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On the quasi-static approximation in the initial boundary value problem of linearised elastodynamics

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Abstract Continuous data dependence estimates are employed to rigorously derive conditions that validate the quasi-static approximation for the initial homogeneous boundary value problem in the theory of small elastic deformations superposed upon large elastic deformations. This theory imposes no sign-definite assumptions on the linearised elastic moduli and in consequence the requisite estimates are established using methods principally motivated by known Lagrange identity arguments.

Keywords Linearised elastodynamics · Quasi-static approximation · Continuous dependence . Lagrange identities arguments . Dirichlet boundary conditions

1 Introduction

The quasi-static approximation assumes that when the inertia is small compared, for example, to the strain and velocity, it may be neglected leading to a simplified problem for which the solution is more easily obtained. The origin of the approximation is attributed by Boley and Weiner [1] to Duhamel as part of his 1837 studies into thermoelasticity [2]. But evidently, thermal damping is not the only cause of comparatively small inertia, if not immediately then after a short period of time. Other causes may be due to viscous damping, and time evolutionary boundary conditions and source terms. Shock waves may be another cause. Typical of numerous applications of the quasi-static approximation are those described by [3–6]. Dafermos [7,8], however, shows that conditions exist in thermoelasticity under which the inertia does not asymptotically vanish in time. See also Lebeau and Zuazua [9]. Such types of long-term behaviour together with several counter-examples caution against universal acceptance of the quasi-static approximation which although intuitively plausible is seldom rigorously established. Without mathematical proof, the reliability of the approximation is kept in doubt irrespective of how expedient its use may

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be in practice. The relatively small literature devoted to rigorous examination of the approximation includes the contributions [10–14] all of which are concerned with linear thermoelastodynamics.

Here the approach generally follows that introduced in [14] which interprets the quasi-static approximation in the context of continuous data dependence. The appropriate strategy has three components:

1. Continuous dependence of inertia upon relevant data or damping mechanisms.
2. Determination of the time interval after which the inertia becomes uniformly spatially negligible compared with other field variables such as displacement, velocity, strain, and temperature.
3. Continuous dependence upon inertia of the difference between the original dynamical solution and its quasi-static approximation.

Rather than the thermoelastic problems discussed in the previously mentioned studies [10–13] which demonstrate that thermal damping leads to negligible inertia independently of initial and boundary conditions, we illustrate the strategy by treating the initial homogeneous Dirichlet boundary value problem for the self-adjoint equations of linearised elastodynamics. Consequently, damping is absent. On the other hand, judicious choice of the underlying large elastic deformation contravenes the sign-definiteness of the linearised elastic moduli, and accordingly our investigation cannot appeal to classical methods based upon sign-definite assumptions. Instead, we employ Lagrange identity techniques variously developed, for example, in [15–17], [18, Chpt 5], and [19]. Characteristic of the approach is the restriction of the solution to certain constrained function classes. Subject to this requirement, we are able to verify Components 1 and 3 and conclude that small non-trivial initial Cauchy data is sufficient for the inertia to be correspondingly small both instantaneously and throughout the interval of existence. This renders the second Component redundant. It is worth remarking that the estimate constructed in the proof of Component 3 involves not only the inertia but also initial data. Of course, continuous dependence of the difference solution upon initial data could be achieved without the involvement of inertia, but its introduction is essential when seeking to justify the quasi-static approximation.

Another consequence of the loss of sign-definiteness is that that while the basic linearised dynamic problem has a unique solution (see, for example, [16]), the quasi-static approximation system may possess several non-trivial solutions as discussed in [20]. It must therefore be decided which of these non-unique solutions is used in the approximation.

Section 2 formulates both the basic initial boundary value problem and its quasi-static approximation. Section 3 employs weighted mean square measures and Lagrange identity methods to construct estimates that establish continuous dependence of the inertia upon the initial Cauchy data. Several bounds of related interest are derived in Sect. 4 leading to estimates between the inertia, displacement, displacement gradient, and velocity. Section 5 explains how the difference between the basic dynamical solution and its quasi-static approximation depends continuously upon the inertia and in what measures. By means of the proposed strategy, proof of the quasi-static approximation is thus achieved for the particular initial boundary value problem of present concern. It is worth repeating that conclusions in Sects. 3 and 5 depend upon the respective solutions belonging to certain constrained function classes. Section 6 contains brief concluding remarks. Two key results taken from [19] are listed in the Appendix to make the paper reasonably self-contained.

Throughout, existence of suitably smooth solutions is assumed on some finite time interval. Generalisation to weak solutions is not considered. The standard conventions are adopted of summation over repeated subscripts and a comma to denote partial differentiation. Latin suffixes have the range 1, 2, 3 apart from the time variable t. Scalar, vector, and tensor quantities are not typographically distinguished.

2 Notation and other preliminaries

Within the theory of small elastic deformations superposed upon large elastic deformations [21], we consider a linearised non-homogeneous anisotropic compressible elastic body that occupies a bounded region Ω of Euclidean three-space. The boundary $\partial\Omega$ is Lipschitz smooth. The body is in motion subject to homogeneous Dirichlet boundary conditions, zero body-force, and prescribed Cauchy initial data. For positive constant T, $[0, T)$ denotes the half-open maximum time interval of existence, while $u(x, t)$, $x \in \Omega$, $t \in [0, T)$ denotes the vector displacement field, where t is the time variable and x is the position vector whose components with respect to a rectangular Cartesian coordinate system are x_1, x_2, x_3. The vector displacement $u(x, t)$ has components u_i, $i = 1, 2, 3$ with respect to same coordinate system and is assumed sufficiently smooth to satisfy all subsequent operations.

