Alternating projection method for sparse model updating problems

Bo Dong · Yan Yu · Dan Dan Tian

Abstract Sparse model updating problems, considered in this paper, focus on updating the constructed second-order finite element model under the sparsity constraint, that is, the updated model should have the desired eigenvalues and eigenvectors, and preserve the symmetry, positive semi-definiteness, and sparsity of the original model. In the process of performing model updating, sparsity of the model, which implies the inner connectivity and other physical properties of the updated system, plays a critically important role in the model updating problems. However, very few results in the earlier literature have paid attention to this important constraint due to the difficulty associated with it. In this paper, an alternating projection method, which is versatile enough to solve a huge class of sparse model updating problems, is presented. A distinct practical feature of this method is that it is easy to design and develop because, in the process of applying this method, one only needs to alternatively find the optimal solutions of some matrix approximation problems arising naturally from the requirement of practical application. And our numerical results demonstrate that alternating projection is an effective tool for sparse model updating problems.

Keywords Alternating projection methods · Connectivity · Model updating problems · Sparsity

Mathematics Subject Classification 65F18 · 15A22 · 93B55

1 Introduction

Many physical applications arising from applied mechanics, circuit analysis, electrical oscillation, vibro-acoustics, or finite element model of PDEs can be mathematically modeled by a second-order ordinary differential system

$$M\ddot{\mathbf{y}} + C\dot{\mathbf{y}} + K\mathbf{y} = \mathbf{f}(t), \tag{1}$$

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where $\mathbf{y}(t) \in \mathbb{R}^n$ is a function of t and $M, C, K \in \mathbb{R}^{n \times n}$ are constant matrices. And if $\mathbf{y}(t) = \mathbf{v}e^{\lambda t}$ is a fundamental solution of (1), then the scalar λ and the vector $\mathbf{v}$ must solve the following quadratic eigenvalue problems:

$$(\lambda^2 M + \lambda C + K) \, \mathbf{v} = 0.$$

Moreover, coefficient matrices M, C, K are often required to be real, symmetric, and positive semi-definite. The model updating problem (MUP) is, for a given quadratic pencil $\lambda^2 M_a + \lambda C_a + K_a$, where M_a, C_a, K_a are matrices in $\mathbb{R}^{n \times n}$ with specified structure, to seek a new quadratic pencil

$$\lambda^2 M + \lambda C + K \tag{2}$$

so that the function $\|(M, C, K) - (M_a, C_a, K_a)\|_F^2$ is minimized subject to the constraints that the resulting quadratic pencil (2) has prescribed eigenpairs, and the coefficient matrices M, C, K have the specified structure.

The MUP with the requirement that M, C, K are symmetric and positive semi-definite has been well studied and there exists a large amount of literature on its solution [1–11]. A good exposition about general principles of model updating and the Lagrange multiplier approach for solving MUPs can be found in the book by Friswell and Mottershead [1]. Bai, Chu and Sun in [2] presented a quadratically convergent Newton-type method which is needed to solve a linear system by the conjugate gradient method in each iteration. In [3], Kuo, Lin and Xu proposed two efficient direct methods via dimension reduction to solve the MUP with the coefficient matrices being symmetric. Liu and Yuan in [4] proposed a gradient-based iterative method. Gauss–Seidel method and the steepest descent method were also employed by Chen in [5] and Ye in [6], respectively.

The MUPs arising from practical applications are often structured due to the inner connectivity of elements in the original physical configuration. It is important from practical view point to keep the sparsity of the coefficient matrices, which implies the partial inner connectivity of the original system. For example, the zero entry c_{ij} in the coefficient matrix C means there is no damping between the ith mass and the jth mass in a vibro-acoustics system. In [12], Bai proposed the tridiagonal case, where M is an identity matrix of size n, C and K are both tridiagonal matrices defined by

$$C = \begin{pmatrix} c_1 & -a_2 & & & \\ -a_2 & c_2 & -a_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_{n-1} & c_{n-1} & -a_n \\ & & & -a_n & c_n \end{pmatrix}, \quad K = \begin{pmatrix} k_1 & -b_2 & & & \\ -b_2 & k_2 & -b_3 & & \\ & \ddots & \ddots & \ddots & \\ & & & k_{n-1} & -b_n \\ & & & -b_n & k_n \end{pmatrix}. \tag{3}$$

In the four-degrees-of-freedom mass–spring system in Fig. 1, see [13], the corresponding coefficient matrices M, C, K take the following zero-pattern:

$$M = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}, \quad C = \begin{pmatrix} * & 0 & * & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \quad K = \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

on the assumption that the restoring force follows Hooke's law and that the damping is negatively proportional to the velocity.

Now, for given matrices $M_a, C_a, K_a \in \mathbb{R}^{n \times n}$ and eigenpairs $(X, \Lambda) \in \mathbb{R}^{n \times p} \times \mathbb{R}^{p \times p}$, denoted in the following real-valued form:

$$\Lambda := \text{diag}\left\{ \begin{pmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{pmatrix}, \cdots, \begin{pmatrix} \alpha_{k_c} & \beta_{k_c} \\ -\beta_{k_c} & \alpha_{k_c} \end{pmatrix}, \lambda_{2k_c+1}, \cdots, \lambda_p \right\} \in \mathbb{R}^{p \times p},$$

$$X := [\mathbf{x}_{1R}, \mathbf{x}_{1I}, \cdots, \mathbf{x}_{k_c R}, \mathbf{x}_{k_c I}, \mathbf{x}_{2k_c+1}, \cdots, \mathbf{x}_p] \in \mathbb{R}^{n \times p},$$

Fig. 1 A four-degrees-of-freedom mass–spring system
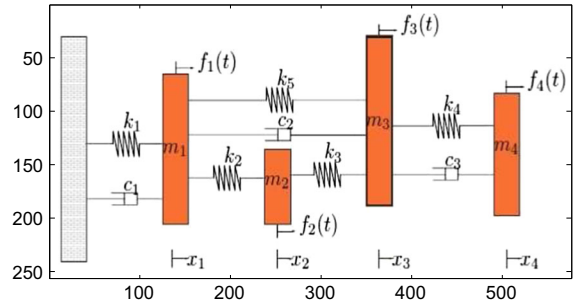
as was characterized in [14], the sparse MUP is to find the optimal solution $(M, C, K) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ of the following optimization problem:

$$\min \quad \|M - M_a\|_F^2 + \|C - C_a\|_F^2 + \|K - K_a\|_F^2 \quad \text{such that} \quad \begin{cases} MX\Lambda^2 + CX\Lambda + KX = 0, \\ M = M^\top, C = C^\top, K = K^\top, \\ M \geq 0, C \geq 0, K \geq 0, \\ \text{sparsity}(M, C, K) = \text{sparsity}(M_a, C_a, K_a), \end{cases} \tag{4}$$

where "$M \geq 0, C \geq 0, K \geq 0$" means M, C, K are all positive semi-definite matrices, and the constraint "sparsity(M, C, K) = sparsity(M_a, C_a, K_a)" means the matrices M, C, K have the same sparsity as the matrices M_a, C_a, K_a, respectively.

Of all the above-mentioned articles or the most in the literature, however, very few have paid attention to the sparsity of the coefficient matrices. In [13], Chu, Buono and Yu dealt with the quadratic inverse eigenvalue problems generated from a physical system with a special structure. In [15], the quadratic inverse eigenvalue problem, where the coefficient matrices M, C, K are all symmetric tridiagonal, with the constraint of nonnegative physical parameters was discussed. The existent methods aim at special structures and can hardly be generalized to other systems. In [16], a general purpose and robust numerical approach for solving the structured quadratic inverse eigenvalue problems, in which the coefficient matrices require to be sparse, are presented; however, it ignores the constraint that the coefficient matrices are positive semi-definite. In this paper, we consider to solve the sparse MUP by the well-known alternating projection technique through alternatively finding the projections of a given matrix onto some sets. In [17], the alternating projection method has been applied to solve the MUP with the fixed mass matrix M; however, it ignores the constraints that the coefficient matrices are sparse and positive semi-definite, which ensure the availability of the updated model.

The organization of this paper is as follows. The essentials and related work of the alternating projection method are introduced in Sect. 2. The details on applying the alternating projection method to solve sparse MUPs are presented in Sect. 3. Numerical examples are given in Sect. 4 to demonstrate the efficiency of our proposed algorithms.

