

Recovering Quantum Logic Within an Extended Classical Framework

Claudio Garola · Sandro Sozzo

Received: 29 March 2011 / Accepted: 11 November 2011 / Published online: 13 December 2011
© Springer Science+Business Media B.V. 2011

Abstract We present a procedure which allows us to recover classical and non-classical logical structures as *concrete logics* associated with physical theories expressed by means of classical languages. This procedure consists in choosing, for a given theory \mathcal{T} and classical language \mathcal{L} expressing \mathcal{T} , an observative sublanguage L of \mathcal{L} with a notion of truth as correspondence, introducing in L a derived and theory-dependent notion of *C-truth* (*true with certainty*), defining a *physical preorder* \prec induced by C-truth, and finally selecting a set of sentences ϕ_V that are *verifiable* (or *testable*) according to \mathcal{T} , on which a *weak complementation* $^\perp$ is induced by \mathcal{T} . The triple $(\phi_V, \prec, ^\perp)$ is then the desired concrete logic. By applying this procedure we recover a classical logic and a standard quantum logic as concrete logics associated with classical and quantum mechanics, respectively. The latter result is obtained in a purely formal way, but it can be provided with a physical meaning by adopting a recent interpretation of quantum mechanics that reinterprets quantum probabilities as conditional on detection rather than absolute. Hence quantum logic can be considered as a mathematical structure formalizing the properties of the notion of verification in quantum physics. This conclusion supports the general idea that some nonclassical logics can coexist without conflicting with classical logic (*global pluralism*) because they formalize metalinguistic notions that do not coincide with the notion of truth as correspondence but are not alternative to it either.

C. Garola (✉)

Dipartimento di Fisica, dell'Università del Salento and Sezione INFN,
via Arnesano, 73100 Lecce, Italy
e-mail: Garola@le.infn.it

S. Sozzo

Center Leo Apostel (CLEA), Vrije Universiteit Brussel (VUB),
Krijkskondestraat 33, 1150 Brussels, Belgium
e-mail: Sozzo@le.infn.it; ssozzo@vub.ac.be

1 Introduction

After the birth of modern formal logic in the nineteenth century many nonstandard logics were proposed, from Heyting's *intuitionistic logic* (Heyting 1934, 1956) and Łukasiewicz's (1920) *many valued logic* to the recent *relevance logic* (Anderson and Belnap 1975; Anderson et al. 1992) and *linear logic* (Girard 1987).

In many cases these logics are interpreted as formalizing the features of notions of truth that are alternative to the classical notion of truth as correspondence formalized by Tarski (1933, 1944), and many logicians maintain that the *plurality of logics* should be considered as an important achievement of the twentieth century, which parallels the *plurality of geometries* which constitutes one of the revolutionary results of the mathematical research in the nineteenth century. Other logicians and philosophers instead uphold the thesis that nonstandard logics can be recovered as fragments of a suitable extension of classical logic (CL), in a unified view (*global pluralism*) which restores the unity of logic and avoids many quarrels about the notion of truth (Haack 1974, 1978). In this perspective, for instance, it has been proven that intuitionistic logic can be recovered as a part of a pragmatic extension of CL, intended to formalize the features of the notion of logical proof rather than the features of an intuitionistic nonclassical notion of truth (Dalla Pozza and Garola 1995).

In the physical realm, many nonstandard logics were propounded after the birth of quantum mechanics (QM) (Jammer 1974). One only of them, however, acquired growing importance in the literature, that is, the quantum logic (QL) propounded by Birkhoff and von Neumann (1936) as the underlying logic of QM, now often called *standard (sharp) QL*. This proposal indeed became rather popular because it seems to spring out directly from the mathematical formalism of QM. Moreover, many scholars maintain that some known paradoxes of QM could follow from an improper use of CL for dealing with the basic notions of QM because this theory would implicitly introduce a nonclassical notion of truth (*quantum truth* or, briefly, *Q-truth*) whose features are formalized by standard QL (Rédei 1998; Dalla Chiara et al. 2004). Hence a huge literature was produced on this topic, which is still flourishing nowadays.

Also in the specific case of standard QL, however, one may wonder whether a perspective of global pluralism can be adopted which allow to recover standard QL within an extended classical framework. In particular, one of us has observed in some previous papers that standard QL could be seen as the mathematical structure resulting from selecting a subset of sentences that are testable according to QM in the set of all sentences of a suitable classical language, that is, as the structure formalizing the features of the metalinguistic notion of *testability* in QM rather than a notion of Q-truth (Garola 1992, 2008). The aim of the present paper is to generalize and implement this view. To be precise, we intend to illustrate a procedure for obtaining a *concrete (theory-dependent) logic* associated with a physical theory \mathcal{T} expressed by means of a classical language \mathcal{L} with a notion of truth as correspondence. This procedure consists of four steps.

- (i) Consider an observational sublanguage L of \mathcal{L} and introduce a derived notion of C-truth (*true/false with certainty*) in L , defined in terms of classical truth but depending on \mathcal{T} .

- (ii) Define a *physical preorder* \prec on L , induced by the notion of C-truth.
- (iii) Introduce a notion of *verification* in L by selecting a subset ϕ_V of sentences of L that are *verifiable*, or *testable*, according to \mathcal{T} .
- (iv) Define a *weak complementation* $^\perp$ induced by \mathcal{T} on (ϕ_V, \prec) .

The structure $(\phi_V, \prec, ^\perp)$ is then the required concrete logic.

The above procedure is conceptually and philosophically relevant because it can explain the origins of some important nonstandard logics. Indeed one can obtain different concrete logics by changing some of its three basic elements, that is, \mathcal{T} , L and the notion of verification.

We illustrate our procedure in Sects. 2 and 3 by introducing a very simple observational (pretheoretical) language $L(x)$ which is suitable for expressing basic notions and relations in a wide class of physical theories. Then we consider two fundamental theories of modern physics, *i.e.*, classical mechanics (CM) and QM. We obtain in Sect. 4 a classical logical structure as the concrete logic associated with CM if all sentences of $L(x)$ are considered verifiable (in principle), as usual in CM. But we also prove by means of an example in Sect. 5 that one can obtain a concrete logic that exhibits a non-Boolean lattice structure in a macroscopic domain in which CM holds if one introduces a suitable extension $L^*(x)$ of $L(x)$ and selects a proper subset of sentences of $L^*(x)$ that are considered verifiable. Coming to QM we then show in Sect. 6 that our procedure allows us to recover standard QL as the concrete logic associated with QM (up to an equivalence relation) if a standard notion of verification according to QM is adopted. Therefore we can construct a quantum language $L_Q(x)$ in which quantum connectives occur and every sentence has both a truth value and a C-truth value. These values generally do not coincide, and the notion of C-truth can be interpreted as a derived notion which coexists with the classical notion of truth as correspondence. Finally we compare our language $L_Q(x)$ in Sect. 7 with the language L_Q constructed by means of standard procedures that do not make reference to classical truth (Rédei 1998; Dalla Chiara et al. 2004), and show that the notion of Q-truth introduced in L_Q is a *verificationist* (hence theory dependent) notion, while truth and verification are carefully distinguished in our approach.

Let us add now some comments and remarks on the results resumed above.

Firstly, we note that our approach does not conflict with *logical localism* (Dalla Chiara 1974; Putnam 1968), for it entails that many nonstandard logics can be obtained as concrete logics, depending both on the theory \mathcal{T} and the notion of verification that is considered. But it also supports global pluralism, because it implies that different concrete logics can coexist which do not clash with classical logic.

Secondly, we note that our procedure has to cope with an important objection. Indeed, a quantum physicist could argue that it recovers standard QL in a classical framework in a purely formal way because QM cannot be expressed by a classical language. More specifically, he would observe that we associate a set with every property of a physical system (the *extension* of the property) whose elements are interpreted as individual examples of the system possessing the property, while such a set cannot be defined according to the orthodox interpretation of QM. Physical properties are in fact *nonobjective* according to this interpretation, which means that

the set of individual examples of the physical system in a given state that display a given property whenever this property is measured depends on the set of measurements that are actually performed and is not prefixed (hence one cannot say that these examples “possess” the property before the measurement). To overcome this objection we preliminarily note that our formal derivation of standard QL is interesting anyway, for it is usually maintained that any derivation of this kind is impossible. More important, the objection vanishes if one accepts the point of view of some alternative interpretations of QM in which objectivity of physical properties is recovered. In particular, an extensive criticism of the theorems that aim to prove the *contextuality* and the *nonlocality* of QM [mainly the Bell–Kochen–Specker (Bell 1966; Kochen and Specker 1967) and the Bell (1964) theorems], hence nonobjectivity of physical properties, has been carried out by one of us together with other authors. It has been shown that the proofs of these theorems implicitly introduce an epistemological assumption which is problematic from the point of view of QM. If this assumption is relaxed, the proofs cannot be completed (Garola and Solombrino 1996a, b; Garola and Pykacz 2004). Basing on this conclusion, a new theoretical proposal called *extended semantic realism* (ESR) *model* has been recently worked out. The ESR model reinterprets quantum probabilities as conditional on detection rather than absolute and embodies the mathematical formalism of QM into a broader formalism which admits a local and noncontextual physical interpretation (Garola and Sozzo 2009, 2010, 2011a, b), recovering objectivity of physical properties. If this new perspective is adopted, our derivation of standard QL is not purely formal because its basic elements are provided with a physical interpretation.