The motion of the elastic body is described by the system (cf. [21]):

$$(c_{ijkl}u_{k,l})_{,j} = \rho u_{i,tt}, \qquad (x, t) \in \Omega \times [0, T), \tag{2.1}$$

$$u_i(x, t) = 0, \qquad (x, t) \in \partial\Omega \times [0, T), \tag{2.2}$$

$$u_i(x, 0) = u_i^{(0)}(x), \quad u_{i,t}(x, 0) = u_i^{(1)}(x), \qquad x \in \Omega, \tag{2.3}$$

where $0 < \rho(x) < \infty$ is the time-independent mass density, and $u^{(0)}$, $u^{(1)}$ are initial data. The suitably smooth Cartesian components c_{ijkl} of the linearised elastic modulus tensor are supposed time independent and to possess the major symmetry

$$c_{ijkl} = c_{klij}, \tag{2.4}$$

but otherwise are of unrestricted symmetry.

Note that the initial value of the inertia is known since the equations of motion (2.1) are assumed valid at $t = 0$. Note also that the symmetry (2.4) implies the equations are self-adjoint.

We emphasise that the major symmetry (2.4) is the sole restriction on the elastic moduli apart from smoothness and time-independence. The underlying large elastic deformation may be chosen to violate sign-definite conditions on the elastic moduli and therefore such assumptions have no part in the problem under consideration. It can be proved, however, either by logarithmic convexity or by Lagrange identity arguments that condition (2.4) is sufficient for uniqueness of the solution to the initial boundary value problems of linearised elastodynamics including the system (2.1)–(2.3); see, for example, [16,20]. Lagrange identity arguments are used in [19] to derive general continuous data dependence results including that of the solution $u(x, t)$ to (2.1) and (2.2) upon initial data (2.3) of particular relevance to the present problem. Indeed, methods developed in [19] motivate the approach adopted in Sects. 3 and 5.

The usual quasi-static approximation to (2.1)–(2.3) asserts that when the inertia in (2.1) is sufficiently small in magnitude it may be neglected and that the displacement vector $u(x, t)$ is satisfactorily approximated by the solution $v(x)$ to the simplified system

$$(c_{ijkl}v_{k,l})_{,j} = 0, \qquad x \in \Omega, \tag{2.5}$$

$$v_i = 0, \qquad x \in \partial\Omega, \tag{2.6}$$

where the components of the linearised elastic moduli continue to satisfy only the major symmetry condition (2.4). The vector field $v(x)$ is assumed to be a smooth function of position independent of the time variable which is not present even as a parameter. Unlike the uniqueness of the solution to the initial boundary value problem (2.1)–(2.3), the boundary value problem (2.5) and (2.6) subject to (2.4) does not possess a unique solution (cf. [20]). Consequently, the quasi-static approximation may depend for its validity on the particular choice of the non-unique solution to (2.5) and (2.6). This aspect is further discussed in Sect. 5.

Remark 2.1 The classical theory of linear elasticity requires the Cartesian components d_{ijkl} of the elastic modulus tensor in addition to the major symmetry condition (2.4) to also satisfy the minor symmetries

$$d_{ijkl} = d_{jikl}. \tag{2.7}$$

Furthermore, it is customary to suppose that the elastic modulus tensor is positive-definite in the sense that there exists a positive constant d_0 such that

$$d_0\xi_{ij}\xi_{ij} \le d_{ijkl}\xi_{ij}\xi_{kl} \qquad \forall\xi_{ij} = \xi_{ji}, \tag{2.8}$$

for all symmetric second order tensors ξ. Then in accordance with the procedure proposed in Sect. 1, the quasi-static approximation is established in [14] for the initial homogeneous boundary value problem of the coupled theory of linear thermoelastodynamics with zero source terms. In the corresponding isothermal theory the approximation is proved for the initial traction boundary value problem with zero body-force and prescribed time-dependent surface traction [22].

In the next section, we consider the first component of the strategy stated in Sect. 1 for the validation of the quasi-static approximation.

3 Dependence of inertia upon initial Cauchy data

Continuous dependence of the inertia upon initial Cauchy data is established by means of a method that modifies arguments developed in [19, para.4]. In consequence, continuous dependence is measured with respect to time weighted mean square norms. Although the time interval of existence is $[0, T)$, where $0 < T \leq \infty$, it is convenient to restrict the time variable t to the interval $[0, T/2]$.

Then for any positive integer m, eventually chosen to satisfy $m \geq 4$, we have

$$\int_0^t \int_{\Omega(\eta)} \rho u_{i,\eta\eta} u_{i,\eta\eta} \,\mathrm{d}x\,\mathrm{d}\eta \leq \int_0^{T/2} \int_{\Omega(\eta)} \rho u_{i,\eta\eta} u_{i,\eta\eta} \,\mathrm{d}x\,\mathrm{d}\eta \tag{3.1}$$

$$\leq \left(\frac{2}{T}\right)^m \int_0^{T/2} \int_{\Omega(\eta)} (T-\eta)^m \rho u_{i,\eta\eta} u_{i,\eta\eta} \,\mathrm{d}x\,\mathrm{d}\eta, \tag{3.2}$$

where η denotes a time variable and $\Omega(t)$ signifies that terms in the corresponding integrand are evaluated at time t. Inequality (3.2) follows on observing that $0 \leq \eta \leq t \leq T/2$ and therefore $0 \leq (T/2 - \eta) = T - \eta - T/2$ so that

$$T/2 \leq (T-\eta). \tag{3.3}$$

Rewrite inequality (3.2) as

$$\int_0^t \int_{\Omega(\eta} \rho u_{i,\eta\eta} u_{i,\eta\eta} \,\mathrm{d}x\,\mathrm{d}\eta \leq \frac{1}{2}\left(I_1 + I_2\right), \qquad 0 \leq t \leq T/2, \tag{3.4}$$

where

$$I_1 = \left(\frac{2}{T}\right)^m \int_0^T \int_{\Omega(\eta)} (T-\eta)^m \left[\rho u_{i,\eta\eta} u_{i,\eta\eta} + c_{ijkl} u_{i,j\eta} u_{k,l\eta}\right] \mathrm{d}x\,\mathrm{d}\eta, \tag{3.5}$$

$$I_2 = \left(\frac{2}{T}\right)^m \int_0^T \int_{\Omega(\eta)} (T-\eta)^m \left[\rho u_{i,\eta\eta} u_{i,\eta\eta} - c_{ijkl} u_{i,j\eta} u_{k,l\eta}\right] \mathrm{d}x\,\mathrm{d}\eta. \tag{3.6}$$