2 Alternating projection method

The alternating projection method is a numerical algorithm which alternatively finds the optimal solution of a constrained optimization problem. There exists an extensive literature on the alternating projection method and its variants. In [18], von Neumann presented the 2-subspace version, that is, for a given point P_0, if the sets S_1 and S_2 are closed, linear subspaces of the Hilbert space, then the alternating projection of P_0 between these two sets converges to the approximation in $S_1 \cap S_2$ nearest to the point P_0. Halperin [19] proved the m-subspace version, where the sets $S_i (1 \leq i \leq m)$ are required to be closed subspaces in the Hilbert space. It can be described as follows:

Theorem 1 [19] *If $M_1, M_2, \ldots, M_r$ are closed subspaces in a Hilbert space H, then*

$$\lim_{n \to \infty} \|(P_{M_r} P_{M_{r-1}} \cdots P_{M_1})^n(x) - P_M(x)\| = 0, \quad \forall x \in H,$$

where $M = \bigcap_{i=1}^r M_i$, and $P_{M_i}(x) \, (1 \le i \le r)$ is the projection of x onto the set M_i.

Also there are fruitful results on the projection onto closed, convex subsets of a Hilbert space [20], see [21] for the 2-subspace version and [22,23] for m-subspace version. The theoretical results on the alternating projection between non-convex sets are still very weak, see [24] for some weak convergence results and applications.

The alternating projection method also have been successfully applied to solve many problems in a wide variety of applications, see, for example, the computation of channel capacity and rate-distortion functions [25], image reconstruction [26], the nearest correlation matrix [27], linear systems arising in the discretization of partial differential equations [28], and the other applications in articles [29–33] and the extensive references collected therein.

In this paper, we mainly consider to apply the alternating projection method to find the approximation, which lies in the intersection of several sets $S_i \, (1 \le i \le m)$ and is nearest to the given coefficient matrices M_a, C_a, K_a.

3 Alternating projection method for sparse MUPs

In this section, we consider solving the sparse MUPs. Before going in detail, it is essential to transform the sparse MUPs to another simpler form. Define

$$T = \begin{pmatrix} M & & \\ & C & \\ & & K \end{pmatrix}, \quad T_a = \begin{pmatrix} M_a & & \\ & C_a & \\ & & K_a \end{pmatrix}, \quad A = (I, I, I), \quad B = \begin{pmatrix} X\Lambda^2 \\ X\Lambda \\ X \end{pmatrix}.$$

Then the problem (4) is reduced to the problem of finding the solution T to the optimization problem

$$\min \quad \|T - T_a\|_F \quad \text{such that} \quad \begin{cases} ATB = 0, \\ T = T^\top \ge 0, \\ \text{sparsity}(T) = \text{sparsity}(T_a). \end{cases} \tag{5}$$

Applying the alternating projection method to solve the optimization problem (5), we find the projections of $T_a = (t_{ij}^{(a)})_{3n \times 3n}$ onto the following sets:

$$S_1 = \left\{ T \in \mathbb{R}^{3n \times 3n} \mid T = T^\top, \ ATB = 0 \right\},$$

$$S_2 = \left\{ T \in \mathbb{R}^{3n \times 3n} \mid T = T^\top \ge 0 \right\},$$

$$S_3 = \left\{ T \in \mathbb{R}^{3n \times 3n} \mid T = T^\top, \ \text{sparsity}(T) = \text{sparsity}(T_a) \right\}.$$

S_1 and S_3 are linear subspaces in the Hilbert space, and S_2 is a closed and convex set. Therefore, the convergence theory of the alternating projection method guarantees that the alternating projections among these three sets S_1, S_2, S_3 can converge to the optimal solution of the problem in (5). Therefore, the main work on applying the alternating projection method to solve sparse MUPs is to find the projections of T_a onto these three sets S_1, S_2, S_3.

Firstly, it is not difficult to find that the projection of the matrix T_a onto the set S_3 is $T = (t_{ij})_{3n \times 3n}$, where

$$t_{ij} = \begin{cases} (t_{ij}^{(a)} + t_{ji}^{(a)})/2 & t_{ij}^{(a)} \neq 0, \\ 0 & t_{ij}^{(a)} = 0. \end{cases}$$

Secondly, we consider the projection of the matrix T_a onto the set S_2, which is to find the nearest approximation of T_a in the set of symmetric and positive semi-definite matrices. Let

$$S = \frac{T_a + T_a^\top}{2} = P\text{diag}(\alpha_1, \ldots, \alpha_n)P^\top,$$

where P is orthogonal. In [34], Higham presented the unique projection:

$$T = P\text{diag}(\beta_1, \ldots, \beta_n)P^\top,$$

where $\beta_i = \max\{\alpha_i, 0\}$.

Lastly, we come to the problem of finding the projection of the given matrix T_a onto the set S_1. We firstly apply the canonical correlation decomposition to find the general form of the elements in the set S_1 and then find the projection of the matrix T_a onto the set S_1.

Without loss of generality, we suppose the matrix X is nonsingular. In (5), $\text{rank}(A) = n$, $\text{rank}(B) = p$, and we suppose $n \geq p$. Here, we must stress that the assumption of $n \geq p$ is just for the convenience of discussion; in fact, if $n < p$, then in the following discussion, $B^\top$, $A^\top$, $B^\top T A^\top = 0$ play the same roles as A, B and $ATB = 0$, respectively. Let the canonical correlation decomposition of matrix pair (A, B) be

$$A^\top = Q\Sigma_A X_A^{-1}, \quad B = Q\Sigma_B X_B^{-1}, \tag{6}$$

where $Q \in \mathbb{R}^{3n \times 3n}$ is orthogonal, $X_A \in \mathbb{R}^{n \times n}$ and $X_B \in \mathbb{R}^{p \times p}$ are nonsingular, $\Sigma_A \in \mathbb{R}^{3n \times n}$ and $\Sigma_B \in \mathbb{R}^{3n \times p}$ are of the form

$$\Sigma_A = \begin{pmatrix} I_A^{(1)} & 0 & 0 \\ 0 & \Gamma_A & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Delta_A & 0 \\ 0 & 0 & I_A^{(2)} \end{pmatrix}, \quad \Sigma_B = \begin{pmatrix} I_B \\ 0 \end{pmatrix}, \tag{7}$$

where $I_A^{(1)}$, $I_A^{(2)}$, I_B are the identity matrices of size r_1, r_3, p, respectively, Γ_A and Δ_A are diagonal matrices of size r_2, and satisfy $\Gamma_A^2 + \Delta_A^2$ which is an identity matrix of size r_2. Here,

$$r_1 = \text{rank}(A) + \text{rank}(B) - \text{rank}(A^\top, B),$$

$$r_2 = \text{rank}(A^\top, B) + \text{rank}(AB) - \text{rank}(A) - \text{rank}(B),$$

$$r_3 = \text{rank}(A) - \text{rank}(AB).$$

Substituting (6) into $ATB = 0$, we have

$$X_A^{-\top}\Sigma_A^\top Q^\top T Q \Sigma_B X_B^{-1} = 0. \tag{8}$$

Partition the matrices $\widehat{T} = Q^\top T Q$ and Σ_B into two block matrices:

$$\widehat{T} = \begin{pmatrix} \overset{r_1}{\widehat{T}_{11}} & \overset{r_2}{\widehat{T}_{12}} & \overset{r_3}{\widehat{T}_{13}} & \overset{r_4}{\widehat{T}_{14}} & \overset{r_2}{\widehat{T}_{15}} & \overset{r_3}{\widehat{T}_{16}} \\ \widehat{T}_{21} & \widehat{T}_{22} & \widehat{T}_{23} & \widehat{T}_{24} & \widehat{T}_{25} & \widehat{T}_{26} \\ \widehat{T}_{31} & \widehat{T}_{32} & \widehat{T}_{33} & \widehat{T}_{34} & \widehat{T}_{35} & \widehat{T}_{36} \\ \widehat{T}_{41} & \widehat{T}_{42} & \widehat{T}_{43} & \widehat{T}_{44} & \widehat{T}_{45} & \widehat{T}_{46} \\ \widehat{T}_{51} & \widehat{T}_{52} & \widehat{T}_{53} & \widehat{T}_{54} & \widehat{T}_{55} & \widehat{T}_{56} \\ \widehat{T}_{61} & \widehat{T}_{62} & \widehat{T}_{63} & \widehat{T}_{64} & \widehat{T}_{65} & \widehat{T}_{66} \end{pmatrix} \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_2 \\ r_3 \end{matrix}, \quad \Sigma_B = \begin{pmatrix} I_B^{(1)} & 0 & 0 \\ 0 & I_B^{(2)} & 0 \\ 0 & 0 & I_B^{(3)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

where $r_4 = 3n - r_1 - 2r_2 - 2r_3$ and $\widehat{T}_{ji} = \widehat{T}_{ij}^\top$ for $1 \leq i < j \leq 6$, $I_B^{(k)}$ is a matrix of the form

$$I_B^{(k)} = \begin{pmatrix} I \\ 0 \end{pmatrix} \begin{matrix} n_k \\ r_k - n_k \end{matrix},$$

where I is the identity matrix, for $k = 1, 2, 3$, and $n_k = \min\left\{r_k, p - \sum_{j=0}^{k-1} r_j\right\}, r_0 = 0$.

