Erkenn (2011) 75:183–202
DOI 10.1007/s10670-011-9290-2

ORIGINAL ARTICLE

Verisimilitude and Belief Change for Conjunctive Theories

Gustavo Cevolani · Vincenzo Crupi · Roberto Festa

Received: 1 February 2010 / Accepted: 13 November 2010 / Published online: 22 June 2011

Abstract Theory change is a central concern in contemporary epistemology and philosophy of science. In this paper, we investigate the relationships between two ongoing research programs providing formal treatments of theory change: the (post-Popperian) approach to verisimilitude and the AGM theory of belief change. We show that appropriately construed accounts emerging from those two lines of epistemological research do yield convergences relative to a specified kind of theories, here labeled "conjunctive". In this domain, a set of plausible conditions are identified which demonstrably capture the verisimilitudinarian effectiveness of AGM belief change, i.e., its effectiveness in tracking truth approximation. We conclude by indicating some further developments and open issues arising from our results.

G. Cevolani (✉)
Department of Philosophy, University of Bologna, Bologna, Italy
e-mail: g.cevolani@gmail.com

Present Address:
G. Cevolani
Piazza Sassatelli 26, 41057 Spilamberto, MO, Italy

V. Crupi
Center for Mathematical Philosophy, Ludwig Maximilian University, Munich, Germany
e-mail: vincenzo.crupi@lrz.unimuenchen.de

R. Festa
Department of Philosophy, University of Trieste, Trieste, Italy
e-mail: festa@units.it

1 Introduction

In many forms of inquiry theory change is a plain fact, most vividly illustrated by the history of science. Accordingly, an issue of increasing interest among epistemology scholars in modern and contemporary times has been how to interpret theory change in terms of progress towards some paramount cognitive goal, often meant as the truth. Karl Popper, for one, famously identified progressive change as a crucial distinctive trait of scientific inquiry. In Popper's words: "science is one of the very few human activities—perhaps the only one—in which [...] we can speak clearly and sensibly about making progress [...]. In most other fields of human endeavour there is change, but rarely progress" (Popper 1963, pp. 216–217). And even in philosophical quarters as far from Popperian realism as Peirce's pragmatism, approaching the truth still features as the driving force of scientific inquiry (e.g., Kelly and Glymour 1989).

In this paper, we will investigate the relationships between two ongoing research programs in epistemology, tracing back to the late twentieth century and providing formal accounts of theory change: the (post-Popperian) theories of verisimilitude (or truth approximation)[1] and so-called AGM theory of belief change.[2]

Theories of verisimilitude and AGM theory stem from very different perspectives on the goals of rational inquiry. On one side, the notion of truth clearly retains a central role in the formal treatment of truth approximation. AGM theorists, by contrast, have traditionally regarded "the concepts of truth and falsity" as "irrelevant for the analysis of belief systems" (Gärdenfors 1988, p. 20).[3] Our present purpose is to explore the possibility, scope and limitations of a convergence between the two approaches. The guiding question will thus be whether and to what extent AGM theory change effectively tracks truth approximation—an issue originally raised by Niiniluoto (1999).

Our discussion will proceed as follows. First we will define the framework of (propositional) "conjunctive theories" and advocate a fundamental theoretical principle as motivated within a "basic feature" approach to verisimilitude (Sect. 2). Accordingly, we introduce both a simple qualitative definition of "more verisimilar" and a continuum of verisimilitude measures, here called "contrast measures" of verisimilitude. In Sect. 3, after presenting the main principles underlying the AGM approach to theory change, we will derive a relevant and convenient application of the "belief base" version of this approach (Sect. 3.1) to conjuntive theory change

[1] The first full-fledged account of verisimilitude was provided by Karl Popper (1963). Later, Miller (1974) and Tichý (1974) showed that Popper's account was untenable, thus opening the way to post-Popperian theories of verisimilitude, emerging ever since 1975. For an excellent survey of the modern history of verisimilitude, see Niiniluoto (1998).

[2] In the literature, the labels "belief dynamics" and "belief revision" are also often employed to denote AGM theory, named after Alchourrón et al. (1985). Gärdenfors (1988) and Hansson (1999) represent the first dedicated monograph and the first textbook presentation, respectively.

[3] There are exceptions, however: Hans Rott, for instance, recently remarked that AGM theorists "should worry more about truth" meant as one of the basic aims of scientific inquiry; see Rott (2000, p. 513, 518 and ff., and in particular note 38).

(Sect. 3.2). Finally (Sect. 4), we will explore and discuss the convergence between the basic feature approach and AGM conjunctive theory change. More specifically, a set of plausible conditions will be identified which demonstrably capture the effectiveness of AGM operations to track verisimilitude as defined in both qualitative and quantitative terms within the basic feature approach.

2 The Basic Feature Approach to Verisimilitude

In some recent papers, the present authors have developed a "basic feature approach"—for short, "BF-approach"—to verisimilitude.[4] In this section, we shall briefly outline the BF-approach and the corresponding notion of truth approximation, which will then be applied to the analysis of the verisimilitudinarian effectiveness of AGM theory change.

Suppose that the *basic features* of the domain under inquiry $\mathcal{U}$ ("the world") are described by a language $\mathcal{L}$. Then, "the whole truth"—or simply "the truth"—about $\mathcal{U}$ in $\mathcal{L}$ can be construed as *the most complete true description* (*in* $\mathcal{L}$) *of the basic features* of $\mathcal{U}$. Given a theory T in $\mathcal{L}$, the *basic content* of T can be seen as the information conveyed by T about the basic features of $\mathcal{U}$. Then, according to the BF-approach, the *verisimilitude* of T is interpreted in terms of the balance of true and false information transmitted by T about the basic features of $\mathcal{U}$.

The BF-approach may be easily illustrated assuming that the world $\mathcal{U}$ is described by a propositional language $\mathcal{L}_n$ with n atomic sentences $p_1, \ldots, p_n$. The possible basic features of $\mathcal{U}$ may then be described by the so-called *literals* of $\mathcal{L}_n$. A literal is either an atomic sentence p_i or the negation $\neg p_i$ of an atomic sentence; thus, for any atomic sentence p_i, there is a pair of literals $(p_i, \neg p_i)$ whose elements are said to be the *dual* of each other. Of course, the set $\mathcal{B} = \{p_1, \neg p_1, \ldots, p_n, \neg p_n\}$ of the literals of $\mathcal{L}_n$ contains $2n$ members. A literal of $\mathcal{L}_n$ will be denoted by "$\pm p_i$", where $\pm$ is either empty or "$\neg$". Following established terminology, a *constituent* C of $\mathcal{L}_n$ is defined as a conjunction of n literals, one for each atomic sentence.[5] A constituent will thus have the following form:

$$\pm p_1 \wedge \ldots \pm p_n \tag{1}$$

One can check that the set $\mathcal{C}$ of the constituents of $\mathcal{L}_n$ contains $q = 2^n$ elements; moreover, there is only one *true* constituent in $\mathcal{C}$, denoted by "$C_\star$", which can be identified with "the (whole) truth" in $\mathcal{L}_n$. A *conjunctive theory* T of $\mathcal{L}_n$—"c-theory" for short—is a conjunction of k literals concerning k different atomic sentences. A c-theory will thus have the following form:

$$\pm p_{i_1} \wedge \ldots \wedge \pm p_{i_k} \tag{2}$$

[4] See Festa (2007) for early motivation and Cevolani et al. (2011b) for a complete exposition; for discussion of some applications see Cevolani and Festa (2009), Cevolani et al. (2010), and Cevolani and Calandra (2010). Some of the main ideas of the BF-approach were anticipated by Kuipers' notion of "descriptive verisimilitude" (Kuipers 1982).

[5] See for instance Hintikka (1973, p. 152).

where $0 \leq k \leq n$. A tautological c-theory, denoted by "$\top$", has no conjuncts, whereas a c-theory with n conjuncts is a constituent.[6] The class of c-theories expressible within $\mathscr{L}_n$ contains 3^n members.