Thirdly, we note that the elementary observational sublanguage $L(x)$ introduced in Sect. 2 takes into account only pure states and physical properties (*exact effects*) and does not contain logical quantifiers, to avoid clouding the conceptual aspects of our approach with technical complications. In principle, however, it can be extended to account for any kind of state, effect and empirical law.

2 The Language $L(x)$

Let \mathcal{T} be a physical theory in which the following notions are introduced (Beltrametti and Cassinelli 1981; Ludwig 1983).

Physical system.

Physical property of a physical system, operationally defined as a class of dichotomic registering devices that are physically equivalent according to \mathcal{T} .

Pure state of a physical system, operationally defined as a class of preparing devices that are physically equivalent according to \mathcal{T} .

Physical object, or individual example of a physical system, operationally defined as the activation of a given preparing device.

Furthermore, let us denote by c , \cap , \cup and \setminus set theoretical complementation, meet, join and difference, respectively. Then we introduce a *classical formal language* $L(x)$, intended to express basic notions and relations in \mathcal{T} , by means of standard definitions in CL, as follows.

Definition 2.1 The *alphabet* of $L(x)$ consists of the following elements.

Two disjoint sets of monadic predicates, $\mathcal{E} = \{E, F, \dots\}$ (intended interpretation: physical properties) and $\mathcal{S} = \{S, T, \dots\}$ (intended interpretation: pure states).

Individual variable x .

Connectives $\neg, \wedge, \vee, \rightarrow$.

Parentheses $(,)$.

Definition 2.2 The set $\psi(x)$ of all *well formed formulas* (wffs) of $L(x)$ is the set obtained by applying recursively *standard formation rules* in CL (to be precise, for every $A \in \mathcal{E} \cup \mathcal{S}$, $A(x) \in \psi(x)$; for every $\alpha(x) \in \psi(x)$, $\neg\alpha(x) \in \psi(x)$; for every $\alpha(x), \beta(x) \in \psi(x)$, $\alpha(x) \wedge \beta(x) \in \psi(x)$, $\alpha(x) \vee \beta(x) \in \psi(x)$, $\alpha(x) \rightarrow \beta(x) \in \psi(x)$). Furthermore, we put $\mathcal{E}(x) = \{E(x) \mid E \in \mathcal{E}\}$, $\mathcal{S}(x) = \{S(x) \mid S \in \mathcal{S}\}$ and call *atomic* wff of $\psi(x)$ every $\alpha(x) \in \mathcal{E}(x) \cup \mathcal{S}(x)$.

Definition 2.3 The *semantics* of $L(x)$ consists of the following elements.

Universe \mathcal{U} (intended interpretation: set of physical objects).

Injective mapping $ext : A \in \mathcal{E} \cup \mathcal{S} \mapsto ext(A) \in \mathcal{P}(\mathcal{U})$ (power set of \mathcal{U}).

Boolean lattice $\langle ext(\mathcal{E} \cup \mathcal{S}) \rangle$ generated by $ext(\mathcal{E} \cup \mathcal{S})$ via $^c, \cap, \cup$.

Recursive definition of the surjective mapping (still called ext by abuse of language)

$$ext : \alpha(x) \in \psi(x) \mapsto ext(\alpha(x)) \in \langle ext(\mathcal{E} \cup \mathcal{S}) \rangle$$

(to be precise, for every $E \in \mathcal{E}$, $ext(E(x)) = ext(E)$; for every $S \in \mathcal{S}$, $ext(S(x)) = ext(S)$; for every $\alpha(x) \in \psi(x)$, $ext(\neg\alpha(x)) = \mathcal{U} \setminus ext(\alpha(x)) = (ext(\alpha(x)))^c$; for every $\alpha(x), \beta(x) \in \psi(x)$, $ext(\alpha(x) \wedge \beta(x)) = ext(\alpha(x)) \cap ext(\beta(x))$, $ext(\alpha(x) \vee \beta(x)) = ext(\alpha(x)) \cup ext(\beta(x))$, $ext(\alpha(x) \rightarrow \beta(x)) = (ext(\alpha(x)))^c \cup ext(\beta(x))$).

Interpretation of the variable $\sigma : x \in \{x\} \mapsto \sigma(x) \in \mathcal{U}$.

Set Σ of all interpretations of the variable.

For every $\sigma \in \Sigma$, truth assignment

$$v_\sigma : \alpha(x) \in \psi(x) \mapsto v_\sigma(\alpha(x)) \in \{T, F\}$$

such that $v_\sigma(\alpha(x)) = T$ (F) iff $\sigma(x) \in ext(\alpha(x))$ ($\sigma(x) \in (ext(\alpha(x)))^c$).

Definition of truth (falsity): for every $\alpha(x) \in \psi(x)$, $\alpha(x)$ is *true* (*false*) in σ iff $v_\sigma(\alpha(x)) = T$ (F).

Definition 2.4 The binary relations of *logical preorder* \leq and *logical equivalence* \equiv on $\psi(x)$ are defined by setting, for every $\alpha(x), \beta(x) \in \psi(x)$,

$$\alpha(x) \leq \beta(x) \text{ iff (for every } \sigma \in \Sigma, v_\sigma(\alpha(x)) = T \text{ implies } v_\sigma(\beta(x)) = T)$$

and

$$\alpha(x) \equiv \beta(x) \text{ iff (for every } \sigma \in \Sigma, v_\sigma(\alpha(x)) = T \text{ iff } v_\sigma(\beta(x)) = T).$$

One can then prove some statements that are useful to compare the logical notions introduced in this section with the physical notions introduced in the following sections.

Proposition 2.1 (i) $ext(\psi(x)) = \langle ext(\mathcal{E} \cup \mathcal{S}) \rangle$, hence $(ext(\psi(x)), \cap, \cup, {}^c)$ is a Boolean algebra; equivalently, $(ext(\psi(x)), \subset, {}^c)$ is a Boolean lattice.

(ii) For every $\alpha(x), \beta(x) \in \psi(x)$,

$$\alpha(x) \leq \beta(x) \text{ iff } ext(\alpha(x)) \subset ext(\beta(x))$$

and

$$\alpha(x) \equiv \beta(x) \text{ iff } (\alpha(x) \leq \beta(x) \text{ and } \beta(x) \leq \alpha(x)) \text{ iff } ext(\alpha(x)) = ext(\beta(x)).$$

(iii) The equivalence relation \equiv is compatible with \neg, \wedge, \vee and \rightarrow , that is, for every $\alpha(x), \beta(x), \gamma(x), \delta(x) \in \psi(x)$, $\alpha(x) \equiv \beta(x)$ implies $\neg\alpha(x) \equiv \neg\beta(x)$, and $\alpha(x) \equiv \gamma(x)$ and $\beta(x) \equiv \delta(x)$ imply $\alpha(x) \wedge \beta(x) \equiv \gamma(x) \wedge \delta(x)$, $\alpha(x) \vee \beta(x) \equiv \gamma(x) \vee \delta(x)$ and $\alpha(x) \rightarrow \beta(x) \equiv \gamma(x) \rightarrow \delta(x)$.

(iv) The structure $(\psi(x)/\equiv, \wedge', \vee', \neg')$ (where \wedge', \vee' and \neg' denote the operations canonically induced on $\psi(x)/\equiv$ by \wedge, \vee and \neg , respectively) is a Boolean algebra isomorphic to $(ext(\psi(x)), \cap, \cup, {}^c)$ (Lindenbaum–Tarski algebra of $L(x)$); equivalently, $(\psi(x)/\equiv, \leq', \neg')$ (where \leq' is the order canonically induced on $\psi(x)/\equiv$ by the preorder \leq defined on $\psi(x)$) is a Boolean lattice isomorphic to $(ext(\psi(x)), \subset, {}^c)$.

3 General Physical Conditions on $L(x)$

The notions introduced in this section hold for any physical theory \mathcal{T} of the kind considered at the beginning of Sect. 2, are suggested by the intended interpretation of $L(x)$ and are partially illustrated by the drawing in Fig. 1. For the sake of brevity we shall understand the word “pure” when referring to states in the following.

Axiom P $\{ext(S) \mid S \in \mathcal{S}\}$ is a partition of \mathcal{U} .