Substituting the expressions (7) and (9) into Eq. (8), we get

$$
\begin{pmatrix}
\widehat{T}_{11} I_B^{(1)} & \widehat{T}_{12} I_B^{(2)} & \widehat{T}_{13} I_B^{(3)} \\
(\Gamma_A \widehat{T}_{21} + \Delta_A \widehat{T}_{51}) I_B^{(1)} & (\Gamma_A \widehat{T}_{22} + \Delta_A \widehat{T}_{52}) I_B^{(2)} & (\Gamma_A \widehat{T}_{23} + \Delta_A \widehat{T}_{53}) I_B^{(3)} \\
\widehat{T}_{61} I_B^{(1)} & \widehat{T}_{62} I_B^{(2)} & \widehat{T}_{63} I_B^{(3)}
\end{pmatrix} = 0;
$$

thus, the submatrices in $\widehat{T}$ should satisfy

$$
\begin{aligned}
&\widehat{T}_{11} I_B^{(1)} = 0, & &\widehat{T}_{12} I_B^{(2)} = 0, & &\widehat{T}_{13} I_B^{(3)} = 0, \\
&(\Gamma_A \widehat{T}_{21} + \Delta_A \widehat{T}_{51}) I_B^{(1)} = 0, & &(\Gamma_A \widehat{T}_{22} + \Delta_A \widehat{T}_{52}) I_B^{(2)} = 0, & &(\Gamma_A \widehat{T}_{23} + \Delta_A \widehat{T}_{53}) I_B^{(3)} = 0, \quad (10) \\
&\widehat{T}_{61} I_B^{(1)} = 0, & &\widehat{T}_{62} I_B^{(2)} = 0 & &\widehat{T}_{63} I_B^{(3)}
\end{aligned}
$$

Therefore, the following theorem holds.

Theorem 2 *The general form of the matrices in the set S_1 can be expressed as $Q^{-\top} \widehat{T} Q^{-1}$, where $\widehat{T}$ has the form (9), and the block entries satisfy (10). In the special case of $p = n$, the elements in the set S_1 can be expressed as*

$$
\begin{array}{cccccc}
r_1 & r_2 & r_3 & r_4 & r_2 & r_3 \\
\end{array}
$$
$$
\begin{pmatrix}
0 & 0 & 0 & \widehat{T}_{14} & 0 & 0 \\
0 & \widehat{T}_{22} & \widehat{T}_{23} & \widehat{T}_{24} & (-\Delta_A^{-1} \Gamma_A \widehat{T}_{22})^\top & 0 \\
0 & \widehat{T}_{23}^\top & \widehat{T}_{33} & \widehat{T}_{34} & (-\Delta_A^{-1} \Gamma_A \widehat{T}_{23})^\top & 0 \\
\widehat{T}_{14}^\top & \widehat{T}_{24}^\top & \widehat{T}_{34}^\top & \widehat{T}_{44} & \widehat{T}_{45} & \widehat{T}_{46} \\
0 & -\Delta_A^{-1} \Gamma_A \widehat{T}_{22} & -\Delta_A^{-1} \Gamma_A \widehat{T}_{23} & \widehat{T}_{45}^\top & \widehat{T}_{55} & \widehat{T}_{56} \\
0 & 0 & 0 & \widehat{T}_{46}^\top & \widehat{T}_{56}^\top & \widehat{T}_{66}
\end{pmatrix}
\begin{array}{c}
r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_2 \\ r_3
\end{array},
$$

where $\widehat{T}_{22}, \widehat{T}_{33}, \widehat{T}_{44}, \widehat{T}_{55}, \widehat{T}_{66}$ are all symmetric square matrices, and $\widehat{T}_{14}, \widehat{T}_{23}, \widehat{T}_{24}, \widehat{T}_{34}, \widehat{T}_{45}, \widehat{T}_{46}, \widehat{T}_{56}$ are arbitrary real matrices.

Next, for a given matrix T_a, we consider finding its nearest matrix in the set S_1. Partitioning the matrix $Q^\top T_a Q$ into a block matrix $(\widehat{T}_{ij}^0)_{6 \times 6}$ consistently according to blocks of $Q^\top T Q$, we get

$$
\begin{aligned}
\|T - T_a\|_F^2 &= \|Q^\top T Q - Q^\top T_a Q\|_F^2 \\
&= \|\widehat{T}_{11} - \widehat{T}_{11}^0\|_F^2 + \|\widehat{T}_{12} - \widehat{T}_{12}^0\|_F^2 + \|\widehat{T}_{12}^\top - \widehat{T}_{21}^0\|_F^2 + \|\widehat{T}_{13} - \widehat{T}_{13}^0\|_F^2 + \|\widehat{T}_{13}^\top - \widehat{T}_{31}^0\|_F^2 \\
&\quad + \|\widehat{T}_{51} - \widehat{T}_{51}^0\|_F^2 + \|\widehat{T}_{51}^\top - \widehat{T}_{15}^0\|_F^2 + \|\widehat{T}_{61} - \widehat{T}_{61}^0\|_F^2 + \|\widehat{T}_{61}^\top - \widehat{T}_{16}^0\|_F^2 \\
&\quad + \|\widehat{T}_{62} - \widehat{T}_{62}^0\|_F^2 + \|\widehat{T}_{62}^\top - \widehat{T}_{26}^0\|_F^2 + \|\widehat{T}_{63} - \widehat{T}_{63}^0\|_F^2 + \|\widehat{T}_{63}^\top - \widehat{T}_{36}^0\|_F^2 \\
&\quad + \|\widehat{T}_{22} - \widehat{T}_{22}^0\|_F^2 + \|\widehat{T}_{52} - \widehat{T}_{52}^0\|_F^2 + \|\widehat{T}_{52}^\top - \widehat{T}_{25}^0\|_F^2 \\
&\quad + \|\widehat{T}_{23} - \widehat{T}_{23}^0\|_F^2 + \|\widehat{T}_{23}^\top - \widehat{T}_{32}^0\|_F^2 + \|\widehat{T}_{53} - \widehat{T}_{53}^0\|_F^2 + \|\widehat{T}_{53}^\top - \widehat{T}_{35}^0\|_F^2 \\
&\quad + \|\widehat{T}_{33} - \widehat{T}_{33}^0\|_F^2 + \|\widehat{T}_{44} - \widehat{T}_{44}^0\|_F^2 + \|\widehat{T}_{55} - \widehat{T}_{55}^0\|_F^2 + \|\widehat{T}_{66} - \widehat{T}_{66}^0\|_F^2 \\
&\quad + \|\widehat{T}_{14} - \widehat{T}_{14}^0\|_F^2 + \|\widehat{T}_{14}^\top - \widehat{T}_{41}^0\|_F^2 + \|\widehat{T}_{24} - \widehat{T}_{24}^0\|_F^2 + \|\widehat{T}_{24}^\top - \widehat{T}_{42}^0\|_F^2 \\
&\quad + \|\widehat{T}_{34} - \widehat{T}_{34}^0\|_F^2 + \|\widehat{T}_{34}^\top - \widehat{T}_{43}^0\|_F^2 + \|\widehat{T}_{45} - \widehat{T}_{45}^0\|_F^2 + \|\widehat{T}_{45}^\top - \widehat{T}_{54}^0\|_F^2 \\
&\quad + \|\widehat{T}_{46} - \widehat{T}_{46}^0\|_F^2 + \|\widehat{T}_{46}^\top - \widehat{T}_{64}^0\|_F^2 + \|\widehat{T}_{56} - \widehat{T}_{56}^0\|_F^2 + \|\widehat{T}_{56}^\top - \widehat{T}_{65}^0\|_F^2.
\end{aligned}
$$