These expressions are separately discussed. Subsequent to an integration by parts, we have for I_1 the alternative representations

$$I_1 = -\left(\frac{2}{T}\right)^m \frac{1}{(m+1)} \int_0^T \int_{\Omega(\eta)} \left[(T-\eta)^{m+1}\right]_{,\eta} \left[\rho u_{i,\eta\eta} u_{i,\eta\eta} + c_{ijkl} u_{i,j\eta} u_{k,l\eta}\right] \mathrm{d}x\,\mathrm{d}\eta$$

$$= \frac{2^m T}{(m+1)} \int_{\Omega(0)} \left[\rho u_{i,tt} u_{i,tt} + c_{ijkl} u_{i,jt} u_{k,lt}\right] \mathrm{d}x \tag{3.7}$$

$$+ \frac{2}{(m+1)} \left(\frac{2}{T}\right)^m \int_0^T \int_{\Omega(\eta)} (T-\eta)^{m+1} \left[\rho u_{i,\eta\eta} u_{i,\eta\eta\eta} + c_{ijkl} u_{i,j\eta} u_{k,l\eta\eta}\right] \mathrm{d}x\,\mathrm{d}\eta. \tag{3.8}$$

But (2.1) and (2.2) together with the assumed smoothness of the solution $u(x, t)$ leads to

$$\int_0^T \int_{\Omega(\eta)} (T-\eta)^{m+1} \left[\rho u_{i,\eta\eta} u_{i,\eta\eta\eta} + c_{ijkl} u_{i,j\eta} u_{k,l\eta\eta}\right] \mathrm{d}x\,\mathrm{d}\eta = 0,$$

and we conclude that I_1 through (3.7) depends only upon T and the initial data. (Recall that (2.1) is assumed valid at $t = 0$.)

Discussion of the second term I_2 commences with the rearrangement

$$I_2 = \left(\frac{2}{T}\right)^m \int_0^T \int_{\Omega(\eta)} (T-\eta)^m \left[(\rho u_{i,\eta\eta} u_{i,\eta})_{,\eta} - \rho u_{i,\eta\eta\eta} u_{i,\eta} - (c_{ijkl} u_{i,\eta} u_{k,l\eta})_{,j} + (c_{ijkl} u_{k,l\eta})_{,j} u_{i,\eta}\right] \mathrm{d}x\, \mathrm{d}\eta,$$

which after appeal to (2.1), (2.2) and (2.4) and integration by parts may be successively expressed as

$$\begin{aligned} I_2 &= \left(\frac{2}{T}\right)^m \int_0^T \int_{\Omega(\eta)} (T-\eta)^m (\rho u_{i,\eta\eta} u_{i,\eta})_{,\eta}\, \mathrm{d}x\, \mathrm{d}\eta \\ &= -2^m \int_{\Omega(0)} \rho u_{i,tt} u_{i,t}\, \mathrm{d}x + m \left(\frac{2}{T}\right)^m \int_0^T \int_{\Omega(\eta)} (T-\eta)^{m-1} \rho u_{i,\eta\eta} u_{i,\eta}\, \mathrm{d}x\, \mathrm{d}\eta \\ &= -2^m \int_{\Omega(0)} \rho u_{i,tt} u_{i,t}\, \mathrm{d}x + \frac{m}{2} \left(\frac{2}{T}\right)^m \int_0^T \int_{\Omega(\eta)} (T-\eta)^{m-1} (\rho u_{i,\eta} u_{i,\eta})_{,\eta}\, \mathrm{d}x\, \mathrm{d}\eta \\ &= -2^m \int_{\Omega(0)} \rho u_{i,tt} u_{i,t}\, \mathrm{d}x - \frac{2^{m-1} m}{T} \int_{\Omega(0)} \rho u_{i,t} u_{i,t}\, \mathrm{d}x \\ &\quad + \left(\frac{2}{T}\right)^m \frac{m(m-1)}{2} \int_0^T \int_{\Omega(\eta)} (T-\eta)^{m-2} \rho u_{i,\eta} u_{i,\eta}\, \mathrm{d}x\, \mathrm{d}\eta \\ &= D_2 + I_3, \end{aligned} \tag{3.9}$$

where the initial data term D_2 is defined by

$$D_2 = -2^m \int_{\Omega(0)} \rho u_{i,tt} u_{i,t}\, \mathrm{d}x - \frac{2^{m-1} m}{T} \int_{\Omega(0)} \rho u_{i,t} u_{i,t}\, \mathrm{d}x, \tag{3.10}$$

and the integral I_3 by

$$I_3 = \left(\frac{2}{T}\right)^m \frac{m(m-1)}{2} \int_0^T \int_{\Omega(\eta)} (T-\eta)^{m-2} \rho u_{i,\eta} u_{i,\eta}\, \mathrm{d}x\, \mathrm{d}\eta. \tag{3.11}$$

Definition (3.11) for I_3 is now decomposed to give

$$I_3 = \frac{1}{2} (Q_1 + Q_2), \tag{3.12}$$

where

$$Q_1 = \left(\frac{2}{T}\right)^m \frac{m(m-1)}{2} \int_0^T \int_{\Omega(\eta)} (T-\eta)^{m-2} \left[\rho u_{i,\eta} u_{i,\eta} + c_{ijkl} u_{i,j} u_{k,l}\right] \mathrm{d}x\, \mathrm{d}\eta, \tag{3.13}$$

$$Q_2 = \left(\frac{2}{T}\right)^m \frac{m(m-1)}{2} \int_0^T \int_{\Omega(\eta)} (T-\eta)^{m-2} \left[\rho u_{i,\eta} u_{i,\eta} - c_{ijkl} u_{i,j} u_{k,l}\right] \mathrm{d}x\, \mathrm{d}\eta. \tag{3.14}$$