A literal $\pm p_i$ occurring as a conjunct of a c-theory T is a *basic claim*—"b-claim" for short—of T.[7] The set T^b of all the b-claims of a c-theory T will be referred to as the *basic content* ("b-content" from now on) of T. One can also define the quantitative notion of *degree of b-content* $cont(T)$ of T, as follows:

$$cont(T) \stackrel{\text{df}}{=} \frac{|T^b|}{n}, \tag{3}$$

where $|T^b|$ is the number of b-claims of T. We shall denote by "T^d" the set formed by the duals of the b-claims of T; of course, $|T^b| = |T^d|$. Finally, the set of the atomic sentences p_i of $\mathscr{L}_n$ such that neither p_i nor $\neg p_i$ is a b-claim of T will be denoted by "$T^?$". In other words, the members of $T^?$ correspond to the basic features of $\mathscr{U}$ about which T does not say anything, or remains silent.[8] The c-theory $\widetilde{T}$, given by the conjunction of the duals of the b-claims of T, will be called the *specular of* T.[9] Note that the b-content of $\widetilde{T}$ is T^d; it follows that $cont(T) = cont(\widetilde{T})$.

Given a constituent C and a c-theory T, we say that a b-claim $\pm p_i$ of T is *true in* C just in case $\pm p_i \in C^b$; otherwise, it is *false in* C. Accordingly, T^b can be partitioned into two subsets: (1) the set $(T, C) \stackrel{\text{df}}{=} T^b \cap C^b$ of the b-claims of T which are true in C, and (2) the set $f(T, C) \stackrel{\text{df}}{=} T^b \setminus C^b$ of the b-claims of T which are false in C. We shall say that $t(T, C)$ is the *true b-content of* T *with respect to* C, while $f(T, C)$ is the *false b-content of* T *with respect to* C. The corresponding quantitative notions of *degree of true b-content* $cont_t(T, C)$ and *degree of false b-content* $cont_f(T, C)$ *of* T *with respect to* C can then be introduced as follows:

$$cont_t(T, C) \stackrel{\text{df}}{=} \frac{|t(T, C)|}{n} \quad \text{and} \quad cont_f(T, C) \stackrel{\text{df}}{=} \frac{|f(T, C)|}{n} \tag{4}$$

Note that $0 \leq cont_t(T, C), cont_f(T, C) \leq 1$ and $cont_t(T, C) + cont_f(T, C) = cont(T)$. Recalling the definition of $\widetilde{T}$ as the specular of T, it is easy to see that $cont_t(\widetilde{T}, C) = cont_f(T, C)$ and $cont_f(\widetilde{T}, C) = cont_t(T, C)$. Given a non-tautological c-theory T, we will say that T is (*completely*) *true in* C just in case $t(T, C) = T^b$, that T is *false in* C when T is not true in C and that T is *completely false in* C when $t(T, C) = \emptyset$. Clearly, if T is true then $\widetilde{T}$ is completely false, and vice versa.

Informally, the verisimilitude of T can be construed as the similarity or closeness of T to the true constituent $C_\star$. The key intuition underlying the BF-approach is that T is highly verisimilar if T tells many things about the basic features of $\mathscr{U}$, as described by $C_\star$, and many of those things are true. In other words, T is highly

[6] A c-theory is called "descriptive statement" (or "D-statement") by Kuipers (1982, pp. 348–349) and "(propositional) quasi-constituent" by Oddie (1986, p. 86).

[7] This terminology is inspired by Carnap (1950, p. 67), who calls "basic sentences" the literals of $\mathscr{L}_n$.

[8] The notation "$T^?$" is motivated by the fact that $T^?$ can be seen as the "question mark area" of T.

[9] This concept has been first introduced (with respect to first-order languages) by Festa (1987, p. 153). Cf. also Niiniluoto (1987, p. 319).

verisimilar if T^b contains many true b-claims and few false b-claims. This intuition suggests the following comparative notion of verisimilitude for c-theories:

Definition 1 Given two c-theories T_1 and T_2, T_2 is *more verisimilar* than T_1—in symbols, $T_2 >_{vs} T_1$—iff at least one of the following conditions holds:

(M_t) $t(T_2, C_\star) \supset t(T_1, C_\star)$ and $f(T_2, C_\star) \subseteq f(T_1, C_\star)$
(M_f) $t(T_2, C_\star) \supseteq t(T_1, C_\star)$ and $f(T_2, C_\star) \subset f(T_1, C_\star)$.

Conditions (M_t) and (M_f) may be called "monotonicity" conditions for the verisimilitude of c-theories. In fact, (M_t) and (M_f) say, respectively, that verisimilitude is monotonically *increasing* with respect to the addition of *true* literals, and monotonically *decreasing* with respect to the addition of *false* literals. Definition 1 summarizes a number of "conjunctive intuitions" which are rather widespread in the literature and often stated, more or less explicitly, by many verisimilitude theorists.[10]

It should be noted that Definition 1 doesn't allow to compare two arbitrary c-theories with respect to their verisimilitude, but only those c-theories whose true and false b-contents are set-theoretically comparable in the terms of the definition. In other words, Definition 1 identifies an ordering relation which is only partially defined over the whole class of c-theories. For instance, if $C_\star = p_1 \wedge p_2 \wedge p_3$ is the truth in language $\mathscr{L}_3$, then we cannot say that $T_2 \equiv p_1 \wedge p_2$ is more verisimilar than $T_1 \equiv \neg p_1 \wedge p_3$, since $t(T_1, C_\star) = \{p_3\}$ and $t(T_2, C_\star) = \{p_1, p_2\}$ are not comparable according to Definition 1. To compare the degrees of verisimilitude of two arbitrary c-theories, one needs to introduce a *verisimilitude measure Vs*. In principle, one can define a number of quite different measures for the verisimilitude of c-theories. However, we will maintain that such measures should be "conjunctively monotonic"—or "c-monotonic" for short—in the following sense:

Definition 2 A verisimilitude measure *Vs* is *c-monotonic* just in case *Vs* satisfies the following condition:

C-monotonicity. Given two c-theories T_1 and T_2, if $T_2 >_{vs} T_1$ then $Vs(T_2) > Vs(T_1)$.

A simple kind of c-monotonic measures is represented by *contrast measures* of verisimilitude.[11] The underlying intuition is that, given a constituent C and a c-theory T, the (degree of) similarity of T to C depends on the number of true b-claims (the "matches") and of false b-claims (the "mistakes") of T in C. Accordingly, $cont_t(T, C)$ may be construed as the overall *prize* attributed to the matches of T and

[10] Cf. Schurz and Weingartner (2010, section 2) and, in particular, the discussion of "Popper's intuitions" on pp. 417–418. Indeed, one may note that Definition 1 is structurally identical to Popper's comparative definition of verisimilitude (Popper 1963, p. 233). The crucial difference is that, instead of the "true b-content" and "false b-content" of c-theories, Popper's definition concerns the "truth-content" and the "falsity-content" of logically closed theories, defined, respectively, as the classes of their true and false classical logical consequences.

[11] The expression "contrast measures" refers to the fact that these measures can be seen as an application of the "contrast model" of similarity introduced by Tversky (1977) in his study of the similarity of psychological stimuli. Contrast measures have been introduced (without using this name) by Cevolani and Festa (2009) and Cevolani et al. (2010) and are fully discussed in Cevolani et al. (2011b).

$-cont_f(T, C)$ as the overall *penalty* attributed to the mistakes of T. A *contrast measure of similarity* between T and C is a weighted average of the prize due to T's degree of true b-content and of the penalty due to T's degree of false b-content:

$$s_\phi(T, C) \overset{\text{df}}{=} cont_t(T, C) - \phi cont_f(T, C) \tag{5}$$

where $\phi > 0$. Intuitively, different values of ϕ reflect the relative weight assigned to truth and falsity. If $\phi = 1$, the prize obtained by endorsing a truth and the penalty obtained by endorsing a falsity about C are valued in a perfectly balanced way. If $\phi < 1$, then one will care more endorsing a truth than one suffers from endorsing a falsity, while the opposite holds if $\phi > 1$. Finally, the (degree of) of verisimilitude of T can be defined as the similarity of T to the truth, i.e., to the true constituent $C_\star$ of $\mathscr{L}_n$:

$$\begin{aligned} Vs_\phi(T) &\overset{\text{df}}{=} s_\phi(T, C_\star) \\ &= cont_t(T, C_\star) - \phi cont_f(T, C_\star). \end{aligned} \tag{6}$$

One can easily check that Vs_ϕ is a c-monotonic measure of the verisimilitude of c-theories. Moreover, it is interesting to note that Vs_ϕ provides a sort of "core theory" of verisimilitude, i.e., it agrees, as far as c-theories are concerned, with many different verisimilitude measures discussed in the literature. Indeed, Vs_ϕ turns out to be a special case of a number of existing measures, among which those proposed by Kuipers (1982, 2011b), Oddie (1986), Schurz and Weingartner (1987, 2010), Brink and Heidema (1987) and Gemes (2007).[12] Another c-monotonic measure, which turns out to be identical to Vs_ϕ as far as c-theories are concerned, is the well-known "min-max" verisimilitude measure introduced by Niiniluoto (1987). It should be noticed, however, that a few known verisimilitude measures are not c-monotonic. Notably, these include Niiniluoto's favored "min-sum" measure; also, the quantitative definition of "refined verisimilitude" proposed by Zwart (2001) implicitly defines a measure which is not c-monotonic, i.e., violates the comparative ordering of Definition 1.[13]

3 AGM Conjunctive Theory Change

3.1 The Belief Base (BB-)Version of the AGM Approach

Theory change is analyzed within the AGM approach by studying the changes of the *beliefs* of an *ideal agent*. In the last thirty years, a number of different versions of

[12] In other words, not only all these measures are c-monotonic, but, when applied to the evaluation of the verisimilitude of c-theories, they also turn out to be ordinally equivalent to (and sometimes identical with) Vs_ϕ. A measure Vs is ordinally equivalent to another measure Vs' just in case, for any pair of theories T_1 and T_2, $Vs(T_1) \gtreqless Vs(T_2)$ iff $Vs'(T_1) \gtreqless Vs'(T_2)$.