Lemma 1 *For the given real matrices* $X_1 = (x_{1ij})_{s \times s}$, $X_2 = (x_{2ij})_{s \times s}$, $Y_1 = (y_{1ij})_{s \times s}$, $Y_2 = (y_{2ij})_{s \times s}$, $D_1 = \mathrm{diag}(d_{111}, \ldots, d_{1ss})$, *and* $D_2 = \mathrm{diag}(d_{211}, \ldots, d_{2ss})$, *where* D_1 *and* D_2 *are diagonal matrices, there exists a unique matrix*

$$T = (2I + D_1^2 + D_2^2)^{-1} (X_1 + X_2 + D_1 Y_1 + D_2 Y_2),$$

such that

$$\|T - X_1\|_F^2 + \|T - X_2\|_F^2 + \|D_1 T - Y_1\|_F^2 + \|D_2 T - Y_2\|_F^2 = \min.$$

Furthermore, if the matrix T *is required to be symmetric, then the matrix can be expressed as*

$$T = \Phi \odot (X_1 + X_1^\top + X_2 + X_2^\top + D_1 Y_1 + Y_1^\top D_1 + D_2 Y_2 + Y_2^\top D_2),$$

where $\odot$ *denotes the Hadamard product of two matrices, and*

$$\Phi = (\phi_{jk}), \quad \phi_{jk} = \frac{1}{4 + d_{1jj}^2 + d_{1kk}^2 + d_{2jj}^2 + d_{2kk}^2}.$$

Proof If the matrix T is not symmetric, then

$$\|T - X_1\|_F^2 + \|T - X_2\|_F^2 + \|D_1 T - Y_1\|_F^2 + \|D_2 T - Y_2\|_F^2$$
$$= \sum_{i=1}^{s} \left((t_{ii} - x_{1ii})^2 + (t_{ii} - x_{2ii})^2 + (d_{1ii}t_{ii} - y_{1ii})^2 + (d_{2ii}t_{ii} - y_{2ii})^2 \right)$$
$$+ \sum_{1 \leq j \neq k \leq s} \left((t_{jk} - x_{1jk})^2 + (t_{jk} - x_{2jk})^2 + (d_{1jj}t_{jk} - y_{1jk})^2 + (d_{2jj}t_{jk} - y_{2jk})^2 \right),$$

and therefore, applying the first-order condition, we get the expression

$$t_{jk} = \frac{x_{1jk} + x_{2jk} + d_{1jj}y_{1jk} + d_{2jj}y_{2jk}}{2 + d_{1jj}^2 + d_{2jj}^2}.$$

If the matrix T is required to be symmetric, then

$$\|T - X_1\|_F^2 + \|T - X_2\|_F^2 + \|D_1 T - Y_1\|_F^2 + \|D_2 T - Y_2\|_F^2$$
$$= \sum_{i=1}^{s} \left((t_{ii} - x_{1ii})^2 + (t_{ii} - x_{2ii})^2 + (d_{1ii}t_{ii} - y_{1ii})^2 + (d_{2ii}t_{ii} - y_{2ii})^2 \right)$$
$$+ \sum_{1 \leq j < k \leq s} \left((t_{jk} - x_{1jk})^2 + (t_{jk} - x_{1kj})^2 + (t_{jk} - x_{2jk})^2 + (t_{jk} - x_{2kj})^2 \right.$$
$$\left. + (d_{1jj}t_{jk} - y_{1jk})^2 + (d_{1kk}t_{jk} - y_{1kj})^2 + (d_{2jj}t_{jk} - y_{2jk})^2 + (d_{2kk}t_{jk} - y_{2kj})^2 \right),$$

and therefore, applying the first-order condition, we get the expression

$$t_{jk} = \frac{x_{1jk} + x_{1kj} + x_{2jk} + x_{2kj} + d_{1jj}y_{1jk} + d_{1kk}y_{1kj} + d_{2jj}y_{2jk} + d_{2kk}y_{2kj}}{4 + d_{1jj}^2 + d_{1kk}^2 + d_{2jj}^2 + d_{2kk}^2}.$$

Thus the lemma holds. □

All submatrices of $\widehat{T}$ can be divided into eight classes due to conditions (10) they have to satisfy; the following is the classification of submatrices and the procedure for finding all submatrices:

– The submatrix $\widehat{T}_{11}$ satisfying

$$\widehat{T}_{11} = \widehat{T}_{11}^\top, \ \widehat{T}_{11} I_B^{(1)} = 0, \ \|\widehat{T}_{11} - \widehat{T}_{11}^0\|_F^2 = \min.$$

Partition the matrices $\widehat{T}_{11}, \widehat{T}_{11}^0$ as follows:

$$\widehat{T}_{11} = \begin{pmatrix} \overset{n_1}{\widehat{T}_{11}^1} & \overset{r_1-n_1}{\widehat{T}_{11}^2} \\ \widehat{T}_{11}^3 & \widehat{T}_{11}^4 \end{pmatrix} \begin{matrix} n_1 \\ r_1-n_1 \end{matrix}, \quad \widehat{T}_{11}^0 = \begin{pmatrix} \overset{n_1}{\widehat{T}_{11}^{01}} & \overset{r_1-n_1}{\widehat{T}_{11}^{02}} \\ \widehat{T}_{11}^{03} & \widehat{T}_{11}^{04} \end{pmatrix} \begin{matrix} n_1 \\ r_1-n_1 \end{matrix},$$

thus we get

$$\widehat{T}_{11}^1 = 0_{n_1 \times n_1}, \quad \widehat{T}_{11}^2 = (\widehat{T}_{11}^3)^\top = 0_{n_1 \times (r_1-n_1)}, \quad \widehat{T}_{11}^4 = \frac{1}{2}\left(\widehat{T}_{11}^{04} + (\widehat{T}_{11}^{04})^\top\right).$$

– The submatrix $\widehat{T}_{16}, \widehat{T}_{61}$ satisfying

$$\widehat{T}_{61} = (\widehat{T}_{16})^\top, \quad \widehat{T}_{61} I_B^{(1)} = 0, \quad \left\|\widehat{T}_{61} - (\widehat{T}_{16}^0)^\top\right\|_F^2 + \left\|\widehat{T}_{61} - \widehat{T}_{61}^0\right\|_F^2 = \min.$$

Partition the matrices $\widehat{T}_{61}, \widehat{T}_{61}^0$ as follows:

$$\widehat{T}_{ij} = \begin{pmatrix} \overset{n_1}{\widehat{T}_{ij}^1} & \overset{r_1-n_1}{\widehat{T}_{ij}^2} \\ \widehat{T}_{ij}^3 & \widehat{T}_{ij}^4 \end{pmatrix} \begin{matrix} n_1 \\ r_1-n_1 \end{matrix}, \quad \widehat{T}_{ij}^0 = \begin{pmatrix} \overset{n_1}{\widehat{T}_{ij}^{01}} & \overset{r_1-n_1}{\widehat{T}_{ij}^{02}} \\ \widehat{T}_{ij}^{03} & \widehat{T}_{ij}^{04} \end{pmatrix} \begin{matrix} n_1 \\ r_1-n_1 \end{matrix}.$$

From $\widehat{T}_{61} I_B^{(1)} = 0$, we can obtain

$$\widehat{T}_{61}^1 = 0_{n_1 \times n_1}, \quad \widehat{T}_{61}^3 = 0_{n_1 \times (r_1-n_1)}.$$

From $\left\|\widehat{T}_{61} - (\widehat{T}_{16}^0)^\top\right\|_F^2 + \left\|\widehat{T}_{61} - \widehat{T}_{61}^0\right\|_F^2 = \min$, we get

$$\widehat{T}_{61}^2 = \frac{1}{2}\left(\widehat{T}_{61}^{02} + (\widehat{T}_{16}^{03})^\top\right), \quad \widehat{T}_{61}^4 = \frac{1}{2}\left(\widehat{T}_{61}^{04} + (\widehat{T}_{16}^{04})^\top\right).$$

