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DAVID H. GLASS

PROBLEMS WITH PRIORS IN PROBABILISTIC MEASURES OF COHERENCE

ABSTRACT. Two of the probabilistic measures of coherence discussed in this paper take probabilistic dependence into account and so depend on prior probabilities in a fundamental way. An example is given which suggests that this prior-dependence can lead to potential problems. Another coherence measure is shown to be independent of prior probabilities in a clearly defined sense and consequently is able to avoid such problems. The issue of prior-dependence is linked to the fact that the first two measures can be understood as measures of coherence as *striking* agreement, while the third measure represents coherence as agreement. Thus, prior (in)dependence can be used to distinguish different conceptions of coherence.

1. INTRODUCTION

It has proved remarkably difficult to provide a satisfactory definition of coherence even though there is agreement about some of the features such a definition should possess. Since the coherence of a set of beliefs is a matter of degree and involves the relationship between those beliefs, it is tempting to think that a probabilistic account can be given. In recent papers Shogenji (1999), Olsson (2002) and Fitelson (2003) have adopted this strategy and proposed probabilistic measures of coherence, while Akiba (2000), Bovens and Hartmann (2003, Section 2.6)[1] and Siebel (2004) have presented a number of criticisms. An important criterion for any adequate measure of coherence is that it should provide a satisfactory account of the coherence of n beliefs in the general case where $n \geq 2$. However, in this paper the discussion is limited to the case of two beliefs. There are two main reasons for this rather severe restriction. First, there is no agreement concerning coherence measures even in this case. Second, there are two distinct ways of characterising coherence for more than two beliefs: one approach takes into account only the n-way coherence (Shogenji, 1999), while the other approach considers the j-way coherence for all $j \leq n$ (Fitelson, 2003). By focussing on the simpler problem of two beliefs, I hope to clarify some issues that might also be relevant in the general case.

In this paper I concentrate mainly on Fitelson's measure and argue that potential problems arise as a result of the way in which it depends on prior probabilities. I then consider an alternative coherence measure that overcomes these problems since it is independent (in a well-defined sense) of prior probabilities. Further analysis shows that this notion of independence can be used to distinguish two fundamentally different conceptions of coherence.

2. FITELSON'S COHERENCE MEASURE

Intuitively the coherence of two beliefs A and B tells us something about how well they fit together or give support to each other. Taking this idea into account, Fitelson bases his measure of coherence on a modification of Kemeny and Oppenheim's (1952) measure of factual support F. The degree to which B supports A can be written as

$$(1) \qquad F(A,B) = \begin{cases} \frac{P(B|A)-P(B|\sim A)}{P(B|A)+P(B|\sim A)}, & \text{if A is contingent and B is not a necessary falsehood} \\ 1, & \text{if A and B are necessary truths} \\ 0, & \text{if A is a necessary truth and B is contingent} \\ -1, & \text{if B is a necessary falsehood} \end{cases}$$

This expression can then be used to define the coherence measure, C_1, for a set of beliefs, which in the case of two beliefs A and B is

$$(2) \qquad C_1(A,B) = \frac{1}{2}\{F(A,B) + F(B,A)\}.$$

Fitelson intends his measure to be a probabilistic generalisation of logical coherence and as such it should respect the extreme deductive cases. Note that values of C_1 lie in the interval $[-1,1]$ with the value of 1 obtained if A and B are logically equivalent and satisfiable and -1 if they are logically inconsistent. Thus, in addition to capturing the idea of mutual support, Fitelson's measure treats these extreme cases in an appropriate manner.

A further claim made by Fitelson is that a probabilistic generalisation of logical coherence should be "properly sensitive to probabilistic dependence". To meet this requirement he develops an account of independence which, unlike the standard account, is able to deal with non-contingent beliefs. Two beliefs A and B are independent if and only if $F(A,B) = 0$ and $F(B,A) = 0$. Furthermore, they are positively(negatively) dependent if and only if F(A,B) and F(B,A) are both positive(negative). In the case of independence $C_1(A,B) = 0$, while in the case of positive(negative) dependence $C_1(A,B)$ is positive(negative). Thus, there is a close connection between the value of coherence as given by C_1 and the dependence relationship among the beliefs under consideration.[2]

All of the features mentioned so far make C_1 a very plausible candidate as a measure of coherence. The features of mutual support and the proper treatment of the deductive cases seem to be reasonable requirements for a coherence measure, especially if coherence is to be understood as a probabilistic generalisation of logical coherence. It also seems perfectly reasonable to expect that such a measure should include the notion of probabilistic (in)dependence. However, while the notion of probabilistic (in)dependence is central to Fitelson's account, it does give rise to potential problems as discussed below.