[13] For a comparison between contrast measures of verisimilitude and the other measures mentioned above, see Cevolani et al. (2011b).

this approach have been developed. In the "standard" or "belief set" version of the AGM approach—for short, "BS-version"—the beliefs accepted by agent $\mathcal{X}$ at any given time are represented by (the elements of) $\mathcal{X}$'s *belief set* K.[14] Given a propositional language $\mathcal{L}_n$ and a consequence operation Cn defined on $\mathcal{L}_n$—which may be identified with the classical consequence operation—K can be defined as follows:

$$\text{A set of sentences } K \text{ is a belief set iff } K = \mathrm{Cn}(K). \quad (7)$$

Thus, a belief set K is a logically closed set of sentences, or, in logical parlance, a *theory*. Although the definition in (7) includes also inconsistent belief sets, AGM theorists adopt the following *principle of consistency*:

(C) The belief set K of an ideal agent $\mathcal{X}$ is consistent.

The BS-version of the AGM approach does not distinguish between "basic" (or "explicit") and "derived" (or "implicit") beliefs of $\mathcal{X}$. This distinction, however, is often relevant. To illustrate, note that if $\mathcal{X}$ independently accepts p and $q, \mathcal{X}$ will also accept $p \vee q, p \vee \neg q$ and all other logical consequences of p and q. However, $p \vee q$ (for instance) is in K just because p (or q) is; in other words, $\mathcal{X}$ will accept $p \vee q$ only implicitly, i.e., as a "merely derived" belief. The distinction between basic and derived beliefs plays a central role in the version of the AGM approach which may be called the "belief base" version—"BB-version" for short.[15] According to the BB-version, the belief set K of $\mathcal{X}$ is construed as the logical closure of a *belief base* B containing $\mathcal{X}$'s basic beliefs:

$$\text{A finite set of sentences } B \text{ is a belief base for a belief set } K \text{ iff } K = \mathrm{Cn}(B). \quad (8)$$

The main difference between basic beliefs and merely derived beliefs is that the latter are not worth believing for their own sake. This implies, for instance, that when a basic belief is given up, then derived beliefs may loose their support and be automatically discarded. In the following, we will only be concerned with the BB-version of the AGM approach, which is, at least for our purposes, the most convenient in order to analyze the relationships between AGM theory change and truth approximation.

The main goal of AGM theory is to provide a plausible account of how agent $\mathcal{X}$ should update his beliefs in response to certain inputs coming from some information source. Given a set of sentences $A = \{a_1, \ldots, a_m\}$, with $m \geq 0$, two kinds of *doxastic input* regarding A are considered within the AGM approach:

Additive inputs, which are expressed as orders of the form: "Add (all the elements of) A to your beliefs!".

Eliminative inputs, which are expressed as orders of the form: "Remove (all the elements of) A from your beliefs!".

[14] See Gärdenfors (1988) for a standard introduction to the BS-version.

[15] The largest part of the BB-version has been developed by Hansson (1999).

If A contains only one sentence, it is a *single* input; otherwise, A is a *multiple* input. When $\mathcal{X}$ receives a multiple input A, $\mathcal{X}$ has to update his beliefs with respect to *all* the sentences in A; this kind of belief change is known as "(multiple) package change".[16]

The updating process is performed by applying to $\mathcal{X}$'s belief base B one of the three *change operations* defined within the AGM theory. Such operations, called "expansion", "revision" and "contraction", may be described as follows. Suppose first that $\mathcal{X}$ receives the *additive* input $A = \{a_1, \ldots, a_m\}$. If $\mathcal{X}$ explicitly accepts A also before receiving it—i.e., $A \subseteq B$—then $\mathcal{X}$'s appropriate response is keeping B unchanged. Otherwise, two possibilities arise, depending on whether A is logically compatible with B. If the input is compatible with the beliefs of $\mathcal{X}$—i.e., B does not imply the negation of any a_i—the operation by which $\mathcal{X}$ should update B by the addition of A is called *expansion* and the expanded belief base is denoted by "$B + A$". Alternatively, if A is incompatible with B—i.e., B implies $\neg a_i$ for some a_i in A—the operation by which $\mathcal{X}$ should update B by the addition of A is called *revision*, and the revised belief base is denoted by "$B * A$". Expansion and revision are two different kinds of "additive" change operations. The third AGM operation, contraction, is instead performed when agent $\mathcal{X}$ receives the *eliminative* input A. In this case, if A does not belong to the beliefs of $\mathcal{X}$—i.e., $A \cap \mathrm{Cn}(B) = \varnothing$—then $\mathcal{X}$'s will keep B unchanged. Otherwise, the operation by which $\mathcal{X}$ should update B by the removal of (all the elements of) A is called *contraction*, and the contracted belief base is denoted by "$B - A$".

AGM theorists have made systematic efforts aiming to show how, given a belief base B and a sentence A, an ideal agent $\mathcal{X}$ could specify the updated belief bases $B + A$, B*A and $B - A$—and consequently the updated belief sets $\mathrm{Cn}(B + A)$, $\mathrm{Cn}(B * A)$ and $Cn(B - A)$. A basic intuition underlying the AGM approach is expressed by the following general principle of rationality, known as the *principle of minimal change*:[17]

(MC) When the belief base B of an ideal agent $\mathcal{X}$ is updated in response to a given doxastic input, a *minimal change* of B is accomplished. This means that $\mathcal{X}$ keeps believing as many of the old beliefs as possible and starts to believe as few new beliefs as possible.

All three AGM change operations should be defined in accordance with the general principles of consistency and minimal change. As far as expansion is concerned, there is no special difficulty in defining the corresponding operation in agreement with (C) and (MC):[18]

[16] Package change, along with other kinds of multiple change, has been studied, in particular, by Fuhrmann and Hansson (1994); see also Hansson (1999, pp. 134–139 and 258–261). Multiple change—i.e., change with respect to a input consisting of more than one sentence—should not be conflated with *iterated change*, where a belief base is repeatedly modified with respect to a number of single inputs.

[17] For a detailed critical examination of this principle see Rott (2000).

[18] As a remark, one should note that the definition in (9) includes, for the sake of generality, also the case where A logically contradicts B, and hence $B + A$ is inconsistent. In the following, we shall always assume, in agreement with (C) and with the informal notion of expansion, that expansion is performed

$$\text{Given a belief base } B \text{ and an input } A, B + A \stackrel{\text{df}}{=} B \cup A. \tag{9}$$

The definition of revision and contraction is more problematic, since both operations require that some elements of B are withdrawn. The contraction of a belief base B by A can be defined with the help of the notion of so-called "remainders" of B, which are maximal subsets of B which don't imply any element of A. More formally:[19]

Definition 3 Given a belief base B and an input A, the *remainder set* of B modulo A is the set of sentence sets $B \perp A$ such that $X \in B \perp A$ iff:

1. $X \subseteq B$;
2. $\text{Cn}(X) \cap A = \varnothing$;
3. if $X \subset Y \subseteq B$ then $\text{Cn}(Y) \cap A \neq \varnothing$.