– The submatrices $\widehat{T}_{21}, \widehat{T}_{51}$ satisfying

$$\widehat{T}_{12} = \widehat{T}_{21}^\top, \quad \widehat{T}_{12} I_B^{(2)} = 0, \quad (\Gamma_A \widehat{T}_{21} + \Delta_A \widehat{T}_{51}) I_B^{(1)} = 0,$$
$$\left\|\widehat{T}_{21} - \widehat{T}_{21}^0\right\|_F^2 + \left\|\widehat{T}_{12} - \widehat{T}_{12}^0\right\|_F^2 + \left\|\widehat{T}_{51} - \widehat{T}_{51}^0\right\|_F^2 + \left\|\widehat{T}_{15} - \widehat{T}_{15}^0\right\|_F^2 = \min.$$

1. If $n_2 \neq 0$, then $I_B^{(1)} = I_{r_1 \times r_1}$. Partition matrices $\widehat{T}_{21}, \widehat{T}_{21}^0, \widehat{T}_{51}, \widehat{T}_{51}^0, \widehat{T}_{12}^0, \widehat{T}_{15}^0, \Delta_A^{-1}\Gamma_A$ in the following form:

$$\widehat{T}_{ij} = \begin{pmatrix} \overset{n_2}{\widehat{T}_{ij}^1} & \overset{r_2-n_2}{\widehat{T}_{ij}^2} \\ \widehat{T}_{ij}^3 & \widehat{T}_{ij}^4 \end{pmatrix} \begin{matrix} n_2 \\ r_2-n_2 \end{matrix}, \quad \widehat{T}_{ij}^0 = \begin{pmatrix} \overset{n_2}{\widehat{T}_{ij}^{01}} & \overset{r_2-n_2}{\widehat{T}_{ij}^{02}} \\ \widehat{T}_{ij}^{03} & \widehat{T}_{ij}^{04} \end{pmatrix} \begin{matrix} n_2 \\ r_2-n_2 \end{matrix},$$

$$\Delta_A^{-1}\Gamma_A = \begin{pmatrix} \overset{n_2}{\Delta_1} & \overset{r_2-n_2}{0} \\ 0 & \Delta_2 \end{pmatrix} \begin{matrix} n_2 \\ r_2-n_2 \end{matrix}.$$

From $\widehat{T}_{12} I_B^{(2)} = 0$ and $(\Gamma_A \widehat{T}_{21} + \Delta_A \widehat{T}_{51}) I_B^{(1)} = 0$, we get

$$\widehat{T}_{21}^1 = 0, \quad \widehat{T}_{21}^2 = 0, \quad \widehat{T}_{51} = -\Delta_A^{-1}\Gamma_A \widehat{T}_{21}.$$

From $\left\|\widehat{T}_{21} - \widehat{T}_{21}^0\right\|_F^2 + \left\|\widehat{T}_{12} - \widehat{T}_{12}^0\right\|_F^2 + \left\|\widehat{T}_{51} - \widehat{T}_{51}^0\right\|_F^2 + \left\|\widehat{T}_{15} - \widehat{T}_{15}^0\right\|_F^2 = \min$, we get

$$\left\|\widehat{T}_{21}^3 - \widehat{T}_{21}^{03}\right\|_F^2 + \left\|\widehat{T}_{21}^3 - (\widehat{T}_{12}^{02})^\top\right\|_F^2 + \left\| -\Delta_2 \widehat{T}_{21}^3 - \widehat{T}_{51}^{03}\right\|_F^2 + \left\| -\Delta_2 \widehat{T}_{21}^3 - (\widehat{T}_{15}^{02})^\top\right\|_F^2 = \min,$$
$$\left\|\widehat{T}_{21}^4 - \widehat{T}_{21}^{04}\right\|_F^2 + \left\|\widehat{T}_{21}^4 - (\widehat{T}_{12}^{04})^\top\right\|_F^2 + \left\| -\Delta_2 \widehat{T}_{21}^4 - \widehat{T}_{51}^{04}\right\|_F^2 + \left\| -\Delta_2 \widehat{T}_{21}^4 - (\widehat{T}_{15}^{04})^\top\right\|_F^2 = \min.$$

Therefore, the matrices $\widehat{T}_{21}^3$, $\widehat{T}_{21}^4$ have the following forms:

$$\widehat{T}_{21}^3 = \left(2I + \Delta_2^2 + \Delta_2^2\right)^{-1} \left(\widehat{T}_{21}^{03} + \left(\widehat{T}_{12}^{02}\right)^\top - \Delta_2\widehat{T}_{51}^{03} - \Delta_2\left(\widehat{T}_{15}^{02}\right)^\top\right),$$

$$\widehat{T}_{21}^4 = \left(2I + \Delta_2^2 + \Delta_2^2\right)^{-1} \left(\widehat{T}_{21}^{04} + \left(\widehat{T}_{12}^{04}\right)^\top - \Delta_2\widehat{T}_{51}^{04} - \Delta_2\left(\widehat{T}_{15}^{04}\right)^\top\right).$$

2. $n_2 = 0$. Partition matrices $\widehat{T}_{21}, \widehat{T}_{21}^0, \widehat{T}_{51}, \widehat{T}_{51}^0, \widehat{T}_{12}^0, \widehat{T}_{15}^0, \Delta_A^{-1}\Gamma_A$ in the following form:

$$\widehat{T}_{ij} = \begin{pmatrix} \widehat{T}_{ij}^1 & \widehat{T}_{ij}^2 \\ \widehat{T}_{ij}^3 & \widehat{T}_{ij}^4 \end{pmatrix} \begin{matrix} n_1 \\ r_1 - n_1 \end{matrix}, \quad \widehat{T}_{ij}^0 = \begin{pmatrix} \widehat{T}_{ij}^{01} & \widehat{T}_{ij}^{02} \\ \widehat{T}_{ij}^{03} & \widehat{T}_{ij}^{04} \end{pmatrix} \begin{matrix} n_1 \\ r_1 - n_1 \end{matrix},$$

$$\Delta_A^{-1}\Gamma_A = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \begin{matrix} n_1 \\ r_1 - n_1 \end{matrix}.$$

From $(\Gamma_A\widehat{T}_{21} + \Delta_A\widehat{T}_{51})I_B^{(1)} = 0$, we get

$$\begin{pmatrix} \widehat{T}_{51}^1 \\ \widehat{T}_{51}^3 \end{pmatrix} = -\Delta_A\Gamma_A \begin{pmatrix} \widehat{T}_{21}^1 \\ \widehat{T}_{21}^3 \end{pmatrix} = -\begin{pmatrix} \Delta_1\widehat{T}_{21}^1 \\ \Delta_2\widehat{T}_{21}^3 \end{pmatrix}.$$

From $\left\|\widehat{T}_{21} - \widehat{T}_{21}^0\right\|_F^2 + \left\|\widehat{T}_{12} - \widehat{T}_{12}^0\right\|_F^2 + \left\|\widehat{T}_{51} - \widehat{T}_{51}^0\right\|_F^2 + \left\|\widehat{T}_{15} - \widehat{T}_{15}^0\right\|_F^2 = \min$, we get

$$\widehat{T}_{21}^2 = \frac{1}{2}\left(\widehat{T}_{21}^{02} + \left(\widehat{T}_{12}^{03}\right)^\top\right), \quad \widehat{T}_{21}^4 = \frac{1}{2}\left(\widehat{T}_{21}^{04} + \left(\widehat{T}_{12}^{04}\right)^\top\right),$$

$$\widehat{T}_{51}^2 = \frac{1}{2}\left(\widehat{T}_{51}^{02} + \left(\widehat{T}_{15}^{03}\right)^\top\right), \quad \widehat{T}_{51}^4 = \frac{1}{2}\left(\widehat{T}_{51}^{04} + \left(\widehat{T}_{15}^{04}\right)^\top\right),$$

$$\left\|\widehat{T}_{21}^1 - \widehat{T}_{21}^{01}\right\|_F^2 + \left\|\widehat{T}_{21}^1 - \left(\widehat{T}_{12}^{01}\right)^\top\right\|_F^2 + \left\| - \Delta_1\widehat{T}_{21}^1 - \widehat{T}_{51}^{01}\right\|_F^2 + \left\| - \Delta_1\widehat{T}_{21}^1 - \left(\widehat{T}_{15}^{01}\right)^\top\right\|_F^2 = \min,$$

$$\left\|\widehat{T}_{21}^3 - \widehat{T}_{21}^{03}\right\|_F^2 + \left\|\widehat{T}_{21}^3 - \left(\widehat{T}_{12}^{02}\right)^\top\right\|_F^2 + \left\| - \Delta_2\widehat{T}_{21}^3 - \widehat{T}_{51}^{03}\right\|_F^2 + \left\| - \Delta_2\widehat{T}_{21}^3 - \left(\widehat{T}_{15}^{02}\right)^\top\right\|_F^2 = \min.$$

