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Equilibrium of elastic lattice shells

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Abstract A model for shells consisting of a continuous distribution of embedded rods is developed in the framework of the direct theory of second-gradient elastic surfaces. The shell is constitutively sensitive to a convenient measure of the gradient of strain in addition to the metric and curvature of standard shell theory.

Keywords Elastic surfaces · Lattice shells · Strain gradients

1 Introduction

Grid-shells have long been treated as shells that are orthotropic but otherwise conventional, in the sense that the relevant response functions are sensitive to changes in the metric, or strain, of the shell surface and to changes in its embedding geometry, or curvature; orthotropy is conferred by a lattice of densely distributed orthogonal thin rods forming the shell surface. The standard theory [1] accounts for those aspects of fiber bending and twist induced by curvature of the surface in which the lattice is embedded, but does not account for the geodesic bending of the lattice in the tangent plane of the surface. Wang and Pipkin [2,3] developed a theory of plates that accounts for the effects of surface bending and geodesic bending in networks consisting of two families of inextensible fibers, but their model does not accommodate a constitutive sensitivity to fiber twist.

Steigmann and dell'Isola [4] recently developed a model for woven fabric sheets regarded as orthotropic plates. This theory treats the fibers of the sheet as embedded spatial Kirchhoff rods [5]. By constraining the fibers to pivot about the evolving surface normal while remaining congruent to the deformed sheet, it is found that the twist of the fibers may be described entirely by the deformation of the underlying surface. In this way the mechanics of the sheet may be modeled using a single position field. The relevant response functions are sensitive to the first and second surface derivatives of this field: the former yielding a surface strain and the latter including both surface and geodesic curvatures and additional strain-gradient effects. This generalizes a simpler model of elastic networks [6] that accounts only for the stretching elasticity of the constituent fibers. The more general model was used to predict certain unusual deformation patterns observed in experiments on 3D-printed pantographic lattices [7,8].

Theories of this kind fall in the category of second-gradient elasticity, in which the response functions depend explicitly on the first and second derivatives of the deformation with respect to material coordinates [9–13]. This

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framework was extended to elastic surfaces by Cohen and DeSilva [14,15] and Balaban et al. [16]. The work of Cohen and DeSilva begins at a level of generality comparable to that discussed here; but, to achieve a bending theory of the conventional type, suppresses effects—such as geodesic bending—associated with the gradient of strain. The work of Balaban et al. is closer to our present perspective, but its development is confined to special cases that lead to certain force and couple densities being constitutively indeterminate. In contrast, no such indeterminacies arise in the present approach. This is due to our a priori restriction to response functions deduced on the basis of the *order-of-differentiation symmetry* [17] reflected in the second (covariant) derivatives of the deformed position field. This symmetry has important implications for the field equations and the boundary conditions [17–19]. Further, we take geodesic bending resistance into account in a general manner.

Our purpose in the present paper is to outline a complete development of the general theory of second-gradient elasticity for surfaces in equilibrium. Specifically, we consider shells of arbitrary geometry and cast the model in a variational setting, obtaining a concise set of equilibrium conditions and associated boundary conditions. We also adapt a general treatment of material symmetry for elastic surfaces due to Murdoch and Cohen [20] and use it to place our earlier model in the general setting.

We work in the framework of the direct theory of elastic surfaces. Thus, we do not explicitly model the three-dimensional aspects of lattice shells associated with thickness. This is due to the difficulty in identifying a three-dimensional structure that reflects the main features of the two-dimensional lattice comprising the shell. Thus in general, there is no three-dimensional parent model that can be used to effect a dimension reduction procedure leading to a two-dimensional model of the kind envisaged.

2 Geometry of surface deformation

2.1 Surface geometry

We use convected coordinates θ^α to label material points of the shell, regarded as a two-dimensional manifold. The function $\mathbf{x}(\theta^\alpha)$ is an embedding of this manifold into 3-space, and serves to define position of a material point on a fixed reference surface Ω. Position of the same material point on a typical deformed surface ω is denoted by $\mathbf{r}(\theta^\alpha)$. The latter parametrization induces the associated basis elements $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha} \in T_\omega$, the tangent plane to ω at the point with coordinates θ^α; the metric $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$; the dual metric $(a^{\alpha\beta}) = (a_{\alpha\beta})^{-1}$; and the dual tangent basis $\mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta$. These in turn yield the local orientation of ω in terms of its unit normal $\mathbf{n}$, defined by $\epsilon_{\alpha\beta}\mathbf{n} = \mathbf{a}_\alpha \times \mathbf{a}_\beta$, where $\epsilon_{\alpha\beta}$ is the covariant permutation tensor ($\epsilon_{12} = \sqrt{a} = -\epsilon_{21}$, $\epsilon_{11} = \epsilon_{22} = 0$), with $a = \det(a_{\alpha\beta})$. The contravariant permutation tensor is $\epsilon^{\alpha\beta}$, with $\epsilon^{12} = 1/\sqrt{a} = -\epsilon^{21}$ and $\epsilon^{11} = \epsilon^{22} = 0$.

The Gauss and Weingarten equations play a central role in the development of the theory. These are

$$\mathbf{r}_{,\alpha\beta} = \Gamma^\lambda_{\alpha\beta}\mathbf{a}_\alpha + b_{\alpha\beta}\mathbf{n} \quad \text{and} \quad \mathbf{n}_{,\alpha} = -b_{\alpha\beta}\mathbf{a}^\beta, \tag{1}$$

where $\Gamma^\lambda_{\alpha\beta}$ are the Levi-Civita connection coefficients induced by the coordinates on ω and $b_{\alpha\beta}$ is the covariant curvature tensor (the coefficients of the second fundamental form). Their counterparts on Ω are

$$\mathbf{e}_{\alpha,\beta} = \Gamma^\lambda_{(\Omega)\alpha\beta}\mathbf{e}_\lambda + B_{\alpha\beta}\mathbf{N} \quad \text{and} \quad \mathbf{N}_{,\alpha} = -B_{\alpha\beta}\mathbf{e}^\beta, \tag{2}$$

where $\mathbf{N}$ is the unit normal to Ω; $\mathbf{e}^\alpha$ are the duals on T_Ω of the basis elements $\mathbf{e}_\alpha$ induced by the coordinates via $\mathbf{e}_\alpha = \mathbf{x}_{,\alpha}$; and $B_{\alpha\beta}$ and $\Gamma^\lambda_{(\Omega)\alpha\beta}$, respectively are the associated curvature and connection coefficients.

The deformation gradient $\mathbf{F} = \nabla\mathbf{r}$ is given by

$$\mathbf{F} = \mathbf{a}_\alpha \otimes \mathbf{e}^\alpha, \tag{3}$$

and the Cauchy–Green deformation tensor is

$$\mathbf{C} = \mathbf{F}^t\mathbf{F} = a_{\alpha\beta}\mathbf{e}^\alpha \otimes \mathbf{e}^\beta. \tag{4}$$

The areal dilation induced by the deformation is

$$J = \sqrt{\det \mathbf{C}} = \sqrt{a/e}, \tag{5}$$

where $e = \det(e_{\alpha\beta})$ and $e_{\alpha\beta}(= \mathbf{e}_\alpha \cdot \mathbf{e}_\beta)$ is the metric on Ω, and the dual metric $e^{\alpha\beta}$ is defined in the usual way.