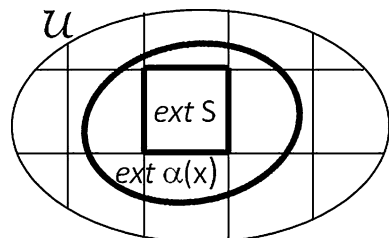
Physical justification. States are defined as equivalence classes of preparations (Sect. 2) and every physical object in \mathcal{U} is prepared by one and only one preparation, hence it belongs to one and only one extension of a state.

The following statement is then an immediate consequence of Axiom P.

Proposition 3.1 For every $\sigma \in \Sigma$, there is one and only one state $S_\sigma \in \mathcal{S}$ such that $\sigma(x) \in ext(S_\sigma)$ (equivalently, $v_\sigma(S_\sigma(x)) = T$).

In a physical theory \mathcal{T} of the kind considered in Sect. 2 it is customary to associate every physical property E with a set of states \mathcal{S}_E such that, if a physical object is prepared by a preparing device belonging to the state $S \in \mathcal{S}_E$, then it

Fig. 1 General representations of the extensions of wffs and states in the universe of all physical objects



displays the property E with certainty whenever a measurement of E is performed. The following definition generalizes this idea.

Definition 3.1 We denote by $\phi(x)$ the subset of wffs of $\psi(x)$ which are constructed by using only atomic wffs in $\mathcal{E}(x)$, hence do not contain predicates denoting states. Then, for every $\alpha(x) \in \phi(x)$, we call *physical proposition* associated with $\alpha(x)$ the set of states

$$p_\alpha = \{S \in \mathcal{S} \mid \text{ext}(S) \subset \text{ext}(\alpha(x))\}$$

and denote by \mathcal{P} the set of all physical propositions associated with wffs of $\phi(x)$.

The proof of the following statement is then straightforward.

Proposition 3.2 For every $\alpha(x) \in \phi(x)$,

$$p_\alpha = \{S \in \mathcal{S} \mid \text{for every } \sigma \in \Sigma, v_\sigma(S(x) \rightarrow \alpha(x)) = T\}.$$

By using the notion of physical proposition one can define a notion of *C-truth* on $\phi(x)$ (which depends on T because, for every $\alpha(x) \in \phi(x)$, the set of states in the physical proposition p_α depends on T).

Definition 3.2 For every $\alpha(x) \in \phi(x)$, let us denote by $p_{\neg\alpha}$ the physical proposition associated with $\neg\alpha(x)$. Then, for every $S \in \mathcal{S}$, we say that $\alpha(x)$ is *C-true* (*certainly true*) in S iff $S \in p_\alpha$, *C-false* (*certainly false*) in S iff $S \in p_{\neg\alpha}$. Furthermore, we say that $\alpha(x)$ has no C-truth value (or that $\alpha(x)$ is *C-indeterminate*) iff $S \notin p_\alpha \cup p_{\neg\alpha}$.¹

One can then easily prove the following statement.

Proposition 3.3 For every $S \in \mathcal{S}$ and $\alpha(x) \in \phi(x)$,

$\alpha(x)$ is C-true in S iff $\text{ext}(S) \subset \text{ext}(\alpha(x))$ iff (for every $\sigma \in \Sigma$, $S(x)$ is true in σ implies $\alpha(x)$ is true in σ) iff for every $\sigma \in \Sigma$, $v_\sigma(S(x) \rightarrow \alpha(x)) = T$,
and

$\alpha(x)$ is C-false in S iff $\text{ext}(S) \subset (\text{ext}(\alpha(x)))^c$ iff (for every $\sigma \in \Sigma$, $S(x)$ is true in σ implies $\neg\alpha(x)$ is true in σ) iff for every $\sigma \in \Sigma$ $v_\sigma(S(x) \rightarrow \neg\alpha(x)) = T$.

By using the notion of C-truth one can introduce new binary relations on $\phi(x)$, as follows.

Definition 3.3 We call *physical preorder* \prec and *physical equivalence* \approx the binary relations defined on $\phi(x)$ by setting, for every $\alpha(x), \beta(x) \in \phi(x)$,

$$\alpha(x) \prec \beta(x) \text{ iff (for every } S \in \mathcal{S}, \alpha(x) \text{ is C-true in } S \text{ implies } \beta(x) \text{ is C-true in } S)$$

and

$$\alpha(x) \approx \beta(x) \text{ iff (for every } S \in \mathcal{S}, \alpha(x) \text{ is C-true in } S \text{ iff } \beta(x) \text{ is C-true in } S).$$

¹ Of course, truth and C-truth are different notions in our approach. But it must be noted that the identification of *true* with *certain*, or *certainly true*, is basic in some approaches to the foundations of QM (in particular, in the Geneva–Brussels approach (Piron 1976; Aerts 1999)). We show in Sect. 4 that *true* and *certainly true* coincide in CM if pure states only are considered, which may lead one to overlook the deep difference between the two notions.

Furthermore, we denote by \prec' the order canonically induced on $\phi(x)/\approx$ by the preorder \prec defined on $\phi(x)$.

The proofs of the following statements are then straightforward.

Proposition 3.4 (i) For every $\alpha(x), \beta(x) \in \phi(x)$,

$$\alpha(x) \prec \beta(x) \text{ iff } p_\alpha \subset p_\beta$$

and

$$\alpha(x) \approx \beta(x) \text{ iff } p_\alpha = p_\beta \text{ iff } (\alpha(x) \prec \beta(x) \text{ and } \beta(x) \prec \alpha(x)).$$

(ii) $(\phi(x)/\approx, \prec')$ is order-isomorphic to (\mathcal{P}, \subset) .

(iii) Let $\alpha(x), \beta(x) \in \phi(x)$. Then

$$\alpha(x) \leq \beta(x) \text{ implies } \alpha(x) \prec \beta(x)$$

and

$$\alpha(x) \equiv \beta(x) \text{ implies } \alpha(x) \approx \beta(x).$$

We observe now that the intended interpretation of $L(x)$ implies that a wff $\alpha(x) \in \phi(x)$ can be verified whenever a dichotomic registering device exists which, for every $\sigma \in \Sigma$, may perform a measurement on $\sigma(x)$ specifying whether $\alpha(x)$ is true or false in σ . But a dichotomic registering device of this kind characterizes a physical property E in the theory \mathcal{T} (Sect. 2), hence we are led to introduce the following definition.

Definition 3.4 We call *set of verifiable*, or *testable*, wffs of $L(x)$ the subset

$$\phi_V(x) = \{\alpha(x) \in \phi(x) \mid \exists E_x \in \mathcal{E} : \alpha(x) \equiv E_x(x)\} \subset \phi(x)$$

and call *set of verifiable*, or *testable*, propositions of $L(x)$ the subset

$$\mathcal{P}_V = \{p_\alpha \in \mathcal{P} \mid \alpha(x) \in \phi_V(x)\} \subset \mathcal{P}.$$

Basing on Definition 3.4 one can prove at once the following statements.

Proposition 3.5 (i) $\phi_V(x) = \{\alpha(x) \in \phi(x) \mid \exists E_x \in \mathcal{E} : \text{ext}(\alpha(x)) = \text{ext}(E_x(x))\}$.

(ii) The following order structures and isomorphisms can be introduced basing on the logical and physical orders.

$$(\phi(x), \leq).$$

$$\phi(x)/\equiv, \leq', \text{ order-isomorphic to } (\text{ext}(\phi(x)), \subset).$$

$$(\phi_V(x), \leq).$$

$$\phi_V(x)/\equiv, \leq', \text{ order-isomorphic to } (\text{ext}(\mathcal{E}), \subset).$$

$$(\phi(x), \prec).$$

$$(\phi(x)/\approx, \prec'), \text{ order-isomorphic to } (\mathcal{P}, \subset).$$

$$(\phi_V(x), \prec).$$

$$(\phi_V(x)/\approx, \prec'), \text{ order-isomorphic to } (\mathcal{P}_V, \subset).$$

The crucial notion of logic associated with a physical theory \mathcal{T} , distinguished from (but related to) the logical structure of the language by means of which \mathcal{T} is expressed, can now be introduced by means of the following definition.

Definition 3.5 Whenever the theory \mathcal{T} induces a *weak complementation* $^\perp$ on $(\phi_V(x), \prec)$, i.e., a mapping $^\perp : \alpha(x) \in \phi_V(x) \mapsto (\alpha(x))^\perp \in \phi_V(x)$ such that $(\alpha(x))^{\perp\perp} \approx \alpha(x)$ and $\alpha(x) \prec \beta(x)$ implies $(\beta(x))^\perp \prec (\alpha(x))^\perp$, we say that the set $(\phi_V(x), \prec, ^\perp)$ is the concrete logic of \mathcal{T} .