Each element X of $B \perp A$ is called a *remainder* of B by A. Intuitively, $B \perp A$ contains all the possible minimal contractions of B by A. Indeed, one can prove that each remainder of B by A is a belief base which does not imply any element of A and that—according to (MC)—is "maximal", i.e., includes as much as possible of the content of B while not implying A (as guaranteed by the third condition of the definition). It is worth noting, however, that Definition 3 alone cannot fully determine the result of any contraction. The reason is that there are in general many different remainders of B by A, i.e., many alternative ways to perform a contraction of B by A.[20] The choice between these alternatives will depend on the relative "importance" that $\mathcal{X}$ attaches to the sentences in B—i.e., on extra-logical considerations.[21]

As far as revision is concerned, AGM theorists follow Levi (1980) in maintaining that $B * A$ is defined in terms of expansion and contraction. The idea naturally arises when A is a single input, i.e., $A = \{a\}$. Suppose in fact that $\mathcal{X}$ has to revise B by the single sentence a; this means that a is incompatible with B, i.e., that $\neg a \in \text{Cn}(B)$. If now $\mathcal{X}$ performs a contraction of B by $\neg a$, the resulting belief base $B - \neg a$ will be obviously compatible with a, since, according to Definition 3, $B - \neg a$ will not imply $\neg a$. Consequently, $\mathcal{X}$ can consistently expand $B - \neg a$ by a. The result of these two-steps process is the required revision of B by a. In order to generalize this procedure to the case of revision by a *multiple* input, one needs to introduce

Footnote 18 continued
only with respect to compatible inputs. When A is incompatible with B, then a *revision*, not an expansion, of B by A should be performed.

[19] Hansson (1999, p. 37).

[20] Suppose for instance that $B = \{p, p \to q, q \to r\}$ and that $\mathcal{X}$ receives the eliminative input $A = \{q, r\}$. In order to remove A from his beliefs, $\mathcal{X}$ has to withdraw some elements of B, since both $q \in \text{Cn}(B)$ and $r \in \text{Cn}(B)$. However, one can check that there are *two* remainders of B by A, i.e., $\{p \to q, q \to r\}$ and $\{p, q \to r\}$. Thus, $\mathcal{X}$ may remove either p or $p \to q$ in order to perform the required contraction. Philosophers of science are familiar with discussions of this issue under the heading of "the Duhem problem."

[21] In the AGM literature, relative importance is usually represented by the degree of "entrenchment" of $\mathcal{X}$'s beliefs: when withdrawing some sentences from $B, \mathcal{X}$ will choose the less entrenched in agreement with appropriate selection rules. See Hansson (1999, pp. 11–15) for a discussion of the basic idea.

Fig. 1 Overlapping, conflicting and excess parts of c-theories T and c-inputs A

$$T \equiv \overbrace{p_1}^{T_{xA}} \wedge \overbrace{\neg p_2}^{T_{oA}} \wedge \overbrace{\neg p_3 \wedge p_4}^{T_{cA}}$$

$$A \equiv \underbrace{\neg p_2}_{A_{oT}} \wedge \underbrace{p_3 \wedge \neg p_4}_{A_{cT}} \wedge \underbrace{\neg p_5}_{A_{xT}}$$

the notion of the negation of a set of sentences A.[22] Given a finite set of sentences A, the negation $\neg A$ of A is defined as $\neg \bigwedge A$, where $\bigwedge A$ denotes the conjunction of all the members of A. (If A is empty, we can safely assume that $\neg A$ is equivalent to $\bot$.) It follows that, if $A = \{a_1, \ldots, a_m\}$, then $\neg A = \neg a_1 \vee \ldots \vee \neg a_m$. The revision of B by A can then be defined according to the so-called *Levi identity*:

$$\text{Given a belief base } B \text{ and an input } A, B * A \overset{\text{df}}{=} (B - \neg A) + A. \tag{10}$$

According to (10), the revision of B by A is construed as the contraction of B by the disjunction of all the elements of A, followed by an expansion of the resulting set by A itself.

3.2 The BB-Version of the AGM Approach Applied to Conjunctive Theory Change

The BB-version of the AGM approach outlined above can be conveniently applied to the analysis of *conjunctive change*. Conjunctive change—or *c-change* for short—is the change of c-theories with respect to *conjunctive inputs*, where A is a conjunctive input—or "c-input"—just in case A is a c-theory. The idea is that if agent $\mathcal{X}$ accepts c-theory T, his basic or explicit beliefs are represented by the b-claims of T; or, which is the same, his belief base is T^b, i.e., the b-content of T. When $\mathcal{X}$ receives an additive or eliminative c-input A, A^b may be construed as a multiple input prompting $\mathcal{X}$ to change his mind. This means that $\mathcal{X}$ has either to add to his beliefs all the b-claims of an additive c-input A or to remove from his beliefs all the b-claims of an eliminative c-input A. It follows that, given a c-theory T and a c-input A, the expansion of T by A will be the c-theory $T + A$ such that $(T + A)^b = T^b + A^b$; and, likewise, $(T * A)^b = T^b * A^b$ and $(T - A)^b = T^b - A^b$, respectively, for the revision and the contraction of T by A.

One crucial property of c-change is that the result of an expansion, revision or contraction of c-theory T by c-input A is *uniquely* determined by the logical relations between T and A. This means that there is no problem of choosing one amongst many alternative ways of performing a given change. Suppose in fact that $\mathcal{X}$, who accepts c-theory T, receives a c-input A. With the help of Fig. 1, one can check that the way in which $\mathcal{X}$ should update T in response to A depends on how A^b relates to the partition $\{T^b, T^d, T^?\}$ in set-theoretical terms. To see this, it is useful to

[22] Hansson (1999, p. 259) calls this the "sentential negation" of A.

introduce the notions of “overlapping”, “conflicting” and “excess” parts of A with respect to T, as follows:

Definition 4 Given a c-theory T and a c-input A, the following “parts” of A are defined: (i) the *overlapping part of A w.r.t. T*, i.e., the conjunction of the elements of $A^b \cap T^b$, denoted by A_{oT}; (ii) the *conflicting part of A w.r.t. T*, i.e., the conjunction of the elements of $A^b \cap T^d$, denoted by A_{oT}; (iii) the *excess part of A w.r.t. T*, i.e., the conjunction of the elements of $A^b \cap T^?$, denoted by A_{xT}.

Note that the three sets $A^b \cap T^b$, $A^b \cap T^d$ and $A^b \cap T^?$ form a partition of A^b. Hence, A can be written as $A_{oT} \wedge A_{cT} \wedge A_{xT}$. Similarly, T can be written as $T_{oA} \wedge T_{cA} \wedge T_{xA}$. The following properties of A_{oT}, A_{cT} and A_{xT} are worth noting. First, A_{oT} is identical to T_{oA}, by definition. Moreover, it is easy to see that $A_{cT} = \widetilde{T_{cA}}$ and $T_{cA} = \widetilde{A_{cT}}$—i.e., that the conflicting part of A with respect to T is the specular of the conflicting part of T with respect to A, and vice versa. Finally, A_{xT} and T_{xA} share by definition no common conjunct.

Having introduced these notions, the following results about AGM c-change can be proved (see the “Appendix” for a proof):

Theorem 1 *Given a c-theory T and a c-input A*:

(i) $T + A = T \wedge A_{xT}$;
(ii) $T * A = T_{xA} \wedge A$;
(iii) $T - A = T_{cA} \wedge T_{xA}$.

Theorem 1 shows that the expansion of T by A consists in the addition to T of the excess part of A, revision in the addition of A to the excess part of T and contraction in the removal of the overlapping part of A from T. As a technical comment, note that Theorem 1 also takes into account cases of “vacuous” change (Hansson 1999, p. 66). Given an additive c-input A, we say that the expansion and the revision of T by A are *vacuous* in case $A^b \subseteq T^b$—or, equivalently, $A_{oT} \equiv A$—i.e., when the information transmitted by A is already contained in T. In such case, it follows immediately from Theorem 1 that $T + A = T * A = T$. Likewise, the contraction of T by A is vacuous when the eliminative input A requires the removal of no information in T—i.e., when $T^b \cap A^b = \varnothing$ or, equivalently, $A^b_{oT} \equiv \top$. Again, in this case we have that $T - A = T$. Since vacuous change always leaves T unmodified, this case will be set aside in the following treatment.

