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# Stability of active muscle tissue

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**Abstract** The notion of material stability is examined in the context of active muscle tissue modeling, where the nonlinear constitutive law is dependent both on the physiologically driven muscle contraction and finite mechanical deformation. First, the governing equations and constitutive laws for a general active-elastic material with a single preferred direction are linearized about a homogeneous underlying configuration with respect to both the active contraction and deformation gradient. In order to obtain mathematical restrictions analogous to those found in elastic materials, stability conditions are derived based on the propagation of homogeneous plane waves with real wave speeds, and the generalized acoustic tensor is obtained. Focusing on 2D motions, and considering a simplified, decoupled transversely isotropic energy function, the restriction on the active acoustic tensor is recast in terms of a generally applicable constitutive law, with specific attention paid to the fiber contribution. The implication of the material stability conditions on material parameters, active contraction, and elastic stretch is investigated for prototype material models of muscle tissue.

**Keywords** Active strain · Finite deformation · Muscle tissue · Plane waves · Stability

## 1 Introduction

Muscles are involved in a multitude of essential physiological functions such as respiration, locomotion, and circulation, and possess the ability to change conformation in the presence of a stimulus. This unique characteristic, along with recent developments in tissue engineering, has made it possible to utilize cultured muscle cells as actuators in engineered systems [1]. Desire to understand and predict muscle's nonlinear behavior, both in biological and engineered systems, has lead to a multitude of studies, particularly in the context of continuum mechanics, where models are readily implemented into a numerical framework [2].

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Mathematical descriptions of muscle behavior generally account for the influence of fiber activation by one of two approaches [3,4]. The first class of models includes an additive active stress component in overall stress tensor, either directly [5–10] or through a decoupled energy function [11–14]. Here, we consider an alternative approach, which incorporates the underlying micromechanical behavior of muscle contraction by adopting a multiplicative decomposition of the deformation gradient, and is referred to as active strain [2,4,15–25]. Specifically, the macroscopic deformation is considered the result of both the elastic strain and the shortening of the sarcomere, the basic motor unit of the muscle. A similar approach has been applied to softening in polymeric materials [26] and growth in soft biological tissues [27]. It is noted that these two approaches are not necessarily mutually exclusive; however, for the purposes of the current study, we consider material models which include an explicit multiplicative decomposition of the deformation gradient to be of an active strain type.

As material modeling and computational simulation related to active muscle tissue continue, it is desirable to develop mathematical restrictions on constitutive formulations, parallel to those found in elastic materials. For instance, the concept of material stability, in the purely mechanical case, can impose bounds on the mathematical model to ensure it maintains physical relevance [28]. Furthermore, when a particular model is implemented in numerical context, stability conditions can help avoid undesirable behavior [29]. Strong ellipticity is a commonly employed stability condition, see for instance [29–35], which ensures the propagation of small-amplitude plane waves superimposed on the underlying configuration. Many criteria exist which may be suitable for an active, fibrous biological material; see Bertoldi and Gei [36] for various stability criteria applied to electroactive materials and Holzapfel et al. [29] for comparison of alternative conditions for purely elastic biological tissues. The strong ellipticity condition is appealing as the formulation is independent of boundary conditions and has a clear physical interpretation while not being excessively restrictive [22].

Prior studies have investigated strong ellipticity conditions for active stress and active strain muscle models. Pathmanathan et al. [9] show loss of ellipticity occurring for a specific active stress model, with material failure possible for sufficiently large fiber compression. Specific to our current study, Ambrosi and Pezzuto [3] provide strong ellipticity conditions for active strain constitutive models. However, they only consider incremental perturbations in the total deformation and do not account for variations in the active fiber contraction. Similarly, Rossi et al. [22] investigate the stability of myocardium constitutive relations with an active strain formulation. They use an exponential energy function that is strongly elliptic, but again neglect incremental changes in the active contraction. These investigations of purely mechanical stability conditions neglect the unique and characteristic behavior of active muscle fibers. Therefore, in the following, we seek a generalization of the strong ellipticity condition for a class of materials described by an active strain approach, which includes a variation in the active contraction. A similar approach has been taken for other non-purely mechanical material models, such as thermoelastics [37], electroelastics [36,38], and magnetoelastics [39].

To this end, in Sect. 2, we provide a brief summary of the kinematic quantities used to describe both the mechanical and physiological deformations, including the incompressibility condition normally adopted for biological tissue. Furthermore, we derive the balance laws based on the virtual power formulation developed by Stålhand et al. [25] and introduce constitutive relations for a non-dissipative muscle model. Incremental equations resulting from variations in the deformation and active contraction, superimposed on the underlying configuration, are provided in Sect. 3. Here, the Lagrangian version of the incremental equation of motion is derived and then pushed forward into the current configuration. We introduce the fourth-order elasticity tensor, the second-order coupling tensor, both in material and spacial configurations, and the scalar coefficient related to the increment in the active contraction. Additionally, homogeneous plane waves are considered as a solution to the governing incremental equations subject to generalized dead loading. The stability condition is obtained by requiring the resulting modified acoustic tensor, referred to here as the active acoustic tensor, to be positive-definite, ensuring waves propagate with real speeds. Section 4 introduces invariant representation of the energy function to satisfy objectivity, and Sect. 5 specializes the analysis to 2D motions for a decoupled energy function, simplifying the stability condition and allowing for explicit expressions of the quantities related to the active acoustic tensor. We introduce a specific form of the decoupled energy function in Sect. 6 and consider three different cases of an overlap parameter, a unique component of muscle models, which accounts for the relative position of the actin and myosin proteins. The set of critical deformations and active contractions for which stability is lost is then determined. Section 7 contains concluding remarks.

## 2 Basic equations

### 2.1 Kinematics

To describe the motion of a deformable body, we identify a reference configuration $\mathcal{B}_\mathrm{r}$ with boundary $\partial\mathcal{B}_\mathrm{r}$ and the location of a generic material point $P$ is given by the position vector $\mathbf{X}$ relative to some origin $O$. The configuration occupied by the body at time $t$ is denoted by $\mathcal{B}_t$ with boundary $\partial\mathcal{B}_t$ and the point $P$ by the position vector $\mathbf{x}$ relative to some origin $o$. The vector field $\boldsymbol{\chi}(\mathbf{X}, t)$ describes the motion of the body such that

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \quad \text{for all} \quad \mathbf{X} \in \mathcal{B}_\mathrm{r}, \quad \mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t) \quad \text{for all} \quad \mathbf{x} \in \mathcal{B}_t. \tag{2.1}$$

The velocity and acceleration of the particle $P$ are given, respectively, by

$$\mathbf{v} = \mathbf{x}_{,t} = \frac{\partial}{\partial t}\boldsymbol{\chi}(\mathbf{X}, t), \quad \mathbf{a} = \mathbf{x}_{,tt} = \frac{\partial^2}{\partial t^2}\boldsymbol{\chi}(\mathbf{X}, t), \tag{2.2}$$

where $\partial/\partial t$ is the rate of change at fixed $\mathbf{X}$, which is also represented by the shorthand notation $_{,t}$. The deformation gradient $\mathbf{F}$ is defined by

$$\mathbf{F} = \mathrm{Grad}\,(\mathbf{x}), \tag{2.3}$$

where Grad is the gradient operator with respect to $\mathcal{B}_\mathrm{r}$. The Cartesian components are $F_{i\alpha} = \partial x_i/\partial X_\alpha$, where $x_i$ and $X_\alpha$ are the components of $\mathbf{x}$ and $\mathbf{X}$, respectively, with $i, \alpha \in \{1, 2, 3\}$. Roman indices are associated with $\mathcal{B}_t$ and Greek indices with $\mathcal{B}_\mathrm{r}$. We also adopt the standard notation

$$J = \det(\mathbf{F}) = \frac{\mathrm{d}v}{\mathrm{d}V} > 0, \tag{2.4}$$

where $\mathrm{d}V$ is a volume element in $\mathcal{B}_\mathrm{r}$ and $\mathrm{d}v$ is the corresponding volume element in $\mathcal{B}_t$. For an incompressible material we have

$$J = \det(\mathbf{F}) = 1. \tag{2.5}$$

The right and left Cauchy–Green tensors are given by $\mathbf{C} = \mathbf{F}^\mathrm{T}\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^\mathrm{T}$, respectively. For a detailed discussion of the kinematics of solid continua, we refer to, for example, Ogden [40] and Dorfmann and Ogden [41].

In this paper, we follow a phenomenological approach to model the behavior of muscle tissue and do not attempt to capture each biological process and microscopic interaction involved in muscle contraction. We consider the muscle to be composed of fibers, which provide the active contraction, and connective tissue. The reference configuration $\mathcal{B}_\mathrm{r}$ of passive muscle is defined to be stress-free (natural configuration) in the absence of mechanical loads. Focusing on the fibers only, when subjected to a stimulus, again, in the absence of surface traction and body forces, in general, the material will deform. The resulting configuration, denoted by $\mathcal{B}_\mathrm{a}$, is taken as the natural, stress-free configuration of the active fibers. Note, the connective tissue is not stress-free in $\mathcal{B}_\mathrm{a}$. For other active materials, the mathematical representation of these stresses is given by Dorfmann and Ogden [42,43].

The change in configuration is conveniently described by a tensor field $\mathbf{F}_\mathrm{a}$, similar to the approach taken by Stålhand et al. [25] for smooth muscle; Ambrosi et al. [15], Cherubini et al. [16], and Nardinocchi and Teresi [18,19] for cardiac tissue; and Hernández-Gascón et al. [17] and Paetsch et al. [4] for skeletal muscle. We assume activation occurs isochorically and therefore require $\det(\mathbf{F}_\mathrm{a}) = 1$. It is noted that the tensor field $\mathbf{F}_\mathrm{a}$ is a function of active fiber contraction, defined by the scalar $\lambda_\mathrm{a}$, and not the result of a gradient operation. Fiber contraction implies that $\lambda_\mathrm{a} \leq 1$ with minimum value, $\lambda_{\mathrm{a,min}}$, such that $\lambda_{\mathrm{a,min}} > 0$. In general, the active contraction $\lambda_\mathrm{a}$ may depend on physiological factors, such as calcium ion concentration or electric potential, allowing for the current framework of active muscle tissue to incorporate coupled models of skeletal muscles, such as Böl et al. [44].

After the application of mechanical loads, the cross-bridges between the interacting actin and myosin filaments are deformed, and the muscle tissue occupies the current configuration $\mathcal{B}_t$. This two-step process, activation and mechanical deformation, gives rise to the multiplicative decomposition:

$$\mathbf{F} = \mathbf{F}_{\mathrm{e}}\mathbf{F}_{\mathrm{a}}, \tag{2.6}$$

where $\mathbf{F}_{\mathrm{e}}$ maps material points from the active natural configuration $\mathcal{B}_{\mathrm{a}}$ to the current configuration $\mathcal{B}_t$ and provides the elastic deformation of the actomyosin cross-bridges; see Paetsch et al. [4] for further details.

## 2.2 Balance laws and constitutive equations

We use the principle of virtual power to derive the governing equations of active muscle tissue. This has been used by, for example, Stålhand et al. [25] for smooth muscle in 3D, Sharifimajd and Stålhand [45] for skeletal muscle in 1D, and Hernández-Gascón et al. [17] for skeletal muscle in 3D. Let $\delta\mathbf{v}$ and $\delta\lambda_{\mathrm{a},t}$ denote the arbitrary, admissible virtual velocity field and the virtual rate of fiber contraction, respectively. The principle of *virtual power* for a dynamic system has the form

$$\hat{\mathcal{P}}_{\mathrm{ext}} = \hat{\mathcal{P}}_{\mathrm{int}} + \hat{\mathcal{P}}_{\mathrm{inertial}}, \tag{2.7}$$

where $\hat{\mathcal{P}}_{\mathrm{ext}}$, $\hat{\mathcal{P}}_{\mathrm{int}}$, and $\hat{\mathcal{P}}_{\mathrm{inertial}}$ depend on $\delta\mathbf{v}$ and $\delta\lambda_{\mathrm{a},t}$. The external virtual power, $\hat{\mathcal{P}}_{\mathrm{ext}}$, consists of the virtual mechanical power and the virtual power related to the fiber contraction. Specifically, we write

$$\hat{\mathcal{P}}_{\mathrm{ext}} = \int_{\partial\mathcal{B}_{\mathrm{r}}} \mathbf{T}\cdot\delta\mathbf{v}\,\mathrm{d}S + \int_{\mathcal{B}_{\mathrm{r}}} T_{\mathrm{a}}\delta\lambda_{\mathrm{a},t}\,\mathrm{d}V, \tag{2.8}$$

where the vector $\mathbf{T}$ is the mechanical traction and the scalar $T_{\mathrm{a}}$, power conjugate to $\delta\lambda_{\mathrm{a},t}$, is the external thermodynamic force related to the active contraction of the muscle fibers. For simplicity, mechanical body forces have been neglected. The internal virtual power, $\hat{\mathcal{P}}_{\mathrm{int}}$, is given by

$$\hat{\mathcal{P}}_{\mathrm{int}} = \int_{\mathcal{B}_{\mathrm{r}}} \mathrm{tr}(\mathbf{S}\,\delta\mathbf{F}_{,t})\mathrm{d}V + \int_{\mathcal{B}_{\mathrm{r}}} P_{\mathrm{a}}\delta\lambda_{\mathrm{a},t}\mathrm{d}V, \tag{2.9}$$

where $\mathbf{S}$ is the nominal stress, $\delta\mathbf{F}_{,t}$ is the material gradient of the virtual velocity, and $P_{\mathrm{a}}$ is the internal counterpart to $T_{\mathrm{a}}$. In Eq. (2.9), we introduced the trace operator acting on two second-rank tensors. For arbitrary tensor fields $\mathbf{A}$, $\mathbf{B}$, the trace operator $\mathrm{tr}(\mathbf{AB})$ is written in component form as $A_{ij}B_{ji}$. The virtual power of the inertial force has the form

$$\hat{\mathcal{P}}_{\mathrm{inertial}} = \int_{\mathcal{B}_{\mathrm{r}}} \rho_{\mathrm{r}}\mathbf{a}\cdot\delta\mathbf{v}\,\mathrm{d}V, \tag{2.10}$$

where $\rho_{\mathrm{r}}$ denotes the mass density in the reference configuration $\mathcal{B}_{\mathrm{r}}$. For an isochoric deformation $J = 1$ and, from the mass conservation equation, it follows that $\rho_{\mathrm{r}} = \rho$, where $\rho$ is the density in the current configuration $\mathcal{B}_t$.