We stress that the set $\phi_V(x)$ is selected by adopting a standard notion of verification that holds both in CM and in QM. Of course, different choices of $\phi_V(x)$ may lead one to associate different concrete logics with \mathcal{T} .

4 Classical Mechanics and Classical Logic

We intend to show in this section that the concrete logic of CM within the general classical approach sketched in Sects. 2 and 3 has the structure of a classical logic. To this end we preliminarily state the following fundamental axiom which establishes the mathematical representation of states and physical properties in CM.

Axiom CM1 Every physical system Ω is represented in CM by a triple $(\mathcal{F}, \varphi, \chi)$, where \mathcal{F} , φ and χ are defined as follows.

\mathcal{F} is a phase space associated with the physical system Ω .

$$\varphi : S \in \mathcal{S} \mapsto \varphi(S) \in \mathcal{F}.$$

$$\chi : E \in \mathcal{E} \mapsto \chi(E) \in \mathcal{P}(\mathcal{F}),$$

where $\mathcal{P}(\mathcal{F})$ is the power set of \mathcal{F} .

The mappings φ and χ are bijective.

We introduce now two axioms that can be justified in CM on the basis of the intended interpretation provided in Sect. 2. The first of them relates the mathematical representation in Axiom CM1 with the set-theoretical representations of states and physical properties provided in Sect. 2.

Axiom CM2 For every $E \in \mathcal{E}$ and $S \in \mathcal{S}$,

$$\varphi(S) \in \chi(E) \text{ iff } \text{ext}(S) \subset \text{ext}(E),$$

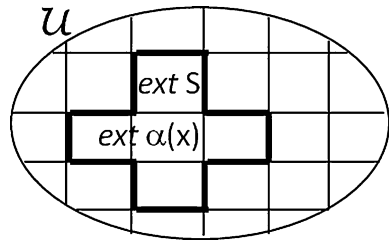
$$\varphi(S) \in \mathcal{F} \setminus \chi(E) \text{ iff } \text{ext}(S) \subset \mathcal{U} \setminus \text{ext}(E).$$

Physical justification. All physical objects in a given state (that is, prepared by preparations that are physically equivalent in the sense established by CM) either possess or do not possess a given physical property according to CM.

By using Axiom CM2 one can easily prove the following statements (which are partially illustrated by the drawing in Fig. 2).

Proposition 4.1 (i) For every $E \in \mathcal{E}$ and $S \in \mathcal{S}$, either $\text{ext}(S) \subset \text{ext}(E)$ or $\text{ext}(S) \subset \mathcal{U} \setminus \text{ext}(E) = (\text{ext}(E))^c$.

Fig. 2 Representations of the extensions of wffs and states in the universe of all physical objects in classical mechanics



(ii) For every $\alpha(x) \in \phi(x)$ and $S \in \mathcal{S}$, either $ext(S) \subset ext(\alpha(x))$ or $ext(S) \subset U \setminus ext(\alpha(x)) = (ext(\alpha(x)))^c$ (hence $ext(\alpha(x)) = \cup\{ext(S) \mid ext(S) \subset ext(\alpha(x))\}$).

(iii) For every $\sigma \in \Sigma$ and $\alpha(x) \in \phi(x)$,

$\alpha(x)$ is true (false) in σ iff $\alpha(x)$ is C-true (C-false) in S_σ iff $ext(S_\sigma) \subset ext(\alpha(x))$ ($ext(S_\sigma) \subset (ext(\alpha(x)))^c$) iff $S_\sigma \in p_x$ ($S_\sigma \in \mathcal{S} \setminus p_x$).

(iv) For every $\alpha(x), \beta(x) \in \phi(x)$,

$$\alpha(x) \leq \beta(x) \text{ iff } \alpha(x) \prec \beta(x)$$

and

$$\alpha(x) \equiv \beta(x) \text{ iff } \alpha(x) \approx \beta(x).$$

It is important to note that Proposition 4.1, (iii), shows that truth and C-truth coincide in CM. Indeed this coincidence explains why no distinction between the two notions of truth is made in CM.

The last axiom then characterizes the notion of verification in CM.

Axiom CM3 The set $\phi_V(x)$ of all verifiable wffs of $\phi(x)$ coincides with $\phi(x)$.

Physical justification. All physical statements about physical objects are testable, in principle, according to CM.

By using Axiom CM3 one can easily prove the following statements.

Proposition 4.2 (i) For every $\sigma \in \Sigma$ and $\alpha(x) \in \phi(x)$, $\alpha(x)$ is true (false) in σ iff $E_\alpha(x)$ is true (false) in σ iff $ext(S_\sigma) \subset ext(E_\alpha)$ ($ext(S_\sigma) \subset (ext(E_\alpha))^c$).

(ii) For every $\alpha(x), \beta(x) \in \phi(x)$,

$$\alpha(x) \leq \beta(x) \text{ iff } E_\alpha(x) \leq E_\beta(x),$$

and

$$\alpha(x) \equiv \beta(x) \text{ iff } E_\alpha(x) \equiv E_\beta(x).$$

(iii) $(\phi(x), \leq)$, $(\phi_V(x), \leq)$ $(\phi(x), \prec)$ and $(\phi_V(x), \prec)$ can be identified.

(iv) $(\phi(x)/\equiv, \leq')$, $(\phi_V(x)/\equiv, \leq')$, $(\phi(x)/\approx, \prec')$ and $(\phi_V(x)/\approx, \prec')$ can be identified, and they are order-isomorphic to $(ext(\phi(x)), \subset)$, $(ext(\mathcal{E}), \subset)$, (\mathcal{P}, \subset) and (\mathcal{P}_V, \subset) .

(v) The mapping $\perp : \alpha(x) \in \phi_V(x) \mapsto \neg\alpha(x) \in \phi_V(x)$

is a weak complementation on $(\phi_V(x), \prec)$.

(vi) The structure $(\phi_V(x), \prec, \perp)$ is the concrete logic of CM and coincides with $(\phi(x), \leq, \perp)$.

(vii) The structure $(\phi_V(x)/\approx, \prec', \perp') = (\phi(x)/\equiv, \leq', \perp')$ (where \perp' is the complementation canonically induced by \perp on $\phi_V(x)/\approx = \phi(x)/\equiv$) is a Boolean lattice that can be identified with the Boolean algebra $(\phi(x)/\equiv, \wedge', \vee', \neg')$.

The result in Proposition 4.2, (vi), explains from our present standpoint the common statement in the literature that “the logic of a classical physical system is classical logic”. We stress, however, that this statement follows from Axioms CM2 and CM3 and could not be proven should these axioms not hold.

For the sake of completeness we add some statements which establish further links between the mathematical representation in Axiom CM1 and the set-theoretical representations of states and physical properties in Sect. 2.

Proposition 4.3 (i) For every $E, F \in \mathcal{E}$,

$$\chi(E) \subset \chi(F) \text{ iff } \text{ext}(E) \subset \text{ext}(F).$$

(ii) For every $\sigma \in \Sigma$ and $\alpha(x) \in \phi(x)$,

$$\alpha(x) \text{ is true (false) in } \sigma \text{ iff } \varphi(S_\sigma) \in \chi(E_\alpha) \ (\varphi(S_\sigma) \in \mathcal{F} \setminus \chi(E_\alpha)).$$

(iii) For every $\alpha(x) \in \phi(x)$,

$$p_\alpha = \{S \in \mathcal{S} \mid \varphi(S) \in \chi(E_\alpha)\}.$$

5 Non-Boolean Structures in Classical Mechanics

Proposition 4.2, (vi), states that $(\phi_V(x)/\approx, \prec', \perp')$ is a Boolean lattice in CM which can be identified with the Boolean algebra $(\phi(x)/\equiv, \wedge', \vee', \neg')$. It is then important to observe that non-Boolean algebras can be obtained in CM if the testability criteria are suitably restricted. For instance, let us assume that not all wffs in $\phi_V(x)$, which are testable in principle, are testable in practice, and let us introduce the subset $\phi_V^P(x) \subset \phi_V(x)$ of wffs of $L(x)$ that are *practically testable*. In this case we can consider the preordered set $(\phi_V^P(x), \leq)$ and the quotient set $(\phi_V^P(x)/\equiv, \leq')$. Yet, the latter generally is not a Boolean lattice, and it can be equipped with different algebraic structures by choosing $\phi_V^P(x)$ in different ways (Garola 1992).

It is also interesting to note that there are examples of physical systems in the literature in which quantum structures are obtained in a macroscopic domain where CM holds. These examples are relevant because they falsify the widespread belief that QL characterizes QM, hence indirectly support our position in this paper. Let us therefore present briefly one of them, that is, Aerts’ *quantum machine* (Aerts 1988, 1991, 1995, 1998, 1999).