Consider Q_1. We have

$$\begin{aligned} Q_1 &= -\frac{m}{2} \left(\frac{2}{T}\right)^m \int_0^T \int_{\Omega(\eta)} \left[(T-\eta)^{m-1}\right]_{,\eta} \left[\rho u_{i,\eta} u_{i,\eta} + c_{ijkl} u_{i,j} u_{k,l}\right] \mathrm{d}x\, \mathrm{d}\eta \\ &= \left(\frac{2^{m-1} m}{T}\right) \int_{\Omega(0)} \left[\rho u_{i,t} u_{i,t} + c_{ijkl} u_{i,j} u_{k,l}\right] \mathrm{d}x \\ &= D_3, \end{aligned} \tag{3.15}$$

where D_3 depends only upon T and initial data.

It remains to treat Q_2 defined by (3.14). The method is similar to that applied to I_2 and likewise appeals to (2.1),(2.2) and (2.4). To avoid a circular argument, however, the displacement field $u(x, t)$ is required later to belong to a constrained set of functions specified by (3.22). Accordingly, we rewrite (3.14) as

$$
\begin{aligned}
Q_2 &= \left(\frac{2}{T}\right)^m \frac{m(m-1)}{2} \int_0^T \int_{\Omega(\eta)} (T-\eta)^{m-2} \left[(\rho u_{i,} u_{i,\eta})_{,\eta} - \rho u_i u_{i,\eta\eta} - c_{ijkl} u_{i,j} u_{k,l}\right] \mathrm{d}x\, \mathrm{d}\eta \\
&= -\frac{2^{m-1} m(m-1)}{T^2} \int_{\Omega(0)} \rho u_i u_{i,t}\, \mathrm{d}x \qquad (3.16)\\
&\quad + \left(\frac{2}{T}\right)^m \frac{m(m-1)(m-2)}{2} \int_0^T \int_{\Omega(\eta)} (T-\eta)^{m-3} \rho u_i u_{i,\eta}\, \mathrm{d}x\, \mathrm{d}\eta \\
&= D_4 + D_5 \qquad (3.17)\\
&\quad + \left(\frac{2}{T}\right)^m \frac{m(m-1)(m-2)(m-3)}{4} \int_0^T \int_{\Omega(\eta)} (T-\eta)^{m-4} \rho u_i u_i\, \mathrm{d}x\, \mathrm{d}\eta, \qquad (3.18)
\end{aligned}
$$

where the data terms D_4 and D_5 are defined by

$$
D_4 = -\frac{2^{m-1} m(m-1)}{T^2} \int_{\Omega(0)} \rho u_i u_{i,t}\, \mathrm{d}x, \qquad (3.19)
$$

$$
D_5 = -\frac{2^{m-2} m(m-1)(m-2)}{T^3} \int_{\Omega(0)} \rho u_i u_i\, \mathrm{d}x. \qquad (3.20)
$$

Insertion of definitions (3.5), (3.6), (3.12), (3.10), (3.15), and (3.18)–(3.20) into (3.2) yields for $0 \le t \le T/2$:

$$
\int_0^t \int_{\Omega(\eta)} \rho u_{i,\eta\eta} u_{i,\eta\eta}\, \mathrm{d}x\, \mathrm{d}\eta \le D + \left(\frac{2}{T}\right)^m \frac{m(m-1)(m-2)(m-3)}{4} \int_0^T \int_{\Omega(\eta)} (T-\eta)^{m-4} \rho u_i u_i\, \mathrm{d}x\, \mathrm{d}\eta, \quad (3.21)
$$

where

$$
D = \frac{1}{2}(I_1 + D_2) + \frac{1}{4}(D_3 + D_4 + D_5)
$$

is specified by initial data. The objective, however, is to derive in suitable measure an upper bound for the inertia entirely in terms of the initial data. For this purpose, we assume the displacement $u(x,t)$ belongs to the constrained set given by

$$
\int_0^T \int_{\Omega(\eta)} \rho u_i u_i\, \mathrm{d}x\, \mathrm{d}\eta \le M_1^2, \qquad (3.22)
$$

for some positive constant M_1. On taking $m = 4$, the estimate (A.2) becomes available which after a time integration and substitution in (3.21) yields the required result:

$$
\begin{aligned}
\int_0^t \int_{\Omega(\eta)} \rho u_{i,\eta\eta} u_{i,\eta\eta}\, \mathrm{d}x\, \mathrm{d}\eta &\le D + \frac{3.2^6}{T^4}\left(1 + \frac{T^2}{4} + \frac{M_1 T^{3/2}}{4}\right) \int_\Omega \rho u_i^{(0)} u_i^{(0)}\, \mathrm{d}x \\
&\quad + M_1 \left(\frac{2}{T}\right)^{3/2} \int_\Omega \rho u_i^{(1)} u_i^{(1)}\, \mathrm{d}x, \quad 0 \le 2t \le T. \qquad (3.23)
\end{aligned}
$$

The interval $[0, T/2]$ over which t varies may be iteratively extended to $(1-2^{-n})T,\ n = 1, 2, 3, \ldots$ as described in [19].

4 Subsidiary results

Further results may be extracted from calculations in Sect. 3 and lead to additional estimates for continuous dependence. In particular, they demonstrate that in weighted mean square measure, the inertia does not exceed either the displacement, displacement gradient, or velocity. Proofs of these inequalities are now sketched.