The use of Cauchy's theorem enables $\mathbf{T}$ to be expressed as $\mathbf{T} = \mathbf{S}^{\mathrm{T}}\mathbf{N}$, where $\mathbf{N}$ is the unit normal on the boundary $\partial\mathcal{B}_{\mathrm{r}}$. Using the definitions (2.8)–(2.10) in (2.7), combined with the divergence theorem, yields the equation of motion in the absence of body forces:

$$\mathrm{Div}\,\mathbf{S} = \rho_{\mathrm{r}}\mathbf{x}_{,tt} \quad \text{in} \quad \mathcal{B}_{\mathrm{r}}, \tag{2.11}$$

where Div is the divergence operator in the reference configuration $\mathcal{B}_{\mathrm{r}}$ and $\mathbf{x}_{,tt}$ is defined in Eq. $(2.2)_2$. Furthermore, the principle of virtual power also provides the balance of thermodynamic forces related to the stress generated by the active fiber contraction $\lambda_{\mathrm{a}}$ and is given by

$$P_{\mathrm{a}} = T_{\mathrm{a}} \quad \text{in} \quad \mathcal{B}_{\mathrm{r}}, \tag{2.12}$$

where $P_{\mathrm{a}}$ and $T_{\mathrm{a}}$ are the respective internal and external thermodynamic forces related to the shortening of the muscle fibers. These thermodynamic forces can be interpreted as the complex biological and chemical processes driving the changing state of the muscle fibers, see Stålhand et al. [25], Hernández-Gascán et al. [17], and Sharifimajd and Stålhand [45] for a detailed discussion.

Next, we introduce the dissipation inequality

$$\mathcal{P}_{\text{int}} \geq \int_{\mathcal{B}_\text{r}} \Psi_{,t} \, \mathrm{d}V, \tag{2.13}$$

requiring the *internal power* $\mathcal{P}_{\text{int}}$ to be greater than or equal to the rate of change in the Helmholtz free energy, $\Psi$, integrated over the body in the reference configuration $\mathcal{B}_\text{r}$. For the purpose of our current study, we consider the system to be conservative and neglect dissipation, contrary to the approach proposed by Sharifimajd and Stålhand [45] for skeletal tissue. While this assumption omits some muscle behavior, it allows for a clearer formulation and still captures the primary characteristics of an active biological tissue. Therefore, the inequality in (2.13) is replaced by an equality. An explicit, pointwise expression for $\mathcal{P}_{\text{int}}$ may be obtained by taking the virtual velocities in (2.9) to be real and we write

$$\Psi_{,t} = \operatorname{tr}(\mathbf{S}\mathbf{F}_{,t}) + P_\text{a}\lambda_{\text{a},t}. \tag{2.14}$$

The free energy $\Psi$ is assumed to be a function of the deformation gradient $\mathbf{F}$ and the active fiber contraction tensor $\mathbf{F}_\text{a}$. At this point, we do not include explicit dependence on $\mathbf{F}_\text{e}$, which is connected to $\mathbf{F}$ and $\mathbf{F}_\text{a}$ through (2.6), allowing for a concise expression of the constitutive equations. We also consider the free energy to explicitly depend on the muscle fiber direction, defined by referential unit vector $\mathbf{A}$. This allows $\Psi$ to be written in the form

$$\Psi = \Psi(\mathbf{F}, \mathbf{F}_\text{a}, \mathbf{A}). \tag{2.15}$$

Substituting (2.15) into (2.14) yields

$$\operatorname{tr}\left(\left(\mathbf{S} - \frac{\partial \Psi}{\partial \mathbf{F}} + p\mathbf{F}^{-1}\right)\mathbf{F}_{,t}\right) + \left(P_\text{a} - \operatorname{tr}\left(\frac{\partial \Psi}{\partial \mathbf{F}_\text{a}}\mathbf{F}_\text{a}'\right)\right)\lambda_{\text{a},t} = 0, \tag{2.16}$$

where $\mathbf{F}_\text{a}' = \partial \mathbf{F}_\text{a}/\partial \lambda_\text{a}$ and the incompressibility constraint (2.5) is enforced by introducing the Lagrange multiplier $p$. Equation (2.16) is satisfied for all values of $\mathbf{F}_{,t}$ by taking

$$\mathbf{S} = \frac{\partial \Psi}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \tag{2.17}$$

which provides the constitutive relation for the nominal stress. To clarify notation, we express the nominal stress in component form as

$$S_{\alpha i} = \frac{\partial \Psi}{\partial F_{i\alpha}} - pF_{\alpha i}^{-1}, \tag{2.18}$$

where the switched order of the indices with respect to the non-symmetric tensor $\mathbf{F}$ is noted, consistent with Ogden [40].

The second term of (2.16) is satisfied by taking $P_\text{a}$ as

$$P_\text{a} = \operatorname{tr}\left(\frac{\partial \Psi}{\partial \mathbf{F}_\text{a}}\mathbf{F}_\text{a}'\right), \tag{2.19}$$

which is interpreted in the following manner. Fiber shortening is the result of the thermodynamic force $T_\text{a}$, through $P_\text{a}$, resulting in a change in free energy $\operatorname{tr}((\partial \Psi/\partial \mathbf{F}_\text{a})\mathbf{F}_\text{a}')$, where we have neglected energy dissipation.

## 3 Incremental equations and stability

### 3.1 Incremental motions

In this section, we derive the equations governing incremental deformations and activation superimposed on a known finitely deformed configuration $\mathcal{B}_t$ in the presence of a known active contraction $\lambda_\text{a}$. Following Dorfmann and Ogden [46,47], we use a superposed dot to represent the increment in a variable. Thus, $\dot{\mathbf{x}}(\mathbf{X}, t)$ represents a time-dependent incremental displacement and $\dot{\lambda}_\mathrm{a}(\mathbf{X}, t)$ a time-dependent increment in the active fiber contraction. The corresponding increment of the deformation gradient is given by $\dot{\mathbf{F}} = \mathrm{Grad}\,(\dot{\mathbf{x}})$. Similarly, the increments of $\mathbf{S}$, $T_\mathrm{a}$, $P_\mathrm{a}$ are denoted by $\dot{\mathbf{S}}$, $\dot{T}_\mathrm{a}$, $\dot{P}_\mathrm{a}$, respectively.

We begin with the incremental forms of the balance equations (2.11) and (2.12), defined by

$$\mathrm{Div}\,\dot{\mathbf{S}} = \rho_\mathrm{r}\dot{\mathbf{x}}_{,tt}, \quad \dot{T}_\mathrm{a} = \dot{P}_\mathrm{a}. \tag{3.1}$$

The increment of the nominal stress (2.17) for an incompressible material is then given in component form as

$$\dot{S}_{\alpha i} = \mathcal{A}_{\alpha i \beta j}\dot{F}_{j\beta} + \Gamma_{\alpha i}\,\dot{\lambda}_\mathrm{a} - \overline{pF^{-1}_{\alpha i}}^{\,\boldsymbol{\cdot}}, \tag{3.2}$$

where we recall $\lambda_\mathrm{a}$ is a scalar. The components of the fourth-order and second-order tensors $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\Gamma}$ are given, respectively, by

$$\mathcal{A}_{\alpha i \beta j} = \frac{\partial^2 \Psi}{\partial F_{i\alpha}\partial F_{j\beta}} \tag{3.3}$$

and

$$\Gamma_{\alpha i} = \frac{\partial^2 \Psi}{\partial F_{i\alpha}\partial F_{\mathrm{a}_{B\beta}}} F'_{\mathrm{a}_{B\beta}}, \tag{3.4}$$

where the upper case Roman subscripts are associated with the components of the intermediate configuration $\mathcal{B}_\mathrm{a}$ of the active fibers. The fourth-order tensor possesses the symmetry

$$\mathcal{A}_{\alpha i \beta j} = \mathcal{A}_{\beta j \alpha i}, \tag{3.5}$$

and, in general, no equivalent symmetry exists for the second-order tensor $\boldsymbol{\Gamma}$. The increment of the term related to the Lagrange multiplier in Eq. (3.2) yields

$$\overline{pF^{-1}_{\alpha i}}^{\,\boldsymbol{\cdot}} = \dot{p}F^{-1}_{\alpha i} - pF^{-1}_{\alpha j}\dot{F}_{j\beta}F^{-1}_{\beta i}. \tag{3.6}$$

The increment of the internal thermodynamic force $P_\mathrm{a}$, in component form, is written as

$$\dot{P}_\mathrm{a} = \Gamma_{\alpha i}\dot{F}_{i\alpha} + \mathcal{L}\dot{\lambda}_\mathrm{a}, \tag{3.7}$$

where the scalar $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{\partial^2 \Psi}{\partial F_{\mathrm{a}_{B\beta}}\partial F_{\mathrm{a}_{D\gamma}}} F'_{\mathrm{a}_{B\beta}} F'_{\mathrm{a}_{D\gamma}} + \frac{\partial \Psi}{\partial F_{\mathrm{a}_{B\beta}}} F''_{\mathrm{a}_{B\beta}}, \tag{3.8}$$

and the notation $\mathbf{F}''_\mathrm{a} = \partial^2\mathbf{F}_\mathrm{a}/\partial\lambda^2_\mathrm{a}$ has been used.

Following the approach suggested by Abeyaratne and Knowles [37], Merodio and Ogden [30], and Dorfmann and Ogden [38,46], we assume that the underlying state is homogeneous and time independent. The increments of the equation of motion $(3.1)_1$ are then given in component form as

$$\mathcal{A}_{\alpha i \beta j}\dot{x}_{j,\alpha\beta} + \Gamma_{\alpha i}(\dot{\lambda}_\mathrm{a})_{,\alpha} - \check{p}_i = \rho_\mathrm{r}\dot{x}_{i,tt}, \tag{3.9}$$

where the Greek subscripts following the comma denote differentiation with respect to $\mathbf{X}$. Following the work by Merodio and Ogden [30], we also use the shorthand notation

$$\check{p}_i = (\dot{p}F^{-1}_{\alpha i} - pF^{-1}_{\alpha j}\dot{F}_{j\beta}F^{-1}_{\beta i})_{,\alpha}. \tag{3.10}$$

In what follows it is convenient to use a push forward operation on the increment field variables, i.e., change the reference configuration from $\mathcal{B}_\mathrm{r}$ to $\mathcal{B}_\mathrm{t}$, [38,46]. Using the relation $(2.1)_1$, we write the incremental displacement $\dot{\mathbf{x}}(\mathbf{X}, t)$ and the increment in the active contraction $\dot{\lambda}_\mathrm{a}(\mathbf{X}, t)$ as Eulerian quantities

$$\dot{\mathbf{x}}(\mathbf{X}, t) = \mathbf{u}(\boldsymbol{\chi}(\mathbf{X}), t) = \mathbf{u}(\mathbf{x}, t), \tag{3.11}$$

$$\dot{\lambda}_\mathrm{a}(\mathbf{X}, t) = \phi(\boldsymbol{\chi}(\mathbf{X}), t) = \phi(\mathbf{x}, t). \tag{3.12}$$