2.2 Deformation measures

We assume the existence of a strain-energy density W, per unit area of Ω, that depends on the pointwise values of the first and second derivatives of the deformation given, respectively, by (3) and by $\mathbf{r}_{|(\Omega)\alpha\beta} \otimes \mathbf{e}^\alpha \otimes \mathbf{e}^\beta$, where

$$\mathbf{r}_{|(\Omega)\alpha\beta} = \mathbf{r}_{,\alpha\beta} - \Gamma^\lambda_{(\Omega)\alpha\beta}\mathbf{r}_{,\lambda}, \tag{6}$$

is the second covariant derivative of the deformation with respect to the metric of Ω. The Gauss equation furnishes

$$\mathbf{r}_{|(\Omega)\alpha\beta} = S^\lambda_{\alpha\beta}\mathbf{r}_{,\lambda} + b_{\alpha\beta}\mathbf{n}, \tag{7}$$

where

$$S^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} - \Gamma^\lambda_{(\Omega)\alpha\beta}, \tag{8}$$

in which $\Gamma^\lambda_{\alpha\beta}$ and $\Gamma^\lambda_{(\Omega)\alpha\beta}$, respectively, are the Levi-Civita connection coefficients on ω and Ω. Thus,

$$\mathbf{r}_{|(\Omega)\alpha\beta} \otimes \mathbf{e}^\alpha \otimes \mathbf{e}^\beta = \mathbf{a}_{\boldsymbol{\mu}} \otimes \mathbf{S}^{\boldsymbol{\mu}} - \mathbf{n} \otimes \boldsymbol{\kappa}, \tag{9}$$

where

$$\mathbf{S}^{\boldsymbol{\mu}} = S^{\boldsymbol{\mu}}_{\alpha\beta}\mathbf{e}^\alpha \otimes \mathbf{e}^\beta \quad \text{and} \quad \boldsymbol{\kappa} = -b_{\alpha\beta}\mathbf{e}^\alpha \otimes \mathbf{e}^\beta. \tag{10}$$

The sign in the second equation conforms to a widely used convention in the literature on shell theory [21].

We observe that although the connection coefficients do not possess tensor character, the difference of two sets of connection coefficients induced by a given (convected) coordinate system is a (third-order) tensor [22]. This result is due to Palatini. Further, as is well known, the $\Gamma^\lambda_{\alpha\beta}$ and $\Gamma^\lambda_{(\Omega)\alpha\beta}$ are determined entirely by the metrics induced by the coordinates on ω and Ω, respectively. The relevant formulas are

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2}a^{\lambda\boldsymbol{\mu}}(a_{\alpha\boldsymbol{\mu},\beta} + a_{\beta\boldsymbol{\mu},\alpha} - a_{\alpha\beta,\boldsymbol{\mu}}) \quad \text{and} \quad \Gamma^\lambda_{(\Omega)\alpha\beta} = \frac{1}{2}e^{\lambda\boldsymbol{\mu}}(e_{\alpha\boldsymbol{\mu},\beta} + e_{\beta\boldsymbol{\mu},\alpha} - e_{\alpha\beta,\boldsymbol{\mu}}). \tag{11}$$

Accordingly, the coefficients $S^\lambda_{\alpha\beta}$ account for strain-gradient effects. These, and the $b_{\alpha\beta}$, are easily seen to be Galilean invariant; that is, they are insensitive to rigid-body motions superposed on the given deformation.

Our constitutive assumption implies that the strain energy depends on the list of deformation variables $\{\mathbf{a}_\alpha, \mathbf{S}^\alpha, \mathbf{n}, \boldsymbol{\kappa}\}$. In a superposed rigid-body motion, this lists is transformed to $\{\mathbf{Q}\mathbf{a}_\alpha, \mathbf{S}^\alpha, \mathbf{Q}\mathbf{n}, \boldsymbol{\kappa}\}$, wherein $\mathbf{Q}$ is an arbitrary rotation. Consider, for example, a rotation with axis $\mathbf{n}$, i.e., $\mathbf{Q}\mathbf{n} = \mathbf{n}$. For these, the list transforms to $\{\mathbf{Q}\mathbf{a}_\alpha, \mathbf{S}^\alpha, \mathbf{n}, \boldsymbol{\kappa}\}$. As usual, we suppose the strain energy to be invariant under all superposed rotations, and hence under these in particular, and thus conclude that it depends on the $\mathbf{a}_\alpha$ through the metric $a_{\alpha\beta}$. Returning to the general case, we then have that the energy remains invariant under $\mathbf{n} \to \mathbf{Qn}$ for *any* rotation, and hence that it depends on $\mathbf{n}$ through $|\mathbf{n}|(= 1)$. The energy is thus determined by the list

$$\{\mathbf{C}, \mathbf{S}, \boldsymbol{\kappa}\}, \tag{12}$$

of deformation invariants, where

$$\mathbf{S} = \mathbf{e}_{\boldsymbol{\mu}} \otimes \mathbf{S}^{\boldsymbol{\mu}}. \tag{13}$$

2.3 Change of reference surface

In preparation for the discussion of material symmetry to follow, we inquire into the manner in which the variables in the list (12) are altered by a change of reference surface. Thus, consider an alternative reference surface, $\bar{\Omega}$, described parametrically by the function $\bar{\mathbf{x}}(\theta^\alpha)$. Using this as reference, the relevant version of (7) is

$$\mathbf{r}_{|(\bar{\Omega})\alpha\beta} = \bar{S}^\lambda_{\alpha\beta}\mathbf{r}_{,\lambda} + b_{\alpha\beta}\mathbf{n}, \tag{14}$$

where

$$\bar{S}^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} - \Gamma^\lambda_{(\bar{\Omega})\alpha\beta}. \tag{15}$$

Evidently,

$$\bar{S}^\lambda_{\alpha\beta} = S^\lambda_{\alpha\beta} + \Lambda^\lambda_{\alpha\beta}, \quad \text{where} \quad \Lambda^\lambda_{\alpha\beta} = \Gamma^\lambda_{(\Omega)\alpha\beta} - \Gamma^\lambda_{(\bar{\Omega})\alpha\beta} \tag{16}$$

is the induced change in the referential Levi-Civita connection. Consequently,

$$\mathbf{r}_{|(\bar{\Omega})\alpha\beta} = \mathbf{r}_{|(\Omega)\alpha\beta} + \Lambda^\lambda_{\alpha\beta}\mathbf{a}_\lambda. \tag{17}$$