The bound (3.21) indicates the sense in which the inertia is no greater than the displacement. However, in conjunction with Poincaré's inequality, for which

$$\lambda \int_{\Omega} \rho u_i u_i \, \mathrm{d}x \leq \int_{\Omega} \rho u_{i,j} u_{i,j} \, \mathrm{d}x \tag{4.1}$$

and λ is the first eigenvalue for the fixed membrane problem for Ω, the same estimate (3.21) provides an upper bound for the inertia in terms of initial data and displacement gradients.

On the other hand, insertion of identity (3.9) into inequaltiy (3.4) leads to continuous dependence of the inertia upon initial data and velocity. We obtain for $0 \leq t \leq T/2$:

$$\int_0^t \int_{\Omega(\eta)} \rho u_{i,\eta\eta} u_{i,\eta\eta} \, \mathrm{d}x \, \mathrm{d}\eta \leq \frac{1}{2}(I_1 + D_2) + \left(\frac{2}{T}\right)^m \frac{m(m-1)}{4} \int_0^T \int_{\Omega(\eta)} (T-\eta)^{m-2} \rho u_{i,\eta} u_{i,\eta} \, \mathrm{d}x \, \mathrm{d}\eta. \tag{4.2}$$

The conclusion that the velocity, again in suitable weighted mean-square measure, is itself bounded above by the displacement, and consequently by initial data for displacement fields in the constrained set (3.22), is immediate from the decomposition (3.12) and the expression (3.18) for Q_2. The explicit bound involving the displacement becomes

$$\int_0^T \int_{\Omega(\eta)} (T-\eta)^{m-2} \rho u_{i,\eta} u_{i,\eta} \, \mathrm{d}x \, \mathrm{d}\eta \leq \frac{(m-2)(m-3)}{2} \int_0^T \int_{\Omega(\eta)} (T-\eta)^{m-4} \rho u_i u_i \, \mathrm{d}x \, \mathrm{d}\eta + \left(\frac{T}{2}\right)^m \frac{1}{m(m-1)} \left[D_2 + \frac{1}{2}(D_3 + D_4) \right]. \tag{4.3}$$

An upper bound in mean-square measure for the velocity in terms solely of initial data may be derived from (4.3) on putting $m = 4$, imposing the constraint (3.22) and using inequality (3.23).

The estimate (4.3) establishes the sense in which the velocity is bounded above by the displacement. Nevertheless, a reverse bound is possible. From (3.17) and (3.18), we have

$$\begin{aligned} &\left(\frac{T}{2}\right)^m \frac{4D_5}{m(m-1)(m-2)} + (m-3) \int_0^T \int_{\Omega(\eta)} (T-\eta)^{m-4} \rho u_i u_i \, \mathrm{d}x \, \mathrm{d}\eta \\ &= 2 \int_0^T \int_{\Omega(\eta)} (T-\eta)^{m-3} \rho u_i u_{i,\eta} \, \mathrm{d}x \, \mathrm{d}\eta \\ &\leq \int_0^T \int_{\Omega(\eta)} \alpha (T-\eta)^{m-3} \rho u_i u_i \, \mathrm{d}x \, \mathrm{d}\eta + \int_0^T \int_{\Omega(\eta)} \alpha^{-1} (T-\eta)^{m-3} \rho u_{i,\eta} u_{i,\eta} \, \mathrm{d}x \, \mathrm{d}\eta, \end{aligned} \tag{4.4}$$

where Young's inequality is employed and α is a positive function. For each η choose

$$\alpha = \gamma (T-\eta)^{-1},$$

where the positive constant γ satisfies $0 < \gamma < m - 3$. Then (4.4) reduces to

$$\begin{aligned} &\left(\frac{T}{2}\right)^m \frac{4D_5}{m(m-1)(m-2)} + (m-3-\gamma) \int_0^T \int_{\Omega(\eta)} (T-\eta)^{m-4} \rho u_i u_i \, \mathrm{d}x \, \mathrm{d}\eta \\ &\leq \gamma^{-1} \int_0^T \int_{\Omega(\eta} (T-\eta)^{m-2} \rho u_{i,\eta} u_{i,\eta} \, \mathrm{d}x \, \mathrm{d}\eta, \end{aligned} \tag{4.5}$$

which represents the required reverse bound.

Particular conclusions are obtained when $2\gamma = (m-3)$ or $\gamma = (m-3)$. Other specific choices of the positive constant γ do not appear to yield inequalities of especial significance.

5 Continuous dependence upon inertia

The final step in the proof of the quasi-static approximation for the linearised elastodynamic system (2.1)–(2.4) demonstrates that the difference between the original dynamical solution and its quasi-static approximation depends continuously upon inertia.

In order to describe the problem, let

$$w(x,t) = u(x,t) - v(x), \qquad x \in \Omega,\ t \in [0,T), \tag{5.1}$$

where u and v are the respective solutions to (2.1)–(2.3) and the quasi-static approximation system (2.5) and (2.6) subject to the symmetry relation (2.4). As remarked in Sect. 2, the solution $v(x)$ in general is non-unique and in consequence both trivial and non-trivial solutions may exist to (2.5) and (2.6). Thus, in definition (5.1), there must be a single definite choice for v from among the non-unique solutions which we recall are supposed continuously differentiable and mean-square integrable. Once v is selected, the vector field $w(x,t)$ is likewise continuously differentiable and satisfies the system

$$(c_{ijkl}w_{k,l})_{,j} = \rho w_{i,tt}, \qquad (x,t) \in \Omega \times [0,T), \tag{5.2}$$

$$w_i(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T), \tag{5.3}$$

$$w_i(x,0) = w_i^{(0)} = u_i^{(0)}(x) - v(x),\ w_{i,t}(x,0) = w_i^{(1)}(x) = u_i^{(1)}(x), \qquad x \in \Omega, \tag{5.4}$$

where (see (2.4))

$$c_{ijkl}(x) = c_{klij}(x). \tag{5.5}$$

Before proceeding to establish that the difference displacement vector field $w(x,t)$ depends continuously upon the inertia $\rho u_{i,tt}$ equivalently expressed as $\rho w_{i,tt}$, it is of interest to derive certain continuous dependence bounds that are independent of any governing equations. The proof repeatedly uses the one-dimensional Wirtinger inequality [23] which for real-valued continuously differentiable functions $g(t)$ defined on $[0,t]$ is given by

$$\int_0^t [g(\eta)) - g(0)]^2\,d\eta \le \frac{4t^2}{\pi^2}\int_0^t g_{,\eta}^2\,d\eta. \tag{5.6}$$