The push forward version of the incremental nominal stress $\dot{\mathbf{S}}$, for an incompressible material, has the form

$$\dot{\mathbf{S}}_0 = \mathbf{F}\dot{\mathbf{S}}, \tag{3.13}$$

where the subscript 0 is used to denote a Eulerian quantity. The updated version of the balance equation $(3.1)_1$ is

$$\operatorname{div} \dot{\mathbf{S}}_0 = \rho \mathbf{u}_{,tt}, \tag{3.14}$$

where div defines the divergence operator in the current configuration $\mathcal{B}_t$. In component form, Eq. (3.14) is expressed as

$$\mathcal{A}_{0_{ijkl}} u_{l,ik} + \Gamma_{0_{ij}} \phi_{,i} - \bar{p}_j = \rho_{\mathrm{r}} u_{j,tt}, \tag{3.15}$$

where the Roman subscripts following the comma refers to differentiation with respect to $\mathbf{x}$ and the subscripts $_{,tt}$ on the right-hand side denote the usual time derivatives. The incremental updated variable of the Lagrange multiplier, denoted by $\bar{p}_{,j}$ in Eq. (3.15), has the form

$$\bar{p}_j = (\dot{p}\delta_{ij} - p u_{i,j})_{,i}, \tag{3.16}$$

where $\mathbf{u}$ satisfies the incremental incompressibility condition

$$\operatorname{div} \mathbf{u} = 0. \tag{3.17}$$

The incremental balance equation $(3.1)_2$, using (3.7), can be recast in the updated form

$$\Gamma_{0_{ij}} u_{j,i} + \mathcal{L}\phi = \dot{T}_a. \tag{3.18}$$

The tensors $\mathcal{A}_{0_{ijkl}}$ and $\Gamma_{0_{ij}}$ are defined in index notation by

$$\mathcal{A}_{0_{ijkl}} = F_{i\alpha} F_{k\beta} \mathcal{A}_{\alpha j \beta l} \tag{3.19}$$

and

$$\Gamma_{0_{ij}} = F_{i\alpha} \Gamma_{\alpha j}, \tag{3.20}$$

which possess the symmetries

$$\mathcal{A}_{0_{ijkl}} = \mathcal{A}_{0_{klij}}, \qquad \Gamma_{0_{ij}} = \Gamma_{0_{ji}}. \tag{3.21}$$

## 3.2 Homogeneous plane waves

We now specify the forms of the incremental displacement and the incremental activation superimposed on a homogeneously deformed underlying configuration in the presence of uniform activation. In this case, the tensors $\boldsymbol{\mathcal{A}}_0$ and $\boldsymbol{\Gamma}_0$ are constant as are the corresponding components $\mathcal{A}_{0_{ijkl}}$ and $\Gamma_{0_{ij}}$, which depend on the underlying deformation gradient $\mathbf{F}$ and on the active contraction $\lambda_{\mathrm{a}}$. Specifically, we seek solutions to the incremental equations (3.15) and (3.18) of the form

$$\mathbf{u} = \mathbf{m}\, \mathrm{e}^{\mathrm{i}(k\mathbf{x}\cdot\mathbf{n} - \omega t)}, \tag{3.22}$$

$$\bar{p} = q\, \mathrm{e}^{\mathrm{i}(k\mathbf{x}\cdot\mathbf{n} - \omega t)}, \tag{3.23}$$

$$\phi = g\, \mathrm{e}^{\mathrm{i}(k\mathbf{x}\cdot\mathbf{n} - \omega t)}, \tag{3.24}$$

where $\mathbf{u}$ represents a plane wave with amplitude given by vector $\mathbf{m}$, propagation direction given by unit vector $\mathbf{n}$, wave number given by $k$ and wave speed given by $\omega$. The imaginary unit i is defined by $\mathrm{i}^2 = -1$. The scalar quantities $q$ and $g$ denote the amplitudes of the increments $\bar{p}$ and $\phi$, respectively. Substituting (3.22) into the incompressibility condition (3.17) gives the connection

$$\mathbf{m} \cdot \mathbf{n} = 0, \tag{3.25}$$

which constrains the vectors $\mathbf{m}$ and $\mathbf{n}$ to be orthogonal.

Here, we consider the case where the external thermodynamic force $T_a$ is independent of the motion $\boldsymbol{\chi}$ and of the active fiber contraction $\lambda_a$ and, therefore, $\dot{T}_a = 0$. This assumption is a generalization of the dead-loading boundary conditions considered in stability problems of purely elastic materials [40] and enables us to avoid undue mathematical complexity in the following formulations without being overly restrictive. Using the incremental forms (3.22), (3.23), and (3.24), we write the updated incremental balance equations as

$$k^2\mathbf{Q}(\mathbf{n})\mathbf{m} - \rho\,\omega^2\mathbf{m} + \mathrm{i}k[q\mathbf{n} - g\mathbf{R}(\mathbf{n})] = 0, \tag{3.26}$$

$$\mathrm{i}k\mathbf{R}(\mathbf{n})\cdot\mathbf{m} + \mathcal{L}g = 0, \tag{3.27}$$

where $\mathbf{Q}(\mathbf{n})$ is the mechanical acoustic tensor with components

$$Q_{jl}(\mathbf{n}) = \mathcal{A}_{0_{ijkl}} n_i n_k, \tag{3.28}$$

and the vector $\mathbf{R}(\mathbf{n})$ follows as

$$\mathbf{R}(\mathbf{n}) = \boldsymbol{\Gamma}_0\mathbf{n}. \tag{3.29}$$

Here, we assume $\mathcal{L} \neq 0$ to prevent Eqs. (3.26) and (3.27) from degenerating to the purely mechanical case. An explicit expression for $g$ is obtained by rewriting Eq. (3.27) in the form

$$g = -\mathrm{i}\frac{k}{\mathcal{L}}\mathbf{R}(\mathbf{n})\cdot\mathbf{m}, \tag{3.30}$$

which shows that the amplitude of the incremental change of the active fiber length, $\phi$, is zero unless accompanied by an increment in the mechanical displacement $\mathbf{u}$ (given by the vector $\mathbf{m}$). However, it does not follow, in general, that $\mathbf{u}$ vanishes in the absence of an incremental change in the active fiber stretch, see Eq. (3.26). Substituting (3.30) into (3.26) and contracting with $\mathbf{m}$ results in

$$\mathbf{m}\cdot\mathbf{H}(\mathbf{n})\mathbf{m} = \rho\left(\frac{\omega}{k}\right)^2, \tag{3.31}$$

where the real, symmetric tensor $\mathbf{H}(\mathbf{n})$ is given by

$$\mathbf{H}(\mathbf{n}) = \mathbf{Q}(\mathbf{n}) - \frac{1}{\mathcal{L}}\mathbf{R}(\mathbf{n})\otimes\mathbf{R}(\mathbf{n}), \tag{3.32}$$

and the incompressibility constraint (3.25) is used.

To ensure real wave speeds, or equivalently, for the eigenvalue $\rho_r(\omega/k)^2$ to be positive, we require that $\mathbf{H}(\mathbf{n})$ be positive-definite for all $\mathbf{m}$, which is enforced by the inequality

$$\mathbf{m}\cdot\mathbf{H}(\mathbf{n})\mathbf{m} > 0, \tag{3.33}$$

or, alternatively, may be expressed in terms of the components of $\boldsymbol{\mathcal{A}}_0$ and $\boldsymbol{\Gamma}_0$ as

$$\left(\mathcal{A}_{0_{ijkl}} - \frac{1}{\mathcal{L}}\Gamma_{0_{ij}}\Gamma_{0_{kl}}\right)n_i n_k m_j m_l > 0. \tag{3.34}$$

If the inequality (3.34) holds for a given $\mathbf{F}$, $\lambda_a$, and for all non-zero vectors $\mathbf{m}$ and $\mathbf{n}$, which satisfy the incompressibility constraint (3.25), then the current state is considered stable for a given energy function $\Psi$. Note, Eq. (3.34) is a generalization of the strong ellipticity condition for purely elastic material and tensor $\mathbf{H}(\mathbf{n})$ is the generalized acoustic tensor, here referred to as the active acoustic tensor. Unsurprisingly, the resulting criterion does not lead to a conclusive remark regarding the stability of active strain material models in general. As we will illustrate in Sect. 6, the loss of stability depends on the specific energy function and the underlying state.

The conditions for real wave speeds may alternatively be found by solving the eigenvalue problem (3.31) written as

$$\det\left(\mathbf{H}(\mathbf{n}) - \rho\left(\frac{\omega}{k}\right)^2\mathbf{I}\right) = 0. \tag{3.35}$$

For a 3D system, the three roots are required to be positive for the underlying configuration with active fiber stretch to be stable. In the following sections, we will utilize Eq. (3.34), which allows leveraging previous work related to the stability of passive fibers.

## 4 Invariant-based constitutive relation for active muscle tissue

The energy function $\Psi$ of an incompressible, transversely isotropic material with active fibers may be written in terms of invariants related to the deformation $I_1$, $I_2$, fiber direction $I_4$, $I_5$, active fiber contraction $I_{4_a}$, $I_{5_a}$, and deformation of the cross-bridges $I_{4_e}$, $I_{5_e}$. Accordingly, Eq. (2.15) is replaced with

$$\Psi = \Psi(I_1, I_2, I_4, I_5, I_{4_a}, I_{5_a}, I_{4_e}, I_{5_e}), \tag{4.1}$$

where we note that not all invariants are independent, but are included for a convenient representation of the cross-bridge deformation. The invariants $I_1$ and $I_2$ are defined in terms of $\mathbf{C}$ (equivalently $\mathbf{B}$) by

$$I_1 = \operatorname{tr}(\mathbf{C}), \qquad I_2 = \frac{1}{2}\Big[\operatorname{tr}^2(\mathbf{C}) - \operatorname{tr}(\mathbf{C}^2)\Big]. \tag{4.2}$$

Furthermore, the inclusion of a preferred direction $\mathbf{A}$ gives rise to the invariants

$$I_4 = \mathbf{A}\cdot\mathbf{C}\mathbf{A}, \quad I_5 = \mathbf{A}\cdot\mathbf{C}^2\mathbf{A}, \tag{4.3}$$

and the invariants related to the active fiber stretch are

$$I_{4_a} = \mathbf{A}\cdot\mathbf{C}_a\mathbf{A}, \quad I_{5_a} = \mathbf{A}\cdot(\mathbf{C}_a^2\mathbf{A}), \tag{4.4}$$

where $\mathbf{C}_a = \mathbf{F}_a^T\mathbf{F}_a$. The square root of $I_4$ provides the stretch of the material in the fiber direction, and similarly, the square root of $I_{4_a}$ provides the contraction of the muscle fibers during activation.

The multiplicative decomposition of the deformation tensor (2.6) gives rise to invariants related to the cross-bridge deformation

$$I_{4_e} = \mathbf{A}_e\cdot\mathbf{C}_e\mathbf{A}_e = I_4 I_{4_a}^{-1}, \tag{4.5}$$

$$I_{5_e} = \mathbf{A}_e\cdot\mathbf{C}_e^2\mathbf{A}_e, \tag{4.6}$$

where the unit vector $\mathbf{A}_e$ specifies the fiber direction in $\mathcal{B}_a$ and is given by $\mathbf{A}_e = I_{4_a}^{-1/2}\mathbf{F}_a\mathbf{A}$. Similar to $\mathbf{C}$ and $\mathbf{C}_a$, we define $\mathbf{C}_e = \mathbf{F}_e^T\mathbf{F}_e = \mathbf{F}_a^{-T}\mathbf{C}\mathbf{F}_a^{-1}$. Equation (4.5) shows that the dependence of $\Psi$ on $I_{4_e}$ can, in principle, be replaced by $I_4$ and $I_{4_a}$.

The first derivatives of $I_1$, $I_2$, $I_4$, and $I_5$ with respect to $\mathbf{F}$ are common in the literature and are not repeated here. However, derivatives of those invariants related to the active fiber contraction are non-standard and are repeated from Paetsch et al. [4] here as

$$\frac{\partial I_{4_e}}{\partial\mathbf{F}} = 2I_{4_a}^{-1}\mathbf{A}\otimes\mathbf{F}\mathbf{A}, \tag{4.7}$$

$$\frac{\partial I_{5_e}}{\partial\mathbf{F}} = 2I_{4_a}^{-1}(\mathbf{A}\otimes\mathbf{F}\mathbf{C}_a^{-1}\mathbf{C}\mathbf{A} + \mathbf{C}_a^{-1}\mathbf{C}\mathbf{A}\otimes\mathbf{F}\mathbf{A}), \tag{4.8}$$

and $I_{4_a}$ and $I_{5_a}$ are independent of $\mathbf{F}$. Using Eq. (2.17), combined with the standard transformation law $\boldsymbol{\sigma} = \mathbf{F}\mathbf{S}$, gives the explicit expression of the Cauchy stress for incompressible active muscle as

$$\begin{aligned}\boldsymbol{\sigma} &= 2(\Psi_1 + I_1\Psi_2)\mathbf{B} - 2\Psi_2\mathbf{B}^2 + 2\Psi_4\mathbf{a}\otimes\mathbf{a} + 2\Psi_5(\mathbf{a}\otimes\mathbf{B}\mathbf{a} + \mathbf{B}\mathbf{a}\otimes\mathbf{a}) + 2I_{4_a}^{-1}\Psi_{4_e}\mathbf{a}\otimes\mathbf{a}\\ &\quad + 2I_{4_a}^{-1}\Psi_{5_e}(\mathbf{a}\otimes\mathbf{B}_e\mathbf{a} + \mathbf{B}_e\mathbf{a}\otimes\mathbf{a}) - p\mathbf{I},\end{aligned} \tag{4.9}$$