3. PROBLEMS WITH PRIOR DEPENDENCE

Coherence seems to be concerned with the extent to which beliefs agree with each other. By contrast, it is far from clear whether it should depend on how probable those beliefs are in the first place. This raises a question concerning the role prior probabilities should play in coherence measures. It is important to note that dependence on priors is not an all-or-nothing affair, but can occur in different ways and to different extents. Consider Shogenji's coherence measure, for example, which for two beliefs is given by,

$$(3) \qquad C_2(A,B) = \frac{P(A \wedge B)}{P(A)P(B)}.$$

Fitelson points out an inappropriate dependence on the prior probability in the C_2-measure in the case where the beliefs are logically equivalent. Consider logical equivalence for two beliefs where $P(A) = P(B) = p$. In this case the C_2-measure yields the value $1/p$. Fitelson's C_1-measure yields the maximal value of 1 in this case, which seems correct since logically equivalent beliefs are in complete agreement with each other and so it might be expected that they would be maximally coherent.

Akiba considers the case where A entails B. The C_2-measure yields the value $1/P(B)$, which Fitelson notes is "unintuitive, since it only depends on the unconditional probability of [B]." The problem here is that the C_2-measure depends only on the prior probability of B rather than on the relation between the beliefs. By contrast the C_1-measure yields the value $1/(1+P(B|{\sim}A))$ and so overcomes the problem. This is an improvement on Shogenji's result in this case, and as in the case of logical equivalence, indicates problems that can arise for Shogenji's measure due to the nature of its dependence on prior probabilities.

While Fitelson's C_1-measure does not depend on priors in the same way as Shogenji's measure, that is not to say that it does not depend on priors at all. This raises the question as to whether the remaining prior dependence in the C_1-measure is innocuous. The following example illustrates a potential problem with the C_1-measure, which suggests that it too might suffer from an inappropriate dependence on priors. Consider the example from Akiba's paper, where a die is rolled and the beliefs A and B are,

A: it will come up 2,

B: it will come up 2 or 4.

According to the C_1-measure the coherence is $1/(1+(1/5))=5/6$. Note that in this case $P(B|A)=1$, $P(A|B)=1/2$, $P(A)=1/6$ and $P(B)=1/3$. Now consider the same beliefs, but instead a dodecahedron is rolled. In this case, the C_1-measure yields the result $1/(1+1/11)=11/12$ and so the coherence is greater than in the case of the die.[3] But what has changed? Crucially, the prior probabilities have changed ($P(A)=1/12$ and $P(B)=1/6$), while the conditional probabilities remain the same ($P(B|A)=1$ and $P(A|B)=1/2$). This example suggests that there might be an inappropriate dependence on prior probabilities in Fitelson's account (as there is in Shogenji's[4]). Furthermore, this dependence could not be removed within Fitelson's framework since it relies on the notion of factual support which, as defined, requires the value for $P(B|{\sim}A)$ when considering the factual support for A provided by B. It is this probability (and $P(A|{\sim}B)$) that gives rise to the dependence on the prior probability of A (and B).

One line of response open to Fitelson, as well as Shogenji, in light of this alleged problem, depends on the distinction between coherence as agreement and coherence as *striking* agreement (Bovens and Olsson, 2000). Coherence as striking agreement refers to a conception of coherence that is sensitive to the specificity of the information, while this is not the case for coherence as agreement. As an example, Bovens and Olsson (2000) consider a roulette wheel with one hundred numbers. In the first scenario, Joe says the winning number is 49 or

50 and Amy says it is 50 or 51. In the second, scenario Joe says the winning number is 1, 2, ..., or 70 and Amy says it is 31, 32, ..., or 100. Are the claims of Joe and Amy more coherent in scenario one or scenario two? If coherence is taken to be coherence as agreement, their claims are more coherent in scenario two since the degree of overlap is greater. However, if coherence is taken to be coherence as striking agreement, their claims are more coherent in scenario one since the claims in scenario one are much more specific (even though the degree of overlap is slightly smaller).