Therefore, the matrices $\widehat{T}_{21}^1$, $\widehat{T}_{21}^3$ have the following forms:

$$\widehat{T}_{21}^1 = \left(2I + \Delta_1^2 + \Delta_1^2\right)^{-1} \left(\widehat{T}_{21}^{01} + \left(\widehat{T}_{12}^{01}\right)^\top - \Delta_1\widehat{T}_{51}^{01} - \Delta_1\left(\widehat{T}_{15}^{01}\right)^\top\right),$$

$$\widehat{T}_{21}^3 = \left(2I + \Delta_2^2 + \Delta_2^2\right)^{-1} \left(\widehat{T}_{21}^{03} + \left(\widehat{T}_{12}^{02}\right)^\top - \Delta_2\widehat{T}_{51}^{03} - \Delta_2\left(\widehat{T}_{15}^{02}\right)^\top\right).$$

– The submatrices $\widehat{T}_{13}, \widehat{T}_{62}, \widehat{T}_{63}$
Partition the matrices $\widehat{T}_{13}, \widehat{T}_{13}^0$ as follows:

$$\widehat{T}_{13} = \begin{pmatrix} \widehat{T}_{13}^1 & \widehat{T}_{13}^2 \\ \widehat{T}_{13}^3 & \widehat{T}_{13}^4 \end{pmatrix} \begin{matrix} n_3 \\ r_3 - n_3 \end{matrix}, \quad \widehat{T}_{13}^0 = \begin{pmatrix} \widehat{T}_{13}^{01} & \widehat{T}_{13}^{02} \\ \widehat{T}_{13}^{03} & \widehat{T}_{13}^{04} \end{pmatrix} \begin{matrix} n_3 \\ r_3 - n_3 \end{matrix}.$$

From $\widehat{T}_{13}I_B^{(3)} = 0$, $\left\|\widehat{T}_{13} - \widehat{T}_{13}^0\right\|_F^2 + \left\|\widehat{T}_{13}^\top - \widehat{T}_{31}^0\right\|_F^2 = \min$, we get

$$\widehat{T}_{13}^1 = 0, \quad \widehat{T}_{13}^2 = \frac{1}{2}\left(\widehat{T}_{13}^{02} + \left(\widehat{T}_{31}^{03}\right)^\top\right), \quad \widehat{T}_{13}^3 = 0, \quad \widehat{T}_{13}^4 = \frac{1}{2}\left(\widehat{T}_{13}^{04} + \left(\widehat{T}_{31}^{04}\right)^\top\right).$$

Similarly, we can get $\widehat{T}_{62}, \widehat{T}_{63}$.

– The submatrices $\widehat{T}_{22}$, $\widehat{T}_{52}$ satisfying

$$\widehat{T}_{22} = \widehat{T}_{22}^{\top}, \quad \left(\Gamma_A \widehat{T}_{22} + \Delta_A \widehat{T}_{52}\right) I_B^{(2)} = 0,$$
$$\|\widehat{T}_{22} - \widehat{T}_{22}^{0}\|_F^2 + \|\widehat{T}_{52} - \widehat{T}_{52}^{0}\|_F^2 + \|\widehat{T}_{52}^{\top} - \widehat{T}_{25}^{0}\|_F^2 = \min.$$

Partition the matrices $\widehat{T}_{22}$, $\widehat{T}_{52}$, $\widehat{T}_{22}^0$, $\widehat{T}_{25}^0$, $\widehat{T}_{52}^0$, $I_B^{(2)}$, $\Delta_A^{-1}\Gamma_A$ in the following form:

$$\widehat{T}_{ij} = \begin{pmatrix} \widehat{T}_{ij}^1 & \widehat{T}_{ij}^2 \\ \widehat{T}_{ij}^3 & \widehat{T}_{ij}^4 \end{pmatrix} \begin{matrix} n_2 \\ r_2 - n_2 \end{matrix}, \quad \widehat{T}_{ij}^0 = \begin{pmatrix} \widehat{T}_{ij}^{01} & \widehat{T}_{ij}^{02} \\ \widehat{T}_{ij}^{03} & \widehat{T}_{ij}^{04} \end{pmatrix} \begin{matrix} n_2 \\ r_2 - n_2 \end{matrix},$$

$$\Delta_A^{-1}\Gamma_A = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \begin{matrix} n_2 \\ r_2 - n_2 \end{matrix}.$$

From the condition $\widehat{T}_{22} = \widehat{T}_{22}^{\top}$, we can get

$$\widehat{T}_{22}^2 = \left(\widehat{T}_{22}^3\right)^{\top},$$

From the condition $\left(\Gamma_A \widehat{T}_{22} + \Delta_A \widehat{T}_{52}\right) I_B^{(2)} = 0$, we can get

$$\begin{pmatrix} \widehat{T}_{52}^1 \\ \widehat{T}_{52}^3 \end{pmatrix} = -\Delta_A^{-1}\Gamma_A \begin{pmatrix} \widehat{T}_{22}^1 \\ \widehat{T}_{22}^3 \end{pmatrix} = -\begin{pmatrix} \Delta_1 \widehat{T}_{22}^1 \\ \Delta_2 \widehat{T}_{22}^3 \end{pmatrix}.$$

From the condition $\|\widehat{T}_{22} - \widehat{T}_{22}^{0}\|_F^2 + \|\widehat{T}_{52} - \widehat{T}_{52}^{0}\|_F^2 + \|\widehat{T}_{52}^{\top} - \widehat{T}_{25}^{0}\|_F^2 = \min$, we get

$$\begin{pmatrix} \widehat{T}_{52}^2 \\ \widehat{T}_{52}^4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \widehat{T}_{52}^{02} + \left(\widehat{T}_{25}^{03}\right)^{\top} \\ \widehat{T}_{52}^{04} + \left(\widehat{T}_{25}^{04}\right)^{\top} \end{pmatrix}, \quad \widehat{T}_{22}^4 = \frac{1}{2} \left(\widehat{T}_{22}^{04} + \left(\widehat{T}_{22}^{04}\right)^{\top}\right),$$
$$\|\widehat{T}_{22}^1 - \widehat{T}_{22}^{01}\|_F^2 + \| - \Delta_1 \widehat{T}_{22}^1 - \widehat{T}_{52}^{01}\|_F^2 + \| - \Delta_1 \widehat{T}_{22}^1 - \left(\widehat{T}_{25}^{01}\right)^{\top}\|_F^2 = \min,$$
$$\|\widehat{T}_{22}^3 - \widehat{T}_{22}^{03}\|_F^2 + \|\widehat{T}_{22}^3 - \left(\widehat{T}_{22}^{02}\right)^{\top}\|_F^2 + \| - \Delta_2 \widehat{T}_{22}^3 - \widehat{T}_{52}^{03}\|_F^2 + \| - \Delta_2 \widehat{T}_{22}^3 - \left(\widehat{T}_{25}^{02}\right)^{\top}\|_F^2 = \min.$$

Therefore, the submatrices $\widehat{T}_{22}^1$, $\widehat{T}_{22}^3$ have the following form:

$$\widehat{T}_{22}^3 = \left(2I + \Delta_2^2 + \Delta_2^2\right)^{-1} \left(\widehat{T}_{22}^{03} + \left(\widehat{T}_{22}^{02}\right)^{\top} - \Delta_2 \widehat{T}_{52}^{03} - \Delta_2\left(\widehat{T}_{25}^{02}\right)^{\top}\right),$$
$$\widehat{T}_{22}^1 = \Phi \odot \left(\widehat{T}_{22}^{01} + \left(\widehat{T}_{22}^{01}\right)^{\top} - \Delta_1 \widehat{T}_{52}^{01} - \left(\widehat{T}_{52}^{01}\right)^{\top} \Delta_1 - \widehat{T}_{25}^{01} \Delta_1 - \Delta_1 \left(\widehat{T}_{25}^{01}\right)^{\top}\right),$$

where $\Phi = (\phi_{kl})$, $\phi_{kl} = \dfrac{1}{2 + (\Delta_1)_k^2 + (\Delta_1)_l^2 + (\Delta_1)_k^2 + (\Delta_1)_l^2}$.