where the vector $\mathbf{a} = \mathbf{F}\mathbf{A}$ denotes the fiber direction in the current configuration $\mathcal{B}_t$ and $\mathbf{B}_e = \mathbf{F}_e\mathbf{F}_e^T$. We also used the shorthand notations $\Psi_i = \partial\Psi/\partial I_i$, $i \in \{1, 2, 4, 5\}$, $\Psi_{4_e} = \partial\Psi/\partial I_{4_e}$, $\Psi_{5_e} = \partial\Psi/\partial I_{5_e}$. Expressions for the tensors and scalar quantities in Eq. (3.34) are lengthy, but straightforward. Therefore, an algorithmic expression is provided in place of the explicit form. To that end, we introduce the following notations:

$$\begin{aligned}&J_1 = I_1, \qquad J_2 = I_2, \qquad J_3 = I_4, \qquad J_4 = I_5,\\ &J_5 = I_{4_e}, \qquad J_6 = I_{5_e}, \qquad J_7 = I_{4_a}, \qquad J_8 = I_{5_a},\end{aligned}$$

and the free energy in terms of $J_i$, $i \in \{1, \dots, 8\}$ is denoted as $\Omega = \Omega(J_1, \dots, J_8)$. The components of $\boldsymbol{\mathcal{A}}_0$, $\boldsymbol{\Gamma}$, and $\mathcal{L}$ can now compactly be written as

$$\mathcal{A}_{0_{ijkl}} = F_{i\alpha} F_{k\beta} \left( \sum_{m=1}^{6} \sum_{n=1}^{6} \Omega_{mn} \frac{\partial J_m}{\partial F_{j\alpha}} \frac{\partial J_n}{\partial F_{l\beta}} + \sum_{m=1}^{6} \Omega_m \frac{\partial^2 J_m}{\partial F_{j\alpha} \partial F_{l\beta}} \right), \tag{4.10}$$

$$\Gamma_{0_{ij}} = F_{i\alpha} F'_{\mathrm{a}B\beta} \left( \sum_{m=1}^{6} \sum_{n=5}^{8} \Omega_{mn} \frac{\partial J_m}{\partial F_{j\alpha}} \frac{\partial J_n}{\partial F_{\mathrm{a}B\beta}} + \sum_{m=5}^{6} \Omega_m \frac{\partial^2 J_m}{\partial F_{j\alpha} \partial F_{\mathrm{a}B\beta}} \right), \tag{4.11}$$

$$\mathcal{L} = F'_{\mathrm{a}B\beta} F'_{\mathrm{a}D\gamma} \left( \sum_{m=5}^{8} \sum_{n=5}^{8} \Omega_{mn} \frac{\partial J_m}{\partial F_{\mathrm{a}B\beta}} \frac{\partial J_n}{\partial F_{\mathrm{a}D\gamma}} + \sum_{m=5}^{8} \Omega_m \frac{\partial^2 J_m}{\partial F_{\mathrm{a}B\beta} \partial F_{\mathrm{a}D\gamma}} \right) + F''_{\mathrm{a}B\beta} \sum_{m=5}^{8} \Omega_m \frac{\partial J_m}{\partial F_{\mathrm{a}B\beta}}, \tag{4.12}$$

where $\Omega_m = \partial\Omega/\partial J_m$, $\Omega_{mn} = \partial^2\Omega/\partial J_m \partial J_n$. Expressions for the non-standard first and second derivatives of the invariants with respect to $\mathbf{F}$ and $\mathbf{F}_\mathrm{a}$ are provided in Appendix 1 for reference.

## 5 Two-dimensional motion and activation

In this section, we specialize the equations from Sect. 4 to those governing two-dimensional motion and accompanying activation. Specifically, we focus on a homogeneous underlying deformation and uniform activation restricted to the $(X_1, X_2)$ plane. Thus, the incremental displacement vector $\mathbf{u}$ and the incremental activation strain $\phi$ depend only on $x_1$ and $x_2$, and the out-of-plane component vanishes, $u_3 = 0$. In terms of principal stretches, the out-of-plane stretch is $\lambda_3 = 1$ and the incompressibility constraint becomes $\lambda_1\lambda_2 = 1$. Furthermore, we take the fibers to lie in the plane of deformation and, assuming the principle directions of $\mathbf{F}$ and $\mathbf{F}_\mathrm{a}$ coincide with the Cartesian coordinate system, we write

$$\mathbf{F} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{5.1}$$

$$\mathbf{F}_\mathrm{a} = \lambda_{\mathrm{a}_1} \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_{\mathrm{a}_2} \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{5.2}$$

where $\lambda_{\mathrm{a}_1}$ and $\lambda_{\mathrm{a}_2}$ satisfy the constraint $\lambda_{\mathrm{a}_1}\lambda_{\mathrm{a}_2} = 1$. We note that, while plane motions of an incompressible material may be described in terms of a single variable, kinematic quantities in this section are provided in more general terms of $\lambda_{\mathrm{a}_1}$ and $\lambda_{\mathrm{a}_2}$. Therefore, the invariants defined by (4.2) and (4.3), for the underlying plane strain deformation, become

$$I_1 = I_2 = 1 + \lambda_1^2 + \lambda_2^2 \tag{5.3}$$

and

$$I_4 = A_1^2\lambda_1^2 + A_2^2\lambda_2^2 = a_1^2 + a_2^2, \qquad I_5 = (I_1 - 1)I_4 - 1, \tag{5.4}$$

where $A_\alpha$ and $a_i$, with $\alpha, i \in \{1, 2\}$, are the components of $\mathbf{A}$ and $\mathbf{a}$, respectively. The invariants related to the active fiber stretch given by (4.4), (4.5), and (4.6) have the forms

$$\begin{aligned}
I_{4_\mathrm{a}} &= A_1^2\lambda_{\mathrm{a}_1}^2 + A_2^2\lambda_{\mathrm{a}_2}^2 = a_{\mathrm{a}_1}^2 + a_{\mathrm{a}_2}^2, \\
I_{5_\mathrm{a}} &= A_1^2\lambda_{\mathrm{a}_1}^4 + A_2^2\lambda_{\mathrm{a}_2}^4 = a_{\mathrm{a}_1}^2\lambda_{\mathrm{a}_1}^2 + a_{\mathrm{a}_2}^2\lambda_{\mathrm{a}_2}^2, \\
I_{4_\mathrm{e}} &= I_4 I_{4_\mathrm{a}}^{-1}, \\
I_{5_\mathrm{e}} &= I_{4_\mathrm{a}}^{-1}(A_1^2\lambda_1^4\lambda_{\mathrm{a}_1}^{-2} + A_2^2\lambda_2^4\lambda_{\mathrm{a}_2}^{-2}) = I_{4_\mathrm{a}}^{-1}(a_1^2\lambda_1^2\lambda_{\mathrm{a}_1}^{-2} + a_2^2\lambda_2^2\lambda_{\mathrm{a}_2}^{-2}),
\end{aligned} \tag{5.5}$$

where $a_{\mathrm{a}_i}$, $i \in \{1, 2\}$, are the components of the vector $\mathbf{a}_\mathrm{a} = \mathbf{F}_\mathrm{a}\mathbf{A}$. Because of the specialization, it can be seen from (5.3), (5.4), and (5.5) that there remain only six independent variables. This suggests a reduced form of $\Psi$, denoted by $\hat{\Psi}$, to be defined as

$$\hat{\Psi} = \hat{\Psi}(I_1, I_4, I_{4_\mathrm{e}}, I_{5_\mathrm{e}}, I_{4_\mathrm{a}}, I_{5_\mathrm{a}}), \tag{5.6}$$

and the Cauchy stress given by (4.9) reduces to

$$\boldsymbol{\sigma} = 2\hat{\Psi}_1\mathbf{B} + 2\hat{\Psi}_4\mathbf{a}\otimes\mathbf{a} + 2I_{4_a}^{-1}\hat{\Psi}_{4_e}\mathbf{a}\otimes\mathbf{a} + 2I_{4_a}^{-1}\hat{\Psi}_{5_e}(\mathbf{a}\otimes\mathbf{B}_e\mathbf{a} + \mathbf{B}_e\mathbf{a}\otimes\mathbf{a}) - p\mathbf{I}. \tag{5.7}$$

Equivalently, the energy formulation $\Omega$ defined in terms of $J_i$, $i \in \{1,\dots,8\}$, is replaced by $\hat{\Omega} = \hat{\Omega}(J_1, J_3, J_5, J_6, J_7, J_8)$ and Eqs. (4.10), (4.11), and (4.12) still hold. Unlike in the purely mechanical case, constraining deformations and active contractions to occur in a plane does not greatly reduce the number of terms in the stability criterion. Therefore, we further specialize $\hat{\Psi}$ based on a generalization of constitutive models found in the literature in order to provide concise, yet widely applicable stability criteria.

Following the simplification in Holzapfel et al. [48], Holzapfel and Ogden [49] and Lin et al. [50], we neglect the dependence of $\hat{\Psi}$ on $I_{5_e}$ and $I_{5_a}$ and consider a decoupled form

$$\hat{\Psi} = \hat{\Psi}_{\text{iso}}(I_1) + \hat{\Psi}_{\text{fib}}(I_4, I_{4_e}, I_{4_a}), \tag{5.8}$$

where $\hat{\Psi}_{\text{iso}}$ provides the contribution of the isotropic connective tissue and $\hat{\Psi}_{\text{fib}}$ gives the passive and active fiber responses. This additively decomposed formulation, similar to the decoupled free energy presented by Paetsch et al. [4], Hernández-Gascón et al. [17], and Sharifimajd and Stålhand [45], significantly reduces the number of terms in (4.10), (4.11), and (4.12). Given the free energy (5.8), the Cauchy stress in (5.7) simplifies to

$$\boldsymbol{\sigma} = 2\hat{\Psi}_1\mathbf{B} + 2(\hat{\Psi}_4 + I_{4_a}^{-1}\hat{\Psi}_{4_e})\mathbf{a}\otimes\mathbf{a} - p\mathbf{I}, \tag{5.9}$$

where $\hat{\Psi}_1 = \partial\hat{\Psi}_{\text{iso}}/\partial I_1$, $\hat{\Psi}_4 = \partial\hat{\Psi}_{\text{fib}}/\partial I_4$ and $\hat{\Psi}_{4_e} = \partial\hat{\Psi}_{\text{fib}}/\partial I_{4_e}$. We also provide the explicit expressions of the fourth- and second-order tensor components shown in the stability criterion (3.34) as

$$\mathcal{A}_{0_{ijkl}} = 2\Big[2\hat{\Psi}_{11}B_{ij}B_{lk} + \hat{\Psi}_1 B_{ik}\delta_{jl} + 2(\hat{\Psi}_{44} + 2I_{4_a}^{-1}\hat{\Psi}_{44_e} + I_{4_a}^{-2}\hat{\Psi}_{4_e4_e})a_ia_ja_ka_l + (\hat{\Psi}_4 + I_{4_a}^{-1}\hat{\Psi}_{4_e})a_ia_k\delta_{jk}\Big] \tag{5.10}$$

and

$$\Gamma_{0_{ij}} = 4\Big[\hat{\Psi}_{44_a} + I_{4_a}^{-1}\hat{\Psi}_{4_e4_a} - I_{4_a}^{-2}(\hat{\Psi}_{4_e} + I_4\hat{\Psi}_{44_e} + I_4 I_{4_a}^{-1}\hat{\Psi}_{4_e4_e})\Big]F'_{a_{k\alpha}}A_\alpha a_{a_k}a_ia_j, \tag{5.11}$$

where $\hat{\Psi}_{11} = \partial^2\hat{\Psi}_{\text{iso}}/\partial I_1^2$, $\hat{\Psi}_{44} = \partial^2\hat{\Psi}_{\text{fib}}/\partial I_4^2$, $\hat{\Psi}_{44_e} = \partial^2\hat{\Psi}_{\text{fib}}/\partial I_4\partial I_{4_e}$, $\hat{\Psi}_{4_e4_e} = \partial^2\hat{\Psi}_{\text{fib}}/\partial I_{4_e}^2$, $\hat{\Psi}_{4_e4_a} = \partial^2\hat{\Psi}_{\text{fib}}/\partial I_{4_e}\partial I_{4_a}$, and $\hat{\Psi}_{44_a} = \partial^2\hat{\Psi}_{\text{fib}}/\partial I_4\partial I_{4_a}$. Equation (5.10) is of a similar form to that presented by Merodio and Ogden [30] for fiber-reinforced elastic materials. Furthermore, from Eq. (4.12), we express the scalar $\mathcal{L}$ as

$$\begin{aligned}\mathcal{L} = {} & 4\Big(I_4^2 I_{4_a}^{-4}\hat{\Psi}_{4_e4_e} + 2I_4 I_{4_a}^{-3}\hat{\Psi}_{4_e} - 2I_4 I_{4_a}^{-2}\hat{\Psi}_{4_e4_a} + \hat{\Psi}_{4_a4_a}\Big)(\mathbf{F}'_a\mathbf{A}\cdot\mathbf{a}_a)^2 \\ & + 2\Big(\hat{\Psi}_{4_a} - I_4 I_{4_a}^{-2}\hat{\Psi}_{4_e}\Big)\big[(\mathbf{F}'_a\mathbf{A}\cdot\mathbf{F}'_a\mathbf{A}) + \mathbf{F}''_a\mathbf{A}\cdot\mathbf{a}_a\big],\end{aligned} \tag{5.12}$$

where we have used the notation $\hat{\Psi}_{4_a} = \partial\hat{\Psi}_{\text{fib}}/\partial I_{4_a}$ and $\hat{\Psi}_{4_a4_a} = \partial^2\hat{\Psi}/\partial I_{4_a}\partial I_{4_a}$.