Perhaps the C_1 and C_2-measures are intended to be measures of coherence as striking agreement, whereas the criticism offered in the example given above presupposes a view of coherence as agreement.[5] Furthermore, even if Fitelson and Shogenji both give measures of coherence as striking agreement, there are also further differences in the conceptions of coherence that they have in mind. The purpose in the rest of this paper is to focus on a measure of coherence as agreement and to explore some important differences between it and the measures discussed so far.

4. A PRIOR INDEPENDENT COHERENCE MEASURE

The discussion in Section 3 suggests that a measure of coherence as agreement should be independent of prior probabilities. This, however, seems like an impossible requirement since the only alternative for a probabilistic measure of coherence is that it should depend on conditional probabilities, but these of course depend on priors as Bayes' theorem makes clear. Nevertheless, there is a precise sense in which a coherence measure can be prior independent: if, for two probability distributions P and P′ on A and B, $P(A|B) = P'(A|B)$ and $P(B|A) = P'(B|A)$, then a coherence measure should assign the pair {A,B} the same coherence relative to P and P′.[6] A simple definition of coherence, C_3, that satisfies this requirement has been discussed by Olsson (2002) and Glass (2002) and for two beliefs is given by

$$(4) \qquad C_3(A,B) = \frac{P(A \wedge B)}{P(A \vee B)}$$

provided$P(A \vee B) \neq 0$.[7] Informally, it is the "degree of overlap" that determines the coherence of the beliefs.[8] $C_3(A,B) = 0$ when the probability of the conjunction is zero (i.e. there is no overlap). Whenever A entails B and vice-versa, $C_3(A,B) = 1$. Thus, C_3 yields a measure on the interval [0,1] with the value of 0 for logically inconsistent beliefs and the value of 1 for beliefs that are logically equivalent and satisfiable.

By using Bayes' theorem and assuming that$P(A \wedge B) \neq 0$, Equation (4) can be rewritten as,

$$
\begin{aligned}
C_3(A,B) &= \frac{P(A|B)P(B)}{P(A) + P(B) - P(A|B)P(B)} \\
&= \left[\frac{1}{P(A|B)} + \frac{1}{P(B|A)} - 1\right]^{-1}
\end{aligned} \tag{5}
$$

Thus the expression for coherence can be expressed in terms of conditional probabilities and so the relationship between the beliefs comes across clearly since coherence increases with increase in the conditional probabilities. Note also that no appeal to prior probabilities is required and so the coherence of A and B does not tell us anything about how likely it is that A and B are true in the first place.

Consider again the die/dodecahedron example discussed in Section 3. C_3 should be able to deal with this case since it was the prior dependence in the coherence measures C_1 and C_2 that caused the problem. Note that $C_3(A,B) = P(A|B)$ in cases where A entails B and so yields a value of 1/2 in the case of the dodecahedron as well as in the case of the die. The reason for the agreement in the two scenarios is that coherence as given by C_3 only depends on the conditional probabilities.

This crucial distinction between C_1 and C_2, on the one hand, and C_3 on the other in terms of prior dependence/independence is closely related to another fundamental difference between these measures. In their accounts Fitelson and Shogenji stress that there is a neutral point where the pair of beliefs (and also in the general case of *n* beliefs) are neither coherent nor incoherent and this neutral point is identified as the point of probabilistic independence. Shogenji notes that A and B are probabilistically independent if $P(A|B)/P(A) = 1$ and as a consequence the neutral value of coherence is 1. Fitelson's characterisation of independence differs for non-contingent statements since it requires that $P(A|B) - P(A|{\sim}B) = 0$ and so the neutral value is 0. Since the neutral points depend on probabilistic independence, the fact that Fitelson's measure has negative values while Shogenji's does not is merely conventional. By contrast, the fact that the C_3-measure of coherence is always positive is highly significant for it has no neutral point.