Let $\bar{\mathbf{e}}_\alpha = \bar{\mathbf{x}}_{,\alpha}$ be the natural basis on the tangent plane $T_{\bar{\Omega}}$ at the point with coordinates θ^α, and let $\bar{\mathbf{e}}^\alpha$ be the dual basis. Using these, we construct the tensors

$$\mathbf{R} = \mathbf{e}^{\boldsymbol{\mu}} \otimes \bar{\mathbf{e}}_{\boldsymbol{\mu}} \quad \text{and} \quad \mathbf{H} = \mathbf{e}_{\boldsymbol{\mu}} \otimes \bar{\mathbf{e}}^{\boldsymbol{\mu}}, \tag{18}$$

so that $\mathbf{e}^\alpha = \mathbf{R}\bar{\mathbf{e}}^\alpha$ and $\mathbf{e}_\alpha = \mathbf{H}\bar{\mathbf{e}}_\alpha$. Then

$$\mathbf{r}_{|(\Omega)\alpha\beta} \otimes \mathbf{e}^\alpha \otimes \mathbf{e}^\beta = \mathbf{a}_{\boldsymbol{\mu}} \otimes \mathbf{R}(\bar{\mathbf{S}}^{\boldsymbol{\mu}} - \bar{\boldsymbol{\Lambda}}^{\boldsymbol{\mu}})\mathbf{R}^t - \mathbf{n} \otimes \mathbf{R}\bar{\boldsymbol{\kappa}}\mathbf{R}^t, \tag{19}$$

where

$$\bar{\mathbf{S}}^{\boldsymbol{\mu}} = \bar{S}^{\boldsymbol{\mu}}_{\alpha\beta}\bar{\mathbf{e}}^\alpha \otimes \bar{\mathbf{e}}^\beta, \quad \bar{\boldsymbol{\Lambda}}^{\boldsymbol{\mu}} = \Lambda^{\boldsymbol{\mu}}_{\alpha\beta}\bar{\mathbf{e}}^\alpha \otimes \bar{\mathbf{e}}^\beta, \quad \text{and} \quad \bar{\boldsymbol{\kappa}} = -b_{\alpha\beta}\bar{\mathbf{e}}^\alpha \otimes \bar{\mathbf{e}}^\beta. \tag{20}$$

Comparison with (9) delivers

$$\mathbf{S}^{\boldsymbol{\mu}} = \mathbf{R}(\bar{\mathbf{S}}^{\boldsymbol{\mu}} - \bar{\boldsymbol{\Lambda}}^{\boldsymbol{\mu}})\mathbf{R}^t \quad \text{and} \quad \boldsymbol{\kappa} = \mathbf{R}\bar{\boldsymbol{\kappa}}\mathbf{R}^t. \tag{21}$$

The more familiar formula

$$\mathbf{C} = \mathbf{R}\bar{\mathbf{C}}\mathbf{R}^t, \tag{22}$$

with

$$\bar{\mathbf{C}} = a_{\alpha\beta}\bar{\mathbf{e}}^\alpha \otimes \bar{\mathbf{e}}^\beta, \tag{23}$$

follows directly from (4).

In the event that the tangent planes T_Ω and $T_{\bar{\Omega}}$ coincide at the point in question, we have

$$\mathbf{R} = R_{\boldsymbol{\mu}}^{\cdot\alpha}\bar{\mathbf{e}}^{\boldsymbol{\mu}} \otimes \bar{\mathbf{e}}_\alpha \quad \text{and} \quad \mathbf{H} = H_{\cdot\boldsymbol{\mu}}^{\alpha}\bar{\mathbf{e}}_\alpha \otimes \bar{\mathbf{e}}^{\boldsymbol{\mu}}, \quad \text{with} \quad H_{\cdot\boldsymbol{\mu}}^{\alpha} R_\beta^{\cdot\boldsymbol{\mu}} = \delta_\beta^\alpha, \tag{24}$$

where δ_β^α is the Kronecker delta, yielding

$$\mathbf{H} = \mathbf{R}^{-t}. \tag{25}$$

In this case the determinants of $\mathbf{R}$ and $\mathbf{H}$ are well defined and we have $\mathbf{N} = \pm\bar{\mathbf{N}}$ according as $\det \mathbf{R}$ (or $\det \mathbf{H}$) is positive or negative, respectively.

3 Immersions and material symmetry

3.1 Immersions in 3-space

Murdoch and Cohen [20] extended to material surfaces Noll's concept of material symmetry as a local change of reference configuration that leaves constitutive response unaltered in a given experiment. This differs in principle from the notion of form invariance under distinguished coordinate transformations adopted in many prior works. The two concepts yield the same mathematical consequences for first-gradient models of elasticity, but not for theories in which higher-order gradients figure in the constitutive response. The Murdoch–Cohen conception of material symmetry was used by Steigmann and Ogden [23] and Steigmann [24] to derive the canonical forms of the strain-energy functions for thin solid films and fluid films with bending resistance. We cast the present work in the same setting.

Briefly, in the work of Murdoch and Cohen, an experiment on a material surface Ω is viewed as an immersion $\boldsymbol{\chi}(\mathbf{X})$ of the surface in 3-space. This is defined in the space containing Ω, and its restriction to the latter furnishes the surface deformation $\mathbf{r}$; thus,

$$\mathbf{r} = \boldsymbol{\chi}(\mathbf{x}). \tag{26}$$

The chain rule provides

$$\mathbf{r}_{,\alpha} = \mathbf{A}(\mathbf{x})\mathbf{e}_\alpha \tag{27}$$

and

$$\mathbf{r}_{,\alpha\beta} = \mathbf{G}(\mathbf{x})[\mathbf{e}_\alpha \otimes \mathbf{e}_\beta] + \mathbf{A}(\mathbf{x})\mathbf{e}_{\alpha,\beta}, \tag{28}$$

where

$$\mathbf{A}(\mathbf{X}) = \nabla\boldsymbol{\chi} \quad \text{and} \quad \mathbf{G}(\mathbf{X}) = \nabla\nabla\boldsymbol{\chi} \tag{29}$$

are the first and second gradients of $\boldsymbol{\chi}(\mathbf{X})$. Here we adopt the notation defined by $\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}[\mathbf{d}\otimes\mathbf{e}] = (\mathbf{b}\cdot\mathbf{d})(\mathbf{c}\cdot\mathbf{e})\mathbf{a}$.

From $(2)_1$, (6), and (27), it follows that

$$\mathbf{r}_{|(\Omega)\alpha\beta} = \mathbf{G}(\mathbf{x})[\mathbf{e}_\alpha\otimes\mathbf{e}_\beta] + B_{\alpha\beta}\mathbf{A}(\mathbf{x})\mathbf{N} \tag{30}$$

and

$$\mathbf{r}_{|(\Omega)\alpha\beta}\otimes\mathbf{e}^\alpha\otimes\mathbf{e}^\beta = \mathbf{G}(\mathbf{x})[\mathbf{e}_\alpha\otimes\mathbf{e}_\beta]\otimes\mathbf{e}^\alpha\otimes\mathbf{e}^\beta + \mathbf{A}(\mathbf{x})\mathbf{N}\otimes\mathbf{B}, \tag{31}$$

where

$$\mathbf{B} = B_{\alpha\beta}\mathbf{e}^\alpha\otimes\mathbf{e}^\beta. \tag{32}$$

Some latitude exists in the construction of the function $\boldsymbol{\chi}$. Consider the normal-coordinate parametrizations

$$\mathbf{X}(\theta^\alpha,\varsigma) = \mathbf{x}(\theta^\alpha) + \varsigma\mathbf{N}(\theta^\alpha) \quad \text{and} \quad \boldsymbol{\chi}(\mathbf{X}) = \mathbf{r}(\theta^\alpha) + \varsigma\mathbf{n}(\theta^\alpha) \tag{33}$$

of 3-space in the vicinities of a fixed material point on Ω and ω, respectively. This provides an extension of (26) from Ω to the surrounding 3-space. It is straightforward to compute the gradient $\nabla\boldsymbol{\chi}$ and to show that its restriction to Ω is

$$\mathbf{A}(\mathbf{x}) = \mathbf{F} + \mathbf{n}\otimes\mathbf{N}, \tag{34}$$

where $\mathbf{F}$ is given by (3).

