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On the chaos control of the Qi system

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Abstract This paper is devoted to the problem of controlling chaos in the Qi system. A time-delayed feedback control method is applied to suppress chaos to unstable equilibria or unstable periodic orbits. Using a local stability analysis, we theoretically prove that the Hopf bifurcation occurs. Some numerical simulations are carried out to support the theoretical predictions. Finally, main conclusions are drawn.

Keywords Chaos · Hopf bifurcation · Qi system · Stability · Time-delayed feedback

1 Introduction

Chaotic systems play a critical role in numerous fields such as, for example, information processing, secure communications, and high-performance circuit design for telecommunications [1]. Chaos is a very attractive subject from a theoretical point of view; however, it is quite challenging technically [2]. Over the last decade, many techniques have been proposed to control chaos, and many excellent results have been reported [3–27]. In 2005, Qi et al. [28] investigated the complex dynamical behaviors (e.g., familiar period-doubling route to chaos, Hopf bifurcation) of the following Qi system:

$$
\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + x_2x_3x_4,\\
\dot{x}_2 &= b(x_1 + x_2) - x_1x_3x_4,\\
\dot{x}_3 &= -cx_3 + x_1x_2x_4,\\
\dot{x}_4 &= -dx_4 + x_1x_2x_3,
\end{aligned}
\tag{1}
$$

where x_1, x_2, x_3, and x_4 are the state variables of the system and a, b, c, and d are all positive real constants. Interestingly, system (2) can generate chaotic phenomena (Fig. 1) given the system parameters $a = 30, b = 10, c = 1$, and $d = 10$.

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Since chaos can cause irregular behaviors that are sometimes undesirable in practical systems, in many cases, it is preferable to avoid or eliminate them. Control mechanisms that enable a chaotic system to achieve and maintain a desired dynamical behavior have potential applications in various disciplines [29]. In 2009, Niah and Sunday [29] investigated the chaos control of system (2) by applying a recursive backstepping nonlinear controller. The aim of this paper is to investigate the dynamics of a four-dimensional (4D) chaotic Qi system by considering the effect of delayed feedback. Analyzing the characteristic equation of a linearized system of the Qi model, we theoretically prove that under some suitable conditions, a Hopf bifurcation will occur. Numerical results support theoretical predictions.

2 Controlling chaos via feedback control methods

In this section, we shall apply a conventional feedback method to the dynamical system (2). Our aim is to drag the chaotic trajectories to the equilibria or the periodic orbits. To reflect the dynamical behaviors of the model depending on past information, it is reasonable to incorporate a time delay into this system. The signal error of the current and past states of the continuous time system will be given as feedback to the system itself. Following the idea of Pyragas [30], we consider two cases.

Case I Add the time-delayed force $k_1[x_2 - x_2(t-\tau_1)]$ to the second equation of system (2). In this case, system (2) takes the form

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + x_2x_3x_4,\\
\dot{x}_2 &= b(x_1 + x_2) - x_1x_3x_4 + k_1[x_2(t) - x_2(t-\tau_1)],\\
\dot{x}_3 &= -cx_3 + x_1x_2x_4,\\
\dot{x}_4 &= -dx_4 + x_1x_2x_3.
\end{aligned} \tag{2}$$

Case II Add the time-delayed forces $k_2[x_2(t) - x_2(t-\tau_2)]$ and $k_3[x_3(t) - x_3(t-\tau_2)]$ to the second and third equations of system (2), respectively. In this case, system (2) becomes

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + x_2x_3x_4,\\
\dot{x}_2 &= b(x_1 + x_2) - x_1x_3x_4 + k_2[x_2(t) - x_2(t-\tau_2)],\\
\dot{x}_3 &= -cx_3 + x_1x_2x_4 + k_3[x_3(t) - x_3(t-\tau_2)],\\
\dot{x}_4 &= -dx_4 + x_1x_2x_3.
\end{aligned} \tag{3}$$

Let $E(x_1^*, x_2^*, x_3^*, x_4^*)$ be the equilibrium of systems (2) and (3).

Case 1 Delayed feedback on the first equation [system (2)]:
The linearized system of Eq. (2) around $E(x_1^*, x_2^*, x_3^*, x_4^*)$ is given by

$$\begin{aligned}
\dot{x}_1 &= -ax_1 + (a + x_3^*x_4^*)x_2 + x_2^*x_4^*x_3 + x_2^*x_3^*x_4,\\
\dot{x}_2 &= (b + x_3^*x_4^*)x_1 + k_1x_2 + x_1^*x_4^*x_3 + x_1^*x_3^*x_4 - k_1x_2(t-\tau_1),\\
\dot{x}_3 &= x_2^*x_4^*x_1 + x_1^*x_4^*x_2 - cx_3 + x_1^*x_2^*x_4,\\
\dot{x}_4 &= x_2^*x_3^*x_1 + x_1^*x_3^*x_2 + x_1^*x_2^*x_3 - dx_4.
\end{aligned} \tag{4}$$

The characteristic equation of (4) takes the form

$$\det\begin{pmatrix}
\lambda + a & -(a + x_3^*x_4^*) & -x_2^*x_4^* & -x_2^*x_3^*\\
-(b + x_3^*x_4^*) & \lambda - k_1 + k_1e^{-\lambda\tau_1} & -x_1^*x_4^* & -x_1^*x_3^*\\
-x_2^*x_4^* & -x_1^*x_4^* & \lambda + c & -x_1^*x_2^*\\
-x_2^*x_3^* & -x_1^*x_3^* & -x_1^*x_2^* & \lambda + d
\end{pmatrix} = 0, \tag{5}$$