Then, by standard inequalities, the difference displacement w satisfies the bounds

$$\begin{aligned}
\int_{\Omega(t)} \rho w_i w_i \mathrm{d}x &\le 2\int_{\Omega(t)} \rho(w_i - w_i^{(0)})(w_i - w_i^{(0)})\mathrm{d}x + 2\int_\Omega \rho w_i^{(0)} w_i^{(0)}\mathrm{d}x \\
&= 4\int_0^t\int_{\Omega(\eta)} \rho(w_i - w_i^{(0)})w_{i,\eta}\mathrm{d}x\,\mathrm{d}\eta + 2\int_\Omega \rho w_i^{(0)} w_i^{(0)}\mathrm{d}x \\
&\le 4\Big[\int_0^t\int_{\Omega(\eta)} \rho(w_i - w_i^{(0)})(w_i - w_i^{(0)})\,\mathrm{d}x\,\mathrm{d}\eta \int_0^t\int_{\Omega(\eta)} \rho w_{i,\eta}w_{i,\eta}\mathrm{d}x\,\mathrm{d}\eta\Big]^{1/2} \\
&\quad +2\int_\Omega \rho w_i^{(0)} w_i^{(0)}\,\mathrm{d}x \\
&\le \frac{8t}{\pi}\int_0^t\int_{\Omega(\eta)} \rho w_{i,\eta}w_{i,\eta}\mathrm{d}x\,\mathrm{d}\eta + 2\int_\Omega \rho w_i^{(0)} w_i^{(0)}\,\mathrm{d}x.
\end{aligned} \tag{5.7}$$

The same argument applied to the velocity $w_{i,t}$ leads to an inequality which after a time integration is given by

$$\int_0^t\int_{\Omega(\eta)} \rho w_{i,\eta}w_{i,\eta}\mathrm{d}x\,\mathrm{d}\eta \le \frac{4t^2}{\pi}\int_0^T\int_{\Omega(\eta)} \rho w_{i,\eta\eta}w_{i,\eta\eta}\mathrm{d}x\,\mathrm{d}\eta + 2t\int_\Omega \rho w_i^{(1)} w_i^{(1)}\,\mathrm{d}x, \tag{5.8}$$

where $0 \le t \le T/2$. Substitution of (5.8) in (5.7) leads to

$$\int_{\Omega(t)} \rho w_i w_i\,\mathrm{d}x \le \frac{32t^3}{\pi^2}\int_0^T\int_{\Omega(\eta)} \rho w_{i,\eta\eta}w_{i,\eta\eta}\,\mathrm{d}x\,\mathrm{d}\eta + \frac{16t^2}{\pi}\int_\Omega \rho w_i^{(1)} w_i^{(1)}\,\mathrm{d}x + 2\int_\Omega \rho w_i^{(0)} w_i^{(0)}\,\mathrm{d}x. \tag{5.9}$$

A time integration of (5.9) provides a second form of the required inequality:

$$\int_0^t \int_{\Omega(\eta)} \rho w_i w_i \mathrm{d}x\, \mathrm{d}\eta \leq \frac{8t^4}{\pi^2} \int_0^T \int_{\Omega(\eta)} \rho w_{i,\eta\eta} w_{i,\eta\eta}\, \mathrm{d}x\, \mathrm{d}\eta + \frac{16t^3}{3\pi} \int_\Omega \rho w_i^{(1)} w_i^{(1)}\, \mathrm{d}x + 2t \int_\Omega \rho w_i^{(0)} w_i^{(0)}\, \mathrm{d}x. \tag{5.10}$$

Improved estimates are possible for vectors satisfying the system (5.2)-(5.4) and which additionally belong to certain constraint sets. While such estimates are established, for example, in [19], another derivation is here presented that results in modified inequalities.

The alternative approach for continuous dependence of the solution $w_i(x,t)$ upon the inertia $\rho w_{i,tt}$ starts by rewriting the equation of motion (5.2) as

$$(c_{ijkl} w_{k,l})_{,j} + \rho w_{i,tt} = 2\rho w_{i,tt}, \quad (x,t) \in \Omega \times [0,T), \tag{5.11}$$

subject to the boundary and initial conditions (5.3) and (5.4) and the symmetry (5.5). Set

$$f_i(x,t) = w_{i,tt}, \tag{5.12}$$

so that (5.11) becomes

$$(c_{ijkl} w_{k,l})_{,j} + \rho f_i = 2\rho w_{i,tt}, \quad (x,t) \in \Omega \times [0,T], \tag{5.13}$$

which are the equations of motion for a linearised elastic theory with mass density $2\rho(x)$ and body force $f_i(x,t)/2$ per unit mass.

