal{Q}$ fails to be continuous in the appropriate topologies. Thus, we conclude that the weak topology on $\mathcal{W}$ is, in a sense, incompatible with the classical theory $\mathcal{W}$ derives from. However, there are alternatives that fare better; we show that Berezin quantization, which uses the algebra of compact operators on a Hilbert space as its quantum algebra, succeeds in being weakly continuous.

First, notice that the generators of the Weyl algebra were defined (formally) as functions of position and momentum: $U_a = e^{ia\mathcal{Q}(q)}$ and $V_b = e^{ib\mathcal{Q}(p)}$. We can consider these operators as the quantization of functions $u_a, v_b : \mathcal{M} \rightarrow \mathbb{C}$ in the classical theory, defined analogously by $u_a(q,p) = e^{iaq}$ and $v_b(q,p) = e^{ibp}$.[18] We

[18] Even though $\mathcal{Q}$ is only required to be linear and not to be a homomorphism (i.e., it is not required to preserve multiplication and thus not required to preserve functional relations *in general*), it seems natural to expect $\mathcal{Q}$ to preserve functional relations on abelian subalgebras of its range. In other words, it seems natural to expect $\mathcal{Q}(g(f)) = g(\mathcal{Q}(f))$ for $f \in C_0(\mathcal{M})$ and any Borel function $g : \mathbb{R} \rightarrow \mathbb{R}$, as in the usual functional calculus.

should already be wary because neither u_a nor v_b vanishes at infinity, and thus these observables lie outside of $C_0(\mathcal{M})$. Instead we have $u_a, v_b \in B(\mathcal{M}) \setminus C_0(\mathcal{M})$. Let us form the classical analog $\mathcal{W}_C$ of the Weyl algebra, the smallest commutative C*-algebra generated by the functions u_a, v_b for all $a, b \in \mathbb{R}$.

One especially important feature of the functions u_a, v_b is that they form pointwise continuous one-parameter unitary groups in the sense that $u_a, v_b \to 1$ in the topology of pointwise convergence as $a, b \to 0$, where 1 is the identity element of $B(\mathcal{M})$, i.e. the constant unit function. This fact allows us to reconstruct the observables q and p as the generators of the one-parameter families u_a, v_b. However, in the Weyl algebra, the one-parameter families U_a, V_b fail to be continuous in the weak topology. It is well known that there exists a state ω on $\mathcal{W}$ such that $\omega(U_a) \nrightarrow \omega(1)$ as $a \to 0$, and similarly there exists a state σ on $\mathcal{W}$ such that $\sigma(V_b) \nrightarrow \sigma(1)$ as $a \to 0$ (see Halvorson 2004). This immediately implies the following proposition.

Proposition 2 Let $\mathcal{Q} : \mathcal{W}_C \to \mathcal{W}$ *be a mapping such that*

$$\mathcal{Q}(u_a) = U_a \quad \text{and} \quad \mathcal{Q}(v_b) = V_b$$

Then $\mathcal{Q}$ is not continuous in the topology of pointwise convergence on $\mathcal{W}_C$ and the weak topology on $\mathcal{W}$.

This shows that the weak topology on the Weyl algebra fails to capture the notion of pointwise approximation that was inherent in the weak topology on the classical algebra $C_0(\mathcal{M})$.

One might object that we should not be looking at the topology of pointwise convergence, but rather the weak topology on $\mathcal{W}_C$, which is distinct. But this would miss the point—$\mathcal{W}_C$ is unlike $C_0(\mathcal{M})$ in that it contains 'states at infinity', which can be thought of as idealizations from the pure states in $\mathcal{M}$. The weak topology of $\mathcal{W}_C$ captures a notion of approximation that includes approximation of values of these idealized states. Thus, it is not at all clear that the weak topology on $\mathcal{W}_C$ is physically significant in the classical case. So I suggest that we ought to at least aim for a quantum analog to the classical topology of pointwise convergence, which would capture a notion of approximation of the values of manifestly significant physical states.