We have

$$\mathbf{r}_{|(\Omega)\alpha\beta}\otimes\mathbf{e}^\alpha\otimes\mathbf{e}^\beta = \mathbf{G}(\mathbf{x})[\mathbf{e}_\alpha\otimes\mathbf{e}_\beta]\otimes\mathbf{e}^\alpha\otimes\mathbf{e}^\beta + \mathbf{n}\otimes\mathbf{B}. \tag{35}$$

Comparison with (9) yields

$$\mathbf{S}^\mu = \{\mathbf{A}(\mathbf{x})^{-t}\mathbf{e}^\mu\cdot\mathbf{G}(\mathbf{x})[\mathbf{e}_\alpha\otimes\mathbf{e}_\beta]\}\mathbf{e}^\alpha\otimes\mathbf{e}^\beta \tag{36}$$

and

$$-\boldsymbol{\kappa} = \{\mathbf{A}(\mathbf{x})^{-t}\mathbf{N}\cdot\mathbf{G}(\mathbf{x})[\mathbf{e}_\alpha\otimes\mathbf{e}_\beta]\}\mathbf{e}^\alpha\otimes\mathbf{e}^\beta + \mathbf{B}, \tag{37}$$

where use has been made of the fact that

$$\mathbf{A}(\mathbf{x})^{-t} = \mathbf{a}^\mu\otimes\mathbf{e}_\mu + \mathbf{n}\otimes\mathbf{N}. \tag{38}$$

From (4), we also have

$$\mathbf{C} = \{\mathbf{e}_\alpha\cdot\mathbf{A}(\mathbf{x})^t\mathbf{A}(\mathbf{x})\mathbf{e}_\beta\}\mathbf{e}^\alpha\otimes\mathbf{e}^\beta. \tag{39}$$

Equations (36), (37), and (39) furnish the values of the constitutive variables, listed in (12), generated by a given experiment, i.e., by a given function $\boldsymbol{\chi}(\cdot)$.

3.2 Symmetry transformations

In the Noll–Murdoch–Cohen framework, symmetry transformations are those local changes of reference surface for which the constitutive response to an experiment remains invariant. Here, a local change of reference refers to a map of a neighborhood N of a material point on Ω to a corresponding neighborhood $\bar{N}$ on $\bar{\Omega}$. We suppose this map to be arranged such that $\mathbf{x}$ and $\bar{\mathbf{x}}$ coincide at the material point in question. This is a pivot point of the local transformation. The two neighborhoods N and $\bar{N}$ are connected by a *symmetry* transformation provided that their constitutive responses to an experiment—regarded as a given orientation preserving immersion of each neighborhood into 3-space—are identical. Here constitutive response is measured by the value of a state variable such as the strain energy per unit mass or the strain energy per unit area of ω.

Murdoch and Cohen [20] noticed, in the setting of material surfaces, that a simplification is possible due to the presumed Galilean invariance of the constitutive functions. This implies, as far as constitutive response is concerned, that the local tangent planes T_ω and T_Ω may be assumed to coincide and be similarly oriented at the material point in question. This amounts to imposing $\mathbf{A}(\mathbf{x})^{-t}\mathbf{N} = \mathbf{N}$ and hence to the replacement of (37) with

$$-\boldsymbol{\kappa} = \{\mathbf{N} \cdot \mathbf{G}(\mathbf{x})[\mathbf{e}_\alpha \otimes \mathbf{e}_\beta]\}\mathbf{e}^\alpha \otimes \mathbf{e}^\beta + \mathbf{B}, \tag{40}$$

in the constitutive function.

In the same way, we may identify T_ω with $T_{\bar{\Omega}}$, and hence T_Ω with $T_{\bar{\Omega}}$, at the pivot point. Using obvious notation, the relative curvature of $\bar{\Omega}$ at the pivot point, induced by the *same* experiment, is found to be

$$-\bar{\boldsymbol{\kappa}} = \{\bar{\mathbf{N}} \cdot \mathbf{G}(\boldsymbol{x})[\bar{e}_\alpha \otimes \bar{\mathbf{e}}_\beta]\}\bar{\mathbf{e}}^\alpha \otimes \bar{\mathbf{e}}^\beta + \bar{\mathbf{B}}. \tag{41}$$

Moreover, (24) is operative and yields

$$-\bar{\boldsymbol{\kappa}} = \{\bar{\mathbf{N}} \cdot \mathbf{G}(\mathbf{x})[\mathbf{e}_\alpha \otimes \mathbf{e}_\beta]\}\mathbf{e}^\alpha \otimes \mathbf{e}^\beta + \bar{\mathbf{B}}, \tag{42}$$

with $\bar{\mathbf{N}} = \pm\mathbf{N}$ according as $\det \mathbf{R}$ in (24) is positive or negative, respectively. Thus,

$$\bar{\boldsymbol{\kappa}} = \boldsymbol{\kappa} + \mathbf{B} - \bar{\mathbf{B}}, \tag{43}$$

if T_Ω and $T_{\bar{\Omega}}$ have the same orientation, whereas

$$\bar{\boldsymbol{\kappa}} = -(\boldsymbol{\kappa} + \mathbf{B} + \bar{\mathbf{B}}), \tag{44}$$

if T_Ω and $T_{\bar{\Omega}}$ have opposite orientations. Of course $\mathbf{B}$ and $\bar{\mathbf{B}}$ need not be related as these reflect the generally distinct local embedding geometries of N and $\bar{N}$.

Using (36) and (39) and proceeding similarly, we conclude that

$$\bar{\mathbf{C}} = \mathbf{C}, \tag{45}$$

and that

$$R_{\lambda}^{\cdot\mu}\bar{\mathbf{S}}^{\lambda} = \mathbf{S}^{\mu}; \quad \text{hence,} \quad \bar{\mathbf{S}} = \mathbf{S}. \tag{46}$$

Let ψ be the strain energy per unit mass, and let Ψ and $\bar{\Psi}$, respectively, be the constitutive functions for this energy based on the use of the reference surfaces Ω and $\bar{\Omega}$. Then, because the state of the shell is not sensitive to the choice of reference,

$$\bar{\Psi}(\bar{\mathbf{C}}, \bar{\mathbf{S}}, \bar{\boldsymbol{\kappa}}) = \Psi(\mathbf{C}, \mathbf{S}, \boldsymbol{\kappa}) = \Psi(\mathbf{R}\bar{\mathbf{C}}\mathbf{R}^t, \mathbf{e}_\mu \otimes \mathbf{R}(\bar{\mathbf{S}}^\mu - \bar{\boldsymbol{\Lambda}}^\mu)\mathbf{R}^t, \mathbf{R}\bar{\boldsymbol{\kappa}}\mathbf{R}^t), \tag{47}$$

where use has been made of (21) and (22).