The plane strain specialization of the incompressibility condition (3.17) becomes

$$u_{1,1} + u_{2,2} = 0, \tag{5.13}$$

where $u_1, u_2$ are the components of the incremental displacement vector $\mathbf{u}$ defined in (3.11) and the subscript following a comma indicates partial derivative with respect to $x_i$, $i \in \{1, 2\}$. From (3.22) and (3.25), we deduce that

$$m_1 = n_2, \quad m_2 = -n_1, \tag{5.14}$$

where $m_i, n_i$, $i \in \{1, 2\}$, are the Cartesian components of the vectors $\mathbf{m}$ and $\mathbf{n}$, respectively. The first term of the stability criterion (3.34) can now be written as

$$\mathcal{A}_{0_{ijkl}}n_in_km_jm_l = 4\hat{\Psi}_{11}(\lambda_1^2 - \lambda_2^2)^2 n_1^2 n_2^2 + 2\hat{\Psi}_1(\lambda_1^2 n_1^2 + \lambda_2^2 n_2^2) + 2(n_1a_1 + n_2a_2)^2\Big[H_1(n_2a_1 - n_1a_2)^2 + H_2\Big], \tag{5.15}$$

where the coefficients $H_1$ and $H_2$ are

$$H_1 = 2(\hat{\Psi}_{44} + 2I_{4_a}^{-1}\hat{\Psi}_{44_e} + I_{4_a}^{-2}\hat{\Psi}_{4_e4_e}), \tag{5.16}$$

$$H_2 = \hat{\Psi}_4 + I_{4_a}^{-1}\hat{\Psi}_{4_e}. \tag{5.17}$$

The second term of (3.34) becomes

$$\frac{1}{\mathcal{L}}\Gamma_{0_{ji}}\Gamma_{0_{kl}}n_i n_k m_j m_l = 4\frac{G^2}{\mathcal{L}}(\mathbf{F}_a'\mathbf{A}\cdot\mathbf{a}_a)^2(n_2a_1 - n_1a_2)^2(n_1a_1 + n_2a_2)^2, \tag{5.18}$$

where $G$ has the form

$$G = 2\Big[\hat{\Psi}_{44_a} + I_{4_a}^{-1}\hat{\Psi}_{4_e4_a} - I_{4_a}^{-2}(\hat{\Psi}_{4_e} + I_4\hat{\Psi}_{44_e} + I_4 I_{4_a}^{-1}\hat{\Psi}_{4_e4_e})\Big]. \tag{5.19}$$

We now restrict our attention to the stability analysis of the muscle fibers. Thus, we assume that the isotropic contribution to the energy satisfies the strong ellipticity condition and, from (5.15), it follows that

$$4\hat{\Psi}_{11}(\lambda_1^2 - \lambda_2^2)^2 n_1^2 n_2^2 + 2\hat{\Psi}_1(\lambda_1^2 n_1^2 + \lambda_2^2 n_2^2) > 0. \tag{5.20}$$

Then, the stability requirement (3.34), using (5.15), (5.18), and (5.20), reduces to

$$(\mathbf{a}\cdot\mathbf{n})^2\left[\left(H_1 - 2\frac{G^2}{\mathcal{L}}(\mathbf{F}_a'\mathbf{A}\cdot\mathbf{a}_a)^2\right)(\mathbf{a}\times\mathbf{n})^2 + H_2\right] \geq 0, \tag{5.21}$$

and must hold for all $\mathbf{n} \neq 0$. Note, we have relaxed the strict inequality as the condition (5.20) is assumed to hold. Equation (5.21) is satisfied when the fiber orientation in the current configuration is orthogonal to the wave propagation direction, i.e., $\mathbf{a}\cdot\mathbf{n} = 0$, and the inequality simplifies to

$$\left(H_1 - 2\frac{G^2}{\mathcal{L}}(\mathbf{F}_a'\mathbf{A}\cdot\mathbf{a}_a)^2\right)(\mathbf{a}\times\mathbf{n})^2 + H_2 \geq 0. \tag{5.22}$$

Condition (5.22) suggests the following three possibilities. First, when $\mathbf{a}$ and $\mathbf{n}$ are parallel, the leading term vanishes and the stability condition is simply $H_2 \geq 0$. Second, assuming $H_2$ is non-negative, the condition (5.22) holds for $\mathbf{a}\times\mathbf{n} \neq 0$ given

$$H_1 - 2\frac{G^2}{\mathcal{L}}(\mathbf{F}_a'\mathbf{A}\cdot\mathbf{a}_a)^2 \geq 0, \tag{5.23}$$

although this is not a necessary condition. For the third case, we assume (5.23) does not hold and, following Merodio and Ogden [30], we eliminate the dependence on the arbitrary vector $\mathbf{n}$ by observing

$$\left(H_1 - 2\frac{G^2}{\mathcal{L}}(\mathbf{F}_a'\mathbf{A}\cdot\mathbf{a}_a)^2\right)(\mathbf{a}\times\mathbf{n})^2 \geq \left(H_1 - 2\frac{G^2}{\mathcal{L}}(\mathbf{F}_a'\mathbf{A}\cdot\mathbf{a}_a)^2\right)I_4. \tag{5.24}$$

The sufficient conditions for material stability of active fibers become

$$H_2 \geq 0, \qquad \left(H_1 - 2\frac{G^2}{\mathcal{L}}(\mathbf{F}_a'\mathbf{A}\cdot\mathbf{a}_a)^2\right)I_4 + H_2 \geq 0, \tag{5.25}$$

along with the assumption that the isotropic base satisfies (5.20).

## 6 Illustrative example

### 6.1 Specialize underlying deformation

In many systems, both natural and engineered, muscles perform as linear actuators subject to uniform extension, in the fiber direction, with lateral contraction. Understanding the onset of instabilities under these conditions is, therefore, of particular interest. Thus, for isochoric deformation, the deformation gradient becomes

$$\mathbf{F} = \lambda\,\mathbf{e}_1\otimes\mathbf{e}_1 + \lambda^{-1}\mathbf{e}_2\otimes\mathbf{e}_2 + \mathbf{e}_3\otimes\mathbf{e}_3, \tag{6.1}$$

and assume that the fiber orientation $\mathbf{A}$ coincides with the principal direction $\mathbf{e}_1$. The tensor describing the active fiber contraction, $\mathbf{F}_a$, follows as

$$\mathbf{F}_a = \lambda_a \, \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_a^{-1} \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{6.2}$$

and the relevant invariants can be expressed as

$$I_1 = \lambda^2 + \lambda^{-2} + 1, \quad I_4 = \lambda^2, \quad I_{4_e} = \lambda^2 \lambda_a^{-2}, \quad I_{4_a} = \lambda_a^2. \tag{6.3}$$

The corresponding principal stresses are

$$\sigma_1 = \sigma = \sigma_{\text{iso}} + \sigma_{\text{fib}}, \quad \sigma_2 = \sigma_3 \equiv 0, \tag{6.4}$$

where the Cauchy stress component in the direction of uniaxial tension is given by an isotropic and a fiber contribution. From (5.8) and (5.9), we find

$$\sigma_{\text{fib}} = 2(\hat{\Psi}_4 + \lambda_a^{-2} \hat{\Psi}_{4_e}) \lambda^2. \tag{6.5}$$

Given the deformation described by (6.1), inequality $(5.25)_1$ implies that muscle fibers cannot sustain a compressive stress, i.e., $\sigma_{\text{fib}} > 0$. This requirement mirrors material instability in fiber-reinforced nonlinear elastic solids, see Merodio and Ogden [33]. Given the specific forms of $\mathbf{A}$ and $\mathbf{F}_a$, we can now express the scalar coefficient $\mathcal{L}$ as

$$\mathcal{L} = 2\lambda^2 \lambda_a^{-2} \Big( 3\lambda_a^{-2} \hat{\Psi}_{4_e} + 2\lambda^2 \lambda_a^{-4} \hat{\Psi}_{4_e 4_e} - 4 \hat{\Psi}_{4_e 4_a} \Big) + 2 \Big( \hat{\Psi}_{4_a} + 2\lambda_a^2 \hat{\Psi}_{4_a 4_a} \Big). \tag{6.6}$$

For the given underlying deformation and active fiber contraction, the stability criterion (5.25) can now be written as

$$H_2 \geq 0, \qquad H_2 + H_1 \lambda^2 - 2 \frac{G^2}{\mathcal{L}} \lambda^2 \lambda_a^2 \geq 0. \tag{6.7}$$

## 6.2 Prototype energy function

For purpose of illustration, we now specialize the decoupled form of $\hat{\Psi}$ given in (5.8). In our present study, we are primarily concerned with fiber contribution to the energy. However, for completeness, we give the energy formulation associated with the isotopic connective tissue as

$$\hat{\Psi}_{\text{iso}} = \frac{\mu}{2}(I_1 - 3), \tag{6.8}$$

which is known as the neo-Hookean material for which $\hat{\Psi}_1 = \mu/2$ and $\hat{\Psi}_{11} = 0$. The condition stated in (5.20) requires that $\mu > 0$.

Inequality $(6.7)_2$ provides some immediate insight for selecting a specific form of the fiber contribution $\hat{\Psi}_{\text{fib}}$. If, for example, we consider a formulation $\hat{\Psi}_{\text{fib}}$ that depends on $I_{4_e}$ only, then the stability criterion (6.7) reduces to

$$F' \geq 0, \qquad 3F' + 2I_{4_e} F'' \leq 0, \tag{6.9}$$

where $\hat{\Psi}_{\text{fib}}(I_{4_e}) = F(I_{4_e})$ and the prime indicates differentiation with respect to $I_{4_e}$. The fiber contribution to the Cauchy stress given by (6.5) and its derivative are

$$\sigma_{\text{fib}} = 2I_{4_e} F', \qquad \sigma'_{\text{fib}} = 2F' + 2I_{4_e} F'', \tag{6.10}$$

and Eq. (6.9) can equivalently be written as

$$\sigma_{\text{fib}} \geq 0, \qquad 2I_{4_e} \sigma'_{\text{fib}} + \sigma_{\text{fib}} \leq 0. \tag{6.11}$$

From $(6.11)_2$, it follows that the derivative of $\sigma_{\text{fib}}$ with respect to $I_{4_e}$ must be negative, i.e., $\sigma'_{\text{fib}} < 0$. Thus, a type of model $\hat{\Psi}_{\text{fib}} = F(I_{4_e})$ is not sufficient.

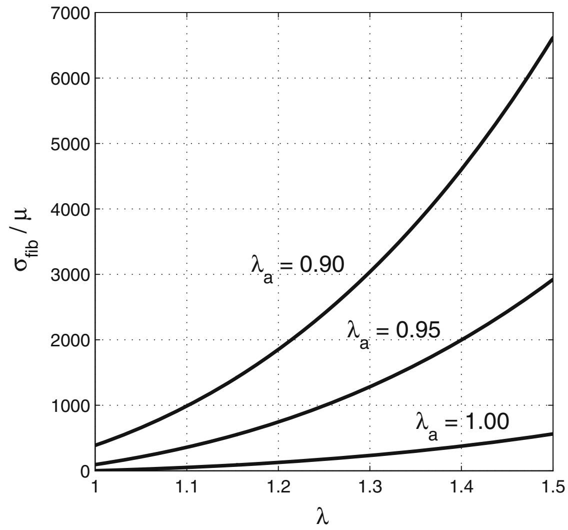



**Fig. 1** (Case 1) The non-dimensionalized fiber contribution to the Cauchy stress, given $\mu_\mathrm{a}$ in Eq. (6.15), for stretch in the fiber direction and different values of active contraction. The material parameters taken are $\mu_\mathrm{p} = 100$, $c_1 = 3{,}000$ and $\lambda_a$ varies as indicated

Consider an alternative formulation of the type $\hat{\Psi}_\mathrm{fib} = \hat{\Psi}_\mathrm{fib}(I_{4_\mathrm{e}}, I_{4_\mathrm{a}})$. Specifically, following the standard reinforcing model, we assume that

$$\hat{\Psi}_\mathrm{fib} = \frac{\mu}{2}\left[\mu_\mathrm{fib}(I_{4_\mathrm{e}} - 1)^2 + \alpha(I_{4_\mathrm{a}} - 1)^2\right], \tag{6.12}$$

where the second term accounts for the change in active contraction that does not result in a change in stress and, as we will see in Sect. 6.4, directly influences the material stability. Additionally, we have introduced the dimensionless material constant $\alpha$. The first term characterizes the change in stress due to mechanical deformation and the active contraction through the dimensionless parameter $\mu_\mathrm{fib}$, which is given by

$$\mu_\mathrm{fib} = \mu_\mathrm{p} + \mu_\mathrm{a}, \tag{6.13}$$

where the parameters $\mu_\mathrm{p}$ and $\mu_\mathrm{a}$ are related to the passive and active fiber contributions, respectively. For the function (6.12), using (6.5), the expression of the fiber contribution to the stress is calculated as follows:

$$\sigma_\mathrm{fib} = \mu\lambda^2\lambda_\mathrm{a}^{-2}\left[\frac{\partial \mu_\mathrm{fib}}{\partial I_{4_\mathrm{e}}}(\lambda^2\lambda_\mathrm{a}^{-2} - 1)^2 + 2\mu_\mathrm{fib}(\lambda^2\lambda_\mathrm{a}^{-2} - 1)\right]. \tag{6.14}$$

Next, we consider three alternative expressions of $\mu_\mathrm{a}$ to illustrate the implications of the derived stability conditions.