Aerts writes in (Aerts 1998):

The machine that we consider consists of a physical entity S that is a point particle P that can move on the surface of a sphere, denoted $surf$, with center O and radius 1. The unit-vector v where the particle is located on $surf$ represents the state p_v of the particle (...). For each point $u \in surf$, we introduce the following measurement e_u . We consider the diametrically opposite point $-u$, and install a piece of elastic of length 2, such that it is fixed with one of its end-points in u and the other end-point in $-u$. Once the elastic is installed, the particle P falls from its original place v orthogonally onto the elastic, and sticks on it (...). Then the elastic breaks and the particle P , attached to one of the two pieces of the elastic (...), moves to one of the two end-points u or $-u$ (...). Depending on whether the particle P arrives in u (...) or in $-u$, we give the outcome o_1^u or o_2^u to e_u . We can easily calculate the probabilities corresponding to the two possible outcomes. Therefore we remark that the particle P arrives in u when the elastic breaks in a point of the interval L_1 (which is the length of the piece of the elastic between $-u$ and the point where the particle has arrived, or $1 + \cos \theta$), [where θ is the angle between u and v] and arrives in $-u$ when it breaks in a point of the interval L_2 ($L_2 = L - L_1 = 2 - L_1$). We make the hypothesis that the elastic breaks uniformly, which means that the probability that the particle, being in state p_v , arrives in u , is given by the length of L_1 divided by the length of the total elastic (which is 2). The probability that the particle in state p_v arrives in $-u$ is the length of L_2 (which is $1 - \cos \theta$) divided by the length of the total elastic. If we denote these probabilities respectively by $P(o_1^u, p_v)$ and $P(o_2^u, p_v)$ we have:

$$P(o_1^u, p_v) = \frac{1 + \cos \theta}{2} = \cos^2 \frac{\theta}{2}$$

$$P(o_2^u, p_v) = \frac{1 - \cos \theta}{2} = \sin^2 \frac{\theta}{2}$$

These transition probabilities are the same as the ones related to the outcomes of a Stern–Gerlach spin measurement on a spin $\frac{1}{2}$ quantum particle, ...

Therefore Aerts concludes:

We can easily see now the two aspects in this quantum machine that we have identified in the hidden measurement approach to give rise to the quantum structure. The state of the particle P is effectively changed by the measuring apparatus (p_v changes to p_u or to p_{-u} under the influence of the measuring process), which identifies the first aspect, and there is a lack of knowledge on the interaction between the measuring apparatus and the particle, namely the lack of knowledge of where exactly the elastic will break, which identifies the second aspect. We can also easily understand now what is meant by the term ‘hidden measurements’. Each time the elastic breaks in one specific point λ , we could identify the measurement process that is carried out afterwards as a hidden measurement $e_{u,\lambda}$. The measurement e_u is then a classical mixture of

the collection of all measurement e_u^λ namely e_u consists of choosing at random one of the e_u^λ and performing this chosen e_u^λ .

Let us qualitatively discuss the above example from the point of view proposed in the present paper. It follows from Aerts' description that the physical system that is considered is a classical particle P of which a complete deterministic description is possible, in principle, in CM. But Aerts introduces hidden variables (the parameter λ) in his measurement processes, which must therefore be considered as *unsharp measurements*. Such measurements occur in actual physical situations, whenever ideal measurements that exactly test a given physical object has a given property do not exist in practice, and cannot be described by means of the simple language $L(x)$ introduced in Sect. 2. One must indeed extend the alphabet of $L(x)$ by adding new monadic predicates denoting *unsharp properties* (or *effects*), which can be considered in CM as new entities, introduced by generalizing the standard notion of physical property. An unsharp property E is a *contextual* property, in the sense that the knowledge that a given physical object belongs to the extension of a given state S is not sufficient to predict whether the object displays E when E is measured because the result of the measurement depends also on hidden variables associated with the measuring apparatus.² Hence Axiom CM2 does not hold and the physical preorder does not coincide with the logical preorder in the extended language $L^*(x)$. Moreover, it is apparent that Aerts considers as verifiable only sentences of $L^*(x)$ that are logically equivalent to atomic sentences containing predicates that denote very peculiar unsharp properties. This implies that the concrete logic that is obtained in this case is not a Boolean algebra. Indeed, Aerts' results entail that it is isomorphic to the QL of a spin- $\frac{1}{2}$ quantum particle, hence to the modular orthocomplemented lattice of all subspaces of \mathbb{C}^2 .

Our informal discussion of this example is thus completed. We add that Aerts and his coworkers have constructed similar models for arbitrary quantum entities (Aerts 1985, 1986, 1987) which can be used to illustrate further our procedures.

6 Quantum Mechanics and Quantum Logic

We intend to show in this section that standard QL can be recovered (up to an equivalence relation) as the concrete logic of QM within the general classical approach sketched in Sects. 2 and 3. To this end we preliminarily state the following fundamental axiom, which establishes the mathematical representation of states and

² A measurement can be described as an interaction between a physical object and a measuring apparatus in CM. In a real measurement the apparatus is in a mixed state because one never knows all its properties at a microscopic level, hence probabilities must be introduced in the theoretical description (which admit an *ignorance interpretation*, hence are epistemic). Unsharp properties and contextuality then occur if one wants to refer to the physical object only, avoiding a complete description of the interaction with the measuring apparatus (the "hidden measurement" processes in the case of Aerts' quantum machine). It must be noted, however, that a deeper form of nonlocal contextuality occurs in QM according to an orthodox view (Bell 1964; Greenberger et al. 1990; Mermin 1993) and that this kind of contextuality is avoided in the ESR model mentioned in Sect. 1 (Garola and Sozzo 2009, 2010, 2011a, b).

physical properties in QM in the case of quantum systems in which superselection rules do not occur.

Axiom QM1 Every physical system Ω is represented in QM by a triple $(\mathcal{H}, \psi, \omega)$, where \mathcal{H} , ψ and ω are defined as follows.

\mathcal{H} is a Hilbert space over the complex field \mathbb{C} associated with the physical system Ω .

$$\psi : S \in \mathcal{S} \mapsto \psi(S) \in \mathcal{L}_1(\mathcal{H}),$$

where $\mathcal{L}_1(\mathcal{H})$ is the set of all one-dimensional subspaces of \mathcal{H} .

$$\omega : E \in \mathcal{E} \mapsto \omega(E) \in \mathcal{L}(\mathcal{H}),$$

where $\mathcal{L}(\mathcal{H})$ is the set of all closed subspaces of \mathcal{H} .

The mappings ψ and ω are bijective.

The following statement is then well known in QM.

Proposition 6.1 The structure $(\mathcal{L}(\mathcal{H}), \subset, \perp)$, where \perp is the mapping which associates every closed subspace of $\mathcal{L}(\mathcal{H})$ with its orthogonal complement, is a complete orthomodular lattice, which is canonically equivalent to the algebra $(\mathcal{L}(\mathcal{H}), \cap, \cup, \perp)$ (where \cap and \cup denote the meet and join, respectively, in the lattice $(\mathcal{L}(\mathcal{H}), \subset, \perp)$).

The lattice $(\mathcal{L}(\mathcal{H}), \subset, \perp)$ or, equivalently, the algebra $(\mathcal{L}(\mathcal{H}), \cap, \cup, \perp)$, plays a crucial role in standard QL. Indeed it allows one to define a lattice structure on \mathcal{E} by means of the following definition.

Definition 6.1 We call *lattice of physical properties* in QM the orthomodular lattice $(\mathcal{E}, \subseteq, \perp)$, where \subseteq denotes the order and \perp the orthocomplementation induced on \mathcal{E} , via ω , by the order \subset and the orthocomplementation \perp , respectively, defined on $\mathcal{L}(\mathcal{H})$, and denote by $(\mathcal{E}, \cap, \cup, \perp)$ the algebra canonically equivalent to $(\mathcal{E}, \subseteq, \perp)$ and isomorphic to $(\mathcal{L}(\mathcal{H}), \cap, \cup, \perp)$.

We can now introduce a further lattice in QM by means of the following definition, which is standard in the foundations of QM (Beltrametti and Cassinelli 1981).

Definition 6.2 Let $E \in \mathcal{E}$. We call *certainly yes domain* of E the subset of states

$$\mathcal{S}_E = \{S \in \mathcal{S} \mid \psi(S) \subset \omega(E)\}$$

and put $\mathcal{P}_\mathcal{E} = \{\mathcal{S}_E \mid E \in \mathcal{E}\}$.

Then, the following proposition holds.