4 Truth Approximation and AGM Conjunctive Theory Change

Coming back to the relationships between truth approximation and theory change, we can now state with greater precision our central problem. Following Niiniluoto (1999, 2009), one ask in which cases AGM theory change is an effective means for approaching the truth, i.e., on which conditions AGM change operations lead closer to the truth. In this connection, the following adequacy conditions might seem plausible at first sight:

(Tr^{+}) For any theory T and input A, if A is true then $T + A$ is more verisimilar than T

(Tr^{-}) For any theory T and input A, if A is true then $T - A$ is less verisimilar than T

(Tr^{*}) For any theory T and input A, if A is true then $T * A$ is more verisimilar than T

(CF^{+}) For any theory T and input A, if A is completely false then $T + A$ is less verisimilar than T

(CF^{-}) For any theory T and input A, if A is completely false then $T - A$ is more verisimilar than T

(CF^{*}) For any theory T and input A, if A is completely false then $T * A$ is less verisimilar than T.

The intuition underlying (Tr^{+})–(CF^{*}) is that the degree of verisimilitude of T should increase when one adds true inputs to T or removes completely false inputs from T; and, vice versa, T's verisimilitude should decrease when additive inputs are completely false or eliminative inputs are true. However, one can argue that, no matter which of the existing verisimilitude measures is taken into account, AGM change operations presumably violate all conditions (Tr^{+})–(CF^{*}). For instance, Niiniluoto (1999) proves that, once his favored "min-sum" verisimilitude measure Vs_N is adopted, expansion, revision and contraction violate the three requirements (Tr^{+}), (Tr^{*}) and (CF^{-}).[23] Niiniluoto does not explicitly consider the remaining three adequacy conditions listed above; one can prove, however, that also (Tr^{-}), (CF^{+}) and (CF^{*}) are not satisfied relative to Vs_N.[24] It seems reasonable to conjecture that Niiniluoto's negative results can be generalized to other measures of verisimilitude. In turn, this suggests that (Tr^{+})–(CF^{*}) are too strong as adequacy conditions for the verisimilitudinarian effectiveness of AGM change. It is then natural to ask on which other conditions, weaker than (Tr^{+})–(CF^{*}), AGM change operations are effective means for truth approximation. As an example, Niiniluoto (1999) considers two further conditions, which are obtained by restricting (Tr^{+}) and (Tr^{-}) above to the case of a *true* theory T:

(t-Tr^{+}) For any true theory T and input A, if A is true then $T + A$ is more verisimilar than T

(t-Tr^{-}) For any true theory T and input A, if A is true then $T - A$ is less verisimilar than T.

Niiniluoto proves that these conditions are satisfied, as far as Vs_N is concerned.[25] We may call (t-Tr^{+}) and (t-Tr^{-}) *t-conditions*, since they are restricted only to *true* theories and inputs.

A different kind of weakening of conditions (Tr^{+})–(CF^{*}) has been instead proposed by Cevolani and Festa (2009). According to this proposal, one may restrict

[23] See Niiniluoto (1999, sections 4 and 5, in particular Eqs. 10, 17 and 20 and the corresponding discussion).

[24] See Cevolani et al (2010a).

[25] See Niiniluoto (1999, Eqs. 11 and 20).

the application of (Tr^{+})–(CF^{*}) to the case of AGM c-change discussed in Sect. 3. As an illustration, the first three conditions may be reformulated as follows:

(c-Tr^{+}) For any c-theory T and c-input A, if A is true then $T + A$ is more verisimilar than T

(c-Tr^{-}) For any c-theory T and c-input A, if A is true then $T - A$ is less verisimilar than T

(c-Tr^{*}) For any c-theory T and c-input A, if A is true then $T * A$ is more verisimilar than T.

We may call (c-Tr^{+})–(c-Tr^{*}) *conjunctive conditions*—*c-conditions* for short—for AGM theory change as tracking truth approximation. Of course, the c-conditions (c-CF^{+})–(c-CF^{*}) corresponding to requirements (CF^{+})–(CF^{*}) can be formulated in a strictly similar fashion. The following results show—see the Appendix for a proof—that conditions (c-Tr^{+})–(c-CF^{*}) suitably capture the verisimilitudinarian effectiveness of AGM theory change:

Theorem 2 *Given a c-theory T and a c-input A, let us exclude the case of vacuous change. If A is true then:* (*i*) $T + A >_{vs} T$, (*ii*) $T - A <_{vs} T$, (*iii*) $T * A >_{vs} T$; *moreover, if A is completely false then:* (*iv*) $T + A <_{vs} T$, (*v*) $T - A >_{vs} T$, (*vi*) $T * A <_{vs} T$.

In turn, Theorem 2 immediately implies (the proof is trivial given Definition 2) that any c-monotonic measure of verisimilitude *Vs* satisfies conditions (c-Tr^{+})–(c-CF^{*}) above:

Theorem 3 *Let Vs be a c-monotonic verisimilitude measure. Then, excluding again cases of vacuous change: if A is true then:* (*i*) $Vs(T + A) > Vs(T)$, (*ii*) $Vs(T - A) < Vs(T)$, (*iii*) $Vs(T * A) > Vs(T)$; *moreover, if A is completely false then:* (*iv*) $Vs(T + A) < Vs(T)$, (*v*) $Vs(T - A) > Vs(T)$, (*vi*) $Vs(T * A) < Vs(T)$.

In words, Theorem 2 guarantees that when a c-theory T is expanded or revised by a true (completely false) c-input its verisimilitude increases (decreases), and that a contraction of T by a true (completely false) c-input will be less (more) verisimilar than T.

The results above can be extended by considering special instances of c-monotonic measures, like the contrast measures introduced in Sect. 2. Once such measures are given, one can analyze the change of c-theories with respect to richer kinds of inputs besides simply true or completely false ones. In fact, most c-inputs are presumably neither true nor completely false: some may be highly verisimilar (thus including many true b-claims and few false b-claims) and others may be very distant from the truth (thus including few true b-claims and many false b-claims). One would then expect that Theorem 2 can be generalized as follows: $Vs_{\phi}(T)$ increases when T is updated with additive c-inputs which are false but highly verisimilar, or with eliminative c-inputs which are false and very distant from the truth; and conversely, $Vs_{\phi}(T)$ decreases when additive c-inputs are distant from the truth and eliminative c-inputs are highly verisimilar. This amounts to considering new c-conditions for AGM change as tracking truth approximation, essentially similar to conditions (c-Tr$^+$)–(c-CF*) above, which will be denoted (c-Vs$^+$)–(c-Ds*). For instance, one may propose the following conditions:

(c-Vs$^+$) For any c-theory T and c-input A, if A is verisimilar then $T + A$ is more verisimilar than T

(c-Ds$^-$) For any c-theory T and c-input A, if A is distant from the truth then $T - A$ is more verisimilar than T

(c-Vs*) For any c-theory T and c-input A, if A is verisimilar then $T * A$ is more verisimilar than T.

Conditions above are *prima facie* very plausible, but, as we will see shortly, can not be accepted. However, more sophisticated, but less simple and general, adequacy conditions can be introduced and met, once the notion of verisimilar input (and of inputs which are distant from the truth) is defined according to contrast measures.

The classificatory notion of verisimilar input can be defined by requiring that the verisimilitude of the input is higher than a certain threshold. As far as contrast measures are concerned, the degree of verisimilitude of a tautology provides a convenient choice. In fact, recalling that a tautological c-theory $\top$ makes no claim at all about the world (i.e., $\top^b = \varnothing$), it is easy to see that $Vs_\phi(\top) = 0$ for any contrast measure Vs_ϕ. For all non-tautological c-theories T, $Vs_\phi(T)$ may be higher than, equal to, or lower than this threshold, depending on the number and weight of the true and false b-claims of T. In the following, we shall assume that c-input A is *verisimilar* if and only if A is strictly more verisimilar than a tautology, and that A is *distant from the truth*—or *t-distant* for short—if and only if A is strictly less verisimilar than a tautology. In other words:

Definition 5 Given a contrast measure Vs_ϕ and a c-input A, A is verisimilar if and only if $Vs_\phi(A) > 0$ and A is t-distant if and only if $Vs_\phi(A) < 0$.

The underlying idea is that a verisimilar c-input is "on the right track", i.e., conveys some valuable information about the world, whereas a t-distant c-input provides misleading information about it. At first sight, one might believe that expanding and revising T by verisimilar inputs A should increase the verisimilitude of T, and that, if A is t-distant, the contraction of T by A should also be more verisimilar than T. This intuition underlies conditions (c-Vs$^+$)–(c-Ds*) above. However, on closer inspection, this cannot be expected in general. To see why, let us consider the expansion of T by a verisimilar c-input A, i.e. the c-theory $T \wedge A_{xT}$ (see Theorem 1). In this case, it may happen that A is verisimilar just because its overlapping part is highly verisimilar, whereas its excess part is actually t-distant. Since what really does matter is only the verisimilitude of the excess part A_{xT} of A, the expansion of T by A will lead to a theory $T \wedge A_{xT}$ which is *less* verisimilar than T. Essentially similar considerations show that all other conditions are inadequate.