### *6.2.1 Case 1*

We take $\mu_\mathrm{a}$ to be given by

$$\mu_\mathrm{a} = c_1(1 - I_{4_\mathrm{a}}), \tag{6.15}$$

where $c_1 > 0$ is a dimensionless parameter. With no activation, $I_{4_\mathrm{a}} = 1$, and thus $\mu_\mathrm{a} = 0$. To illustrate the influence of muscle activation, we provide the dimensionless response curves for various values of $\lambda_\mathrm{a}$ in Fig. 1. We note that in the passive case, the muscle fibers are stress-free in the reference configuration $\mathcal{B}_\mathrm{r}$. However, when activated, $\lambda_\mathrm{a} < 1$ and the stress $\sigma_\mathrm{fib} > 0$ at the reference length. Furthermore, the activation results in a stiffer fiber response.

### *6.2.2 Case 2*

We now augment the form of $\mu_\mathrm{a}$ to include the influence of the actomyosin overlap. For recent studies which include dependence on muscle fiber overlap, see Böl et al. [44], Ehret et al. [51], Grasa et al. [11], Hernández-Gascón et

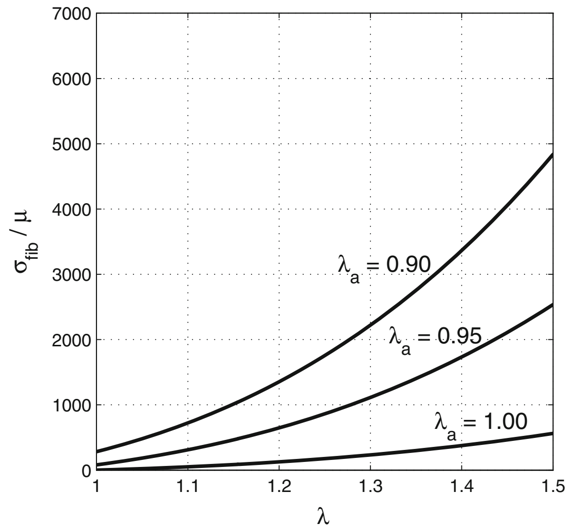



**Fig. 2** (Case 2) The non-dimensionalized fiber contribution to the Cauchy stress, given $\mu_\mathrm{a}$ in Eq. (6.16), for stretch in the fiber direction and different values of active contraction. The material parameters taken are $\mu_\mathrm{p} = 100$, $c_1 = 3{,}000$, $c_2 = 0.5$ and $\lambda_a$ varies as indicated

al. [17], Murtada et al. [13], and Sharifimajd and Stålhand [45], Stålhand et al. [25]. Consider $\mu_\mathrm{a}$ to include an exponential overlap parameter dependent on $I_{4_\mathrm{a}}$

$$\mu_\mathrm{a} = c_1(1 - I_{4_\mathrm{a}})\mathrm{e}^{(I_{4\mathrm{a}}-1)/c_2}, \tag{6.16}$$

where $c_2 \neq 0$ is a dimensionless parameter. The corresponding stress-deformation response, depicted in Fig. 2, shows that the behavior is similar to the previous case. Compared to Fig. 1, however, the magnitude of the response is scaled by introducing the exponential term in (6.16).

### *6.2.3 Case 3*

Paetsch et al. [4] recently proposed an alternative formulation of $\mu_\mathrm{a}$ that accounts for the change in fiber stiffness due to increasing activation and deformation. This is accounted for by specifying the dependence of $\mu_\mathrm{a}$ on $I_{4_\mathrm{a}}$ and $I_{4_\mathrm{e}}$, respectively. We consider the form

$$\mu_\mathrm{a} = c_1(1 - I_{4_\mathrm{a}})\mathrm{e}^{-(I_{4\mathrm{e}}-1)/c_2}, \tag{6.17}$$

where again $c_2 \neq 0$ is a dimensionless parameter. When the muscle fiber is in the passive state, (6.17) recovers $\mu_\mathrm{a} = 0$ and we note the inclusion of the minus sign in the exponent with respect to $I_{4_\mathrm{e}}$. Unlike the formulation (6.16), with constant activation, the magnitude of $\mu_\mathrm{a}$ decreases with increasing deformation, which can be seen from the dimensionless stress-stretch response in Fig. 3. For an experimental verification of (6.17), we refer to [52].

## 6.3 Stability condition given by Eq. $(6.7)_1$

In this section, we examine the stability condition $(6.7)_1$ of active muscle fibers, given (6.1) and (6.2), and bounds on the material properties are explored. Merodio and Ogden [33,34] derive a connection between the stress-deformation response and the stability of fiber-reinforced materials. A corresponding relation for active materials is not possible here, as the stress response is independent of the second term appearing in (6.12), as can be seen from (6.14), but appears in the stability requirement through $\mathcal{L}$, see (9.3).

For the considered deformation, using the formulation of $\hat{\Psi}_\mathrm{fib}$ given by (6.12), we write Eq. (5.17) as

$$H_2 = \mu\lambda_\mathrm{a}^{-2}(\lambda^2\lambda_\mathrm{a}^{-2} - 1)\left[\frac{1}{2}\frac{\partial\mu_\mathrm{fib}}{\partial I_{4_\mathrm{e}}}(\lambda^2\lambda_\mathrm{a}^{-2} - 1) + \mu_\mathrm{fib}\right], \tag{6.18}$$

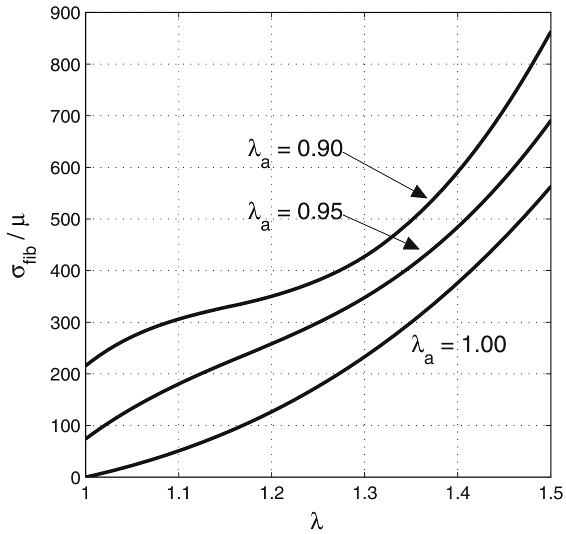



**Fig. 3** (Case 3) The non-dimensionalized fiber contribution to the Cauchy stress, given $\mu_a$ in Eq. (6.17), for stretch in the fiber direction and different values of active contraction. The material parameters taken are $\mu_p = 100$, $c_1 = 3{,}000$, $c_2 = 0.5$ and $\lambda_a$ varies as indicated

where $\partial\mu_{fib}/\partial I_{4_e}$ depends on the particular expression of $\mu_a$. For passive fibers in the undeformed configuration, we have

$$H_2 = 0, \qquad \lambda = 1, \qquad \lambda_a = 1, \tag{6.19}$$

and condition $(6.7)_1$ holds.

We now show that in the deformed state, an instability arises when the fiber stretch is less than the active contraction. Cases 1 and 2 yield $\partial\mu_{fib}/\partial I_{4_e} = 0$, while Case 3 gives $\partial\mu_{fib}/\partial I_{4_e} = -\mu_a/c_2$. As it will be shown, $\mu_p \geq 0$, thus $\mu_{fib} \geq 0$ for all cases and we have

$$H_2 < 0 \qquad \text{for} \qquad \lambda < \lambda_a, \tag{6.20}$$

which corresponds to $\sigma_{fib} < 0$. This, as noted by $(6.11)_1$, is not permitted if the material response is to remain stable.

For fiber extension, $\lambda > \lambda_a$, the stability condition $H_2 \geq 0$ reduces to

$$f(\lambda, \lambda_a) = \frac{1}{2}\frac{\partial\mu_{fib}}{\partial I_{4_e}}(\lambda^2\lambda_a^{-2} - 1) + \mu_p + \mu_a \geq 0. \tag{6.21}$$

For Cases 1 and 2, the above expression becomes $f(\lambda, \lambda_a) = \mu_p + \mu_a$. The minimum value for $\mu_a$ in both cases is zero for the admissible range of $0 < \lambda_a \leq 1$, and a sufficient condition to satisfy the stability condition with fibers in extension is

$$\mu_p \geq 0. \tag{6.22}$$

For Case 3, the condition $(6.7)_1$ has the form

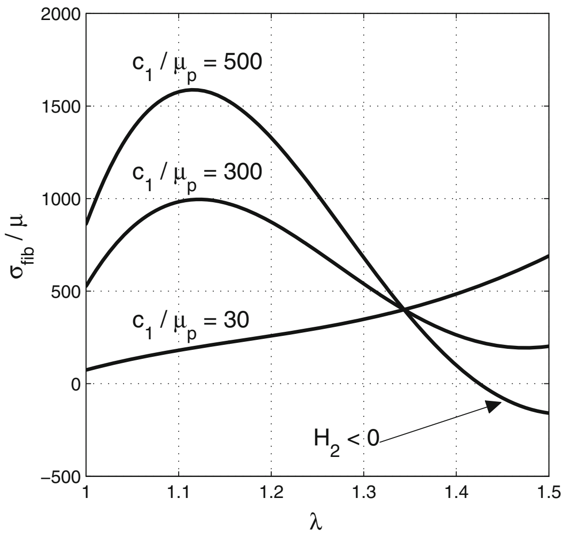
$$f(\lambda, \lambda_a) = \mu_p + \mu_a\left[1 - \frac{1}{2c_2}(\lambda^2\lambda_a^{-2} - 1)\right] \geq 0. \tag{6.23}$$

For the passive state $\mu_a = 0$, $f(\lambda, 1) = \mu_p$ and (6.23) requires $\mu_p \geq 0$. With active fibers, Case 3 does not readily yield a restriction on the material parameters as we have for Cases 1 and 2. However, with some effort, we can find *sufficient* stability conditions for $\lambda > \lambda_a$ that provide bounds on the material properties.

A sufficient restriction for the ratio of $c_1$ to $\mu_p$ can be obtained by noting that the minimum value of a function $\exp(-\xi)(1 - 1/2\,\xi)$ is $-1/2\exp(3) \approx -0.025$, and (6.23) becomes $c_1(1 - \lambda_a^2)/\mu_p \leq 40$. We can eliminate the dependence on the active contraction by requiring the stricter inequality:

$$\frac{c_1}{\mu_p} \leq 40. \tag{6.24}$$

**Fig. 4** The fiber contribution to the stress, non-dimensionalized by $\mu$, for an active stretch of $\lambda_a = 0.95$ and the material parameters taken are $\mu_p = 100$, and $c_2 = 0.5$ and $c_1$ varies as indicated. For sufficiently large ratios of $c_1/\mu_p$, $\sigma_{fib} < 0$ and stability is lost in accordance with $(6.7)_1$




Equation (6.24) stipulates that instabilities with respect to $H_2 \geq 0$ associated with fiber overlap in extension can be eliminated for a sufficiently small ratio of active fiber to passive fiber stiffness. While this ratio of parameters ensures stability for all $\lambda$ and $\lambda_a$ such $\lambda > \lambda_a$, it is not a necessary condition. This point may be illustrated by considering the stress-stretch response for different values of $c_1/\mu_p$.

Figure 4 gives the non-dimensional fiber contribution to the stress, $\sigma_{fib}/\mu$, for various ratios of $c_1$ to $\mu_p$ plotted against the stretch for an active contraction of $\lambda_a = 0.95$ and a fixed value of $\mu_p = 100$. Given $\sigma_{fib}$ in Eq. (6.5) and the expression for $H_2$ of (5.17), $\sigma_{fib} = 2H_2\lambda^2$ and $H_2 < 0$ corresponds to $\sigma_{fib} < 0$. In Fig. 4, we observe $\sigma_{fib} > 0$ with $c_1/\mu_p = 30$ for all $\lambda$. Additionally, for $c_1/\mu_p = 300$, $\sigma_{fib} > 0$ for all $\lambda$. However, it is noted that $\sigma_{fib} > 0$ does not hold for all $\lambda_a$ with $c_1/\mu_p = 300$. Finally, for $c_1/\mu_p = 500$, stability is lost as the stress goes to zero for a sufficiently large stretch.