Lack of continuity in the weak topologies poses real problems. When $\mathcal{Q}$ fails to be continuous in the appropriate topologies, we do not have a unique way of extending $\mathcal{Q}$ to the rest of the classical observables. In particular, in the absence of further constraints, one cannot reconstruct the position and momentum observables because the one-parameter unitary groups U_a and V_b, failing to be weakly continuous, do not have generators in arbitrary representations of the algebra. Similarly, one cannot extend the quantization map in the way described previously to construct quantized propositions associated with position and momentum. This provides *prima facie* reason to look for alternatives to the Weyl algebra that might yield weakly continuous quantization procedures.

We do have other quantization procedures that stand in contrast by using a different quantum algebra for the range of $\mathcal{Q}$. In particular, Landsman (1998, 2006) explicitly uses as the quantum algebra of observables the C*-algebra $\mathfrak{A} = K(\mathcal{H})$ of compact operators on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, i.e. the square integrable functions on $\mathbb{R}$.[19] Landsman (1998, 2006) gives multiple examples of quantization maps $\mathcal{Q} : C_0(\mathcal{M}) \to K(\mathcal{H})$. Here we prove that one of those procedures, called *Berezin quantization*,[20] results in a weakly continuous quantization map $\mathcal{Q}$.

Following Landsman (2006, p. 460), Berezin quantization is given by the mapping $\mathcal{Q}^B : C_0(\mathcal{M}) \to K(\mathcal{H})$, which is defined as

$$\mathcal{Q}^B(f)\Phi(x) = \int_{\mathbb{R}^3} \frac{dp\, dq\, dy}{2\pi} f(q,p)\overline{\Psi_{(q,p)}(y)}\Phi(y)\Psi_{(q,p)}(x)$$

for all $f \in C_0(\mathcal{M})$ and $\Phi(x) \in \mathcal{H}$, where $\Psi_{(q,p)}(x) \in \mathcal{H}$ is a so-called 'coherent state', given by

$$\Psi_{(q,p)}(x) = \pi^{-1/4} e^{-ipq/2} e^{ipx} e^{-(x-q)^2/2}$$

The weak topology on $K(\mathcal{H})$ is familiar—because the pure states on $K(\mathcal{H})$ are precisely the vector states in $\mathcal{H}$, the weak topology corresponds with the weak operator topology. In other words, a net $\{A_i\} \subseteq K(\mathcal{H})$ converges to $A \in K(\mathcal{H})$ in the weak topology iff $\langle \psi, A_i \varphi \rangle \to \langle \psi, A\varphi \rangle$ for all $\psi, \varphi \in \mathcal{H}$. We can use this fact to prove that Berezin quantization is weakly continuous.

Proposition 3 *The Berezin quantization map $\mathcal{Q}^B : C_0(\mathcal{M}) \to K(\mathcal{H})$ is continuous in the topology of pointwise convergence on $C_0(\mathcal{M})$ and the weak topology on $K(\mathcal{H})$.*

Proof It suffices to show that if $\{f_i\} \subseteq C_0(\mathcal{M})$ is a net that converges pointwise to $f \in C_0(\mathcal{M})$, then $\mathcal{Q}^B(f_i)$ converges in the strong operator topology to $\mathcal{Q}^B(f)$, which implies that it converges in the weak operator topology and hence in the weak topology on $K(\mathcal{H})$. Hence, we will show that for all $\Phi(x) \in \mathcal{H}$, $\mathcal{Q}^B(f_i)\Phi(x) \to \mathcal{Q}^B(f)\Phi(x)$ in the norm topology on $\mathcal{H}$. Straightforward calculation shows that

$$\begin{aligned}\mathcal{Q}^B(f_i)\Phi(x) &= \frac{1}{2\pi^{3/2}} \int_{\mathbb{R}^3} dp\, dq\, dy\, f_i(q,p)\Phi(y) e^{ip(x-y)} e^{-(y-q)^2-(x-q)^2/2} \\ &= \frac{1}{2\pi^{3/2}} \int_{\mathbb{R}^3} dp\, dq\, dy\, f_i(q,p) g(x,y,q,p)\end{aligned}$$

where $g(x,y,q,p) = \Phi(y)e^{ip(x-y)}e^{-(y-q)^2-(x-q)^2/2}$ is in $L^2(\mathbb{R}^4)$, and similarly for $\mathcal{Q}^B(f)\Phi(x)$. It follows that