Proposition 6.2 The mapping

$$\rho : \mathcal{S}_E \in \mathcal{P}_\mathcal{E} \mapsto \omega(E) \in \mathcal{L}(\mathcal{H})$$

is bijective and, for every $E, F \in \mathcal{E}$, $\mathcal{S}_E \subset \mathcal{S}_F$ iff $\omega(E) \subset \omega(F)$. Therefore the structure $(\mathcal{P}_\mathcal{E}, \subset, \perp)$, where \perp denotes the orthocomplementation induced on $\mathcal{P}_\mathcal{E}$, via ρ , by the orthocomplementation \perp defined on $\mathcal{L}(\mathcal{H})$, is an orthomodular lattice

isomorphic to $(\mathcal{L}(\mathcal{H}), \subset, \perp)$. Then the algebra $(\mathcal{P}_{\mathcal{E}}, \cap, \cup, \perp)$ canonically equivalent to $(\mathcal{P}_{\mathcal{E}}, \subset, \perp)$ is isomorphic to $(\mathcal{L}(\mathcal{H}), \cap, \cup, \perp)$.

We have thus singled out three isomorphic algebras, that is, $(\mathcal{L}(\mathcal{H}), \cap, \cup, \perp)$, $(\mathcal{E}, \cap, \cup, \perp)$ and $(\mathcal{P}_{\mathcal{E}}, \cap, \cup, \perp)$, which only differ because of their supports and can be identified with the standard QL in the literature (which takes its name from the interpretation of the elements of $\mathcal{L}(\mathcal{H})$ proposed by Birkhoff and von Neumann (1936)).

We introduce now two axioms that can be justified in QM on the basis of the intended interpretation provided in Sect. 2. The first of them relates the mathematical representation in Axiom QM1 with the set-theoretical representation of states and physical properties provided in Sect. 2.

Axiom QM2 For every $E \in \mathcal{E}$ and $S \in \mathcal{S}$,

$$\begin{aligned} \psi(S) \in \omega(E) \text{ (equivalently, } S \in \mathcal{S}_E \text{ iff } \text{ext}(S) \subset \text{ext}(E), \\ \psi(S) \in \omega(E^\perp) \text{ (equivalently, } S \in \mathcal{S}_{E^\perp} \text{ iff } \text{ext}(S) \subset \mathcal{U} \setminus \text{ext}(E). \end{aligned}$$

Physical justification. For every physical property E there exist in QM infinitely many states such that the quantum probability of E in each of them is neither 0 nor 1. Hence there exist preparing devices that can be used to prepare ensembles of physical objects such that, for every ensemble, some elements display E whenever a measurement of E is performed and some do not. It follows that one cannot assert in QM that, for every state S and property E , a physical object prepared by a preparing device belonging to S either possesses or does not possess the property E . Therefore one can only characterize the sets of states for which one of the two alternatives occur.

By using Axiom QM2 one can easily prove the following statement.

Proposition 6.3 For every $\alpha(x) \in \phi_V(x)$ and $S \in \mathcal{S}$,

$$\alpha(x) \text{ is C-true (C-false) in } S \text{ iff } S \in \mathcal{S}_{E_\alpha} (S \in (\mathcal{S}_{E_\alpha})^\perp).$$

The last axiom then distinguishes the notion of verification in QM from the notion of verification in CM.

Axiom QM3 The set $\phi_V(x)$ of all verifiable wffs of $\phi(x)$ is strictly included in $\phi(x)$.

Physical justification. A nontrivial compatibility relation exists in QM which prohibits testing sentences as $E(x) \wedge F(x)$ if the physical properties E and F are not compatible.

The proof of the following statements is then straightforward.

Proposition 6.4 (i) For every $\alpha(x) \in \phi_V(x)$, $p_\alpha = \mathcal{S}_{E_\alpha}$, hence $\mathcal{P}_V = \mathcal{P}_{\mathcal{E}}$.

(ii) For every $E, F \in \mathcal{E}$,

$$E = F \text{ iff } E(x) \equiv F(x) \text{ iff } E(x) \approx F(x),$$

hence the equivalence relations \equiv and \approx coincide on $\phi_V(x)$.

(iii) The mapping

$$\perp^o : \alpha(x) \in \phi_V(x) \mapsto \alpha^{\perp^o}(x) = E_x^{\perp}(x) \in \phi_V(x)$$

is a weak complementation on $(\phi_V(x), \prec)$, hence $(\phi_V(x), \prec, \perp^o)$ is the concrete logic of QM.

(iv) For every $\alpha(x) \in \phi_V(x)$, the equivalence class $[\alpha(x)]_{\approx}$ contains one and only one wff of $\mathcal{E}(x)$, that is, $E_x(x)$, hence the mapping

$$\perp'^o : [\alpha(x)]_{\approx} \in \phi_V(x)/_{\approx} \mapsto [\alpha(x)]_{\approx}^{\perp'^o} = [\alpha^{\perp^o}(x)]_{\approx} = [E_x^{\perp}(x)]_{\approx} \in \phi_V(x)/_{\approx}$$

is well defined and bijective.

(v) The mapping

$$\xi : [\alpha(x)]_{\approx} \in \phi_V(x)/_{\approx} \mapsto p_x \in \mathcal{P}_{\mathcal{E}}$$

is an order isomorphism of $(\phi_V(x)/_{\approx}, \prec', \perp'^o)$ onto $(\mathcal{P}_{\mathcal{E}}, \subset, \perp)$ which maps $[\alpha(x)]_{\approx}^{\perp'^o}$ on p_x^{\perp} , hence $(\phi_V(x)/_{\approx}, \prec', \perp'^o)$ is a standard QL.

Proposition 6.4 shows, in particular, that the concrete logic of QM, obtained within a classical framework by means of the procedures expounded in Sects. 2 and 3, can be identified with a standard QL, up to an equivalence relation.

One can now construct a quantum language $L_Q(x)$ by enlarging $\phi_V(x)$ by means of new quantum connectives, as follows.

Definition 6.3 (i) The alphabet of $L_Q(x)$ consists of the following symbols.

All wffs of $\phi_V(x)$.

Quantum connectives $\neg_Q \wedge_Q$ and \vee_Q .

Parentheses $(,)$.

(ii) The set $\phi_Q(x)$ of all wffs of $L_Q(x)$ is the set obtained by applying recursively the following formation rules.

$$\phi_V(x) \subset \phi_Q(x).$$

$$\text{For every } \alpha(x) \in \phi_Q(x), \neg_Q \alpha(x) \in \phi_Q(x).$$

$$\text{For every } \alpha(x), \beta(x) \in \phi_Q(x) \alpha(x) \wedge_Q \beta(x) \in \phi_Q(x) \text{ and } \alpha(x) \vee_Q \beta(x) \in \phi_Q(x).$$

(iii) A truth and a C-truth assignment are introduced on $\phi_Q(x)$ by assuming that every $\alpha(x) \in \phi_Q(x)$ is logically equivalent to an atomic wff of $\mathcal{E}(x)$ obtained by applying recursively the replacement rule and the following semantic rules.

$$\text{For every } \alpha(x) \in \phi_V(x), \neg_Q \alpha(x) \equiv E_x^{\perp}(x).$$

$$\text{For every } \alpha(x), \beta(x) \in \phi_V(x), \alpha(x) \wedge_Q \beta(x) \equiv (E_x \cap E_{\beta})(x) \text{ and } \alpha(x) \vee_Q \beta(x) \equiv (E_x \cup E_{\beta})(x).$$

(iv) A logical preorder \leq and a logical equivalence \equiv are introduced on $\phi_Q(x)$ by referring to the truth assignment, proceeding as in Definition 2.4.

A physical preorder \prec and a physical equivalence \approx are introduced on $\phi_Q(x)$ by referring to the C-truth assignment, proceeding as in Definition 3.3.

The following statements can then be proved.

Proposition 6.5 (i) The equivalence relations \equiv and \approx coincide on $\phi_Q(x)$.

(ii) The equivalence relation \approx is compatible with \neg_Q, \wedge_Q and \vee_Q , that is, for

every $\alpha(x), \beta(x), \gamma(x), \delta(x) \in \phi_Q(x)$, $\alpha(x) \approx \beta(x)$ implies $\neg_Q \alpha(x) \approx \neg_Q \beta(x)$, and $\alpha(x) \approx \gamma(x)$ and $\beta(x) \approx \delta(x)$ imply $\alpha(x) \wedge_Q \beta(x) \approx \gamma(x) \wedge_Q \delta(x)$ and $\alpha(x) \vee_Q \beta(x) \approx \gamma(x) \vee_Q \delta(x)$.

(iii) The algebra $(\phi_Q(x)/\approx, \wedge'_Q, \vee'_Q, \neg'_Q)$ (where \wedge'_Q, \vee'_Q and \neg'_Q denote the operations canonically induced on $(\phi_Q(x)/\approx, \wedge_Q, \vee_Q$ and \neg_Q , respectively) is a standard QL isomorphic to $(\mathcal{P}_{\mathcal{E}}, \cap, \cup, \perp)$, $(\mathcal{E}, \cap, \cup, \perp)$ and $(\mathcal{L}(\mathcal{H}), \cap, \cup, \perp)$.