The following theorem (see the "Appendix" for a proof) already suggests how conditions (c-Vs$^+$)–(c-Ds*) should be refined:

Theorem 4 *Given a c-theory T and a c-input* :A, *then*:

1. $Vs_\phi(T + A) > Vs_\phi(T)$ *iff* $Vs_\phi(A_{xT}) > 0$

2. $Vs_{\phi}(T - A) > Vs_{\phi}(T)$ *iff* $Vs_{\phi}(A_{oT}) < 0$
3. $Vs_{\phi}(T * A) > Vs_{\phi}(T)$ *iff* $Vs_{\phi}(A_{xT}) > Vs_{\phi}(\widetilde{A_{cT}}) - Vs_{\phi}(A_{cT})$.

The intuitive content of the first two clauses of Theorem 4 can be expressed by saying that $T + A$ is more verisimilar than T if and only if the excess part of A is verisimilar, and, conversely, that $T - A$ is more verisimilar than T if and only if the overlapping part of A is t-distant. The more complex case, corresponding to the third clause of the theorem, is that of revision, which implies that the conflicting part of T with respect to A is replaced by its specular (cf. Theorem 1 and Fig. 1). Hence, the way in which revision by A affects the verisimilitude of T is determined by the verisimilitude of both the excess and the conflicting part of T with respect to A, and the conflicting part of A with respect to T. In particular, $T * A$ will be more verisimilar than T if and only if the increase in verisimilitude due to the excess part of A outweighs the possible decrease of verisimilitude due to the substitution of the conflicting part of T with its specular, i.e., with the conflicting part of A.

Theorem 4 shows that only specific parts of a c-input affect the verisimilitude of a given theory, leading to a more or less verisimilar theory. In particular, it is easy to see that the following conditions are immediately fulfilled:

(c-Vs_x^+) If the excess part of A is verisimilar, then $Vs_{\phi}(T + A) > Vs_{\phi}(T)$
(c-Ds_x^+) If the excess part of A is t-distant, then $Vs_{\phi}(T + A) < Vs_{\phi}(T)$
(c-Vs_o^-) If the overlapping part of A is verisimilar, then $Vs_{\phi}(T - A) < Vs_{\phi}(T)$
(c-Ds_o^-) If the overlapping part of A is t-distant, then $Vs_{\phi}(T - A) > Vs_{\phi}(T)$

Interestingly, conditions as simple as those above are not available for revision. However, one can prove that (see the "Appendix" for a proof):

Theorem 5 *Given a c-theory T and a partially verisimilar c-input A:*

1. *If $\phi \geq 1$ and if both the conflicting and excess parts of A are verisimilar, then $Vs_{\phi}(T * A) > Vs_{\phi}(T)$*
2. *If $\phi \leq 1$ and if both the conflicting and excess parts of A are t-distant, then $Vs_{\phi}(T * A) < Vs_{\phi}(T)$.*

Thus, the verisimilitudinarian effectiveness of revision essentially depends on the relative weight of truth and falsity, as represented by the value of ϕ. Indeed, depending on this value, the revision of T by A may sometimes be less verisimilar than T, even if the conflicting and excess parts of A are verisimilar. Conversely, it may happen that the revision of T by A is more verisimilar than T, even if the conflicting and excess parts of A are t-distant.

5 Conclusions

In the preceding section, we identified a set of plausible conditions which, as far as c-theories are concerned, demonstrably capture the verisimilitudinarian effectiveness of AGM belief change, i.e., its effectiveness in tracking truth approximation. The results above immediately suggest some possible extensions of the basic feature approach. Due to space limitations, we shall only briefly mention these extensions, which will be fully discussed elsewhere (cf. Cevolani et al. 2011b).

First, Theorem 3 is stated in very general terms, i.e., is formulated with respect to all c-monotonic measures of verisimilitude, which include most (though not all) measures discussed in the literature. In contrast with Theorem 3, Theorem 4 and its corollaries are stated for a specific kind of c-monotonic measures, i.e., the contrast measure of verisimilitude Vs_{ϕ}, and for a particular definition of verisimilar and t-distant input. Then, one may ask whether similar or stronger results can be proved for other verisimilitude measures and/or some for alternative definitions of verisimilar input.

Second, one must concede that c-theories represent only a specific kind of theories, and, consequently, that c-change is only a very special case of AGM theory change. Thus, further research should explore the possibility that different forms of verisimilitudinarian effectiveness hold also for other kinds of theories and inputs, besides c-theories and c-inputs. This would lead to the formulation of new verisimilitudinarian requirements, analogous to c-conditions (c-Tr^{+})–(c-CF*), but concerning different kinds of theories. For some promising results in this direction, see Schurz (2011).

Finally, so far we have only considered theories—and, more specifically, c-theories—expressed in propositional languages. However, one may argue that propositional languages are insufficient or inadequate for the analysis of truth approximation and theory change in scientific research.[26] Consequently, it would be desirable that the above results could be generalized to richer kinds of language, such as monadic and polyadic first-order languages, modal languages, causal languages, "nomic" languages, higher-order languages, and so on. In this connection, one should note that the BF-approach is not limited to propositional languages, since the notion of c-theory can be appropriately defined within many other languages.[27] For instance, c-theories are definable within first-order languages (Festa 2007), nomic languages (Kuipers 2011b; Cevolani et al. 2011a), and statistical languages (Cevolani et al. 2011b; Festa 2007). More precisely, our approach can be generalized to any language characterized by a suitable notion of *constituent*—where a constituent is informally defined as a maximally informative conjunction of "elementary claims" about the world. In such languages, in fact, a c-theory can be conveniently defined as a "fragment" of a constituent, i.e., in the terminology adopted by Oddie (1986), as a *quasi-constituent*. Here, we will only hint at two important applications of our approach to languages of this kind. The first concerns theories expressed as generalizations within a first-order monadic language.[28] A (quasi-)constituent of this language is a conjunction of existential and

[26] By the way, one may note that the AGM analysis of theory change has so far been limited only to propositional theories; on this, cf. also Niiniluoto (2010).

[27] Moreover, the BF-approach can be applied to any kind of non-propositional and non-conjunctive theory *T*, for instance a logically closed set of sentences in a first-order monadic language. Even in cases like this, in fact, one is often interested only in what *T* entails about the basic features of the world, e.g., about which prototypical properties characterize a certain class of entities. For other relevant examples of this kind, see Kuipers (2011b).

[28] See (Niiniluoto 1987, Ch. 2 and 11) and Festa (2007).

non-existential claims, each of which affirming or denying that a given "kind of individuals" is instantiated in the target domain (or, equivalently, that a given Carnapian Q-predicate of the language is instantiated). The second application addresses "propositional nomic languages", i.e., languages describing the conceptual possibilities characterizing a given scientific inquiry.[29] Here, (quasi-)constituents are conjunctions specifying whether a given conceptual possibility is also a nomic (e.g., physical) possibility of the target domain or is instead a nomic impossibility. In both cases above, the basic feature approach can be applied to evaluating the verisimilitude of monadic and nomic (quasi-)constituents and the verisimilitudinarian effectiveness of AGM theory change in those contexts. For further results in this direction, see Kuipers (2011a) and Niiniluoto (2011).

Acknowledgments The authors wish to thank Theo Kuipers, Ilkka Niiniluoto, Gerhard Schurz, and all the participants in the workshop on *Probability, Confirmation, and Verisimilitude* (Trieste, September 2009) and in the EPSA2009-symposium on *Belief Revision Aiming at Truth Approximation* (Amsterdam, October 2009) for their comments. Two anonymous referees suggested a number of improvements which we gratefully acknowledge. Vincenzo Crupi also acknowledges support by a grant from the Spanish Department of Science and Innovation (FFI2008-01169/FISO).

Appendix: Proofs

Proof of Theorem 1 The following remark will be useful in proof:

Remark 1 If X is set of literals and x is a literal of $\mathscr{L}_n$, then $x \in \mathrm{Cn}(X)$ iff $x \in X$.