[19] This is meant only for the case of finitely many degrees of freedom on a simply connected phase space. It is well known that for systems with infinitely many degrees of freedom or with superselection rules, one requires a different algebra.

[20] Landsman (1998, 2006) also discusses another quantization map that he calls *Weyl quantization*. This map does *not* have the Weyl algebra as its image, and so it should be distinguished from the quantization map defined above from $\mathcal{W}_C$ to $\mathcal{W}$. It is worth reiterating that what is at issue in this paper is the quantum algebra that is the image of a quantization map, but we have not discussed the details of the different mappings one might use once we have fixed some quantum algebra.

$$||\mathcal{Q}^B(f_i)\Phi(x) - \mathcal{Q}^B(f)\Phi(x)||^2 = \frac{1}{2\pi^{3/2}}\int_{\mathbb{R}^4} dx\,dp\,dq\,dy\,|f_i(q,p) - f(q,p)|^2|g(x,y,q,p)|^2$$

The dominated convergence theorem for L^p spaces (Simonnet 1996, Thm. 5.2.2, p. 100) implies that the expression on the right hand side approaches zero as $i \to \infty$, which shows that $\mathcal{Q}^B(f_i)$ converges in the strong operator topology, and hence in the weak topology, to $\mathcal{Q}^B(f)$. □

This shows that the weak topology on $K(\mathcal{H})$ preserves the notion of approximation from the classical theory we began with. Or in other words, this shows that the states on $K(\mathcal{H})$ have physical significance because we can trace the topology they define back to the topology defined by the manifestly physically significant classical states in $\mathcal{M}$. Thus, the states on $K(\mathcal{H})$ have a much more secure status than states on $\mathcal{W}$.

Moreover, we can use the weak topology on $K(\mathcal{H})$ to do real work. The completion of $K(\mathcal{H})$ in the weak topology is $K(\mathcal{H})^{**} = \mathcal{B}(\mathcal{H})$, the collection of all bounded operators on $\mathcal{H}$. Thus, the quantization map $\mathcal{Q}^B$ extends uniquely to a map $\tilde{\mathcal{Q}}^B : B(\mathcal{M}) \to \mathcal{B}(\mathcal{H})$ that is continuous in the topology of pointwise convergence on $B(\mathcal{M})$ and the weak operator topology on $\mathcal{B}(\mathcal{H})$. This extension of the Berezin quantization map has in its image the projections associated with the position and momentum observables, which do not appear in the Weyl algebra. In fact, representations of the Weyl algebra will not in general contain projections associated with position and momentum unless one imposes further constraints. But the Hilbert space of a representation of the compact operators—the image of the Berezin quantization map—will always contain these projections, and so will always allow us to reconstruct the position and momentum observables.

This brings us back to the familiar situation for the quantum mechanics of a single particle, where our bounded observables correspond to all bounded operators on a Hilbert space. It is my contention that the weak continuity of the quantization map $\mathcal{Q}$ plays a fundamental role in this story of how we construct quantum theories. My suggestion is that even in more complex quantum theories, including ones with infinitely many degrees of freedom like field theories, we ought to prefer weakly continuous quantizations in the absence of arguments to the contrary.