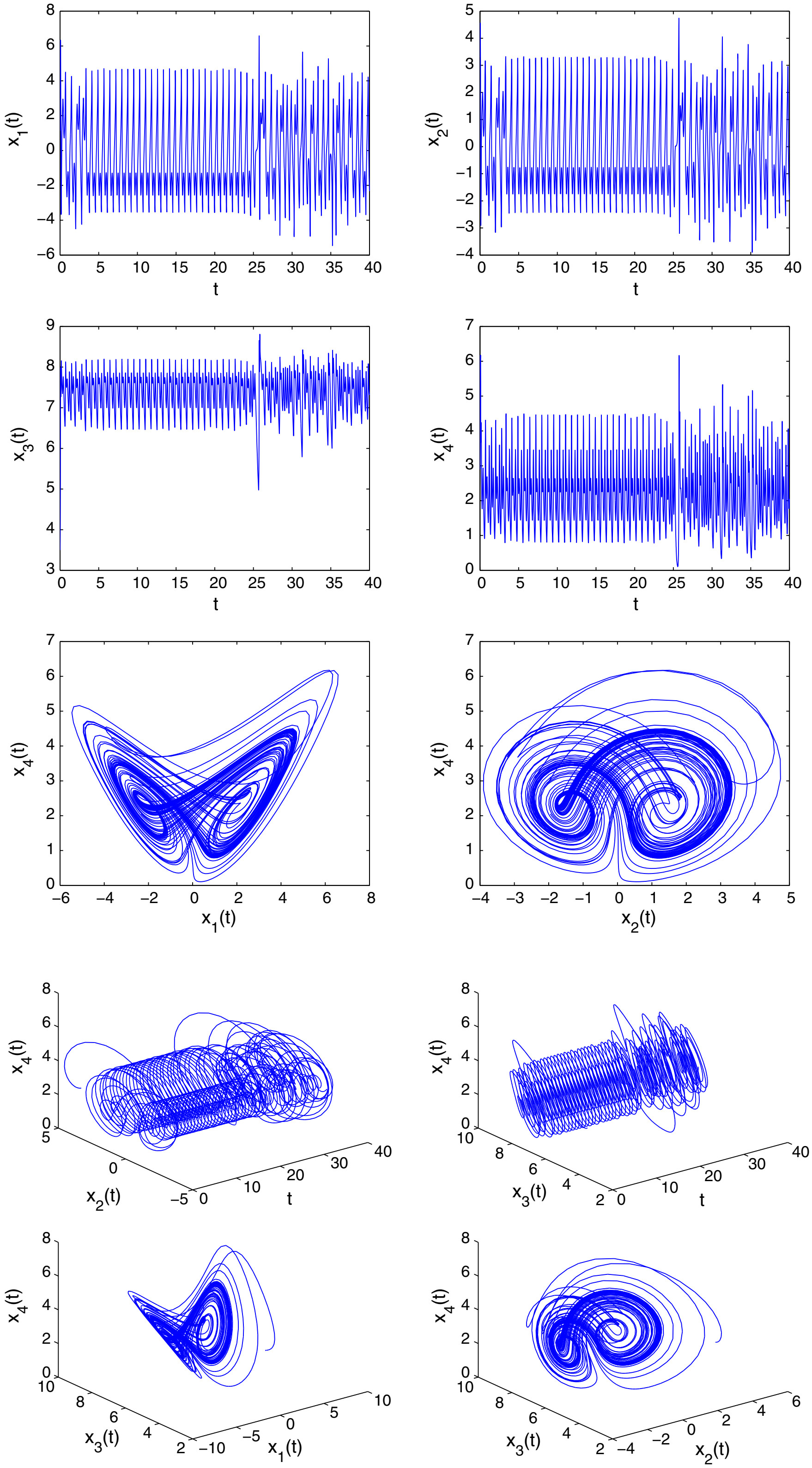

Fig. 1 Chaotic attractor of system (2) with $a = 30$, $b = 10$, $c = 1$, and $d = 10$

that is,

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0)e^{-\lambda\tau_1} = 0, \tag{6}$$

where

$$\begin{aligned}
a_0 &= cdk_1 + 2(x_1^*)^2x_2^*x_3^*x_4^* + c(x_1^*x_3^*)^2 + d(x_1^*x_4^*)^2 + k_1(x_1^*x_2^*)^2\\
&\quad +(b + x_3^*x_4^*)[cd(a + x_3^*x_4^*) - 2(x_1^*x_2^*)^2x_3^*x_4^* - cx_1^*x_2^*(x_3^*)^2\\
&\quad -dx_1^*x_2^*(x_4^*)^2 - (a + x_3^*x_4^*)(x_1^*x_2^*)^2] - x_2^*x_4^*[(a + x_3^*x_4^*)x_1^*x_4^*d\\
&\quad +(x_1^*x_3^*)^2x_2^*x_4^* - k_1x_1^*x_4^*(x_2^*)^2 + (x_1^*)^2x_2^*x_3^*(a + x_3^*x_4^*)]\\
&\quad -x_2^*x_4^*[(a + x_3^*x_4^*)(x_1^*x_4^*)^2 + (x_1^*x_4^*)^2x_2^*x_3^* - k_1x_2^*x_3^*\\
&\quad -(x_1^*x_4^*)^2x_2^*x_3^* + k_1x_1^*x_2^*(x_4^*)^2 + c(a + x_3^*x_4^*)x_1^*x_3^*],\\
a_1 &= a[cd - k_1(c + d) - (x_1^*x_3^*)^2 - (x_1^*x_4^*)^2 - (x_1^*x_2^*)^2]\\
&\quad -[cdk_1 + 2(x_1^*)^2x_2^*x_3^*x_4^* + c(x_1^*x_3^*)^2 + d(x_1^*x_4^*)^2 + k_1(x_1^*x_2^*)^2]\\
&\quad +(b + x_3^*x_4^*)[(c + d)(a + x_3^*x_4^*) - x_1^*x_2^*((x_3^*)^2 + (x_4^*)^2)]\\
&\quad -x_2^*x_4^*[(a + x_3^*x_4^*)x_1^*x_4^* + x_1^*x_4^*(x_2^*)^2 + (d - k_1)x_2^*x_4^*]\\
&\quad -x_2^*x_3^*[((c - k_1)x_2^*x_3^* + (a + x_3^*x_4^*)x_1^*x_3^* + x_1^*x_2^*(x_4^*)^2],\\
a_2 &= a(c + d - k_1) + cd - k_1(c + d) - (x_1^*x_3^*)^2 - (x_1^*x_4^*)^2 - (x_1^*x_2^*)^2\\
&\quad -(a + x_3^*x_4^*)(b + x_3^*x_4^*) - (x_2^*x_4^*)^2 - (x_2^*x_3^*)^2,\\
a_3 &= a + c + d - k_1,\\
b_0 &= k_1acd - k_1a(x_1^*x_4^*)^2 - k_1x_1^*(x_4^*)^3(x_2^*)^3 - k_1d(x_2^*x_4^*)^2 - x_2^*x_3^*[x_2^*x_3^*k_1c + x_1^*x_2^*(x_4^*)^2k_1],\\
b_1 &= k_1a(c + d) + k_1cd - k_1(x_1^*x_2^*)^2 - (x_2^*x_4^*)^2k_1,\\
b_2 &= k_1(c + d + a),\\
b_3 &= k_3.
\end{aligned}$$

Next, we will discuss the distribution of the roots of the transcendental equation (6).

Lemma 1 [31] *For the transcendental equation*

$$\begin{aligned}
P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}) &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} + \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1} + \cdots\\
&\quad + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m} = 0,
\end{aligned}$$

as $(\tau_1, \tau_2, \tau_3, \ldots, \tau_m)$ *vary, the sum of orders of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ *in the open right half-plane can change, and only a zero appears on or crosses the imaginary axis.*

When $\tau_1 = 0$, equation (6) becomes

$$\lambda^4 + (a_3 + b_3)\lambda^3 + (a_2 + b_2)\lambda^2 + (a_1 + b_1)\lambda + (a_0 + b_0) = 0. \tag{7}$$

We can easily know that all the roots of (7) have a negative real part if the following conditions hold:

$$D_1 = a_3 + b_3 > 0, \tag{8}$$

$$D_2 = \det\begin{pmatrix} a_3 + b_3 & a_1 + b_1\\ 1 & a_2 + b_2\end{pmatrix} = (a_2 + b_2)(a_3 + b_3) - (a_1 + b_1) > 0, \tag{9}$$

$$\begin{aligned}
D_3 &= \det\begin{pmatrix} a_3 + b_3 & a_1 + b_1 & 0\\ 1 & a_2 + b_2 & a_0 + b_0\\ 0 & a_3 + b_3 & a_1 + b_1\end{pmatrix}\\
&= (a_3 + b_3)[(a_2 + b_2)(a_1 + b_1) - (a_3 + b_3)(a_0 + b_0)] - (a_1 + b_1)^2 > 0,
\end{aligned} \tag{10}$$

$$D_4 = \det\begin{pmatrix} a_3+b_3 & a_1+b_1 & 0 & 0 \\ 1 & a_2+b_2 & a_0+b_0 & 0 \\ 0 & a_3+b_3 & a_1+b_1 & 0 \\ 0 & 1 & a_2+b_2 & a_0+b_0 \end{pmatrix} = (a_0+b_0)D_3 > 0. \tag{11}$$

Then the equilibrium point $E(x_1^*, x_2^*, x_3^*, x_4^*)$ is locally asymptotically stable when (8)–(11) hold.

We assume that

(H1) (8)−(11) hold.

For $\omega > 0$, $\mathrm{i}\omega$ is a root of (6) if and only if

$$\omega^4 - a_3\omega^3\mathrm{i} - a_2\omega^2 + a_1\omega\mathrm{i} + a_0 + (-b_3\omega^3\mathrm{i} - b_2\omega^2 + b_1\omega\mathrm{i} + b_0)\mathrm{e}^{-\omega\tau_1\mathrm{i}} = 0.$$

Separating the real and imaginary parts gives

$$\begin{aligned}&(b_0 - b_2\omega^2)\cos\omega\tau_1 + (b_1\omega - b_3\omega^3)\sin\omega\tau_1 = a_2\omega^2 - \omega^4 - a_0,\\ &(b_1\omega - b_3\omega^3)\cos\omega\tau_1 - (b_0 - b_2\omega^2)\sin\omega\tau_1 = a_3\omega^3 - a_1\omega.\end{aligned} \tag{12}$$

It follows from (12) that

$$(b_0 - b_2\omega^2)^2 + (b_1\omega - b_3\omega^3)^2 = (a_2\omega^2 - \omega^4 - a_0)^2 + (a_3\omega^3 - a_1\omega)^2,$$

which is equivalent to

$$\omega^8 + p_3\omega^6 + p_2\omega^4 + p_1\omega^2 + p_0 = 0, \tag{13}$$

where

$$\begin{aligned}&p_0 = a_0^2 - b_0^2, \quad p_1 = a_1^2 - 2a_0a_2 + 2b_0b_2 - b_1^2,\\ &p_2 = a_0^2 - 2a_1a_3 + 2b_1b_3 + 2a_0 - b_2^2, \quad p_3 = a_3^2 - 2a_2 - b_3^2.\end{aligned}$$

Use the notation $z = \omega^2$; then (13) takes the following form:

$$z^4 + p_3z^3 + p_2z^2 + p_1z + p_0 = 0. \tag{14}$$

Let

$$h(z) = z^4 + p_3z^3 + p_2z^2 + p_1z + p_0. \tag{15}$$

Suppose

(H2) (14) has at least one positive real root.