The following remarks are important.

- (i) Proposition 6.5, (iii), implies that the algebra $(\phi_Q(x)/\approx, \wedge'_Q, \vee'_Q, \neg'_Q)$ is canonically equivalent to a lattice which is isomorphic to $(\phi_V(x)/\approx, \wedge', \vee')$.
- (ii) Definition 6.5, (iii), implies that, for every $\sigma \in \Sigma$, every $\alpha(x) \in \phi_Q(x)$ has both a truth value and a C-truth value (if one takes C-indeterminate as a third C-truth value, see Definition 3.2) which, generally do not coincide, even if $\alpha(x)$ is true (false) in σ whenever it is C-true (C-false) in S_σ (see Proposition 3.3). Hence the notion of C-truth can be interpreted as a derived notion, which applies to verifiable sentences only and is not alternative to the classical notion of truth as correspondence.
- (iii) The meaning of the derived and theory-dependent quantum connectives (which is determined by QM) must be clearly distinguished from the meaning of classical connectives, in agreement with the known principle of Quine, “change of logic, change of subject” (Garola 1992; Quine 2006).

We have thus accomplished our task. We recall however from Sect. 1 that our procedure for recovering standard QL can be charged to be purely formal if one adopts the orthodox interpretation of QM. To overcome this objection we have observed that a physical interpretation of our procedure can be given by referring to the recent proposal of a theory (*ESR model*) which generalizes QM embedding its mathematical apparatus into a broader mathematical formalism and reinterpreting quantum probabilities as conditional on detection rather than absolute (Garola and Sozzo 2009, 2010, 2011a, b). We can be more precise here and specify that objectivity of physical properties, which holds in the ESR model, allows one to provide, for every $E \in \mathcal{E}$, a physical interpretation of $ext(E)$ as the set of all physical objects that possess the property E independently of any measurement. This interpretation is obviously impossible if the orthodox interpretation of quantum probabilities is maintained.

7 The Orthodox Approach to Standard Quantum Logic

Because of the objection discussed at the end of Sect. 6, orthodox quantum logicians avoid associating an extension with the predicates denoting physical properties.³ Hence they introduce standard QL by adopting procedures that are mainly based on

³ Note that a similar objection does not occur in the case of predicates denoting states, because the extension $ext(S)$ of a state $S \in \mathcal{S}$ can be interpreted as the set of all physical objects that are actually prepared in the state S .

the mathematical representation of states and effects in QM (Rédei 1998; Dalla Chiara et al. 2004). To allow a comparison with our approach we sketch in this section an introduction to standard QL which particularizes and simplifies the methods described in (Dalla Chiara et al. 2004).

Let us consider a quantum system Ω and the notions of pure state, physical property and physical object as defined in QM (Beltrametti and Cassinelli 1981; Ludwig 1983). Then, let us denote by L_Q a formal language, intended to express basic notions and relations in QM, constructed as follows.

Definition 7.1 The *alphabet* of L_Q consists of the following elements.

A set $\mathcal{E} = \{E, F, \dots\}$ of atomic sentences (intended interpretation: physical properties).

Connectives \neg_Q, \wedge_Q, \vee_Q .

Parentheses $(,)$.

Definition 7.2 The set ϕ_Q of all *well formed formulas* (wffs) of L_Q is the set obtained by applying recursively standard *formation rules* (to be precise, for every $E \in \mathcal{E}, E \in \phi_Q$; for every $\alpha \in \phi_Q, \neg_Q \alpha \in \phi_Q$; for every $\alpha, \beta \in \phi_Q, \alpha \wedge_Q \beta \in \phi_Q$ and $\alpha \vee_Q \beta \in \phi_Q$).

Definition 7.3 By referring to the algebraic structure $(\mathcal{P}_{\mathcal{E}}, \cap, \cup, \perp)$ introduced in Sect. 6, Proposition 6.2, a *physical proposition* p_α is associated with every $\alpha \in \phi_Q$, recursively defined as follows.

For every $E \in \mathcal{E}, p_E = \mathcal{S}_E$.

For every $\alpha \in \phi_Q, p_{\neg_Q \alpha} = (p_\alpha)^\perp$.

For every $\alpha, \beta \in \phi_Q, p_{\alpha \wedge_Q \beta} = p_\alpha \cap p_\beta, p_{\alpha \vee_Q \beta} = p_\alpha \cup p_\beta$.

The proof of the following statement is then straightforward.

Proposition 7.1 For every $\alpha \in \phi_Q$ there exists a unique $E_\alpha \in \mathcal{E}$ such that $p_\alpha = p_{E_\alpha}$, and the set \mathcal{P}_Q of all physical propositions associated with wffs of ϕ_Q coincides with $\mathcal{P}_{\mathcal{E}}$.

The notion of Q-truth can now be introduced by means of the following definition.

Definition 7.4 For every $\alpha \in \phi_Q$ and $S \in \mathcal{S}, \alpha$ is *Q-true* (quantum true) in S iff $S \in p_\alpha, \alpha$ is *Q-false* (quantum false) in S iff $S \in (p_\alpha)^\perp, \alpha$ has no Q-truth value (equivalently, α is *Q-indeterminate*) iff $S \notin p_\alpha \cup (p_\alpha)^\perp$.

One can then prove the following nontrivial statement.

Proposition 7.2 For every $\alpha \in \phi_Q$ and $S \in \mathcal{S},$

(α has a Q-truth value in S) iff $\psi(S) \subset \omega(E_\alpha) \cup (\omega(E_\alpha))^\perp$ iff (the probability of the property E_α in the state S is either 1 or 0) iff (a measurement of the property E_α exists which does not modify the state S).

Proof The first equivalence follows from Definition 7.4, Axiom QM1 and Proposition 6.2. The second equivalence follows from the standard rules of QM. The third equivalence is proven in (Garola and Sozzo 2004).

Definition 7.5 The binary relations of *quantum preorder* \leq_Q and *quantum equivalence* \equiv_Q on ϕ_Q are defined by setting, for every $\alpha, \beta \in \phi_Q$,

- $\alpha \leq_Q \beta$ iff (for every $S \in \mathcal{S}$, α is Q-true in S implies β is Q-true in S)
- and
- $\alpha \equiv_Q \beta$ iff (for every $S \in \mathcal{S}$, α is Q-true in S iff β is Q-true in S).

The proof of the following statements is then straightforward.

Proposition 7.3 (i) For every $\alpha, \beta \in \phi_Q$,

- $\alpha \leq_Q \beta$ iff $p_\alpha \subset p_\beta$ iff (for every $S \in \mathcal{S}$, β is Q-false in S implies α is Q-false in S)
- iff $(p_\beta)^\perp \subset (p_\alpha)^\perp$

and

- $\alpha \equiv_Q \beta$ iff ($\alpha \leq_Q \beta$ and $\beta \leq_Q \alpha$) iff $p_\alpha = p_\beta$ iff (for every $S \in \mathcal{S}$, β is Q-false in S iff α is Q-false in S) iff $(p_\beta)^\perp = (p_\alpha)^\perp$.

(ii) The equivalence relation \equiv_Q is compatible with \neg_Q, \wedge_Q and \vee_Q (to be precise, for every $\alpha, \beta \in \phi_Q$, $\alpha \equiv_Q \beta$ implies $\neg_Q \alpha \equiv_Q \neg_Q \beta$; for every $\alpha, \beta, \gamma, \delta \in \phi_Q$, $\alpha \equiv_Q \gamma$ and $\beta \equiv_Q \delta$ imply $\alpha \wedge_Q \beta \equiv_Q \gamma \wedge_Q \delta$ and $\alpha \vee_Q \beta \equiv_Q \gamma \vee_Q \delta$).

(iii) Let $\wedge'_Q, \vee'_Q, \neg'_Q$ denote the operations canonically induced on ϕ_Q/\equiv_Q by \wedge_Q, \vee_Q, \neg_Q , respectively. Then, the mapping $\zeta : [\alpha]_{\equiv_Q} \in \phi_Q/\equiv_Q \mapsto p_\alpha \in \mathcal{S}_{E_x}$ is an isomorphism of $(\phi_Q/\equiv_Q, \wedge'_Q, \vee'_Q, \neg'_Q)$ onto $(\mathcal{P}_{\mathcal{E}}, \cap, \cup, \perp)$ (standard QL).

We have thus concluded our short presentation of the orthodox approach to QL. By comparing our general methods in Sects. 2 and 3 with this approach we can single out some relevant similarities and differences.

(i) The language L_Q is a propositional logic in which no reference is done to individual examples of physical systems, and states are considered in the orthodox approach as possible worlds of a Kripkean semantics, not as predicates.