There seems to be a good reason for bringing probabilistic independence into the picture since it provides a neutral point between logical equivalence and logical contradiction. However, while this might be appropriate for measures of coherence as striking agreement it is problematic for coherence as agreement since it gives rise to the problems associated with the die/dodecahedron example. The reason for this is that there does not seem to be any way of characterising independence that does not depend on prior probabilities in a fundamental way. By contrast, the C_3-measure does not include the prior probabilities and so does not contain any information about the probabilistic dependence of the beliefs. Consequently, it is able to avoid the problem raised by the example. Thus, it seems that the presence or absence of prior-dependence and probabilistic (in)dependence might provide ways of characterising the difference between these two conceptions of coherence.

5. THE BOVENS–OLSSON CONDITION

Bovens and Olsson (2000) set out what they describe as a minimal sufficient condition for the relation "more coherent than" for a set of beliefs {A,B}. Considering two probability distributions P and P′ on A and B satisfying the condition that $P(A|B) > P'(A|B)$ and $P(B|A) > P'(B|A)$, they claim that {A,B} is more coherent on distribution P than on distribution P′. Clearly, the Bovens–Olsson condition and the notion of prior independence, as defined in Section 4, are closely related. In fact, prior independence can be understood as a natural extension of the Bovens–Olsson condition to the case where the relevant conditional probabilities are equal. It can be seen from expression (5) that increasing both the conditional probabilities necessarily increases coherence according to the C_3-measure so that $C_3(A,B) > C_3'(A,B)$ i.e. that {A,B} is more C_3-coherent on probability distribution P than on probability distribution P′. Thus, in addition to being prior independent, the C_3-measure satisfies the Bovens–Olsson condition. By using their condition Bovens and Olsson are able to establish a partial ordering of information pairs, whereas the C_3-measure provides a total ordering.

Given that the C_1 and C_2-measures are prior dependent, it might be expected that they would fail to satisfy the Bovens–Olsson condition and this is indeed the case as the following counterexample shows. Consider a die being rolled and the beliefs:

A: the die will come up 1 or 2,

B: the die will come up 2 or 3.

Let P′ be the distribution for an unbiased die. For this distribution we find that $P'(A|B) = 1/2$ and $P'(B|A) = 1/2$ and that the coherence measures yield the values $C_1'(A,B) = 1/3$, $C_2'(A,B) = 3/2$ and $C_3'(A,B) = 1/3$. Let P be the distribution for a biased die such that $P(1) = 1/5$, $P(2) = 2/5$, $P(3) = 1/5$, $P(4) = P(5) = P(6) = 1/15$ and so $P(A|B) = 2/3$ and $P(B|A) = 2/3$. For this distribution we find that $C_1(A,B) = 1/7$, $C_2(A,B) = 10/9$ and $C_3(A,B) = 1/2$. Although the conditional probabilities are higher for distribution P than they are for distribution P′, C_1 and C_2 are lower for distribution P. Thus, C_1 and C_2 fail to satisfy the Bovens–Olsson condition.

Bovens and Olsson note that one response to their condition is that it applies to coherence as agreement, but not to coherence as striking agreement. This is consistent with our earlier discussion and brings out the connection between three features of coherence measures. The coherence measures C_1 and C_2 depend on prior probabilities, incorporate the idea of probabilistic (in)dependence and fail to satisfy the Bovens–Olsson condition. By contrast C_3 is independent of prior probabilities, does not incorporate any notion of probabilistic (in)dependence and does satisfy the Bovens–Olsson condition. These three features are further linked with the fact that C_3 is a measure of coherence as agreement whereas C_1 and C_2 are perhaps better considered as measures of coherence as striking agreement.

6. THE PROBLEM OF CONJUNCTION

A final point needs to be taken into account since it is a more general criticism of coherence measures. Akiba (2000) points out a very serious concern regarding the C_2-measure, but it applies equally to the C_1 and C_3-measures. To quote Akiba,

> ... for any two things [A] and [B] we believe, we can also believe one thing, their conjunction, $[A \wedge B]$. Obviously the coherence of two beliefs [A] and [B] should be no different from the coherence of one conjunctive belief $[A \wedge B]$; that is, $[C(A,B) = C(A \wedge B)]$. (Akiba, 2000, p. 358)

If correct, it is a "devastating problem" since none of the coherence measures being considered satisfies Akiba's condition. Here I attempt to show that Akiba's argument is incorrect.[9]

Consider again the example of the die discussed in Section 5. Coherence, C(A,B), describes the relationship between two beliefs.