If all the coefficients of system (2) are given, then we can easily calculate the roots of (14). Since $\lim_{z\to\infty} h(z) = \infty$, we can conclude that if $p_0 < 0$, then (14) has at least one positive real root. Without loss of generality, we assume that (14) has four positive real roots, defined by z_1, z_2, z_3, z_4. Then (13) has four positive roots:

$$\omega_1 = \sqrt{z_1}, \quad \omega_2 = \sqrt{z_2}, \quad \omega_3 = \sqrt{z_3}, \quad \omega_4 = \sqrt{z_4}.$$

By (12), we derive

$$\cos\omega_k\tau_1 = \frac{(a_2\omega_k^2 - \omega_k^4 - a_0)(b_0 - b_2\omega_k^2) + (a_3\omega_k^3 - a_1\omega_k)(b_1\omega_k - b_3\omega_k^3)}{(b_0 - b_2\omega_k^2)^2 + (b_1\omega_k - b_3\omega_k^3)^2}.$$

Thus, if we use the notation

$$\tau_{1k}^{(j)} = \frac{1}{\omega_k}\left\{\arccos\left(\frac{(a_2\omega_k^2 - \omega_k^4 - a_0)(b_0 - b_2\omega_k^2) + (a_3\omega_k^3 - a_1\omega_k)(b_1\omega_k - b_3\omega_k^3)}{(b_0 - b_2\omega_k^2)^2 + (b_1\omega_k - b_3\omega_k^3)^2}\right) + 2j\pi\right\}, \tag{16}$$

where $k = 1, 2, 3, 4$; $j = 0, 1, 2, \ldots$, then $\pm\mathrm{i}\omega_k$ are a pair of imaginary roots of Eq. (6) when $\tau_1 = \tau_{1k}^{(j)}$. Define

$$\tau_{1_0} = \tau_{1k0}^{(0)} = \min_{k\in\{1,2,3,4\}}\{\tau_{1k}^{(0)}\}, \quad \omega_0 = \omega_{k0}. \tag{17}$$

Let $\lambda(\tau_1) = \alpha(\tau_1) + \mathrm{i}\omega(\tau_1)$ be a root of (6) around $\tau_1 = \tau_{1k}^{(j)}$, where $\alpha(\tau_1)$ and $\omega(\tau_1)$ satisfy $\alpha(\tau_{1k}^{(j)}) = 0$ and $\omega(\tau_{1k}^{(j)}) = \omega_k$. Differentiating both sides of (6) with respect to τ_1 yields

$$\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1}\right]^{-1} = -\frac{(4\lambda^3 + 3a_3\lambda^2 + 2a_2\lambda + a_0)\mathrm{e}^{\lambda\tau_1}}{\lambda(b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0)} + \frac{3b_3\lambda^2 + 2b_2\lambda + b_1}{\lambda(b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0)} - \frac{\tau_1}{\lambda}.$$

Letting $\lambda = \mathrm{i}\omega_k$, $\tau_1 = \tau_{1k}^{(j)}$, we have

$$\mathrm{Re}\left\{\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1}\right]^{-1}\Bigg|_{\lambda=i\omega_k,\tau_1=\tau_{1k}^{(j)}}\right\} = \frac{\rho_1\cos\omega_k\tau_{1k}^{(j)} + \rho_2\sin\omega_k\tau_{1k}^{(j)} + \rho_3}{\Lambda},$$

where

$$\begin{aligned}
\rho_1 &= (a_0 - 3a_3\omega_k^2)(b_3\omega_k - b_1) + (4\omega_k^2 + 2a_0)(b_0 - b_2\omega_k^2),\\
\rho_2 &= (a_0 - 3a_3\omega_k^2)(b_0 - b_2\omega_k^2) - (4\omega_k^3 + 2a_2\omega_k)(b_3\omega_k - b_1),\\
\rho_3 &= (b_1 - 3b_3\omega_k^2)(b_3\omega_k^2 - b_1) + 2b_2(b_0 - b_2\omega_k^2)\\
\Lambda &= (b_3\omega_k^2 - b_1\omega_k)^2 + (b_0 - b_2\omega_k^2)^2.
\end{aligned}$$

Suppose that the following condition holds:

(H3) $\quad \rho_1\cos\omega_k\tau_{1k0}^{(j)} + \rho_2\sin\omega_k\tau_{1k}^{(j)} + \rho_3 \neq 0.$

According to the preceding analysis and the results of Kuang [32] and Hale [33], we have the following theorem.

Theorem 2 *If* (H1) *and* (H2) *hold, then the equilibrium* $E(x_1^*, x_2^*, x_3^*, x_4^*)$ *of system* (2) *is asymptotically stable when* $\tau_1 \in [0, \tau_{10})$. *In addition to* (H1) *and* (H2), *if* (H3) *holds, then system* (2) *undergoes a Hopf bifurcation at the equilibrium* $E(x_1^*, x_2^*, x_3^*, x_4^*)$ *when* $\tau_1 = \tau_{1k}^{(j)}$, $k = 1, 2, 3, 4$, $j = 0, 1, 2, \ldots$.

Remark 3 It is shown that if (H1) and (H2) are fulfilled, then the states $x_i (i = 1, 2, 3, 4)$ of system (2) will tend to x_i^* when $\tau_1 \in [0, \tau_{10})$. If (H1), (H2), and (H3) hold, then the states $x_i (i = 1, 2, 3, 4)$ of system (2) may coexist and remain in an oscillatory mode near the equilibrium $E(x_1^*, x_2^*, x_3^*, x_4^*)$. Thus, chaos vanishes, which means that chaos can be controlled.

Case 2 Delayed feedback on second and third equations [system (3)]
The linearized system of Eq. (3) around $E(x_1^*, x_2^*, x_3^*, x_4^*)$ is given by

$$\begin{aligned}
\dot{x}_1 &= -ax_1 + (a + x_3^*x_4^*)x_2 + x_2^*x_4^*x_3 + x_2^*x_3^*x_4,\\
\dot{x}_2 &= (b + x_3^*x_4^*)x_1 + k_2x_2 + x_1^*x_4^*x_3 + x_1^*x_3^*x_4 - k_2x_2(t - \tau_2),\\
\dot{x}_3 &= x_2^*x_4^*x_1 + x_1^*x_4^*x_2 + (k_3 - c)x_3 + x_1^*x_2^*x_4 - k_3x_3(t - \tau_2),\\
\dot{x}_4 &= x_2^*x_3^*x_1 + x_1^*x_3^*x_2 + x_1^*x_2^*x_3 - dx_4.
\end{aligned} \tag{18}$$