(i) Recall first that we are assuming that A is logically compatible with T. By the definition in (9), $(T + A)^b = T^b + A^b = T^b \cup A^b$. According to Definition 4, T^b may be written as $T^b_{oA} \cup T^b_{cA} \cup T^b_{xA}$ and, likewise, $A^b = A^b_{oT} \cup A^b_{cT} \cup A^b_{xT}$. Since by hypothesis A is compatible with T, $T^b_{cA} = A^b_{cT} = \varnothing$; moreover $T^b_{oA} = A^b_{oT}$ by Definition 4. Hence, $T^b = T^b_{oA} \cup T^b_{xA}$ and $A^b = T^b_{oA} \cup A^b_{xT}$. It follows that $T^b \cup A^b = T^b \cup A^b_{xT}$, i.e., $T + A = T \wedge A_{xT}$.

(ii) According to the definition in (10), $T^b * A^b = ((T^b - \neg(A^b)) + A^b)$, where $\neg(A^b)$ is the negation of A^b. Note that, since A is a c-input, any $a \in A^b$ is a literal. We start by proving that $T^b_{oA} \cup T^b_{xA}$ is the unique remainder of T^b by $\neg(A^b)$ (see Definition 3), and hence the unique result of the contraction of T^b by $\neg(A^b)$. By Definition 4, $T^b_{oA} \cup T^b_{xA} \subseteq T^b$. Moreover, $T^b_{oA} \cup T^b_{xA}$ does not imply $\neg(A^b)$. In fact, by Definition 4, each element of $T^b_{oA} \cup T^b_{xA}$ is either the negation of a disjunct of $\neg(A^b)$ or a logically independent literal. Finally, any set Y such that $T^b_{oA} \cup T^b_{xA} \subset Y \subseteq T^b$ will contain some element of T_{oA}, again by Definition 4. Any such element is a negation of an element of A^b and then implies $\neg(A^b)$. Thus, $T^b_{oA} \cup T^b_{xA}$ is a remainder of T^b by A^b. To see that $T^b_{oA} \cup T^b_{xA}$ is the *unique* remainder, note that any other subset X of T^b is such that either $X \subseteq T^b_{oA} \cup T^b_{xA}$ or X overlaps T_{cA} and then implies $\neg(A^b)$. In other words, either X is not maximal or implies $\neg(A^b)$. It follows that $(T^b - \neg(A^b)) = T^b_{oA} \cup T^b_{xA}$.

[29] See Kuipers (2000, 2011a, b) and Cevolani et al. (2011a).

According to definition 9, $(T^b - \neg(A^b)) + A^b = T^b_{oA} \cup T^b_{xA} \cup A^b$. Since, by Definition 4, $T^b_{oA} = A^b_{oT} \subseteq A^b_{oT}$, it follows that $T^b_{oA} \cup T^b_{xA} \cup A^b = T^b_{xA} \cup A^b$, i.e., $T * A = T_{xT} \wedge A$.

iii) To prove the theorem it will suffice to prove that $T^b_{cA} \cup T^b_{xA}$ is the unique remainder of T^b by A^b (cf. Definition 3), and hence the unique result of the contraction of T^b by A^b. First, let us prove that $T^b_{cA} \cup T^b_{xA} \in T^b \perp A^b$. By Definition 4, $T^b_{cA} \cup T^b_{xA} \subseteq T^b$. Moreover, by Remark 1 and Definition 4, $T^b_{cA} \cup T^b_{xA}$ implies no element of A^b. Finally, any set Y such that $T^b_{cA} \cup T^b_{xA} \subset Y \subseteq T^b$ will contain some element of $T_{cA} = A_{oT}$ and then will imply some element of A^b. Thus, $T^b_{cA} \cup T^b_{xA}$ is a remainder of T^b by A^b. To see that $T^b_{cA} \cup T^b_{xA}$ is the *unique* remainder, note that any other subset X of T^b is such that either $X \subseteq T^b_{cA} \cup T^b_{xA}$ or X overlaps $T_{oA} = A_{oT}$ and then implies some element of A^b. In other words, X is not maximal or implies some element of A^b. Thus, $T^b - A^b = T^b_{cA} \cup T^b_{xA}$ and then $T - A = T_{cA} \wedge T_{xA}$.

Proof of Theorem 2 First, note that if A is true then A_{oT}, A_{cT} and A_{xT} are also true, and that if A is completely false then A_{oT}, A_{cT} and A_{xT} are also completely false. Moreover, recall that we leave cases of vacuous change aside.

(i) By hypothesis, A is compatible with T, i.e., $A^b_{cT} = T^b_{cA} = \varnothing$. By Theorem, $T + A = T \wedge A_{xT} = T_{oA} \wedge T_{xA} \wedge A_{xT}$. Thus, $T^b \subset (T + A)^b = T^b \cup A^b_{xT}$; moreover, since A_{xT} is true by hypothesis, $t(T, C_\star) \subset t(T + A, C_\star)$, whereas $f(T, C_\star) = f(T + A, C_\star)$. It follows from condition (M_t) of Definition 1 that $Vs(T + A) > Vs(T)$.

(ii) By Theorem 1, $T - A = T_{cA} \wedge T_{xA}$. Thus, $(T - A)^b \subset T^b$; moreover, since A_{oT} is true by hypothesis, $t(T - A, C_\star) \subset t(T, C_\star)$, whereas $f(T, C_\star) = f(T - A, C_\star)$. It follows from condition (M_t) of Definition 1 that $Vs(T) > Vs(T - A)$.

(iii) By Theorem, $T * A = A \wedge T_{xA}$. Note that T can be written as $A_{oT} \wedge \widetilde{A_{cT}} \wedge T_{xA}$ and $T * A$ as $A_{oT} \wedge A_{cT} \wedge A_{xT} \wedge T_{xA}$. In the limiting case $A^b_{cT} = \varnothing$ then $T * A = A_{oT} \wedge A_{xT} \wedge T_{xA} = T \wedge A_{xT} = T + A$; thus $Vs(T * A) > Vs(T)$ by the first clause of the present theorem (proved above). Otherwise, note first that since A_{cT} is true by hypothesis, $\widetilde{A_{cT}}$ is completely false by Definition 4. It follows that $t(T, C_\star) \subset t(T + A, C_\star) = t(T, C_\star) \cup (A_{oT} \wedge A_{xT})^b$, and that $f(T * A, C_\star) \subset f(T, C_\star) = f(T * A, C_\star) \cup (\widetilde{T_{oT}})^b$. Then, from condition ($M_f$) of Definition 1 that $Vs(T * A) > Vs(T)$.

The proofs of the remaining three clauses of the theorem can be obtained from the ones above in a straightforward manner.

Proof of Theorem 4 Let us first state without proof the following useful lemma:

Lemma 1 *Given a c-theory T, $Vs_\phi(T) = \sum_{x \in T^b} Vs_\phi(x)$. It follows that, given a c-input A, $Vs_\phi(T) = Vs_\phi(T_{oA}) + Vs_\phi(T_{cA}) + Vs_\phi(T_{xA})$ and $Vs_\phi(A) = Vs_\phi(A_{oT}) + Vs_\phi(A_{cT}) + Vs_\phi(A_{xT})$.*

Then:

1. $Vs_{\phi}(T + A) > Vs_{\phi}(T)$ iff (by Theorem 1) $Vs_{\phi}(T\wedge\ A_{xT}) > Vs_{\phi}(T)$ iff (by Definition 4 and Lemma 1) $Vs_{\phi}(T) + Vs_{\phi}(A_{xT}) > Vs_{\phi}(T)$ iff $Vs_{\phi}(A_{xT}) > 0$.
2. $Vs_{\phi}(T - A) > Vs_{\phi}(T)$ iff (by Theorem 1) $Vs_{\phi}(T_{cA}\wedge\ T_{xA}) > Vs_{\phi}(T)$ iff (by Definition 4 and Lemma 1) $Vs_{\phi}(T_{cA}) + Vs_{\phi}(T_{xA}) > Vs_{\phi}(T_{oA}) + Vs_{\phi}(T_{cA}) + Vs_{\phi}(T_{xA})$ iff $Vs_{\phi}(A_{oT}) = Vs_{\phi}(T_{oA}) < 0$.
3. $Vs_{\phi}(T\ *\ A) > Vs_{\phi}(T)$ iff (by Theorem 1) $Vs_{\phi}(T_{xA}\wedge\ A) > Vs_{\phi}(T)$ iff (by Definition 4 and Lemma 1) $Vs_{\phi}(T_{xA}) + Vs_{\phi}(A_{oT}) + Vs_{\phi}(A_{cT}) + Vs_{\phi}(A_{xT}) > Vs_{\phi}(T_{oA}) + Vs_{\phi}(T_{cA}) + Vs_{\phi}(T_{xA})$ iff (*recalling the definition of specular*) $Vs_{\phi}(A_{xT}) > Vs_{\phi}(\widetilde{A_{cT}}) - Vs_{\phi}(A_{cT})$.