As explained in [19, eqn. (2.17)], introduction of the system adjoint to (5.13), (5.3) and (5.5) leads to (A.1) which in our case for mass density 2ρ and body-force $f_i/2 = w_{i,tt}/2$ per unit mass becomes the identity

$$4 \int_{\Omega(t)} \rho w_i w_{i,t}\, \mathrm{d}x = A(t) + B(t), \qquad 0 \leq 2t \leq T, \tag{5.14}$$

where

$$A(t) = \int_0^t \int_{\Omega(\eta)} \rho [w_{i,\eta\eta}(\eta) w_i(2t-\eta) - w_i(\eta) w_{i,\eta\eta}(2t-\eta)]\, \mathrm{d}x\, \mathrm{d}\eta, \tag{5.15}$$

$$B(t) = 2 \int_{\Omega(t)} \rho [w_i(2t) w_i^{(1)} + w_i^{(0)} w_{i,t}(2t)]\, \mathrm{d}x. \tag{5.16}$$

Explicit dependence upon the spatial variables is not displayed. A further time integration of (5.14) produces

$$2 \int_{\Omega(t)} \rho w_i w_i\, \mathrm{d}x = \int_0^t [A(\eta) + B(\eta)]\, \mathrm{d}\eta + 2 \int_\Omega \rho w_i^{(0)} w_i^{(0)}\, \mathrm{d}x. \tag{5.17}$$

Let the vector displacement field $u(x,t)$ belong to the constraint set

$$\max_{[0,T]} \int_{\Omega(t)} \rho u_i u_i\, \mathrm{d}x \leq M_2^2 \tag{5.18}$$

for some positive constant M_2, which implies that u satisfies the constraint (3.22) for finite T. The difference vector field w_i consequently belongs to the set

$$\max_{[0,T]} \int_{\Omega(t)} \rho w_i w_i\, \mathrm{d}x \leq 2M_2^2 + 2 \int_\Omega \rho v_i v_i \mathrm{d}x \leq M_3^2, \tag{5.19}$$

where M_3 is a positive constant.

Consider the time integrals appearing on the right of (5.17) and let $0 \le 2s \le t \le T/2$. An application of Schwarz's inequality leads to

$$\begin{aligned}
\int_0^t A(\eta)\,\mathrm{d}\eta &= \int_0^t \int_0^s \int_{\Omega(\eta)} \rho[w_{i,\eta\eta}(\eta)w_i(2s-\eta) - w_i(\eta)w_{i,\eta\eta}(2s-\eta)]\,\mathrm{d}x\,\mathrm{d}\eta\,\mathrm{d}s \\
&\le \left(\int_0^t \int_0^s \int_{\Omega(\eta)} \rho[w_i(\eta)w_i(\eta) + w_i(2s-\eta)w_i(2s-\eta)]\,\mathrm{d}x\,\mathrm{d}\eta\,\mathrm{d}s\right)^{1/2} \\
&\quad \times \left(\int_0^t \int_0^s \int_{\Omega(\eta)} \rho[w_{i,\eta\eta}(\eta)w_{i,\eta\eta}(\eta) + w_{i,\eta\eta}(2s-\eta)w_{i,\eta\eta}(2s-\eta)]\,\mathrm{d}x\,\mathrm{d}\eta\,\mathrm{d}s\right)^{1/2} \\
&= \left(\int_0^t \int_0^s \int_{\Omega(\eta)} \rho w_i(\eta)w_i(\eta)\,\mathrm{d}x\,\mathrm{d}\eta\,\mathrm{d}s + \int_0^t \int_s^{2s} \int_{\Omega(\eta)} \rho w_i(\eta)w_i(\eta)\,\mathrm{d}x\,\mathrm{d}\eta\,\mathrm{d}s\right)^{1/2} \\
&\quad \times \left(\int_0^t \int_0^s \int_{\Omega(\eta)} \rho w_{i,\eta\eta}(\eta)w_{i,\eta\eta}(\eta)\,\mathrm{d}x\,\mathrm{d}\eta\,\mathrm{d}s + \int_0^t \int_s^{2s} \int_{\Omega(\eta)} \rho w_{i,\eta\eta}(\eta)w_{i,\eta\eta}(\eta)\,\mathrm{d}x\,\mathrm{d}\eta\,\mathrm{d}s\right)^{1/2} \\
&= \left(\int_0^t \int_0^{2s} \int_{\Omega(\eta)} \rho w_i(\eta)w_i(\eta)\,\mathrm{d}x\,\mathrm{d}\eta\,\mathrm{d}s \int_0^t \int_0^{2s} \int_{\Omega(\eta)} \rho w_{i,\eta\eta}(\eta)w_{i,\eta\eta}(\eta)\,\mathrm{d}x\,\mathrm{d}\eta\,\mathrm{d}s\right)^{1/2} \\
&\le \frac{TM_3}{2} \left(\int_0^t \int_0^{2s} \int_{\Omega(\eta)} \rho w_{i,\eta\eta}(\eta)w_{i,\eta\eta}(\eta)\,\mathrm{d}x\,\mathrm{d}\eta\,\mathrm{d}s\right)^{1/2} \\
&\le \left(\frac{T^{3/2}}{2}\right) M_3 \left(\int_0^{2t} \int_{\Omega(\eta)} \rho w_{i,\eta\eta}(\eta)w_{i,\eta\eta}(\eta)\,\mathrm{d}x\,\mathrm{d}\eta\right)^{1/2}, \quad 0 \le t \le T/2. \qquad (5.20)
\end{aligned}$$