In the next section, we select values of $\mu_p$ and $c_1$ such that (6.24) holds, so that $H_2 \geq 0$ for all $\lambda > \lambda_a$.

## 6.4 Stability related to Eq. $(6.7)_2$

In this section, we separately consider the formulations of $\mu_a$, given by (6.15), (6.16), and (6.17), and determine the corresponding critical values of the extension $\lambda$ and the active contraction $\lambda_a$ that violate the stability condition $(6.7)_2$. These are denoted by $\lambda_{crit}$ and $\lambda_{a,crit}$. We again focus on the fiber contribution (6.12) and evaluate the response for selected values of model parameters. The explicit expressions of the coefficients required to evaluate the stability condition $(6.7)_2$ are shown in Appendix 2.

We evaluate the stability criterion $(6.7)_2$ using the energy (6.12) and the Case 1 expression of $\mu_a$ given by (6.15). We consider the fixed values of $\mu_p = 100$ and $c_1 = 3{,}000$. The corresponding results for selected values of $\alpha = 3c_1$, $6c_1$, and $9c_1$ are shown in Fig. 5. For clarity of presentation, we do not consider fiber compression and limit the extension to $0 \leq \lambda \leq 1.5$. Starting from the reference length $\lambda = 1$, for fibers in the passive state ($\lambda_a = 1$), an increase in the magnitude of $\alpha$ has a stabilizing effect, i.e., for increasing values of $\alpha$, fibers become unstable at large values of $\lambda$. Figure 5 also shows that, for each value of $\alpha$, fiber activation further stabilizes the response when $0.95 < \lambda_a < 1$. For $\lambda_a < 0.95$, fiber activation has a destabilizing influence. The graph corresponding to $\alpha = 3c_1$, for example, shows that the amount of stretch $\lambda$ required to induce instability monotonically reduces with increased activation. For $\lambda_a < 0.78$, the configuration becomes unstable for all values of $\lambda$. The critical curves corresponding to $\alpha = 6c_1$ and $\alpha = 9c_1$ show that increasing values of $\alpha$ clearly enhances the stability of the response.

**Fig. 5** (Case 1) Critical stretches and critical active contractions for loss of stability given in (6.15). Material parameters $\mu_p = 100$ and $c_1 = 3{,}000$ are fixed while $\alpha = 3c_1,\ 6c_1,\ 9c_1$

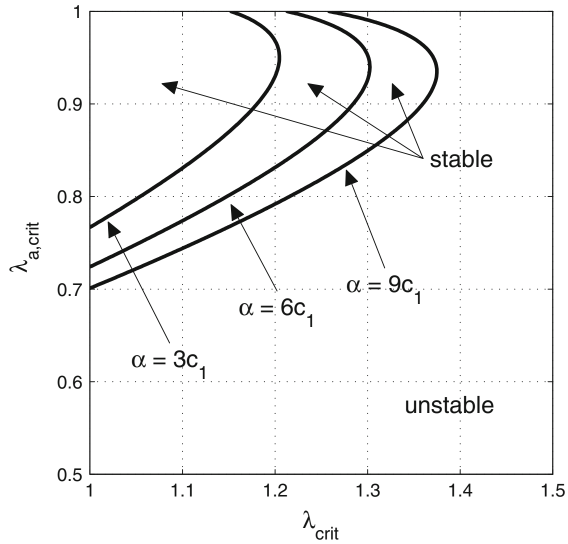
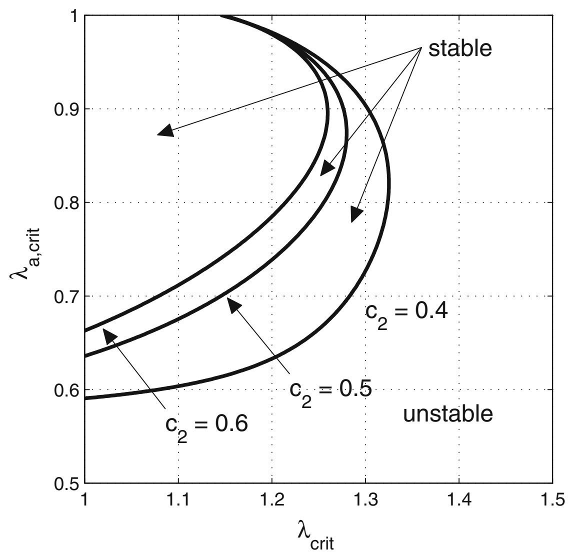



**Fig. 6** (Case 2) Critical stretches and critical active contractions for loss of stability given in (6.16). Material parameters $\mu_p = 100$, $c_1 = 3{,}000$ and $\alpha = 3c_1$ are fixed while $c_2 = 0.4,\ 0.5,\ 0.6$




Next, we study the stability of the energy (6.12) using the expression of the active fiber contribution $\mu_a$ given by (6.16). We again consider the fixed values of $\mu_p = 100$ and $c_1 = 3{,}000$ and restrict attention to $\alpha = 3c_1$. To evaluate the effect of the exponential term on the overall response, we select values of $c_2 = 0.4, 0.5$, and 0.6 and show the corresponding results in Fig. 6. The critical curves for the values of $c_2 = 0.4, 0.5$, and 0.6 are similar to the results shown in Fig. 5. The passive fiber becomes unstable for values of $\lambda$ greater than approximately 1.15. Active contraction has a very similar effect as discussed in the previous case. Figure 6 further shows that an increasing value of $c_2$ has a destabilizing influence when $\lambda_a < 1$. For example, $\lambda_a = 0.8$ the critical stretch becomes smaller with increasing values of $c_2$.

In the final illustration, we use the Case 3 expression of $\mu_a$ given by (6.17) to assess the stability of the energy function (6.12) by comparing with the results shown in Figs. 5 and 6. The material parameters considered are again $\mu_p = 100$, $c_1 = 3{,}000$ and $\alpha = 3c_1$. The critical curves for selected values of $c_2 = 0.4, 0.5$, and 0.6 are shown in Fig. 7a and b, and the results are quite different from the other cases of $\mu_a$ considered. Loss of stability occurs in the passive case for a stretch of 4.9, which is much larger when compared with the other cases, similar to the

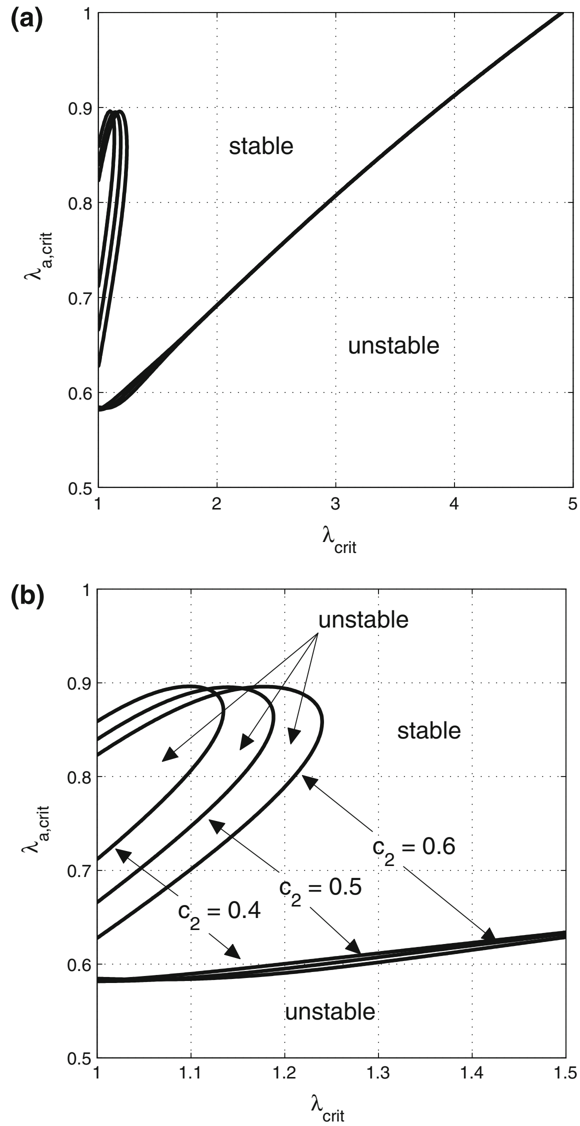



**Fig. 7** (Case 3) Critical stretches and critical active contractions for loss of stability given in (6.17) with **a** maximum stretch of 5, **b** maximum stretch of 1.5. Material parameters $\mu_p = 100$, $c_1 = 3{,}000$ and $\alpha = 3c_1$ are fixed while $c_2 = 0.4,\ 0.5,\ 0.6$

difference in the stiffness observed in the stress response for a given set of material parameters, see Figs. 2 and 3. We observe in Fig. 7(b) an additional instability mode for stretches less than 1.25 and values of $\lambda_a$ larger than 0.6. For example, for the stretch $\lambda = 1.1$, the passive state is a stable configuration. However, for the case of $c_2 = 0.4$, the fibers become unstable when $\lambda_a$ takes a value less than 0.9. It can be shown that this additional instability mode arises due to the inclusion of a minus sign in the exponential overlap parameter with respect to the invariant.

## 7 Concluding remarks

Driven by recent interests in modeling muscle tissue using an active strain approach, we have developed stability criteria based on a generalization of the strong ellipticity condition of purely elastic materials. We analyze homogeneous plane waves in a homogeneously deformed body subjected to a uniform active contraction. Material stability is based on waves propagating with real speed and is enforced by requiring the active acoustic tensor to be positive-definite.

The restriction on the generalized acoustic tensor does not lead to a conclusive statement regarding the stability of the active strain approach, rather, stability depends on the specific form of the material model and the underlying configuration. We specialize the stability criteria using a decoupled energy function and focus on two-dimensional motion. Attention is restricted to the fiber contribution by assuming that the isotropic part to the overall energy satisfies the stability requirements. Specifically, we show that the fiber energy contribution cannot solely depend on the elastic deformation of the muscle cross-bridges.

To illustrate implications of the derived criteria, we introduce three forms of the fiber energy function. For Cases 1 and 2, a sufficient condition to satisfy the first stability condition is $\lambda > \lambda_a$ and the passive fiber stiffness $\mu_p \geq 0$. For Case 3, a requirement for stability is that the ratio of active to passive fiber stiffnesses $c_1/\mu_p \leq 40$. Instabilities then occur for sufficiently large deformations and active contractions, based on the selected material parameters. The critical values of $\lambda$ and $\lambda_a$ are highly dependent on the parameter $\alpha$, which scales the non-stress generating term of the energy function. Additionally, we find that the inclusion of a fiber overlap function (Case 3) has a destabilizing effect and gives rise to a failure mode not observed in the other two cases.

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## Appendix 1: Derivatives related to invariants

We provide the derivatives of the invariants introduced in this study related to the deformation gradient and the active strain tensor. The second derivatives with respect to the deformation tensor are, in component form, provided as

$$\frac{\partial^2 I_1}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}\delta_{\alpha\beta}, \tag{8.1}$$

$$\frac{\partial^2 I_2}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\big(2F_{j\beta}F_{i\alpha} - F_{j\alpha}F_{i\beta} + I_1\delta_{ij}\delta_{\alpha\beta} - B_{ij}\delta_{\alpha\beta} - C_{\alpha\beta}\delta_{ij}\big), \tag{8.2}$$

$$\frac{\partial^2 I_4}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}A_\alpha A_\beta, \tag{8.3}$$

$$\frac{\partial^2 I_5}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\big(\delta_{ij}A_\alpha C_{\beta\eta}A_\eta + A_\alpha F_{i\beta}F_{j\eta}A_\eta + A_\alpha B_{ij}A_\beta + \delta_{\alpha\beta}F_{i\eta}A_\eta F_{j\gamma}A_\gamma$$
$$+ A_\beta F_{j\alpha}F_{i\eta}A_\eta + \delta_{ij}A_\beta C_{\alpha\gamma}A_\gamma\big), \tag{8.4}$$

$$\frac{\partial^2 I_{4_e}}{\partial F_{i\alpha}\partial F_{j\beta}} = 2I_{4_a}^{-1}\delta_{ij}A_\alpha A_\beta, \tag{8.5}$$

$$\frac{\partial^2 I_{5_e}}{\partial F_{i\alpha}\partial F_{j\beta}} = 2I_{4_a}^{-1}\big(\delta_{ij}A_\alpha C_{a_{\beta\eta}}^{-1}C_{\eta\phi}A_\phi + A_\alpha F_{i\gamma}C_{a_{\gamma\beta}}^{-1}F_{j\phi}A_\phi + A_\alpha F_{i\gamma}C_{a_{\gamma\eta}}^{-1}F_{j\eta}A_\beta + C_{a_{\alpha\beta}}^{-1}F_{j\eta}A_\eta F_{i\phi}A_\phi$$
$$+ C_{a_{\alpha\gamma}}^{-1}F_{j\gamma}A_\beta F_{i\phi}A_\phi + \delta_{ij}C_{a_{\alpha\gamma}}^{-1}C_{\gamma\eta}A_\eta A_\beta\big). \tag{8.6}$$