The characteristic equation of (19) takes the form

$$\det\begin{pmatrix}
\lambda + a & -(a + x_3^*x_4^*) & -x_2^*x_4^* & -x_2^*x_3^*\\
-(b + x_3^*x_4^*) & \lambda - k_2 + k_2\mathrm{e}^{-\lambda\tau_2} & -x_1^*x_4^* & -x_1^*x_3^*\\
-x_2^*x_4^* & -x_1^*x_4^* & \lambda - k_2 + c + k_2\mathrm{e}^{\lambda\tau_2} & -x_1^*x_2^*\\
-x_2^*x_3^* & -x_1^*x_3^* & -x_1^*x_2^* & \lambda + d
\end{pmatrix} = 0, \tag{19}$$

that is,

$$\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 + (d_3\lambda^3 + d_2\lambda^2 + d_1\lambda + d_0)\mathrm{e}^{-\lambda\tau_2} + e_0\mathrm{e}^{-2\lambda\tau_2} = 0, \tag{20}$$

where

$$
\begin{aligned}
c_0 &= a[dk_2(c-k_2) - 2x_2^*x_3^*x_4^*(x_1^*)^2 + (x_1^*x_3^*)^2(c-k_2) + (x_1^*x_4^*)^2 d \\
&\quad -k_2(x_1^*x_2^*)^2] + (b + x_3^*x_4^*)[d(a + x_3^*x_4^*)(k_2 - c) - x_1^*(x_2^*)^2x_3^*x_4^* \\
&\quad -(x_1^*x_2^*)^2x_3^*x_4^* - (c-k_2)x_1^*x_2^*(x_3^*)^2 + dx_1^*x_2^*(x_4^*)^2 - (a + x_3^*x_4^*)(x_1^*x_2^*)^2] \\
&\quad +x_2^*x_4^*[(a + x_3^*x_4^*)x_1^*x_4^*d - (x_1^*x_3^*)^2x_2^*x_4^* + k_2x_1^*x_3^*(x_2^*)^2 + (x_1^*x_3^*)^2x_2^*x_4^* \\
&\quad -dk_2x_2^*x_4^* + (x_1^*)^2x_2^*x_3^*(a + x_3^*x_4^*)] - x_2^*x_3^*[(a + x_3^*x_4^*)(x_1^*)^2x_2^*x_4^* \\
&\quad -k_2x_2^*x_3^*(c-k_2) - k_2x_1^*(x_2^*)^2x_4^* + (c-k_2)(a + x_3^*x_4^*)x_1^*x_3^*, \\
c_1 &= dk_2(c-k_2) - 2(x_1^*)^2x_2^*x_3^*x_4^* + (x_1^*x_3^*)(c-k_2) + d(x_1^*x_4^*) - k_2(x_1^*x_2^*)^2 \\
&\quad +a[k_2(c-k_2) + d(c-2k_2) - (x_1^*x_3^*)^2 + (x_1^*x_4^*)^2 + (x_1^*x_2^*)^2] \\
&\quad -(b + x_3^*x_4^*)[(a + x_3^*x_4^*)(c + d - k_2) + x_1^*x_2^*(x_3^*)^2 - x_1^*x_2^*(x_4^*)^2] \\
&\quad +x_2^*x_4^*[(a + x_3^*x_4^*)x_1^*x_4^* - x_1^*x_4^*(x_2^*)^2 + (d-k_2)x_2^*x_4^*] \\
&\quad +x_2^*x_3^*[(2k_2 - c)x_2^*x_3^* - x_1^*x_4^*(x_2^*)^2 - (a + x_3^*x_4^*)x_1^*x_3^*], \\
c_2 &= k_2(c-k_2) + d(c-2k_2) - (x_1^*x_3^*)^2 + (x_1^*x_4^*)^2 + a(d + c - 2k_2) \\
&\quad -(b + x_3^*x_4^*)(a + x_3^*x_4^*) - (x_2^*x_2^*)^2 - (x_2^*x_3^*)^2, \\
c_3 &= a + b + c - 2k_2, \\
d_0 &= a(ck_2 - k_2^2)d - ak_2(x_1^*x_3^*)^2 + ak_2(x_1^*x_2^*)^2, \\
&\quad -(b + x_3^*x_4^*)[dk_2(a + x_3^*x_4^*) + x_1^*x_2^*(x_3^*)^2k_2] \\
&\quad +x_2^*x_4^*[dk_2x_2^*x_4^* - k_2x_1^*x_3^*(x_2^*)^2] + x_2^*x_3^*[ck_2 - k_2x_1^*(x_2^*)^2x_4^* + k_2(a + x_3^*x_4^*)x_1^*x_3^*], \\
d_1 &= (k_2 - 2k_2^2)d - k_2(x_1^*x_3^*)^2 + k_2(x_1^*x_2^*)^2 + a(ck_2 - 2k_2^2 + 2k_2d) \\
&\quad -(b + x_3^*x_4^*)(a + x_3^*x_4^*)k_2 + x_2^*x_4^*(k_2x_2^*x_4^* - k_2x_1^*x_3^*(x_2^*)^2) - 2k_2(x_2^*x_3^*), \\
d_2 &= ck_2 - 2k_2^2 + 2k_2d + 2k_2a, \\
d_3 &= 2k_3, \\
e_0 &= ak_2^2.
\end{aligned}
$$

Multiplying $\mathrm{e}^{\lambda\tau_2}$ on both sides of (20), we have

$$
(\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0)\mathrm{e}^{\lambda\tau_2} + (d_3\lambda^3 + d_2\lambda^2 + d_1\lambda + d_0) + e_0\mathrm{e}^{-\lambda\tau_2} = 0. \tag{21}
$$

Next, we will focus on the distribution of the roots of the transcendental equation (21).

When $\tau_2 = 0$, (21) reads as

$$
\lambda^4 + (c_3 + d_3)\lambda^3 + (c_2 + d_2)\lambda^2 + (c_1 + d_1)\lambda + c_0 + d_0 + e_0 = 0. \tag{22}
$$

All the roots of (22) have a negative real part if the following conditions hold:

$$
\bar{D}_1 = c_3 + d_3 > 0, \tag{23}
$$

$$
\bar{D}_2 = \det\begin{pmatrix} c_3 + d_3 & c_1 + d_1 \\ 1 & c_2 + d_2 \end{pmatrix} = (c_2 + d_2)(c_3 + d_3) - (c_1 + d_1) > 0, \tag{24}
$$

$$
\begin{aligned}
\bar{D}_3 &= \det\begin{pmatrix} c_3 + d_3 & c_1 + d_1 & 0 \\ 1 & c_2 + d_2 & c_0 + d_0 + e_0 \\ 0 & c_3 + c_3 & c_1 + c_1 \end{pmatrix} \\
&= (c_3 + d_3)[(c_2 + d_2)(c_1 + d_1) - (c_3 + d_3)(c_0 + d_0 + e_0)] - (c_1 + d_1)^2 > 0,
\end{aligned} \tag{25}
$$

$$\bar{D}_4 = \det\begin{pmatrix} c_3+d_3 & c_1+d_1 & 0 & 0 \\ 1 & c_2+d_2 & c_0+d_0+e_0 & 0 \\ 0 & c_3+d_3 & c_1+d_1 & 0 \\ 0 & 1 & c_2+d_2 & c_0+d_0+e_0 \end{pmatrix} = (c_0+d_0+e_0)D_3 > 0. \tag{26}$$

Then the equilibrium point $E(x_1^*, x_2^*, x_3^*, x_4^*)$ is locally asymptotically stable when (23)–(26) hold.

We assume that

(H4) (23)−(26) hold.

For $\tilde{\omega} > 0$, $\mathrm{i}\tilde{\omega}$ is a root of (21) if and only if

$$(-\tilde{\omega}^4 - c_3\tilde{\omega}^3\mathrm{i} - c_2\tilde{\omega}^2 + c_1\tilde{\omega}\mathrm{i}c_0)\mathrm{e}^{\tilde{\omega}\tau_2} + (-d_3\tilde{\omega}^3\mathrm{i} - d_2\tilde{\omega}^2 + d_1\tilde{\omega}\mathrm{i} + d_0) + e_0\mathrm{e}^{-\tilde{\omega}\tau_2} = 0.$$

Separating the real and imaginary parts gives

$$\begin{aligned} &(\tilde{\omega}^4 - c_2\tilde{\omega}^2 + c_0 + e_0)\cos\tilde{\omega}\tau_2 + (c_3\tilde{\omega}^3 - c_1\tilde{\omega})\sin\tilde{\omega}\tau_2 = d_2\tilde{\omega}^2 - d_0, \\ &(c_1\tilde{\omega} - c_3\tilde{\omega}^3)\cos\tilde{\omega}\tau_2 + (\tilde{\omega}^4 - c_2\tilde{\omega}^2 + c_0 - e_0)\sin\tilde{\omega}\tau_2 = d_3\tilde{\omega}^3 - d_1\tilde{\omega}. \end{aligned} \tag{27}$$