Recall that N and $\bar{N}$ are related by a symmetry transformation if their constitutive responses to a given experiment coincide. In view of (43)–(45), this entails the restrictions

$$\Psi(\mathbf{C}, \mathbf{e}_\mu \otimes \mathbf{S}^\mu, \boldsymbol{\kappa}) = \Psi(\mathbf{R}\mathbf{C}\mathbf{R}^t, \mathbf{e}_\mu \otimes \mathbf{R}(\bar{\mathbf{S}}^\mu - \bar{\boldsymbol{\Lambda}}^\mu)\mathbf{R}^t, \mathbf{R}(\boldsymbol{\kappa} + \mathbf{B} - \bar{\mathbf{B}})\mathbf{R}^t) \quad \text{if} \quad \det \mathbf{R} > 0 \tag{48}$$

and

$$\Psi(\mathbf{C}, \mathbf{e}_\mu \otimes \mathbf{S}^\mu, \boldsymbol{\kappa}) = \Psi(\mathbf{R}\mathbf{C}\mathbf{R}^t, \mathbf{e}_\mu \otimes \mathbf{R}(\bar{\mathbf{S}}^\mu - \bar{\boldsymbol{\Lambda}}^\mu)\mathbf{R}^t, -\mathbf{R}(\boldsymbol{\kappa} + \mathbf{B} + \bar{\mathbf{B}})\mathbf{R}^t) \quad \text{if} \quad \det \mathbf{R} < 0, \tag{49}$$

where

$$\bar{\mathbf{S}}^\mu = H^\mu_{\cdot\alpha}\mathbf{S}^\alpha. \tag{50}$$

Remark In an addendum to their work [20], Murdoch and Cohen observed that considerable simplification is achieved in their treatment by using a representation equivalent to (34) above for the restriction to Ω of the gradient of the immersion $\boldsymbol{\chi}$. To justify this simplification, they observed that the derivative of $\boldsymbol{\chi}$ in the direction normal to Ω may be specified arbitrarily because only the tangential surface derivatives of $\boldsymbol{\chi}$ are involved in the constitutive theory. In [20] and [23], it was shown that any alternative immersion yields the non-physical conclusion that the energy takes the same value at infinitely many values of the relative curvature tensor.

4 Example: plane orthogonal lattice

In [4] we proposed the strain-energy function

$$W = w(\lambda, \mu, J) + \frac{1}{2}A_g\left(|\mathbf{g}_L|^2 + |\mathbf{g}_M|^2\right) + \frac{1}{2}A_\Gamma|\boldsymbol{\Gamma}|^2 + \frac{1}{2}k\left(K_L^2 + K_M^2\right) + \frac{1}{2}\bar{k}T^2, \tag{51}$$

for a plate consisting of extensible crossed elasticae, initially arranged in a uniform orthogonal grid characterized by fixed orthonormal vectors $\mathbf{L}$ and $\mathbf{M}$ lying on a plane. We identify Ω with a portion of this plane. The energy per unit mass is $\Psi = \rho_\Omega^{-1}W$, where ρ_Ω, the mass density on Ω, is assumed to be constant. Here

$$\lambda = \sqrt{\mathbf{L}\cdot\mathbf{C}\mathbf{L}}, \quad \mu = \sqrt{\mathbf{M}\cdot\mathbf{C}\mathbf{M}} \quad \text{and } J = \sqrt{\det \mathbf{C}}, \tag{52}$$

respectively, are the fiber stretches and the areal stretch, and the remaining terms are associated with the orthogonal decomposition

$$\mathbf{r}_{|\alpha\beta} = L_\alpha L_\beta(\mathbf{g}_L + K_L\mathbf{n}) + M_\alpha M_\beta(\mathbf{g}_M + K_M\mathbf{n}) + (L_\alpha M_\beta + M_\alpha L_\beta)(\boldsymbol{\Gamma} + T\mathbf{n}), \tag{53}$$

of the second gradient of position. In particular,

$$K_L = -\mathbf{L}\cdot\boldsymbol{\kappa}\mathbf{L}, \quad K_M = -\mathbf{M}\cdot\boldsymbol{\kappa}\mathbf{M} \quad \text{and } T = -\mathbf{L}\cdot\boldsymbol{\kappa}\mathbf{M}, \tag{54}$$

represent bending and twist of the fibers embedded in the surface, and

$$\mathbf{g}_L = L^\alpha L^\beta S^\mu_{\alpha\beta}\mathbf{a}_\mu, \quad \mathbf{g}_M = M^\alpha M^\beta S^\mu_{\alpha\beta}\mathbf{a}_\mu \quad \text{and } \boldsymbol{\Gamma} = L^\alpha M^\beta S^\mu_{\alpha\beta}\mathbf{a}_\mu = M^\alpha L^\beta S^\mu_{\alpha\beta}\mathbf{a}_\mu \tag{55}$$

represent the combined effects of the gradients of fiber stretch and geodesic bending.

The quadratic dependence of the energy on $\mathbf{r}_{|\alpha\beta}$ may be understood in terms of a local length scale, which may be the sheet thickness, the mesh spacing of the lattice, or the diameters of the constituent fibers. If any of these is used as the unit of length, then the norms of the non-dimensionalized vectors $\mathbf{g}_L$, $\mathbf{g}_M$, and $\boldsymbol{\Gamma}$ are typically so small that their contributions to the energy may be approximated by quadratic functions; linear terms vanish if the couple stresses and bending/twisting moments are zero when the fibers are straight and untwisted.

The coefficients A_g, A_Γ, k, and $\bar{k}$ may be functions of λ, μ, and J; here, for simplicity, we take them to be constants. Other forms are possible. For example, we could separate out the effects of geodesic curvature and tangential stretch gradient in $\mathbf{g}_L$ or $\mathbf{g}_M$ and assign different elastic moduli to each.

The energy W is easily shown to exhibit orthotropic symmetry in the sense that (48) and (49) are satisfied with

$$\mathbf{R} \in \{\pm\mathbf{L}\otimes\mathbf{L} \pm \mathbf{M}\otimes\mathbf{M}\}, \tag{56}$$

at all points of Ω and with any combination of signs, with $\mathbf{B} = \mathbf{0}$, $\bar{\mathbf{B}} = \mathbf{0}$, and $\bar{\boldsymbol{\Lambda}}^\mu = \mathbf{0}$. Here both Ω and $\bar{\Omega}$ occupy the same plane, while $\mathbf{R}$ is uniform and orthogonal ($\mathbf{R}^{-1} = \mathbf{R}^t$); the metrics $e_{\alpha\beta}$ and $\bar{e}_{\alpha\beta}$ therefore coincide identically, implying (cf. $(16)_2$) that all $\Lambda^\mu_{\alpha\beta}$ vanish.

For example, let $I = |\mathbf{g}_L|^2$; then

$$I = (\mathbf{L}\cdot\mathbf{S}^\alpha\mathbf{L})(\mathbf{L}\cdot\mathbf{S}^\beta\mathbf{L})\mathbf{e}_\alpha\cdot\mathbf{C}\mathbf{e}_\beta. \tag{57}$$

Replacing $\mathbf{C}$ by $\mathbf{RCR}^t$ and $\mathbf{S}^\alpha$ by $\mathbf{R}\bar{\mathbf{S}}^\alpha\mathbf{R}^t$ in accordance with (48) and (49), we compute

$$\begin{aligned}\bar{I} &= (\mathbf{R}^t\mathbf{L}\cdot\bar{\mathbf{S}}^\alpha\mathbf{R}^t\mathbf{L})(\mathbf{R}^t\mathbf{L}\cdot\bar{\mathbf{S}}^\beta\mathbf{R}^t\mathbf{L})\mathbf{R}^t\mathbf{e}_\alpha\cdot\mathbf{C}\mathbf{R}^t\mathbf{e}_\beta\\ &= (\mathbf{L}\cdot\bar{\mathbf{S}}^\alpha\mathbf{L})(\mathbf{L}\cdot\bar{\mathbf{S}}^\beta\mathbf{L})\bar{\mathbf{e}}_\alpha\cdot\mathbf{C}\bar{\mathbf{e}}_\beta,\end{aligned} \tag{58}$$

where use has been made of (24) and (56). Using (24) in the form $\bar{\mathbf{e}}_\alpha = R_\alpha^{\cdot\mu}\mathbf{e}_\mu$, we reduce this to

$$\bar{I} = R_\alpha^{\cdot\mu}R_\beta^{\cdot\lambda}(\mathbf{L}\cdot\bar{\mathbf{S}}^\alpha\mathbf{L})(\mathbf{L}\cdot\bar{\mathbf{S}}^\beta\mathbf{L})\mathbf{e}_\mu\cdot\mathbf{C}\mathbf{e}_\lambda, \tag{59}$$

and $(46)_1$ finally yields $\bar{I} = I$. Thus, $|\mathbf{g}_L|^2$ satisfies the symmetry conditions (48) and (49), for $\mathbf{R}$ as in (56). Similarly, each term in (51) may be shown to satisfy the same symmetry condition.