Additionally, we provide the derivatives of the invariants related to the active stretch:

$$\frac{\partial I_{4_e}}{\partial F_{a_{B\beta}}} = -2I_4 I_{4_a}^{-2}A_\beta F_{a_{B\eta}}A_\eta, \tag{8.7}$$

$$\frac{\partial I_{5_e}}{\partial F_{a_{B\beta}}} = -2\big(I_{4_a}^{-2}A_\beta F_{a_{B\gamma}}A_\gamma A_\phi C_{\phi\eta}C_{a_{\eta\mu}}^{-1}C_{\mu\alpha}A_\alpha + I_{4_a}^{-1}C_{a_{\beta\pi}}^{-1}C_{\pi\eta}A_\eta A_\gamma C_{\gamma\psi}F_{a_{\psi B}}^{-1}\big), \tag{8.8}$$

$$\frac{\partial I_{4_a}}{\partial F_{a_{B\beta}}} = 2A_\beta F_{a_{B\eta}}A_\eta, \tag{8.9}$$

$$\frac{\partial I_{5_a}}{\partial F_{a_{B\beta}}} = 2\big(A_\beta F_{a_{B\phi}}C_{a_{\phi\eta}}A_\eta + C_{a_{\beta\eta}}A_\eta F_{a_{B\gamma}}A_\gamma\big). \tag{8.10}$$

Next, the mixed derivatives with respect to $\mathbf{F}$ and $\mathbf{F}_{\mathrm{a}}$ are written in component form

$$\frac{\partial^2 I_{4_\mathrm{e}}}{\partial F_{i\alpha}\partial F_{\mathrm{a}_{B\beta}}} = -4 I_{4_\mathrm{a}}^{-2} A_\alpha F_{i\gamma} A_\gamma A_\beta F_{\mathrm{a}_{B\eta}} A_\eta, \tag{8.11}$$

$$\frac{\partial^2 I_{5_\mathrm{e}}}{\partial F_{i\alpha}\partial F_{\mathrm{a}_{B\beta}}} = -2\Big[2 I_{4_\mathrm{a}}^{-2}\big(A_\alpha F_{i\gamma} C^{-1}_{\mathrm{a}_{\gamma\eta}} C_{\eta\phi} A_\phi + C^{-1}_{\mathrm{a}_{\alpha\gamma}} C_{\gamma\eta} A_\eta F_{i\phi} A_\phi\big) A_\beta F_{\mathrm{a}_{B\mu}} A_\mu + I_{4_\mathrm{a}}^{-1}\big(A_\alpha F_{i\gamma} C^{-1}_{\mathrm{a}_{\gamma\beta}} A_\phi C_{\phi\eta} F^{-1}_{\mathrm{a}_{\eta B}} + A_\alpha F_{i\gamma} F^{-1}_{\mathrm{a}_{\gamma B}} C^{-1}_{\mathrm{a}_{\beta\eta}} C_{\eta\phi} A_\phi + F^{-1}_{\mathrm{a}_{\alpha B}} F_{i\phi} A_\phi C^{-1}_{\mathrm{a}_{\beta\gamma}} C_{\gamma\eta} A_\eta + C^{-1}_{\mathrm{a}_{\alpha\beta}} F_{i\phi} A_\phi A_\eta C_{\eta\gamma} F^{-1}_{\mathrm{a}_{\gamma B}}\big)\Big]. \tag{8.12}$$

The second derivatives with respect to $\mathbf{F}_{\mathrm{a}}$ of the relevant invariants are given:

$$\frac{\partial^2 I_{4_\mathrm{e}}}{\partial F_{\mathrm{a}_{B\beta}}\partial F_{\mathrm{a}_{D\gamma}}} = 2 I_4 I_{4_\mathrm{a}}^{-2} A_\beta A_\gamma (4 I_{4_\mathrm{a}}^{-1} F_{\mathrm{a}_{B\phi}} A_\phi F_{\mathrm{a}_{D\eta}} A_\eta - \delta_{BD}), \tag{8.13}$$

$$\frac{\partial^2 I_{5_\mathrm{e}}}{\partial F_{\mathrm{a}_{B\beta}}\partial F_{\mathrm{a}_{D\gamma}}} = 2\Big[I_{4_\mathrm{a}}^{-2}\Big(A_\phi C_{\phi\eta} C^{-1}_{\mathrm{a}_{\eta\mu}} C_{\mu\alpha} A_\alpha (4 I_{4_\mathrm{a}}^{-1} A_\beta F_{\mathrm{a}_{B\phi}} A_\phi A_\gamma F_{\mathrm{a}_{D\mu}} A_\mu - \delta_{BD} A_\beta A_\gamma) + 2 C^{-1}_{\mathrm{a}_{\beta\lambda}} C_{\lambda\mu} A_\mu A_\phi C_{\phi\psi} F^{-1}_{\mathrm{a}_{\psi B}} A_\gamma F_{\mathrm{a}_{D\rho}} A_\rho + 2 A_\beta F_{\mathrm{a}_{B\theta}} A_\theta C^{-1}_{\mathrm{a}_{\gamma\mu}} C_{\mu\alpha} A_\alpha A_\phi C_{\phi\eta} F^{-1}_{\mathrm{a}_{\eta D}}\Big) + I_{4_\mathrm{a}}^{-1}\Big(C_{\lambda\phi} A_\phi A_\mu C_{\mu\psi} F^{-1}_{\mathrm{a}_{\psi D}} (C^{-1}_{\mathrm{a}_{\beta\gamma}} F^{-1}_{\mathrm{a}_{\lambda B}} + C^{-1}_{\mathrm{a}_{\beta\lambda}} F^{-1}_{\mathrm{a}_{\gamma B}}) + F^{-1}_{\mathrm{a}_{\beta D}} A_\phi C_{\phi\lambda} F^{-1}_{\mathrm{a}_{\lambda B}} C^{-1}_{\mathrm{a}_{\gamma\psi}} C_{\psi\mu} A_\mu\Big)\Big], \tag{8.14}$$

$$\frac{\partial^2 I_{4_\mathrm{a}}}{\partial F_{\mathrm{a}_{B\beta}}\partial F_{\mathrm{a}_{D\gamma}}} = 2\delta_{BD} A_\beta A_\gamma, \tag{8.15}$$

$$\frac{\partial^2 I_{5_\mathrm{a}}}{\partial F_{\mathrm{a}_{B\beta}}\partial F_{\mathrm{a}_{D\gamma}}} = 2\Big[A_\beta \delta_{BD} C_{\mathrm{a}_{\gamma\eta}} A_\eta + A_\beta F_{\mathrm{a}_{B\gamma}} F_{\mathrm{a}_{D\eta}} A_\eta + A_\beta F_{\mathrm{a}_{B\phi}} A_\gamma F_{\mathrm{a}_{D\phi}} + \delta_{\beta\gamma} F_{\mathrm{a}_{B\phi}} A_\phi F_{\mathrm{a}_{D\eta}} A_\eta + F_{\mathrm{a}_{D\beta}} F_{\mathrm{a}_{B\phi}} A_\phi A_\gamma + C_{\mathrm{a}_{\beta\eta}} A_\eta \delta_{BD} A_\gamma\Big]. \tag{8.16}$$

## Appendix 2: Coefficients related to stability

The coefficients appearing in (6.7) given (6.12) are

$$H_1 = \mu\lambda_\mathrm{a}^{-4}\left[\frac{\partial^2 \mu_{\mathrm{fib}}}{\partial I_{4_\mathrm{e}}^2}(\lambda^2\lambda_\mathrm{a}^{-2}-1)^2 + 4\frac{\partial \mu_{\mathrm{fib}}}{\partial I_{4_\mathrm{e}}}(\lambda^2\lambda_\mathrm{a}^{-2}-1) + 2\mu_{\mathrm{fib}}\right], \tag{9.1}$$

$$G = \mu\lambda_\mathrm{a}^{-2}(\lambda^2\lambda_\mathrm{a}^{-2}-1)\left[\frac{\partial^2 \mu_{\mathrm{fib}}}{\partial I_{4_\mathrm{e}}\partial I_{4_\mathrm{a}}}(\lambda^2\lambda_\mathrm{a}^{-2}-1) - \frac{\partial^2 \mu_{\mathrm{fib}}}{\partial I_{4_\mathrm{e}}^2}\lambda^2\lambda_\mathrm{a}^{-4}(\lambda^2\lambda_\mathrm{a}^{-2}-1) + 2\frac{\partial \mu_{\mathrm{fib}}}{\partial I_{4_\mathrm{a}}} - \frac{\partial \mu_{\mathrm{fib}}}{\partial I_{4_\mathrm{e}}}\lambda_\mathrm{a}^{-2}(5\lambda^2\lambda_\mathrm{a}^{-2}-1) - 2\mu_{\mathrm{fib}}\lambda_\mathrm{a}^{-2}\right] - 2\mu\mu_{\mathrm{fib}}\lambda^2\lambda_\mathrm{a}^{-6}, \tag{9.2}$$

$$\mathcal{L} = 2\mu\lambda^2\lambda_\mathrm{a}^{-2}\left\{\lambda_\mathrm{a}^{-2}(\lambda^2\lambda_\mathrm{a}^{-2}-1)\left[\frac{\partial^2 \mu_{\mathrm{fib}}}{\partial I_{4_\mathrm{e}}^2}\lambda^2\lambda_\mathrm{a}^{-2}(\lambda^2\lambda_\mathrm{a}^{-2}-1) + \frac{\partial \mu_{\mathrm{fib}}}{\partial I_{4_\mathrm{e}}}\left(\frac{11}{2}\lambda^2\lambda_\mathrm{a}^{-2} - \frac{3}{2}\right) + 3\mu_{\mathrm{fib}} - 2\lambda_\mathrm{a}^2\left(\frac{\partial^2 \mu_{\mathrm{fib}}}{\partial I_{4_\mathrm{e}}\partial I_{4_\mathrm{a}}}(\lambda^2\lambda_\mathrm{a}^{-2}-1) + 2\frac{\mu_{\mathrm{fib}}}{\partial I_{4_\mathrm{a}}}\right)\right] + 2\mu_{\mathrm{fib}}\lambda^2\lambda_\mathrm{a}^{-4}\right\} + \mu\left[(\lambda^2\lambda_\mathrm{a}^{-2}-1)^2\left(2\frac{\partial^2 \mu_{\mathrm{fib}}}{\partial I_{4_\mathrm{a}}^2}\lambda_\mathrm{a}^2 + \frac{\partial \mu_{\mathrm{fib}}}{\partial I_{4_\mathrm{a}}}\right) + 2\alpha(3\lambda_\mathrm{a}^2-1)\right], \tag{9.3}$$

where $H_2$ is given by (6.18). Next, we provide the derivatives related to $\mu_{\mathrm{fib}}$ for each case of $\mu_\mathrm{a}$.

Derivatives for Case 1

For $\mu_\mathrm{a} = c_1(1 - I_{4_\mathrm{a}})$, we have

$$\frac{\partial \mu_\mathrm{fib}}{\partial I_{4_\mathrm{a}}} = -c_1, \tag{9.4}$$

where all other derivatives are zero.

Derivatives for Case 2

For $\mu_\mathrm{a} = c_1(1 - I_{4_\mathrm{a}})\exp((I_{4_\mathrm{a}} - 1)/c_2)$ we have

$$\frac{\partial \mu_\mathrm{fib}}{\partial I_{4_\mathrm{a}}} = c_1\left[\frac{1}{c_2}(1 - I_{4_\mathrm{a}}) - 1\right]\mathrm{e}^{(I_{4_\mathrm{a}}-1)/c_2}, \tag{9.5}$$

$$\frac{\partial^2 \mu_\mathrm{fib}}{\partial I_{4_\mathrm{a}}^2} = \frac{c_1}{c_2}\left[\frac{1}{c_2}(1 - I_{4_\mathrm{a}}) - 2\right]\mathrm{e}^{(I_{4_\mathrm{a}}-1)/c_2}, \tag{9.6}$$

with omitted derivatives equal to zero.

Derivatives for Case 3

For $\mu_\mathrm{a} = c_1(1 - I_{4_\mathrm{a}})\exp(-(I_{4_\mathrm{e}} - 1)/c_2)$, we have

$$\frac{\partial \mu_\mathrm{fib}}{\partial I_{4_\mathrm{e}}} = -\frac{\mu_\mathrm{a}}{c_2}, \tag{9.7}$$

$$\frac{\partial \mu_\mathrm{fib}}{\partial I_{4_\mathrm{a}}} = -c_1\mathrm{e}^{-(I_{4_\mathrm{e}}-1)/c_2}, \tag{9.8}$$

$$\frac{\partial^2 \mu_\mathrm{fib}}{\partial I_{4_\mathrm{e}}^2} = \frac{\mu_\mathrm{a}}{c_2^2}, \tag{9.9}$$

$$\frac{\partial^2 \mu_\mathrm{fib}}{\partial I_{4_\mathrm{e}}\partial I_{4_\mathrm{a}}} = \frac{c_1}{c_2}\mathrm{e}^{-(I_{4_\mathrm{e}}-1)/c_2}, \tag{9.10}$$

along with $\partial^2\mu_\mathrm{fib}/\partial I_{4_\mathrm{a}}^2 = 0$.

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