It follows from (27) that

$$\cos\tilde{\omega}\tau_2 = \frac{(d_2\tilde{\omega}_k^2 - d_0)(\tilde{\omega}_k^4 - c_2\tilde{\omega}_k^2 + c_0 - e_0) - (d_3\tilde{\omega}_k^3 - d_1\tilde{\omega}_k)(c_3\tilde{\omega}_k^3 - c_1\tilde{\omega}_k)}{(\tilde{\omega}_k^4 - c_2\tilde{\omega}_k^2 + c_0)^2 - e_0^2 + (c_3\tilde{\omega}_k^3 - c_1\tilde{\omega}_k^2)^2},$$

$$\sin\tilde{\omega}\tau_2 = \frac{(d_3\tilde{\omega}_k^2 - d_1\tilde{\omega}_k)(\tilde{\omega}_k^4 - c_2\tilde{\omega}_k^2 + c_0 + e_0) - (d_2\tilde{\omega}_k^2 - d_0)(c_1\tilde{\omega}_k - c_3\tilde{\omega}_k^3)}{(\tilde{\omega}_k^4 - c_2\tilde{\omega}_k^2 + c_0)^2 - e_0^2 + (c_3\tilde{\omega}_k^3 - c_1\tilde{\omega}_k^2)^2},$$

which leads to

$$\delta_{12}\tilde{\omega}^{12} + \delta_{11}\tilde{\omega}^{11} + \delta_{10}\tilde{\omega}^{10} + \delta_9\tilde{\omega}^9 + \delta_8\tilde{\omega}^8 + \delta_7\tilde{\omega}^7 + \delta_6\tilde{\omega}^6 + \delta_5\tilde{\omega}^5 + \delta_4\tilde{\omega}^4 + \delta_3\tilde{\omega}^3 + \delta_2\tilde{\omega}^2 + \delta_1 = 0, \tag{28}$$

where

$$\begin{aligned}
\delta_1 &= d_0^2(e_0 - c_0)^2 - c_0^2, \\
\delta_2 &= 2d_0(c_0d_2 - e_0d_2 + c_2d_0 - c_1d_1)(e_0 - c_0) + 2_0c_2 + (c_1d_0 - c_0d_1 - e_0d_1)^2, \\
\delta_3 &= 2d_3(c_0 + e_0)(c_1d_0 - c_0d_1 - e_0d_1), \\
\delta_4 &= (c_0d_2 - e_0d_2 + c_2d_0 - c_1d_1)^2 + 2d_0(e_0 - c_0)(c_1d_3 - c_2d_2 + c_1d_1 - d_0) \\
&\quad + 2(c_2d_1 - d_2c_1 - c_3d_0)(c_1d_0 - c_0d_1 - e_0d_1) + (c_0 + d_0)^2d_3^2 - 2c_0 - c_1^2 - c_2^2, \\
\delta_5 &= 2d_3(c_2d_1 - d_2c_1 - c_3d_0)(c_0 + e_0) - 2c_2d_3(c_1d_0 - c_0d_1 - e_0d_1) + 2c_1c_3, \\
\delta_6 &= 2d_0(d_2 - c_3d_3)(e_0 - c_0) + (c_2d_1 - d_2c_1 - c_3d_0)^2 + 2c_2 - c - 3^2 \\
&\quad + 2(c_1d_3 - c - 2d_2 + c_3d_1 - d_0)(c_0d_2 - e_0d_2 + c_2d_0 - c_1d_1) - 2c_2d_3^2(c_0 + e_0), \\
\delta_7 &= 2d_3(c_1d_0 - c_0d_1 - e_0d_1) + 2d_3(d_2c_3 - d_1)(c_0 + e_0) - 2c_2d_3(c_2d_1 - d_2c_1 - c_3d_0), \\
\delta_8 &= (c_1d_3 - c_2d_2 + c_3d_1 - d_0)^2 + 2(d_2 - c_3d_3)(c_0d_2 - e_0d_2 + c_2d_0 - c_1d_1) \\
&\quad + (c_2d_3)^2 + 2d_3^2(c_0 + e_0) + 2(d_2c_3 - d_1)(c_2d_1 - d_2c_1 - c_3d_0) - 1, \\
\delta_9 &= 2d_3(c_2d_1 - d_2c_1 - c_3d_0) - 2c_2d_3(d_2c_3 - d_1), \\
\delta_{10} &= 2(d_2 - c_3d_3)(c_1d_3 - c_2d_2 + c_3d_1 - d_0) + (d_2c_3 - d_1)^2 - 2c_2d_3^2, \\
\delta_{11} &= 2d_3(d_2c_3 - d_1), \\
\delta_{12} &= (d_2 - c_3d_3)^2 + d_3^2.
\end{aligned}$$

Let

$$\tilde{h}(\tilde{\omega}) = \delta_{12}\tilde{\omega}^{12} + \delta_{11}\tilde{\omega}^{11} + \delta_{10}\tilde{\omega}^{10} + \delta_9\tilde{\omega}^9 + \delta_8\tilde{\omega}^8 + \delta_7\tilde{\omega}^7 + \delta_6\tilde{\omega}^6 + \delta_5\tilde{\omega}^5 + \delta_4\tilde{\omega}^4 + \delta_3\tilde{\omega}^3 + \delta_2\tilde{\omega}^2 + \delta_1.$$

We assume that

(H5) (28) has at least one positive real root.

If all the coefficients of system (3) are given, it is not difficult to calculate the roots of (28). Since $\lim_{\tilde{\omega}\to\infty} \tilde{h}(\tilde{\omega}) = \infty$, we can conclude that if $\delta_1 < 0$, then (28) has at least one positive real root.

Suppose that Eq. (28) has positive roots. Without loss of generality, we assume that it has 12 positive roots, denoted by $\tilde{\omega}_k (k = 1, 2, 3 \ldots, 12)$. If we use the notation

$$\tau_{2k}^{(j)} = \frac{1}{\tilde{\omega}_k}\left\{\arccos\left(\frac{(d_2\tilde{\omega}_k^2 - d_0)(\tilde{\omega}_k^4 - c_2\tilde{\omega}_k^2 + c_0 - e_0) - (d_3\tilde{\omega}_k^3 - d_1\tilde{\omega}_k)(c_3\tilde{\omega}_k^3 - c_1\tilde{\omega}_k)}{(\tilde{\omega}_k^4 - c_2\tilde{\omega}_k^2 + c_0)^2 - e_0^2 + (c_3\tilde{\omega}_k^3 - c_1\tilde{\omega}_k^2)^2}\right) + 2j\pi\right\}, \tag{29}$$

where $k = 1, 2, \ldots, 12$, $j = 0, 1, 2, \ldots$, then $\pm i\tilde{\omega}_k$ are a pair of imaginary roots of Eq. (21) when $\tau_2 = \tau_{2k}^{(j)}$. Define

$$\tau_{2_0} = \tau_{2k0}^{(0)} = \min_{k\in\{1,2,3,\ldots,12\}}\left\{\tau_{2k}^{(0)}\right\}, \tilde{\omega}_0 = \tilde{\omega}_{k0}. \tag{30}$$

Let $\lambda(\tau_2) = \tilde{\alpha}(\tau_2) + i\tilde{\omega}(\tau_2)$ be a root of (21) around $\tau_2 = \tau_{2k}^{(j)}$, and let $\tilde{\alpha}(\tau_{2k}^{(j)}) = 0$ and $\tilde{\omega}(\tau_{2k}^{(j)}) = \tilde{\omega}_k$. Differentiating both sides of (21) with respect to τ_2 yields

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = \frac{(4\lambda^3 + 3c_3\lambda^2 + 2c_2\lambda + c_1)e^{\lambda\tau_2} + 3d_3\lambda^2 + 2d_2\lambda + d_1}{\lambda[e_0 e^{-\lambda\tau_2} - e^{\lambda\tau_2}(\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0)]} - \frac{\tau_2}{\lambda}.$$