Again, we have

$$\begin{aligned}
\int_0^t B(\eta)\,\mathrm{d}\eta &= 2\int_0^t \int_{\Omega(\eta)} \rho[w_i(2\eta)w_i^{(1)} + w_i^{(0)}w_{i,\eta}(2\eta)]\,\mathrm{d}x\,\mathrm{d}\eta, \qquad 0 \le 2t \le T, \\
&= \int_0^{2t} \int_{\Omega(\eta)} \rho w_i(\eta)w^{(1)}\,\mathrm{d}x\,\mathrm{d}\eta + 2\int_0^{2t} \int_{\Omega(\eta)} \rho w_i^{(0)} w_{i,\eta}(\eta)\,\mathrm{d}x\,\mathrm{d}\eta \\
&= \int_0^{2t} \int_{\Omega(\eta)} \rho w_i(\eta)w^{(1)}\,\mathrm{d}x\,\mathrm{d}\eta + 2\int_{\Omega(t)} \rho w_i^{(0)} w_i(2t)\,\mathrm{d}x - 2\int_\Omega \rho w_i^{(0)} w_i^{(0)}\,\mathrm{d}x \\
&\le (T)^{1/2} \left(\int_0^{2t} \int_{\Omega(\eta)} \rho w_i(\eta)w_i(\eta)\,\mathrm{d}x\,\mathrm{d}\eta \int_\Omega \rho w_i^{(1)} w_i^{(1)}\,\mathrm{d}x\right)^{1/2} \\
&\quad +2\left(\int_{\Omega(t)} \rho w_i(2t)w_i(2t)\mathrm{d}x \int_\Omega \rho w_i^{(0)} w_i^{(0)}\,\mathrm{d}x\right)^{1/2} - 2\int_\Omega \rho w_i^{(0)} w_i^{(0)}\,\mathrm{d}x \\
&\le TM_3 \left(\int_\Omega \rho w_i^{(1)} w_i^{(1)}\,\mathrm{d}x\right)^{1/2} + 2M_3 \left(\int_\Omega \rho w_i^{(0)} w_i^{(0)}\,\mathrm{d}x\right)^{1/2} - 2\int_\Omega \rho w_i^{(0)} w_i^{(0)}\,\mathrm{d}x. \qquad (5.21)
\end{aligned}$$

Substitution of (5.20) and (5.21) in (5.17) leads to the upper bound

$$\begin{aligned}
\int_{\Omega(t)} \rho w_i w_i\,\mathrm{d}x &\le \left(\frac{T^{3/2}}{4}\right) M_3 \left(\int_0^{2t} \int_{\Omega(\eta)} \rho w_{i,\eta\eta} w_{i,\eta\eta}\,\mathrm{d}x\,\mathrm{d}\eta\right)^{1/2} + \frac{TM_3}{2}\left(\int_\Omega \rho w_i^{(1)} w_i^{(1)}\,\mathrm{d}x\right)^{1/2} \\
&\quad + M_3 \left(\int_\Omega \rho w_i^{(0)} w_i^{(0)}\,\mathrm{d}x\right)^{1/2}, \qquad 0 \le t \le T/2, \qquad (5.22)
\end{aligned}$$

and continuous dependence of the difference vector w_i upon the inertia $\rho w_{i,tt}$: or equivalently $\rho u_{i,tt}$: is established.

The initial values $w^{(0)}(x)$ and $w^{(1)}(x)$ in mean-square measure should be small compared to the measured inertia in order that the bound (5.22) is meaningful. The magnitude of the inertia, however, is determined by that of the initial values $u^{(0)}(x)$ and $u^{(1)}(x)$. In consequence, the solution $v(x)$ to (2.5) and (2.6) should be selected to ensure its measured difference from $u^{(0)}$ is as small as possible.

In summary, the combined estimates (5.22), (3.21) with $m = 4$, and (3.23) define the sense in which the quasi-static approximation is valid for the initial homogeneous Dirichlet boundary value problem of linearised elastodynamics.

Remark 5.1 Different constraint sets to those previously introduced lead to different estimates as shown in [19], but do not lead to fundamentally different conclusions.

Remark 5.2 As already mentioned, the interval $[0, T/2]$ can be iteratively extended to $(1-2^{-n})T$, $n = 1, 2, 3\ldots$ as shown in [19].

6 Concluding remarks

The comparatively simple initial boundary value problem of linearised elastodynamics under discussion has initial Cauchy conditions as the only non-zero data. The procedure proposed in Sect. 1 is followed to rigorously justify the relevant quasi-static approximation but it remains open whether the corresponding conditions are both necessary and sufficient.

It is of obvious interest to generalise the present treatment to include non-vanishing body-forces and time-dependent Dirichlet, Neumann, and mixed boundary conditions. Equally, methods similar to those developed in this study may be applicable to various linearised coupled theories such as thermoelastodynamics, whose classical counterparts are the subject of the previously cited contributions [10–14], and viscous elastodynamics.

Finally, interpretation of the quasi-static approximation within the broader context of continuous data dependence suggests a possible relationship with continuity of maps and in particular notions of stability. These topics await investigation.

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Appendix A

For ease reference, we list without proof two key results previously derived in [19]. Notation has been altered to that adopted here, and a body-force vector $f(x,t)$ per unit mass is now included in the equation of motion (2.1) . We have for $0 \le 2t \le T$ [19, eqn.(2.17)]:

$$2\int_{\Omega(t)} \rho u_i(t)u_{i,t}(t)\,\mathrm{d}x = \int_0^t \int_{\Omega(\eta)} \rho[u_i(2t-\eta)f_i(\eta) - u_i(\eta)f_i(2t-\eta)]\,\mathrm{d}x\,\mathrm{d}\eta + \int_{\Omega(t)} \rho u_i(2t)u_i^{(1)}\mathrm{d}x + \int_{\Omega(t)} \rho u_{i,t}(2t)u_i^{(0)}\mathrm{d}x. \tag{A.1}$$

Secondly, when $f_i = 0$, the following bound holds [19, eqn.(3.22)] for $0 \le 2t \le T$:

$$\int_{\Omega(t)} \rho u_i(t)u_{i,t}(t)\,\mathrm{d}x \le \frac{T}{4}\int_{\Omega} \rho u_i^{(0)}u_i^{(0)}\mathrm{d}x + \frac{M_1 T^{1/2}}{4}\left[\int_{\Omega} \rho u_i^{(0)}u_i^{(0)}\mathrm{d}x + \frac{T}{\sqrt{2}}\int_{\Omega} \rho u_i^{(1)}u_i^{(1)}\mathrm{d}x\right], \tag{A.2}$$

where M_1 is given by (3.22).

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