(ii) The assignment of a Q-truth value to a wff α of L_Q is given resorting to the physical proposition p_α associated with α , hence it parallels the assignment of a C-truth value to a wff $\alpha(x)$ of $L(x)$ introduced in Definition 3.2 rather than the assignment of a truth value introduced in Definition 2.3.

(iii) Proposition 7.2 shows that the definition of Q-truth introduces a notion of truth as verification, which is considered problematic by many logicians and philosophers (Russell 1940; Pap 1961; Popper 1963; Lycan 2000) and is at odds with our choice of introducing a notion of truth as correspondence in Sects. 2 and 3, carefully distinguishing between truth and verification. Consistently, the set \mathcal{P}_Q of all physical propositions associated with wffs of ϕ_Q coincides with the set $\mathcal{P}_{\mathcal{E}}$ of all physical propositions associated with atomic wffs of ϕ_Q , which implies that only verifiable quantum sentences have been taken into account from the very beginning when constructing L_Q .

Acknowledgments The authors are greatly indebted with Prof. Carlo Dalla Pozza and Dr. Marco Persano for reading the manuscript and providing useful remarks and suggestions.

References

- Aerts, D. (1985). A possible explanation for the probabilities of quantum mechanics and a macroscopic situation that violates Bell inequalities. In Mittelstaedt, P., et al. (Eds.), *Recent developments in quantum logic* (pp. 235–251). Mannheim: Bibliographisches Institut.
- Aerts, D. (1986). A possible explanation for the probabilities of quantum mechanics. *Journal of Mathematical Physics*, 27, 202–210.
- Aerts, D. (1987). The origin of the non-classical character of the quantum probability model. In Blanquiere, A., et al. (Eds.), *Information, complexity and control in quantum physics* (pp. 77–100). New York: Springer.
- Aerts, D. (1988). The physical origin of the EPR paradox and how to violate Bell inequalities by macroscopic systems. In Lahti, P., et al. (Eds.), *Symposium on the foundations of modern physics* (pp. 305–320). Singapore: World Scientific.
- Aerts, D. (1991). A macroscopic classical laboratory situation with only macroscopic classical entities giving rise to a quantum mechanical probability model. In Accardi, L. (Ed.), *Quantum probability and related topics* (pp. 75–85). Singapore: World Scientific.
- Aerts, D. (1995). Quantum structures: An attempt to explain their appearance in nature. *International Journal of Theoretical Physics*, 34, 1165–1186.
- Aerts, D. (1998). The hidden measurement formalism: What can be explained and where quantum paradoxes remain. *International Journal of Theoretical Physics*, 37, 291–304.
- Aerts, D. (1999). Quantum mechanics: Structures, axioms and paradoxes. In Aerts, D., & Pykacz, J. (Eds.), *Quantum physics and the nature of reality* (pp. 141–205). Dordrecht: Kluwer.
- Anderson, A. R., & Belnap, N. D. (1975). *Entailment: The logic of relevance and necessity* (Vol. I). Princeton: Princeton University Press.
- Anderson, A. R., Belnap, N. D., & Dunn, J. M. (1992). *Entailment: The logic of relevance and necessity* (Vol. II). Princeton: Princeton University Press.
- Bell, J. S. (1964). On the Einstein–Podolsky–Rosen paradox. *Physics*, 1, 195–200.
- Bell, J. S. (1966). On the problem of hidden variables in quantum mechanics. *Review of Modern Physics*, 38, 447–452.
- Beltrametti, E. G., & Cassinelli, G. (1981). *The logic of quantum mechanics*. Reading, MA: Addison.
- Birkhoff, G., & von Neumann, J. (1936). The logic of quantum mechanics. *Annals of Mathematics*, 37, 823–843.
- Dalla Chiara, M. L. (1974). *Logica*. Milano: ISEDI.
- Dalla Chiara, M. L., Giuntini, R., & Greechie, R. (2004). *Reasoning in quantum theory*. Dordrecht: Kluwer.
- Dalla Pozza, C., & Garola, C. (1995). A pragmatic interpretation of intuitionistic propositional logic. *Erkenntnis*, 43, 81–109.
- Garola, C. (1992). Truth versus testability in quantum logic. *Erkenntnis*, 37, 197–222.
- Garola, C. (2008). Physical propositions and quantum languages. *International Journal of Theoretical Physics*, 47, 90–103.
- Garola, C., & Pykacz, J. (2004). Locality and measurements within the SR model for an objective interpretation of quantum mechanics. *Foundations of Physics*, 34, 449–475.
- Garola, C., & Solombrino, L. (1996a). The theoretical apparatus of semantic realism: A new language for classical and quantum physics. *Foundations of Physics*, 26, 1121–1164.
- Garola, C., & Solombrino, L. (1996b). Semantic realism versus EPR-like paradoxes: The Furry, Bohm–Aharonov, and Bell paradoxes. *Foundations of Physics*, 26, 1329–1356.
- Garola, C., & Sozzo, S. (2004). A semantic approach to the completeness problem in quantum mechanics. *Foundations of Physics*, 34, 1249–1266.
- Garola, C., & Sozzo, S. (2009). The ESR model: A proposal for a noncontextual and local Hilbert space extension of QM. *Europhysics Letters*, 86, 20009.
- Garola, C., & Sozzo, S. (2010). Embedding quantum mechanics into a broader noncontextual theory: A conciliatory result. *International Journal of Theoretical Physics*, 49, 3101–3117.

- Garola, C., & Sozzo, S. (2011a). Generalized observables, Bell's inequalities and mixtures in the ESR model. *Foundations of Physics*, 41, 424–449.
- Garola, C., & Sozzo, S. (2011b). Extended representations of observables and states for a noncontextual reinterpretation of QM. *ArXiv:1107.2271v2 [quant-ph]*.
- Girard, J. Y. (1987). Linear logic. *Theoretical Computer Science*, 50, 1–102.
- Greenberger, D. M., Horne, M. A., Shimony, A., & Zeilinger, A. (1990). Bell's theorem without inequalities. *American Journal of Physics*, 58, 1131–1143.
- Haack, S. (1974). *Deviant logic*. Cambridge: Cambridge University Press.
- Haack, S. (1978). *Philosophy of logic*. Cambridge: Cambridge University Press.
- Heyting, A. (1934). *Matematische Grundlagenforschung, Intuitionismus, Beweistheorie. Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3, Berlin.
- Heyting, A. (1956). *Intuitionism. An introduction*. Amsterdam: North-Holland.
- Jammer, M. (1974). *The philosophy of quantum mechanics*. New York: Wiley.
- Kochen, S. & Specker, E. P. (1967). The problem of hidden variables in quantum mechanics. *Journal of Mathematics and Mechanics*, 17, 59–87.
- Ludwig, G. (1983). *Foundations of quantum mechanics I*. Berlin: Springer.
- Łukasiewicz, J. (1920). O logice trójwartościowej. *Ruch Filozoficzny*, 5, 169–171; (1970. On three-valued logic. In L. Borkowski (Ed.), *Jan Łukasiewicz, selected works* (pp. 87–88). Amsterdam: North-Holland Publishing Company, trans).
- Lycan, W. (2000). *Philosophy of language: A contemporary introduction*. London: Routledge.
- Mermin, N. D. (1993). Hidden variables and the two theorems of John Bell. *Reviews of Modern Physics*, 65, 803–815.
- Pap, A. (1961). *An introduction to the philosophy of science*. New York: The Free Press.
- Piron, C. (1976). *Foundations of quantum physics*. Reading: W. A. Benjamin, Inc.
- Popper, K. (1963). *Conjectures and refutations*. London: Routledge and Kegan Paul.
- Putnam, H. (1968). Is logic empirical? In Cohen R. S. and Wartofsky, M. W. (Eds.), *Boston studies in the philosophy of science* (Vol. 5, pp. 216–241). Dordrecht: Reidel.
- Quine, W. V. O. (2006). *Philosophy of logic*. Cambridge: Harvard University Press.
- Rédei, M. (1998). *Quantum logic in algebraic approach*. Dordrecht: Kluwer.
- Russell, B. (1940). *An inquiry into meaning and truth*. New York: W. W. Norton & Company.
- Tarski, A. (1933). Pojęcie prawdy w językach nauk dedukcyjnych. *Acta Towarzystwa Naukowego i Literackiego Warszawskiego*, 34, V–16; (1956. The concept of truth in formalized languages. In J. M. Woodger (Ed.), *Logic, semantics, metamathematics* (pp. 152–268). Oxford: Oxford University Press, trans).
- Tarski, A. (1944). The semantic conception of truth and the foundations of semantics. *Philosophy and phenomenological research*, 4, 341–375 (1952. In L. Linsky (Ed.), *Semantics and the philosophy of language* (pp. 13–47). Urbana: University of Illinois Press).