Letting $\lambda = i\tilde{\omega}_k$, $\tau_2 = \tau_{2k}^{(j)}$, we obtain

$$\mathrm{Re}\left\{\left[\frac{d\lambda}{d\tau_2}\right]^{-1}\Bigg|_{\lambda=i\tilde{\omega}_k,\tau_2=\tau_{2k}^{(j)}}\right\} = \frac{\theta_1 + i\theta_2}{\theta_3 + i\theta_4} = \frac{\theta_1\theta_3 + \theta_2\theta_4}{\theta_3^3 + \theta_4^3},$$

where

$$\theta_1 = (c_1 - 3c_3\tilde{\omega}_k^2)\cos\tilde{\omega}_k\tau_{2_0}^{(j)} + (4\tilde{\omega}_k^3 - 2c_2\tilde{\omega}_k)\sin\tilde{\omega}_k\tau_{2_0}^{(j)} + d_1 - 3d_3\tilde{\omega}_k^2,$$

$$\theta_2 = (c_1 - 3c_3\tilde{\omega}_k^2)\sin\tilde{\omega}_k\tau_{2_0}^{(j)} + (2c_2\tilde{\omega}_k - 4\tilde{\omega}_k^3)\cos\tilde{\omega}_k\tau_{2_0}^{(j)} + 2d_2\tilde{\omega}_k,$$

$$\theta_3 = (c_1 - c_3\tilde{\omega}_k^2)\tilde{\omega}_k^2\cos\tilde{\omega}_k\tau_{2_0}^{(j)} + (e_0 - c_0 + c_2\tilde{\omega}_k^2 - \tilde{\omega}_k^4)\tilde{\omega}_k\sin\tilde{\omega}_k\tau_{2_0}^{(j)},$$

$$\theta_4 = (e_0 - c_0 + c_2\tilde{\omega}_k^2 - \tilde{\omega}_k^4)\tilde{\omega}_k\cos\tilde{\omega}_k\tau_{2_0}^{(j)}(c_1\tilde{\omega}_k - c_3\tilde{\omega}_k^3)\omega_k\cos\tilde{\omega}_k\tau_{2_0}^{(j)}.$$

Assume that the following condition holds:

(H6) $\quad\theta_1\theta_3 + \theta_2\theta_4 \neq 0.$

Based on the foregoing analysis and the results of Kuang [32] and Hale [33], we obtain the following theorem.

Theorem 4 *If* (H4) *and* (H5) *hold, then the equilibrium* $E(x_1^*, x_2^*, x_3^*, x_4^*)$ *of system* (3) *is asymptotically stable when* $\tau_2 \in [0, \tau_{2_0})$. *In addition to* (H4) *and* (H5), *if* (H6) *holds, then system* (3) *undergoes a Hopf bifurcation at the equilibrium* $E(x_1^*, x_2^*, x_3^*, x_4^*)$ *when* $\tau_2 = \tau_{2k}^{(j)}$, $k = 1, 2, \ldots, 12$, $j = 0, 1, 2, \ldots$.

Remark 5 It is shown that if (H4) and (H5) are satisfied, then the states $x_i (i = 1, 2, 3, 4)$ of system (2) will tend to x_i^* when $\tau_2 \in [0, \tau_{2_0})$. If (H4), (H5), and (H6) hold, then the states $x_i (i = 1, 2, 3, 4)$ of system (2) may coexist and remain in an oscillatory mode near the equilibrium $E(x_1^*, x_2^*, x_3^*, x_4^*)$. Thus, chaos vanishes, which means that chaos can be controlled.

3 Computer simulations

In this section, we present some numerical results of systems (2) and (3) to verify the analytical predictions obtained in the previous section. Let us consider the following two systems:

$$\begin{aligned}
\dot{x}_1 &= 30(x_2 - x_1) + x_2x_3x_4,\\
\dot{x}_2 &= 10(x_1 + x_2) - x_1x_3x_4 - 5[x_2(t) - x_2(t - \tau_1)],\\
\dot{x}_3 &= -x_3 + x_1x_2x_4,\\
\dot{x}_4 &= -10x_4 + x_1x_2x_3
\end{aligned} \tag{31}$$

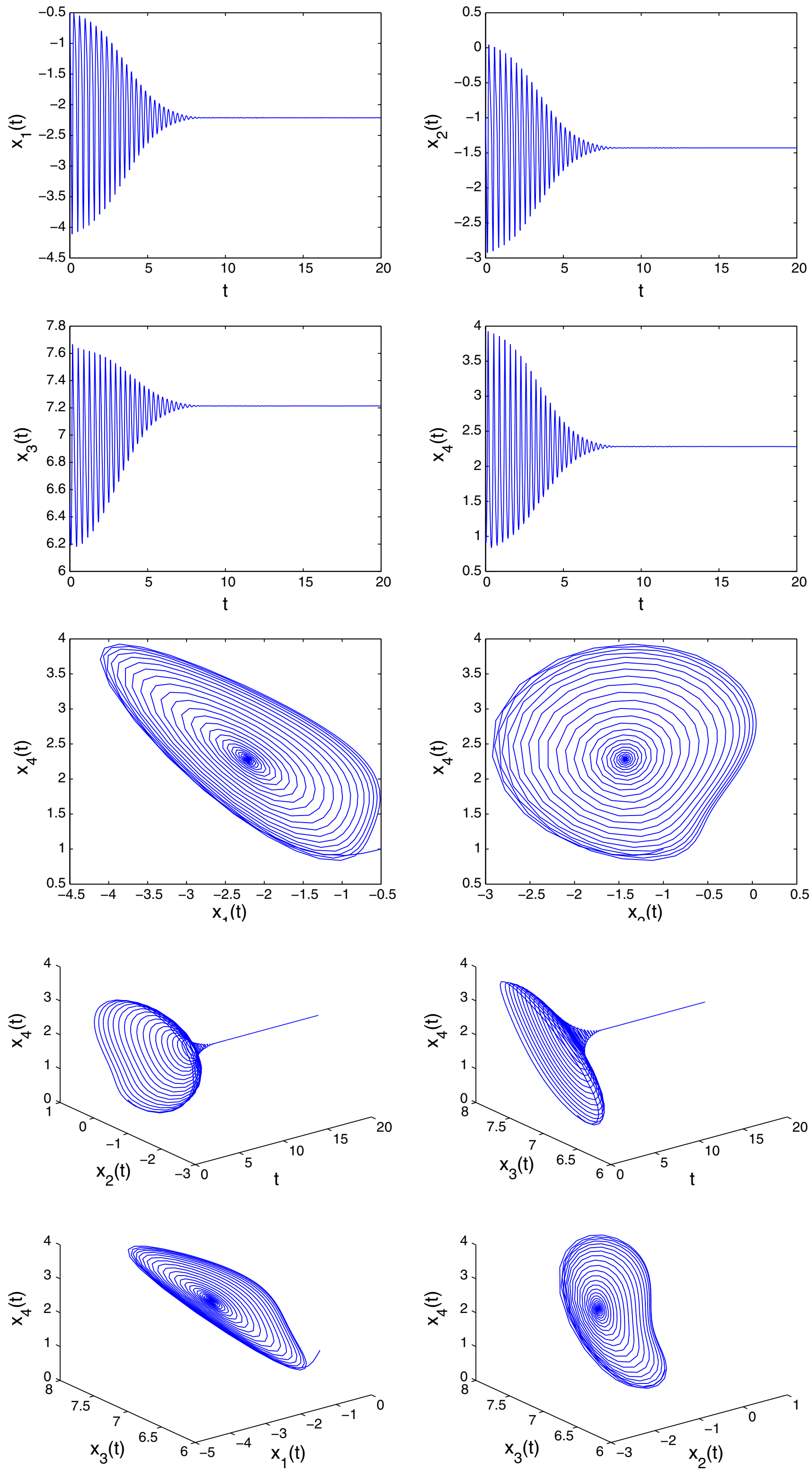

Fig. 2 Chaos vanishes when $\tau_1 = 0.1 < \tau_{1_0} \approx 0.162$. The equilibrium $E(-2.2129, -1.4290, 7.2141, 2.2813)$ is asymptotically stable; the initial value is $(-0.5, -1, 6.5, 1)$