– The submatrices $\widehat{T}_{23}$, $\widehat{T}_{53}$ satisfying

$$\left(\Gamma_A \widehat{T}_{23} + \Delta_A \widehat{T}_{53}\right) I_B^{(3)} = 0,$$
$$\|\widehat{T}_{23} - \widehat{T}_{23}^{0}\|_F^2 + \|\widehat{T}_{23}^{\top} - \widehat{T}_{32}^{0}\|_F^2 + \|\widehat{T}_{53} - \widehat{T}_{53}^{0}\|_F^2 + \|\widehat{T}_{53}^{\top} - \widehat{T}_{35}^{0}\|_F^2 = \min.$$

Partition the matrices $\widehat{T}_{23}, \widehat{T}_{53}, \widehat{T}_{23}^0, \widehat{T}_{35}^0, \widehat{T}_{53}^0, \Delta_A^{-1}\Gamma_A$ in the following form:

$$\widehat{T}_{ij} = \begin{array}{c} \\ \begin{pmatrix} \widehat{T}_{ij}^1 & \widehat{T}_{ij}^2 \\ \widehat{T}_{ij}^3 & \widehat{T}_{ij}^4 \end{pmatrix} \end{array} \begin{array}{c} n_3 \\ r_3 - n_3 \end{array}, \quad \widehat{T}_{ij}^0 = \begin{pmatrix} \widehat{T}_{ij}^{01} & \widehat{T}_{ij}^{02} \\ \widehat{T}_{ij}^{03} & \widehat{T}_{ij}^{04} \end{pmatrix} \begin{array}{c} n_3 \\ r_3 - n_3 \end{array},$$

where columns are labeled n_3 and $r_3 - n_3$.

$$\Delta_A^{-1}\Gamma_A = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \begin{array}{c} n_3 \\ r_3 - n_3 \end{array}.$$

From the condition $\left(\Gamma_A \widehat{T}_{23} + \Delta_A \widehat{T}_{53}\right) I_B^{(3)} = 0$, we get

$$\begin{pmatrix} \widehat{T}_{53}^1 \\ \widehat{T}_{53}^3 \end{pmatrix} = -\Delta_A^{-1}\Gamma_A \begin{pmatrix} \widehat{T}_{23}^1 \\ \widehat{T}_{23}^3 \end{pmatrix} = -\begin{pmatrix} \Delta_1 \widehat{T}_{23}^1 \\ \Delta_2 \widehat{T}_{23}^3 \end{pmatrix}.$$

From $\|\widehat{T}_{23} - \widehat{T}_{23}^0\|_F^2 + \|\widehat{T}_{23}^\top - \widehat{T}_{32}^0\|_F^2 + \|\widehat{T}_{53} - \widehat{T}_{53}^0\|_F^2 + \|\widehat{T}_{53}^\top - \widehat{T}_{35}^0\|_F^2 = \min$, we get

$$\begin{pmatrix} \widehat{T}_{53}^2 \\ \widehat{T}_{53}^4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \widehat{T}_{53}^{02} + (\widehat{T}_{35}^{03})^\top \\ \widehat{T}_{53}^{04} + (\widehat{T}_{35}^{04})^\top \end{pmatrix}, \quad \begin{pmatrix} \widehat{T}_{23}^2 \\ \widehat{T}_{23}^4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \widehat{T}_{23}^{02} + (\widehat{T}_{32}^{03})^\top \\ \widehat{T}_{23}^{04} + (\widehat{T}_{32}^{04})^\top \end{pmatrix},$$

$$\|\widehat{T}_{23}^1 - \widehat{T}_{23}^{01}\|_F^2 + \|\widehat{T}_{23}^1 - (\widehat{T}_{32}^{01})^\top\|_F^2 + \| -\Delta_1 \widehat{T}_{23}^1 - \widehat{T}_{53}^{01}\|_F^2 + \| -\Delta_1 \widehat{T}_{23}^1 - (\widehat{T}_{35}^{01})^\top\|_F^2 = \min,$$

$$\|\widehat{T}_{23}^3 - \widehat{T}_{23}^{03}\|_F^2 + \|\widehat{T}_{23}^3 - (\widehat{T}_{32}^{02})^\top\|_F^2 + \| -\Delta_2 \widehat{T}_{23}^3 - \widehat{T}_{53}^{03}\|_F^2 + \| -\Delta_2 \widehat{T}_{23}^3 - (\widehat{T}_{35}^{02})^\top\|_F^2 = \min.$$

Therefore, the submatrices $\widehat{T}_{23}^1, \widehat{T}_{23}^3$ have the following form:

$$\widehat{T}_{23}^1 = \left(2I + \Delta_1^2 + \Delta_1^2\right)^{-1} \left(\widehat{T}_{23}^{01} + (\widehat{T}_{32}^{01})^\top - \Delta_1 \widehat{T}_{53}^{01} - \Delta_1 (\widehat{T}_{35}^{01})^\top\right),$$

$$\widehat{T}_{23}^3 = \left(2I + \Delta_2^2 + \Delta_2^2\right)^{-1} \left(\widehat{T}_{23}^{03} + (\widehat{T}_{32}^{02})^\top - \Delta_2 \widehat{T}_{53}^{03} - \Delta_2 (\widehat{T}_{35}^{02})^\top\right).$$

– The submatrices $\widehat{T}_{33}, \widehat{T}_{44}, \widehat{T}_{55}, \widehat{T}_{66}$.
The matrix $\widehat{T}_{33}$ satisfies

$$\widehat{T}_{33} = \widehat{T}_{33}^\top, \quad \|\widehat{T}_{33} - \widehat{T}_{33}^0\|_F^2 = \min,$$

thus we get

$$\widehat{T}_{33} = \frac{1}{2}\left(\widehat{T}_{33}^0 + (\widehat{T}_{33}^0)^\top\right).$$

Similarly, we can get $\widehat{T}_{44}, \widehat{T}_{55}, \widehat{T}_{66}$.

– The submatrices $\widehat{T}_{14}, \widehat{T}_{24}, \widehat{T}_{34}, \widehat{T}_{45}, \widehat{T}_{46}, \widehat{T}_{56}$.
The matrix $\widehat{T}_{14}$ satisfies

$$\|\widehat{T}_{14} - \widehat{T}_{14}^0\|_F^2 + \|\widehat{T}_{14}^\top - \widehat{T}_{41}^0\|_F^2 = \min,$$

thus we get

$$\widehat{T}_{14} = \frac{1}{2}\left(\widehat{T}_{14}^0 + (\widehat{T}_{41}^0)^\top\right).$$

Similarly, we can get $\widehat{T}_{24}, \widehat{T}_{34}, \widehat{T}_{45}, \widehat{T}_{46}, \widehat{T}_{56}$.

Through the above discussion, we show how to produce the projections of the given matrix T_a onto the different sets S_1, S_2, S_3 and propose an alternating projection among the sets S_1, S_2, S_3 to solve the sparse MUP.

Table 1 Numerical results of Example (1.1)

n	k	APM				Yalmip			
		Time	It.	$\|T - Ta\|_F$	$\|ATB\|_F$	Time	It.	$\|T - Ta\|_F$	$\|ATB\|_F$
5	2	0	1	6.42×10^{-15}	6.09×10^{-15}	0.16	13	1.04×10^{-11}	2.45×10^{-9}
10	8	0	1	4.67×10^{-15}	9.09×10^{-15}	0.19	13	3.46×10^{-12}	3.06×10^{-9}
20	10	0.05	1	6.30×10^{-15}	3.39×10^{-13}	0.28	14	1.21×10^{-13}	1.28×10^{-9}
50	32	0.09	1	1.58×10^{-14}	2.86×10^{-13}	0.48	16	3.56×10^{-12}	6.74×10^{-10}
100	56	0.53	1	2.02×10^{-14}	8.74×10^{-12}	1.91	16	4.98×10^{-12}	1.57×10^{-9}

Table 2 Numerical results of Example (1.2)

n	k	Time	It.	$\|T - Ta\|_F$	$\|ATB\|_F$
5	2	4.41	10, 000	1.10×10^{-5}	5.69×10^{-4}
10	8	20.72	10, 000	3.84×10^{-5}	2.92×10^{-3}
20	10	70.67	10, 000	4.44×10^{-5}	7.01×10^{-3}
50	32	664.61	10, 000	1.61×10^{-5}	2.17×10^{-2}
100	56	4264.91	10, 000	7.74×10^{-5}	1.1415

4 Numerical experiments

In this section, we implemented the alternating projection method and compared our software with the well-known software Yalmip [35]. Two numerical examples, which were run in MATLAB 7.6.0 on a machine with Windows XP operation system, Intel(R) Core(TM) i3 CPU M370 @2.40GHz processor, and 2GB of memory, will be described to show the efficiency of our method.