The strain-energy function based on the use of an arbitrary reference surface $\bar{\Omega}$, say, may be derived from the representation (51). This is $\bar{W} = W\sqrt{e/\bar{e}}$, where e and $\bar{e}$, respectively, are the determinants of the metrics induced by the convected coordinates on Ω and $\bar{\Omega}$. This may be obtained as an explicit function of the deformation measures based on the use of $\bar{\Omega}$ as reference by substituting (18)–(22) into (51), (52), (54), and (55).

5 Virtual work principle

The derivation of the Euler equations and boundary conditions in second-gradient elasticity is well known [9–13]. We present a version appropriate for surfaces, and use it to establish the constitutive connection between the loads and the deformation. The relevant virtual work statement is

$$\dot{E} = P, \tag{60}$$

where the superposed dot refers to the variational derivative,

$$E = \int_{\Omega} W \mathrm{d}a \tag{61}$$

is the strain energy, and P is the virtual power of the edge loads, the form of which is made explicit in the sequel.

Here and henceforth, Ω may be replaced by any subregion $\Pi \subset \Omega$ without affecting the argument. In different contexts, this fact was already known to Piola and Hellinger. We refer to [25] for historical perspective.

5.1 Variational derivatives and response functions

We have

$$\dot{E} = \int_{\Omega} \dot{W} \mathrm{d}a, \tag{62}$$

where $\dot{W}$ is given by

$$\dot{W} = \frac{\partial W}{\partial a_{\alpha\beta}} \dot{a}_{\alpha\beta} + \frac{\partial W}{\partial S^{\lambda}_{\alpha\beta}} \dot{S}^{\lambda}_{\alpha\beta} + \frac{\partial W}{\partial b_{\alpha\beta}} \dot{b}_{\alpha\beta}. \tag{63}$$

It is necessary to express this in terms of the variation $\mathbf{u} = \dot{\mathbf{r}}$ of the position field. To this end, we use

$$\dot{a}_{\alpha\beta} = \mathbf{a}_{\alpha} \cdot \mathbf{u}_{,\beta} + \mathbf{a}_{\beta} \cdot \mathbf{u}_{,\alpha}, \tag{64}$$

together with

$$\mathbf{u}_{;\alpha\beta} = \mathbf{u}_{,\alpha\beta} - \Gamma^{\lambda}_{\alpha\beta} \mathbf{u}_{,\lambda}, \tag{65}$$

in which subscripts preceded by semicolons identify covariant derivatives on the equilibrium surface ω. This is reduced to

$$\mathbf{u}_{;\alpha\beta} = \dot{S}^{\lambda}_{\alpha\beta} \mathbf{a}_{\lambda} + b_{\alpha\beta} \dot{\mathbf{n}} + \dot{b}_{\alpha\beta} \mathbf{n}, \tag{66}$$

by using the variational derivative of the Gauss equation $(1)_1$. Thus,

$$\dot{b}_{\alpha\beta} = \mathbf{n} \cdot \mathbf{u}_{;\alpha\beta}. \tag{67}$$

To compute $\dot{S}^{\lambda}_{\alpha\beta}$, we use (66) to derive

$$\mathbf{a}^{\mu} \cdot \mathbf{u}_{;\alpha\beta} = \dot{S}^{\mu}_{\alpha\beta} + b_{\alpha\beta} a^{\mu\lambda} \mathbf{a}_{\lambda} \cdot \dot{\mathbf{n}}, \tag{68}$$

together with $\mathbf{a}_{\lambda} \cdot \dot{\mathbf{n}} = -\mathbf{n} \cdot \mathbf{u}_{,\lambda}$ (derived by differentiating $\mathbf{a}_{\lambda} \cdot \mathbf{n} = 0$), obtaining

$$\dot{S}^{\mu}_{\alpha\beta} = \mathbf{a}^{\mu} \cdot \mathbf{u}_{;\alpha\beta} + b_{\alpha\beta} a^{\mu\lambda} \mathbf{n} \cdot \mathbf{u}_{,\lambda}. \tag{69}$$

To express $\dot{b}_{\alpha\beta}$ and $\dot{S}^{\boldsymbol{\mu}}_{\alpha\beta}$ in terms of $\mathbf{u}_{,\alpha}$ and $\mathbf{u}_{|\alpha\beta}$, we use

$$\mathbf{u}_{|\alpha\beta} = \mathbf{u}_{,\alpha\beta} - \Gamma^{\lambda}_{(\Omega)\alpha\beta}\mathbf{u}_{,\lambda}, \tag{70}$$

solve for $\mathbf{u}_{,\alpha\beta}$, and substitute into (65), obtaining

$$\mathbf{u}_{;\alpha\beta} = \mathbf{u}_{|\alpha\beta} - S^{\boldsymbol{\mu}}_{\alpha\beta}\mathbf{u}_{,\boldsymbol{\mu}}. \tag{71}$$

Accordingly,

$$\dot{b}_{\alpha\beta} = \mathbf{n} \cdot \left(\mathbf{u}_{|\alpha\beta} - S^{\lambda}_{\alpha\beta}\mathbf{u}_{,\lambda}\right) \tag{72}$$

and

$$\dot{S}^{\lambda}_{\alpha\beta} = \mathbf{a}^{\lambda} \cdot \mathbf{u}_{|\alpha\beta} + \left(b_{\alpha\beta}a^{\boldsymbol{\mu}\lambda}\mathbf{n} - S^{\boldsymbol{\mu}}_{\alpha\beta}\mathbf{a}^{\lambda}\right) \cdot \mathbf{u}_{,\boldsymbol{\mu}}. \tag{73}$$

Altogether,

$$\dot{W} = \mathbf{N}^{\alpha} \cdot \mathbf{u}_{,\alpha} + \mathbf{M}^{\alpha\beta} \cdot \mathbf{u}_{|\alpha\beta}, \tag{74}$$

with $\mathbf{M}^{\alpha\beta} = \mathbf{M}^{\beta\alpha}$ (the *order-of-differentiation symmetry* identified in Sect. 1) and

$$\mathbf{N}^{\alpha} = N^{\beta\alpha}\mathbf{a}_{\beta} + N^{\alpha}\mathbf{n}, \quad \mathbf{M}^{\alpha\beta} = M^{\lambda\alpha\beta}\mathbf{a}_{\lambda} + M^{\alpha\beta}\mathbf{n}, \tag{75}$$

in which

$$M^{\lambda\alpha\beta} = \frac{1}{2}\left(\frac{\partial W}{\partial S^{\mu}_{\alpha\beta}} + \frac{\partial W}{\partial S^{\mu}_{\beta\alpha}}\right)a^{\mu\lambda}, \quad M^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial W}{\partial b_{\alpha\beta}} + \frac{\partial W}{\partial b_{\beta\alpha}}\right),$$
$$N^{\beta\alpha} = \sigma^{\beta\alpha} - M^{\beta\gamma\mu}S^{\alpha}_{\gamma\mu}, \quad N^{\alpha} = M^{\alpha\mu\gamma}b_{\mu\gamma} - M^{\beta\lambda}S^{\alpha}_{\beta\lambda}, \tag{76}$$

and

$$\sigma^{\beta\alpha} = \frac{\partial W}{\partial a_{\alpha\beta}} + \frac{\partial W}{\partial a_{\beta\alpha}}. \tag{77}$$