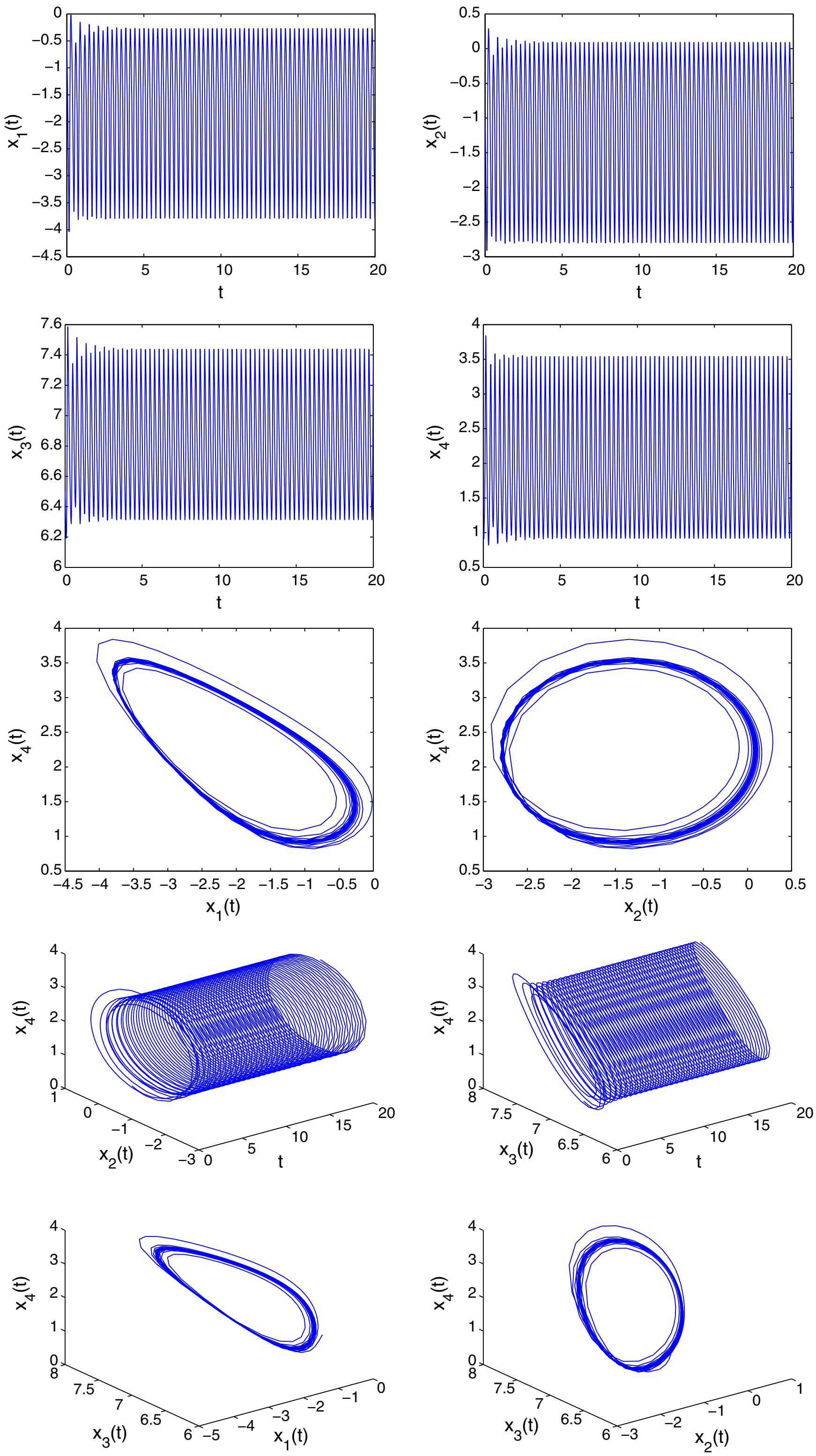

Fig. 3 Chaos vanishes when $\tau_1 = 0.2 > \tau_{1_0} \approx 0.162$. The Hopf bifurcation occurs from the equilibrium $E(-2.2129, -1.4290, 7.2141, 2.2813)$; the initial value is $(-0.5, -1, 6.5, 1)$

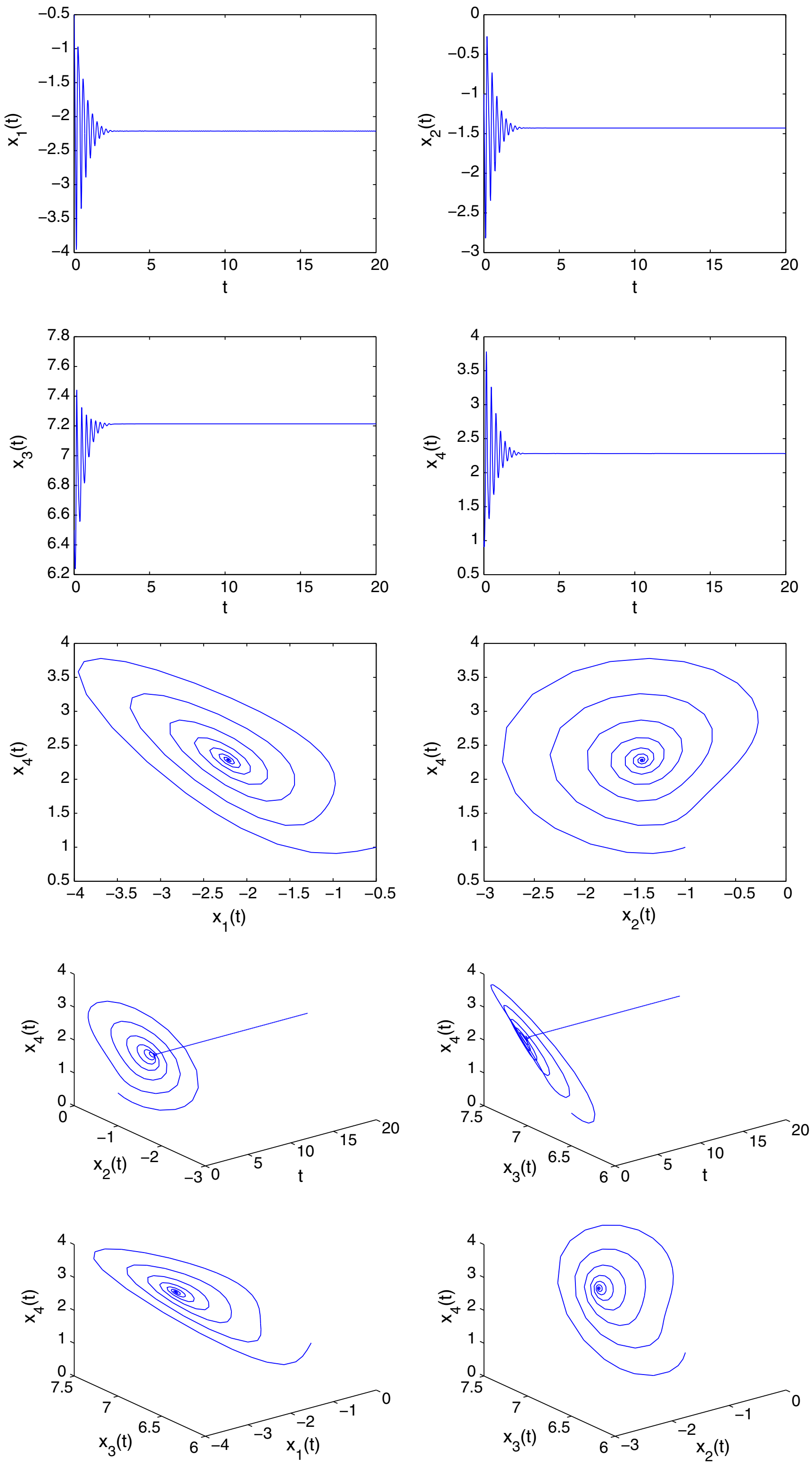

Fig. 4 Chaos vanishes when $\tau_2 = 0.1 < \tau_{2_0} \approx 0.164$. The equilibrium $E(-2.2129, -1.4290, 7.2141, 2.2813)$ is asymptotically stable; the initial value is $(-0.5, -1, 6.5, 1)$