Crucially, it must take into account not only the extent of agreement between A and B, but also the extent of disagreement between them. For example, in the C_3-measure $P(A \wedge B)$ represents the agreement between A and B, while $P(A \vee B)$ also takes into account disagreement if $P(A \vee B) \neq P(A \wedge B)$. By contrast the coherence of the conjunction describes the coherence of a single belief and it is not even obvious that this makes sense since coherence is primarily a relationship between two beliefs. However, perhaps it does make sense to talk about the coherence of a belief with itself in which case Akiba's claim amounts to saying that $C(A,B) = C(A \wedge B, A \wedge B)$, but this does not seem plausible at all. To see this note that in the die/dodecahedron example in Section 2 the conjunctive belief in question is

$A \wedge B$: the die will come up 2.

Since this belief is in complete agreement with itself, we might expect that $C(A \wedge B, A \wedge B)$ should be maximal. There is no good reason to expect it to be the same as $C(A,B)$, which takes into account disagreement between the beliefs A and B, and so Akiba's argument fails.

7. CONCLUSIONS

In this paper I have considered a number of similarities and some important differences between Fitelson's measure of coherence, C_1, and the C_3-measure discussed in Section 4. Both measures are symmetric, treat the extreme deductive cases appropriately and capture the intuitive idea that coherent beliefs fit together well. However, C_1 takes probabilistic (in)dependence into account and as a result is dependent on prior probabilities. This leads to the problem associated with the example in Section 3 and the fact that C_1 fails to satisfy the Bovens–Olsson condition. By avoiding probabilistic (in)dependence, and hence prior dependence, C_3 is able to avoid these problems. This is important for C_3 since it is intended as a measure of coherence as agreement, whereas C_1 is arguably understood better as a measure of coherence as striking agreement.[10]

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NOTES

[1] Bovens and Hartmann (2003) give a detailed discussion of the concept of coherence and present their own probabilistic account, which differs in a number of respects from the approaches considered in this paper.
[2] The nature of this link is not quite as clear as it might appear from the discussion above. It turns out that in the case of two beliefs $C_1(A,B) = 0$ if and only if A and B are independent and $C_1(A,B)$ is positive/negative if and only if A and B are positively/negatively dependent. In the case of more than two beliefs, however, these are necessary but not sufficient conditions for the dependence relationship between the beliefs. C_1 could be greater than 0, for example, and yet the beliefs not be positively dependent. It also turns out to be the case that $C_1(A,B) = 1$ if and only if A and B are logically equivalent and satisfiable and $C_1(A,B) = -1$ if and only if A and B are logically inconsistent. Furthermore, it seems to be the case that these conditions for maximal and minimal values also hold for more than two beliefs.
[3] Fitelson (unpublished) has modified his original definition of coherence, but this change has no effect on the values of coherence in this example.
[4] Shogenji's measure, C_2, yields the value 3 in the case of the die and 6 in the case of the dodecahedron and so the C_2-measure has a much stronger prior dependence than the C_1-measure in this case.
[5] Olsson (2002) also draws attention to this distinction between Shogenji's measure and the C_3-measure given in Section 4 of this paper.
[6] I would like to thank an anonymous referee for helping to clarify the definition of prior independence.
[7] There seem to be two plausible values, 0 or 1, for $C_3(A,B)$ when $P(A \vee B) = 0$. The rationale for selecting the value of 0 is that the coherence measure generally yields a value of 0 when $P(A \wedge B) = 0$, which will be the case when $P(A \vee B) = 0$. The rationale for selecting the value of 1 is that the coherence of a belief with itself is 1 and even a contradictory belief could be considered to cohere maximally with itself.
[8] The idea of 'degree of overlap' becomes very clear if the probabilities of the beliefs are illustrated by a Venn diagram.
[9] Alternative responses to the problem of conjunction are presented by Shogenji (2001) and Olsson (2001).
[10] Other measures that are prior independent, deal adequately with deductive extremes and satisfy the Bovens–Olsson condition are $C_4(A,B) = 1/2\,[P(A|B) + P(B|A)]$ and $C_5(A,B) = P(A|B) \times P(B|A)$.

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School of Computing and Mathematics
University of Ulster
Newtownabbey, Co. Antrim
BT37 0QB
U.K.
E-mail: dh.glass@ulster.ac.uk