For the special strain-energy function (51), the foregoing equations furnish [4]

$$\mathbf{N}^{\alpha} = w_{\lambda}L^{\alpha}\mathbf{l} + w_{\mu}M^{\alpha}\mathbf{m} + Jw_{J}\mathbf{a}^{\alpha} + \{[(A_g - k)(K_L\mathbf{g}_L + K_M\mathbf{g}_M) + (A_{\Gamma} - \bar{k})T\boldsymbol{\Gamma}] \cdot \mathbf{a}^{\alpha}\}\mathbf{n} \tag{78}$$

and

$$\mathbf{M}^{\alpha\beta} = L^{\alpha}L^{\beta}(A_g\mathbf{g}_L + kK_L\mathbf{n}) + M^{\alpha}M^{\beta}(A_g\mathbf{g}_M + kK_M\mathbf{n}) + \frac{1}{2}(L^{\alpha}M^{\beta} + M^{\alpha}L^{\beta})(A_{\Gamma}\boldsymbol{\Gamma} + \bar{k}T\mathbf{n}), \tag{79}$$

where $\mathbf{l}$ and $\mathbf{m}$ are the unit tangents to the deformed fibers, i.e., $\lambda\mathbf{l} = \mathbf{FL}$ and $\mu\mathbf{m} = \mathbf{FM}$. The tangential and normal components in (75), if needed, can be read off from these.

5.2 Reduction

Proceeding with the reduction of (60), as in [4], we define

$$\varphi^{\alpha} = \mathbf{T}^{\alpha} \cdot \mathbf{u} + \mathbf{M}^{\alpha\beta} \cdot \mathbf{u}_{,\beta} \tag{80}$$

with

$$\mathbf{T}^{\alpha} = \mathbf{N}^{\alpha} - \mathbf{M}^{\alpha\beta}_{|\beta} \tag{81}$$

and

$$\mathbf{M}^{\alpha\beta}_{|\beta} = \mathbf{M}^{\beta\alpha}_{,\beta} + \mathbf{M}^{\beta\alpha}\Gamma^{\lambda}_{(\Omega)\lambda\beta} + \mathbf{M}^{\beta\lambda}\Gamma^{\alpha}_{(\Omega)\lambda\beta}. \tag{82}$$

With this, we have

$$\dot{W} = \varphi^{\alpha}_{|\alpha} - \mathbf{u} \cdot \mathbf{T}^{\alpha}_{|\alpha} \tag{83}$$

and Stokes' theorem may then be used to reduce (62) to

$$\dot{E} = \int_{\partial\Omega} \varphi^{\alpha} \nu_{\alpha} \mathrm{d}s - \int_{\Omega} \mathbf{u} \cdot \mathbf{T}^{\alpha}_{|\alpha} \mathrm{d}a, \tag{84}$$

wherein $\boldsymbol{\nu} = \nu_{\alpha}\mathbf{e}^{\alpha}$ is the rightward unit normal to $\partial\Omega$.

A distributed load $\mathbf{g}$, per unit area of Ω, contributes $\int_{\Omega} \mathbf{g} \cdot \mathbf{u} \mathrm{d}a$ to the virtual work of the loads. It follows immediately from (60) that the relevant Euler–Lagrange equation, holding in Ω, is

$$\mathbf{T}^{\alpha}_{|\alpha} + \mathbf{g} = \mathbf{0}. \tag{85}$$

It is conventional to record the tangential and normal equations of equilibrium but we refrain from doing so here as these are greatly complicated, relative to those of standard shell theory, by the presence of strain-gradient effects.

Turning to the boundary terms, a standard integration-by-parts procedure [4] may be used to recast the first integral in (84) as

$$\int_{\partial\Omega} \varphi^{\alpha} \nu_{\alpha} \mathrm{d}s = \int_{\partial\Omega} \left\{ \left(\mathbf{T}^{\alpha}\nu_{\alpha} - (\mathbf{M}^{\alpha\beta}\nu_{\alpha}\tau_{\beta})' \right) \cdot \mathbf{u} + \mathbf{M}^{\alpha\beta}\nu_{\alpha}\nu_{\beta} \cdot \mathbf{u}_{\nu} \right\} \mathrm{d}s - \sum \left[\mathbf{M}^{\alpha\beta}\nu_{\alpha}\tau_{\beta} \right]_{i} \cdot \mathbf{u}_{i}, \tag{86}$$

where $\boldsymbol{\tau} = \tau_{\alpha}\mathbf{e}^{\alpha} = \mathbf{N} \times \boldsymbol{\nu}$ is the unit tangent to $\partial\Omega$; $\mathbf{u}_{\nu} = \nu^{\alpha}\mathbf{u}_{,\alpha}$ is the normal derivative of $\mathbf{u}$, $(\cdot)' = \mathrm{d}(\cdot)/\mathrm{d}s$; and the square bracket refers to the forward jump as a corner of the boundary is traversed. Thus, $[\cdot] = (\cdot)_{+} - (\cdot)_{-}$, where the subscripts $\pm$ identify the limits as a corner located at arc length station s is approached through larger and smaller values of arc length, respectively. The sum accounts for the contributions from all corners. Here we assume the boundary to be piecewise smooth in the sense that its tangent $\boldsymbol{\tau}$ is piecewise continuous.

From (60), it follows that admissible powers have the form

$$P = \int_{\Omega} \mathbf{g} \cdot \mathbf{u} \mathrm{d}a + \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{u} \mathrm{d}s + \int_{\partial\Omega_m} \boldsymbol{\mu} \cdot \mathbf{u}_{\nu} \mathrm{d}s + \sum_{*} \mathbf{f}_i \cdot \mathbf{u}_i, \tag{87}$$

where

$$\mathbf{t} = \mathbf{T}^{\alpha}\nu_{\alpha} - \left(\mathbf{M}^{\alpha\beta}\nu_{\alpha}\tau_{\beta}\right)', \quad \boldsymbol{\mu} = \mathbf{M}^{\alpha\beta}\nu_{\alpha}\nu_{\beta} \quad \text{and} \quad \mathbf{f}_i = -\left[\mathbf{M}^{\alpha\beta}\nu_{\alpha}\tau_{\beta}\right]_i \tag{88}$$

are the edge traction, edge double force, and the corner force at the i^{th} corner, respectively. Here, $\partial\Omega_t$ and $\partial\Omega_m$, respectively, are parts of $\partial\Omega$ where $\mathbf{r}$ and $\mathbf{r}_\nu$ are not assigned, and the starred sum includes only the corners where position is not assigned. We suppose that $\mathbf{r}$ and $\mathbf{r}_\nu$ are assigned on $\partial\Omega\backslash\partial\Omega_t$ and $\partial\Omega\backslash\partial\Omega_m$, respectively, and that position is assigned at the corners not included in the starred sum.