For convenience, it is necessary to explain notations we use in this section. "APM" is the alternating projection method, "Yalmip" is the software Yalmip, n is the size of the problem, k is the number of given eigenpairs, time is the elapsed time, and It. is the number of iterative steps. In our procedure, we set the largest number of iterations to be 10000, and set the initial point to be (M_a, C_a, K_a).

Example 1 Similar to the article [12], in this example, we consider solving the sparse MUP with the constraints that M is an identity matrix, C and K are both tridiagonal matrices as (3). Here, the given symmetric tridiagonal matrices C_a, K_a are all randomly generated. Suppose $(\Lambda, X) \in \mathbb{R}^{k \times k} \times \mathbb{R}^{n \times k}$ is the eigenpairs of the quadratic eigenvalue problem $(\lambda^2 M_a + \lambda C_a + K_a)\mathbf{x} = \mathbf{0}$, the following three tests are performed:

1.1. Applying the exact eigenpairs (Λ, X) to update the coefficient matrices M_a, C_a, K_a;
1.2. Perturbing entries of eigenpairs (Λ, X) with $\varepsilon = 10^{-5}$. Applying the perturbed eigenpairs to update the coefficient matrices M_a, C_a, K_a.
1.3. Perturbing the coefficient matrices M_a, C_a, K_a with $\varepsilon = 10^{-5}$. Applying the eigenpairs (Λ, X) to update the coefficient matrices.

Tables 1, 2, and 3 present the efficiency of the alternating projection method for solving the sparse MUP. From these three tables, we can obtain the following information:

– If (Λ, X) consists of several exact eigenpairs of the quadratic pencil $\lambda^2 M_a + \lambda C_a + K_a$, since the initial point M_a, C_a, K_a is exactly the solution of the problem, then the alternating projection method converges to the required accuracy by only one step. Therefore, the test (1.1) states that this method can exactly reconstruct the quadratic model rapidly based on the exact eigeninformation. The software Yalmip also can find the quadratic model; however, it needs more iterations and CPU time.

Table 3 Numerical results of Example (1.3)

n	k	APM				Yalmip			
		Time	It.	$\|T - Ta\|_F$	$\|ATB\|_F$	Time	It.	$\|T - Ta\|_F$	$\|ATB\|_F$
5	2	0	1	1.34×10^{-5}	4.63×10^{-4}	0.08	6	1.52×10^{-5}	4.23×10^{-4}
10	8	0	2	2.57×10^{-5}	5.62×10^{-4}	0.11	7	2.11×10^{-5}	2.56×10^{-4}
20	10	0.073	1	1.56×10^{-5}	6.12×10^{-4}	0.19	9	1.72×10^{-5}	5.15×10^{-4}
50	32	0.16	2	1.43×10^{-5}	3.12×10^{-4}	0.37	11	1.67×10^{-5}	4.95×10^{-4}
100	56	0.94	2	2.44×10^{-5}	2.91×10^{-4}	1.51	12	2.31×10^{-5}	6.34×10^{-4}

- If (Λ, X) consists of perturbed eigenpairs of the quadratic pencil $\lambda^2 M_a + \lambda C_a + K_a$, still it is not clear at all that any quadratic pencils could have (Λ, X) as its eigenpairs, then maybe there does not exist one solution satisfying the quadratic pencil, and thus the alternating projection method reaches the maximum number of iterations. Therefore, the test (1.2) states that the alternating projection method can provide a numerical justification for the existence of such a quadratic pencil. The software Yalmip can only find a solution approximate to zero matrix, which is meaningless.
- If (Λ, X) consists of several exact eigenpairs of a quadratic pencil $\lambda^2 M + \lambda C + K$, and M_a, C_a, K_a are the perturbed coefficient matrices of M, C, K with $\varepsilon = 10^{-5}$, then the alternating projection method can find a solution, which is of the almost same order 10^{-5} as that of the perturbation to coefficient matrices, in few steps. Therefore, the test (1.3) states that the alternating projection method can solve the sparse MUP and serve as a tool to reconstruct the quadratic pencil with the constraint of sparsity. The software Yalmip also can find the quadratic model; however, it needs more iterations and CPU time.

Example 2 Taking the four-degrees-of-freedom mass–spring system coming from the engineering application into consideration, we randomly generate positive real values as the physical parameters, where

$$M_a = \begin{pmatrix} 0.2967 & 0 & 0 & 0 \\ 0 & 0.3188 & 0 & 0 \\ 0 & 0 & 0.4242 & 0 \\ 0 & 0 & 0 & 0.5097 \end{pmatrix}, \quad C_a = \begin{pmatrix} 0.3480 & 0 & -0.2625 & 0 \\ 0 & 0 & 0 & 0 \\ -0.2625 & 0 & 1.0635 & -0.8010 \\ 0 & 0 & -0.8010 & 0.8010 \end{pmatrix},$$

$$K_a = \begin{pmatrix} 1.5366 & -0.9289 & -0.5785 & 0 \\ -0.9289 & 1.6592 & -0.7303 & 0 \\ -0.5785 & -0.7303 & 1.7975 & -0.4886 \\ 0 & 0 & -0.4886 & 0.4886 \end{pmatrix},$$

the quadratic pencil has 2 real eigenvalues and 3 pairs of complex conjugate eigenvalues

$$-2.2523, -0.9839, -0.1740 \pm 2.7786i, -0.8098 \pm 1.7182i, -0.0266 \pm 0.1337i,$$

and the associated eigenvectors

$$X = \begin{pmatrix} -0.0415 & 0.4091 & -0.1259 \pm 0.2103i & 0.0286 \pm 0.4058i & 0.8619 \mp 0.1081i \\ -0.1066 & 0.4312 & 0.0142 \mp 0.3261i & -0.0422 \pm 0.2251i & 0.8726 \mp 0.1054i \\ -0.4256 & 0.6416 & 0.0070 \pm 0.0803i & 0.1835 \mp 0.1792i & 0.8794 \mp 0.1038i \\ 0.4440 & -1.0000 & 0.0449 \pm 0.0096i & -0.1696 \mp 0.2073i & 0.8972 \mp 0.1028i \end{pmatrix},$$

where $i = \sqrt{-1}$. We consider the model updating problem with different numbers of eigenpairs in Example 2, see Table 4 for the numerical results, which state that our method has no requirement on the number of the prescribed eigenpairs, and is more efficient than the software Yalmip.

Table 4 Numerical results with different numbers of eigenpairs

n	k	APM				Yalmip			
		Time	It.	$\|T - Ta\|_F$	$\|ATB\|_F$	Time	It.	$\|T - Ta\|_F$	$\|ATB\|_F$
5	1	0.033	1	4.13×10^{-15}	6.74×10^{-15}	0.12	12	5.76×10^{-13}	1.77×10^{-10}
5	2	0.043	1	3.29×10^{-15}	3.93×10^{-15}	0.16	12	1.02×10^{-11}	2.78×10^{-9}
5	3	0.047	1	7.15×10^{-15}	5.41×10^{-15}	0.17	13	1.70×10^{-12}	5.69×10^{-9}
5	4	0.052	1	1.32×10^{-14}	3.26×10^{-15}	0.17	12	2.15×10^{-12}	3.95×10^{-9}
5	5	0.053	1	4.64×10^{-15}	2.46×10^{-15}	0.16	12	1.31×10^{-13}	3.10×10^{-10}

From these numerical results, we can make the conclusion that the alternating projection method is an effective tool for sparsity-preserving model updating problems.

5 Conclusions

Sparse damped model updating problems play an important role in the applications arising from engineering. However, very few results have paid attention to this problem due to the difficulties associated with the sparsity constraint. And also the existent work aims at the special sparse structures, such as the tridiagonal, and can hardly be generalized to other structures.

In this paper, we consider applying the alternating projection method, which is powerful and easy to be implemented, and solving the sparse model updating problems. To our convenience, we firstly transform the sparse damped model updating problem into another optimization problem with a simpler form and then solve this transformed problem through alternatively finding the projections of the given matrix onto some convex sets or closed linear subspaces of a Hilbert space. The numerical experiments show that this method is very efficient for sparsity-preserving model updating problems. Furthermore, the so-called no spill-over is very important in many application; however, our method cannot maintain this property and we leave this as our future research topic.

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