Proof of Theorem 5 The following results will be useful in proof:

Lemma 2 *For any c-theory T*:

1. *T is verisimilar iff (by Definition 5)* $Vs_{\phi}(T) > 0$ *iff (by Definition 6)* $cont_t(T, C_{\bigstar}) > \phi cont_f(T, C_{\bigstar})$.
2. $Vs_{\phi}(T) > Vs_{\phi}(\widetilde{T})$ *iff (by Definition 6 and the definition of specular)* $cont_t(T, C_{\bigstar}) - \phi cont_f(T, C_{\bigstar}) > cont_f(T, C_{\bigstar}) - \phi cont_t(T, C_{\bigstar})$ iff $cont_t(T, C_{\bigstar}) > cont_f(T, C_{\bigstar})$.
3. *By the two previous results, it follows that: For* $\phi = 1$, *T is verisimilar iff* $Vs_{\phi}(T) > Vs_{\phi}(\widetilde{T})$ *iff* $\widetilde{T}$ *is t-distant. For* $\phi < 1$, if *T is t-distant then* $Vs_{\phi}(T) < Vs_{\phi}(\widetilde{T})$ *and* $\widetilde{T}$ *is verisimilar. For* $\phi > 1$, *if T is verisimilar then* $Vs_{\phi}(T) > Vs_{\phi}(\widetilde{T})$ *and* $\widetilde{T}$ *is t-distant.*

Accordingly:

1. If A_{cT} and A_{xT} are verisimilar, then $Vs_{\phi}(A_{cT}) > 0$ and $Vs_{\phi}(A_{xT}) > 0$ by Definition 5. Thus, if $\phi \geq 1$, then (by Lemma 2) $Vs_{\phi}(A_{cT}) > Vs_{\phi}(\widetilde{A_{cT}})$ and $Vs_{\phi}(\widetilde{A_{cT}}) - Vs_{\phi}(A_{cT}) < 0 < Vs_{\phi}(A_{xT})$; it follows that $Vs_{\phi}(T^*\ A) > Vs_{\phi}(T)$ by Theorem 4.
2. If A_{cT} and A_{xT} are t-distant, then $Vs_{\phi}(A_{cT}) < 0$ and $Vs_{\phi}(A_{xT}) < 0$ by Definition 5. Thus, if $\phi \leq 1$, then (by Lemma 2) $Vs_{\phi}(A_{cT}) < Vs_{\phi}(\widetilde{A_{cT}})$ and $Vs_{\phi}(\widetilde{A_{cT}}) - Vs_{\phi}(A_{cT}) > 0 > Vs_{\phi}(A_{xT})$; it follows that $Vs_{\phi}(T^*\ A) < Vs_{\phi}(T)$ by Theorem 4.

References

Alchourrón, C., Gärdenfors, P., & Makinson, D. (1985). On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic, 50*, 510–530.

Brink, C., & Heidema, J. (1987). A verisimilar ordering of theories phrased in a propositional language. *British Journal for the Philosophy of Science, 38*, 533–549.

Carnap, R. (1950). *Logical foundations of probability*. Chicago: University of Chicago Press.

Cevolani, G., & Calandra, F. (2010). *Approaching the truth via belief change in propositional languages*. In M. Suárez, M. Dorato, & M. Rèdei, (Eds.), *EPSA epistemology and methodology of science: Launch of the European Philosophy of Science Association, Chap. 5* (pp. 47–62). Dordrecht: Springer.

Cevolani, G., & Festa, R. (2009). Scientific change, belief dynamics and truth approximation. *La Nuova Critica, 51–52*, 27–59.
Cevolani, G., Crupi, V., & Festa, R. (2010). The whole truth about Linda: Probability, verisimilitude and a paradox of conjunction. In: M. D'Agostino, G. Giorello, F. Laudisa, T. Pievani, & C. Sinigaglia (Eds.), *New essays in logic and philosophy of science* (pp. 603–615). London: College Publications.
Cevolani, G., Festa, R., & Kuipers, T. A. F. (2011a). *Verisimilitude and belief change for nomic conjunctive theories*, manuscript.
Cevolani, G., Crupi, V., & Festa, R. (2011b). *Features of verisimilitude* (in preparation).
Festa, R. (1987). Theory of similarity, similarity of theories, and verisimilitude. In T. A. F. Kuipers (Ed.), *What is closer-to-the-truth?* (pp. 145–176). Amsterdam: Rodopi.
Festa, R. (2007). Verisimilitude, qualitative theories, and statistical inferences. In: S. Pihlström, P. Raatikainen, & M. Sintonen (Eds.) *Approaching truth: Essays in honour of Ilkka Niiniluoto* (pp. 143–178). Londra: College Publications.
Fuhrmann, A., & Hansson, S. O. (1994). A survey of multiple contractions. *Journal of Logic, Language and Information, 3*(1), 39–76.
Gärdenfors, P. (1988). *Knowledge in flux: Modeling the dynamics of epistemic states*. Cambridge, MA: MIT.
Gemes, K. (2007). Verisimilitude and content. *Synthese, 154*(2), 293–306.
Hansson S. O. (1999). *A textbook of belief dynamics: Theory change and database updating*. Dordrecht: Kluwer.
Hintikka, J. (1973). *Logic, language-games and information*. Oxford: Oxford University Press.
Kelly, K. T., & Glymour, C. (1989). Convergence to the truth and nothing but the truth. *Philosophy of Science, 56*(2), 185–220.
Kuipers, T. A. F. (1982). Approaching descriptive and theoretical truth. *Erkenntnis, 18*, 343–378.
Kuipers, T. A. F. (2000). *From instrumentalism to constructive realism*. Dordrecht: Kluwer.
Kuipers, T. A. F. (2011a). Basic and refined nomic truth approximation by evidence-guided belief revision in AGM-terms. *Erkenntnis* (this volume).
Kuipers, T. A. F. (2011b). *Dovetailing belief base revision with (basic) truth approximation*. Forthcoming in the proceedings of the Logic, Reasoning and Rationality conference (Gent, September 20–22, 2010).
Levi, I. (1980). *The enterprise of knowledge*. Cambridge, MA: MIT Press.
Miller, D. (1974). Popper's qualitative theory of verisimilitude. *The British Journal for the Philosophy of Science, 25*(2), 166–177.
Niiniluoto, I. (1987). *Truthlikeness*. Dordrecht: Reidel.
Niiniluoto, I. (1998). Verisimilitude: The third period. *The British Journal for the Philosophy of Science, 49*(1), 1–29.
Niiniluoto, I. (1999). Belief revision and truthlikeness. In: B. Hansson, S. Halldén, N. E. Sahlin, & W. Rabinowicz (Eds.), *Internet Festschrift for Peter Gärdenfors*. Lund: Department of Philosophy, Lund University. http://www.lucs.lu.se/spinning.
Niiniluoto, I. (2010). Theory change, truthlikeness, and belief revision. In: M. Suárez, M. Dorato, & M. Rèdei (Eds.), *EPSA Epistemology and Methodology of Science: Launch of the European Philosophy of Science Association* (pp. 189–199). Dordrecht: Springer.
Niiniluoto, I. (2011) Revising beliefs toward the truth. *Erkenntnis* (this volume).
Oddie, G. (1986). *Likeness to truth*. Dordrecht: Reidel.
Popper, K. R. (1963). *Conjectures and refutations: The growth of scientific knowledge* (3rd ed.). London: Routledge and Kegan Paul.
Rott, H. (2000). Two dogmas of belief revision. *Journal of Philosophy, 97*, 503–522.
Schurz G (2011) Verisimilitude and belief revision. Erkenntnis (this volume).
Schurz, G., & Weingartner, P. (1987) Verisimilitude defined by relevant consequence-elements. In: Kuipers T. A. F. (Ed.), *What is closer-to-the-truth*? (pp. 47–77). Amsterdam: Rodopi.
Schurz, G., & Weingartner, P. (2010). Zwart and Franssen's impossibility theorem holds for possible-world-accounts but not for consequence-accounts to verisimilitude. *Synthese 172*, 415–436.
Tichý, P. (1974). On Popper's definitions of verisimilitude. *The British Journal for the Philosophy of Science, 25*(2), 155–160.
Tversky, A. (1977). Features of similarity. *Psychological Review, 84*, 327–352.
Zwart, S. D. (2001). *Refined Verisimilitude*. Dordrecht: Kluwer.