5.3 Overall equilibrium

Consider the special case in which no kinematical data are assigned anywhere on $\partial\Omega$, so that $\partial\Omega_t = \partial\Omega_m = \partial\Omega$ and rigid-body deformations are kinematically admissible. The variational derivative of such a deformation is expressible in the form $\mathbf{u} = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{c}$, where $\mathbf{c}$ and $\boldsymbol{\omega}$ are arbitrary spatially uniform vectors. Because the strain-energy function is invariant under such variations, we have $\dot{E} = 0$ and (60) reduces to $P = 0$, i.e.,

$$\mathbf{c} \cdot \left(\int_{\Omega} \mathbf{g}\,da + \int_{\partial\Omega} \mathbf{t}\,ds + \sum \mathbf{f}_i\right) + \boldsymbol{\omega} \cdot \left\{\int_{\Omega} \mathbf{r} \times \mathbf{g}\,da + \int_{\partial\Omega} (\mathbf{r} \times \mathbf{t} + \mathbf{r}_\nu \times \boldsymbol{\mu})\,ds + \sum \mathbf{r}_i \times \mathbf{f}_i\right\} = 0. \tag{89}$$

We then have

$$\int_{\Omega} \mathbf{g}\,da + \int_{\partial\Omega} \mathbf{t}\,ds + \sum \mathbf{f}_i = \mathbf{0} \quad \text{and} \quad \int_{\Omega} \mathbf{r} \times \mathbf{g}\,da + \int_{\partial\Omega} (\mathbf{r} \times \mathbf{t} + \mathbf{r}_\nu \times \boldsymbol{\mu})\,ds + \sum \mathbf{r}_i \times \mathbf{f}_i = \mathbf{0}, \tag{90}$$

and hence the interpretation of $\mathbf{r}_\nu \times \boldsymbol{\mu}$ as a density of edge couples.

Evidently Eqs. $(90)_{1,2}$ are necessary, but not sufficient, for equilibrium. For, $(88)_2$ involves the entire double force on $\partial\Omega_m$, whereas in $(90)_2$ only the part of the double force orthogonal to $\mathbf{r}_\nu$ is relevant (see [9] for further discussion). This situation stands in contrast to single-gradient theories that do not involve double forces or corner forces. There, the global statements $(90)_{1,2}$ (with double forces and corner forces omitted), when applied to an arbitrary part $\Pi \subset \Omega$, provide a complete characterization of equilibrium.

5.4 Interpretation of the double force

To better understand the nature of the double force, imagine a thin strip, or seam, riveted to the edge $\partial\Omega_m$ of the shell and extending into Ω through a small distance h. Imagine a force density $-\mathbf{f}$ acting on $\partial\Omega_m$, and a force density $\mathbf{f}$, per unit length of $\partial\Omega_m$, acting on the opposite edge of the seam. These make no contribution to the net force and so do not appear in $(88)_1$ or $(90)_1$. Nevertheless, they make a net contribution

$$-\mathbf{f} \cdot \mathbf{u} + \mathbf{f} \cdot [\mathbf{u} + h\mathbf{u}_\nu + o(h)] = \boldsymbol{\mu} \cdot \boldsymbol{u}_\nu + o(h), \tag{91}$$

to the virtual work density, where $\boldsymbol{\mu} = h\mathbf{f}$ is the couple of the forces. For small h, this contribution is represented, at leading order, in (87).

Similarly, the net torque density generated by these forces is

$$\mathbf{r} \times (-\mathbf{f}) + [\mathbf{r} + h\mathbf{r}_\nu + o(h)] \times \mathbf{f} = \mathbf{r}_\nu \times \boldsymbol{\mu} + o(h), \tag{92}$$

which is represented, again at leading order, in $(90)_2$.

Thus the double force may be regarded a self-equilibrating pair of force densities acting over a vanishingly thin strip adjoining the edge of the shell to produce power and torque. This is precisely the mechanism whereby a bending moment may be generated at the edge of an elementary beam.

6 Example: bending a plane to a cylinder

A simple example is furnished by the bending of a plane to a right circular cylinder of radius r, say. We adopt (51)–(55), pertaining to a rectangular lattice. Let $\{\theta^\alpha\}$ be Cartesian coordinates on the initial plane and let $\{r, \theta, z\}$ be polar coordinates. The considered deformation is then described by

$$\mathbf{x} = \theta^\alpha \mathbf{e}_\alpha \quad \text{and} \quad \mathbf{r} = r\mathbf{e}_r(\theta) + z\mathbf{k}, \tag{93}$$

with $\theta^1 = r\theta$ and $\theta^2 = z$. Here, $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{k}\}$ is the usual cylindrical polar basis: $\mathbf{k}(= \mathbf{e}_2)$ is aligned with the axis of the cylinder; $\mathbf{e}_r$ is the exterior unit normal to the cylinder at azimuth θ; and $\mathbf{e}_\theta = \mathbf{k} \times \mathbf{e}_r$ is tangential to a parallel of latitude.

This deformation is an isometry; accordingly, the $S^{\boldsymbol{\mu}}_{\alpha\beta}$ vanish and the fiber stretches and areal stretch are equal to unity. Assuming the derivatives w_λ, w_μ, and w_J to vanish at such a state, we find, from (78) and (79) that $\mathbf{N}^\alpha = \mathbf{0}$ and

$$\mathbf{M}^{\alpha\beta} = M^{\alpha\beta}\mathbf{n}, \quad \text{with} \quad M^{\alpha\beta} = kK_L L^\alpha L^\beta + kK_M M^\alpha M^\beta + \frac{1}{2}\bar{k}T\left(L^\alpha M^\beta + M^\alpha L^\beta\right). \tag{94}$$

Let γ be the fiber orientation on the initial plane, i.e., $L^1 = \cos\gamma$, $L^2 = \sin\gamma$, $M^1 = -\sin\gamma$, and $M^2 = \cos\gamma$. Omitting the details of the simple derivation, we find that

$$K_L = -r^{-1}\cos^2\gamma, \quad K_M = -r^{-1}\sin^2\gamma \qquad \text{and,} \quad T = r^{-1}\sin\gamma\cos\gamma \tag{95}$$

and that

$$\mathbf{T}^\alpha = M^{\alpha\beta} b^\mu_\beta \mathbf{a}_\mu, \tag{96}$$

where $b^{\boldsymbol{\mu}}_\beta$ are the mixed components of the curvature ($b^1_1 = -r^{-1}$; all other components zero). Then

$$\mathbf{T}^\alpha_{|\alpha} = -r^{-2}M\mathbf{n}, \tag{97}$$

where $M = -M^{11}$, i.e.,

$$M = r^{-1}k(\cos^4\gamma + \sin^4\gamma) + r^{-1}\bar{k}\sin^2\gamma\cos^2\gamma. \tag{98}$$

The distributed force required to maintain equilibrium is a pressure $\mathbf{g} = pJ\mathbf{n}$, where $J = 1$ and $p = r^{-2}M$. The force and torque per unit length of the edge with initial unit normal $\boldsymbol{\nu} = \mathbf{e}_1$ are $\mathbf{f} = f\mathbf{e}_\theta$, with $f = r^{-1}M$, and $\mathbf{r}_{,1} \times \boldsymbol{\mu} = M\mathbf{k}$, a pure bending moment directed along the axis of the cylinder. We note that $f = rp$ in accordance with elementary statics. The edge forces, double forces, and corner forces acting on the boundary of any simply connected subregion of the shell may be computed using (88).

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