5 Discussion

We have seen that the weak (respectively, weak*) topology on a C*-algebra (respectively, W*-algebra) of quantum observables can be given physical signif-cance by connecting it to the manifestly physically significant weak topology on $C_0(\mathcal{M})$. That is, a quantization map $\mathcal{Q}$ allows us to capture the notion of pointwise approximation on classical states within the quantum algebra. However, the weak topology on the Weyl algebra fails to capture this notion of approximation, or in other words fails to be related to the classical theory by an appropriate quantization procedure. I believe this provides reason to use an alternative algebra for our quantization procedures, one in which the preservation of the notion of pointwise approximation from classical physics ensures that we focus on states that are physically significant. As we saw, we already possess such an algebra in the form of the compact operators, which succeeds in capturing the topological information at issue, and hence the physicality of certain states.

I admit, though, that none of the considerations in this paper are decisive. In practice, when one uses the Weyl algebra, one restricts attention to Hilbert space representations satisfying a condition of *regularity* (see Petz 1990), which in turn guarantees that the weak operator topology on the Hilbert space has the appropriate relation to the weak topology of the classical theory. The problem is not that using the Weyl algebra constitutes some sort of error, but rather that using the Weyl algebra makes it easy to be misled. For example, one may be misled into thinking that the regularity condition is an unjustified assumption, as Halvorson (2004) claims. From the point of view of this paper, Halvorson's position gets things backwards. The regularity assumption is necessary to recover the topological information from the classical theory we began with. Thus, it would be a mistake to do away with the regularity assumption because one would lose important topological information, and thus important information about which states are physical and which are idealizations. Using the algebra of compact operators rather than the Weyl algebra is a way to safeguard against making this mistake.

I believe that the discussion of this paper demonstrates how mathematical aspects of our current theories can be used to constrain future scientific theorizing. I have assumed that the goal of quantization is to alter a classical theory *as little as possible* in order to make it into a quantum theory. This means that one should preserve, e.g. the topological structure of the classical theory, while altering minimally the algebraic structure by imposing the canonical commutation relations. Of course, one is free to question this assumption; one might assert that theorists should be free to explore whatever avenues seem profitable for constructing new quantum theories. Some have in fact argued that discontinuous quantization procedures are fruitful in developing theories of quantum gravity (Ashtekar 2009; Corichi et al. 2007). I do not wish to argue against such approaches—I only wish to point out that they are far more radical than they might at first seem. Rather than finding unexplored options already available in the standard approach to quantum mechanics, theorists who pursue discontinuous quantization procedures are proposing a major departure from the quantum theories we currently possess.

It should be clear by now that topological considerations are not the only ones that influence and constrain the construction of quantum theories. The topological considerations of this paper are intimately tied with positions on what we should take to be the appropriate state space of our theory. A choice of topology picks out a collection of continuous states in that topology, which one can then take to be the privileged collection of physical states. Part of the goal of this paper was to show that topological constraints can be used to rule out 'states at infinity', or non-regular states and non-regular representations. This can be seen as both a vice and a virtue of the position taken in this paper (after all, one person's modus ponens is another's modus tollens); advocates of non-regular states may take this very same implication as reason to reject the topological constraints discussed here. I believe there is much more to be said concerning the relationship between algebra, topology, and state space, but I must save this for future work.

Moreover, there are other types of constraints on the construction of quantum theories that I have not considered in this paper: for example, the preservation of classical symmetries, geometry, and symplectic structure.[21] While this is not the place for further discussion of these constraints (although see Landsman 1998, 2006), I hope that this paper has demonstrated that there is much for philosophers of physics to engage with, learn from, and contribute on these topics.

The fact that the structure of our current physical theories can be used as a methodological tool for constraining future theorizing, even in the absence of any data, ought to have implications for general philosophy of science. But there are still many open questions that need to be discussed to understand these implications. Does the fact that a future theory is constructed in order to respect the structure of current physics entail that we ought to put more credence in that new theory? Or does the fact that our theorizing relies so heavily on current physics only show that we are more likely to leave important alternatives unconceived? And can new physical theories constitute Kuhnian paradigm shifts if they are constructed directly out of the materials of our current theories? I hope that philosophers of science continue to think about these important questions. I also hope that philosophers of physics engage with these issues in the context of quantization, where their conclusions may make a real difference for the future of physics.

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[21] I thank an anonymous referee for this point.

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