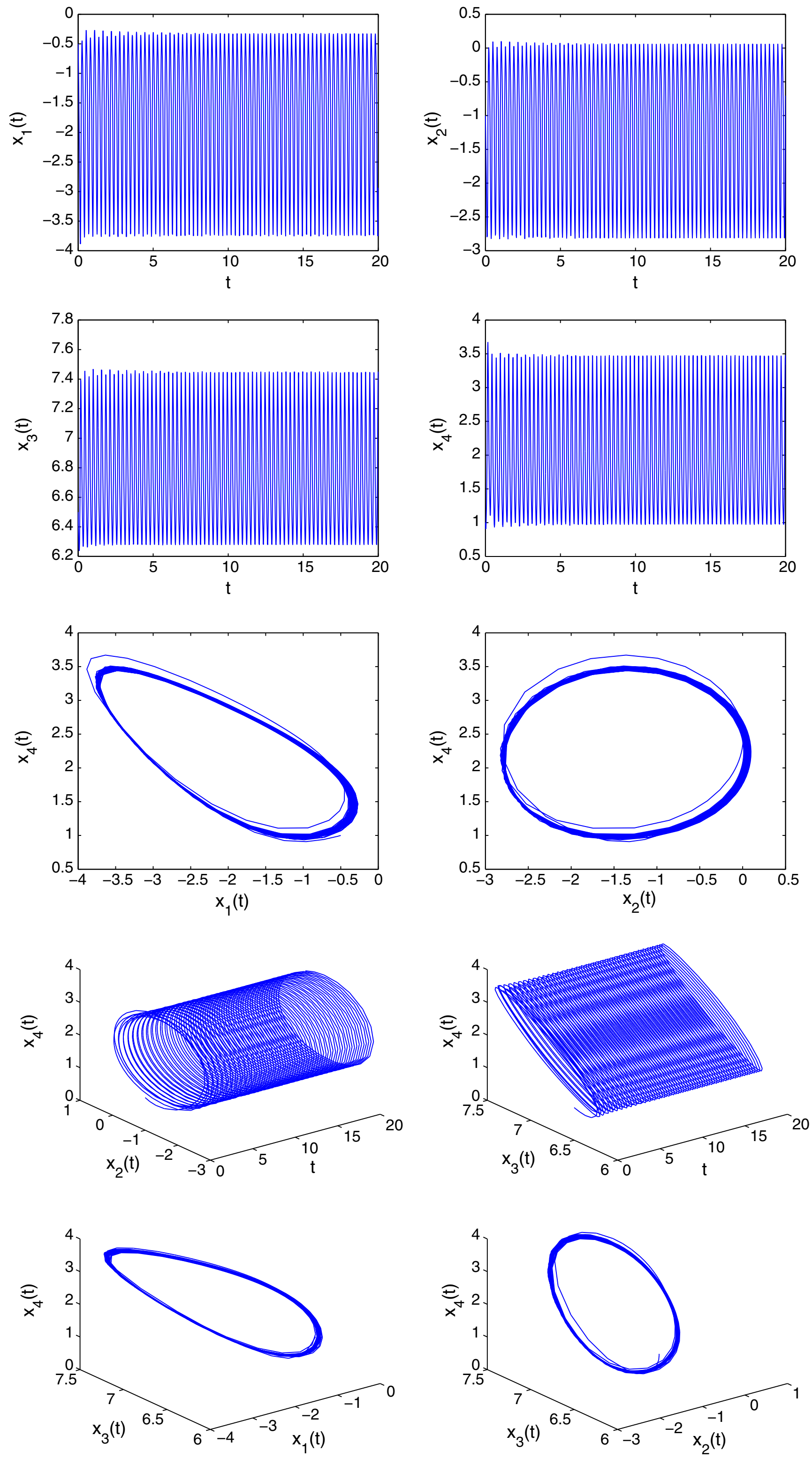

Fig. 5 The chaos vanishes when $\tau_2 = 0.2 > \tau_{2_0} \approx 0.164$. The Hopf bifurcation occurs from the equilibrium $E(-2.2129, -1.4290, 7.2141, 2.2813)$. The initial value is $(-0.5, -1, 6.5, 1)$

and

$$
\begin{aligned}
\dot{x}_1 &= 30(x_2 - x_1) + x_2 x_3 x_4,\\
\dot{x}_2 &= 10(x_1 + x_2) - x_1 x_3 x_4 - 6[x_2(t) - x_2(t - \tau_2)],\\
\dot{x}_3 &= -x_3 + x_1 x_2 x_4 - 4[x_3(t) - x_3(t - \tau_2)],\\
\dot{x}_4 &= -10 x_4 + x_1 x_2 x_3,
\end{aligned}
\tag{32}
$$

respectively. We can easily obtain that systems (33) and (33) have an equilibrium $E(-2.2129, -1.4290, 7.2141, 2.2813)$.

For system (32), we can easily check that (H1)–(H3) are satisfied. We let $j = 0$, and, using MATLAB 7.0 software, we derive $\omega_0 \approx 0.7004$, $\tau_{1_0} \approx 0.162$. Thus, the equilibrium $E(-2.2129, -1.4290, 7.2141, 2.2813)$ is asymptotically stable when $\tau_1 < \tau_{1_0} \approx 0.162$, which is illustrated in Fig. 2. When $\tau_1 = \tau_{1_0} \approx 0.162$, Eq. (32) undergoes a Hopf bifurcation at the equilibrium $E(-2.2129, -1.4290, 7.2141, 2.2813)$, i.e., a small-amplitude periodic solution occurs near $E(-2.2129, -1.4290, 7.2141, 2.2813)$ when τ_1 is close to $\tau_{1_0} \approx 0.162$, which can be shown in Fig. 3.

For system (33), we can check that (H4)–(H6) are satisfied. Then $\tilde{\omega}_0 \approx 0.6809$, $\tau_{2_0} \approx 0.164$. Thus, the equilibrium $E(-2.2129, -1.4290, 7.2141, 2.2813)$ is asymptotically stable when $\tau_2 < \tau_{2_0} \approx 0.164$, which is illustrated in Fig. 4. When $\tau_2 = \tau_{2_0} \approx 0.164$, Eq. (33) undergoes a Hopf bifurcation around the equilibrium $E(-2.2129, -1.4290, 7.2141, 2.2813)$ when τ_2 is close to $\tau_{2_0} \approx 0.164$, which is shown in Fig. 5.

Remark 6 Since the original system (2) is chaotic, there is no stabilized orbit. When we add feedback perturbations to the original system (2), then under some suitable conditions, stabilized orbits will occur. Thus, we can conclude that the stabilized orbits of the original system (2) are delay-induced.

4 Conclusions

In this paper, a feedback control method was applied to suppress the chaotic behavior of a 4D chaotic Qi system. By adding a time-delayed force to the second equation of the 4D chaotic Qi system, we focused on the local stability of the equilibrium $E(x_1^*, x_2^*, x_3^*, x_4^*)$ and local Hopf bifurcation of the 4D delayed chaotic Qi system. It was shown that if (H1) is satisfied, then the 4D delayed chaotic Qi system is asymptotically stable when $\tau_1 \in [0, \tau_{1_0})$. If (H1)–(H3) hold, a sequence of Hopf bifurcations occur around the equilibrium $E(x_1^*, x_2^*, x_3^*, x_4^*)$, that is, a family of periodic orbits bifurcate from the equilibrium $E(x^*, y^*, z^*)$. Adding a time-delayed force to the second and third equations of the 4D chaotic Qi system, we analyzed the local stability of the equilibrium $E(x_1^*, x_2^*, x_3^*, x_4^*)$ and local Hopf bifurcation of the 4D delayed chaotic Qi system. We found that if (H4) is satisfied, then the 4D delayed chaotic Qi system is asymptotically stable when $\tau_2 \in [0, \tau_{2_0})$. If (H4)–(H6) hold, a sequence of Hopf bifurcations occurs around the equilibrium $E(x_1^*, x_2^*, x_3^*, x_4^*)$. All the cases showed that chaos vanishes and can be suppressed. Some numerical simulations were carried out to visualize the theoretical findings.

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