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A plane contact problem for an elastic orthotropic strip

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Abstract The contact of a punch with an elastic orthotropic strip is considered. A singular integral equation is derived for the contact pressure. The analytic expression of the associated kernel is unique for all types of orthotropy. An iterative solution method is developed to investigate a thick strip. A direct asymptotic procedure proposed for a thin strip leads to simple explicit formulae. Numerical examples are presented for various values of the relative strip thickness.

Keywords Asymptotic solution · Iterative method · Logarithmic singularity · Orthotropic material · Plane and parabolic stamp

1 Introduction

Contact problems dealing with the indentation of an elastic half-space or elastic strip have been the subject of numerous publications (see, for example, [1–4]). These problems are usually presented in the form of mixed-type boundary-value problems in the theory of elasticity. Two basic plane-strain problems for an isotropic strip are analysed in the advanced research monograph by Vorovich et al. [5, Chap. 5, Sects. 22–25]. The first of these problems considers a strip hinged along its lower face. The second problem relates to a strip resting on a non-deformable base without friction. These plane problems formulated in terms of displacements represent boundary-value problems for the Lamé equation. Both of them can be reduced to singular integral equations of the first type with logarithmic kernels that can be tackled by the method originated by Tricomi [8, Sect. 1.3, pp. 50ff]. Another approach to the mixed problems in question suggested by Rostovstev [9, pp. 326–332] does not involve singular integral equations. The unknown quantity in both problems is the pressure arising in the contact region.

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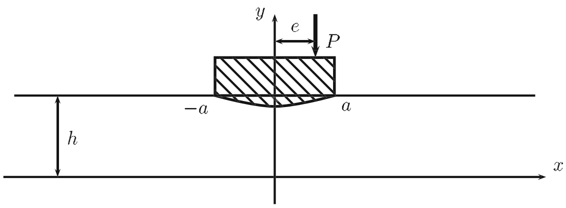

Fig. 1 The geometry of the problem

In this paper, we investigate the pressure resulting from the contact of a punch with an elastic strip composed of an orthotropic material. All the components of strain and stress tensors, as well as the displacement vector, can be readily deduced from the calculated contact pressure. The problem is formulated as a boundary-value problem for the strain and stress components employing Hooke's law for an orthotropic material. On using the Fourier transform, the problem is reduced to a singular integral equation which is unique for all types of orthotropy in contrast to that derived in [10, Eq. 2.7]. Analysis of this singular integral equation was earlier performed in [5] for an isotropic material, making use of asymptotic series in a strip of small or large relative thickness.

Below, we employ an iterative approach for solving the singular integral equation starting from the theorem (see [5]) that guarantees the existence and uniqueness of the solution in the space of continuous functions and also suggests a procedure for constructing the above solution. Furthermore, we present a formula for the lower bound of the relative strip thickness that defines the validity range of the adapted iterative methodology.

For a thin orthotropic strip we develop a direct asymptotic scheme operating with the original partial differential equations of the problem without their reduction to an integral equation. Such a scheme is traditional in asymptotic theories for thin plates and shells (e.g. see [11]) resulting in simple explicit formulae for a punch of arbitrary shape.

In addition, we discuss the results of the numerical computations for different punch shapes and strip thicknesses. It is demonstrated that the combination of the two methodologies provides reasonable accuracy over a wide range of problem parameters.

2 Formulation of the problem

Consider an elastic orthotropic strip occupying the region $|x| < \infty, 0 \le y \le h$ on the plane x, y. Let the lower face of the strip, $y = 0$, be hinged along the x-axis and a punch be applied on the upper face of the strip $y = h$ along the area $|x| \le a$ with no friction forces. We also assume that a force P is applied to the punch at a distance $e < a$. The elastic contact is modelled as

$$\begin{aligned} &v(x,0)=0, \quad \sigma_{xy}(x,0)=0, \quad \sigma_{xy}(x,h)=0; \quad |x|<\infty, \\ &\sigma_y(x,h)=0, \quad |x|>a; \quad v(x,h)=-[\delta+\alpha x-f(x)], \quad |x|\le a, \end{aligned} \tag{1}$$

where v denotes the vertical displacement; σ_{xy} and σ_y are shear and normal stress components, respectively. The applied force P displaces the punch by a distance δ along the vertical; in doing so, the associated moment Pe rotates the punch by a angle α. The shape of the base of the punch is described by the function $f(x)$ (Fig. 1).

In what follows we use the notation

$$g(x) = \delta + \alpha x - f(x). \tag{2}$$

In case of plane strain, Hooke's law for an orthotropic material takes the form (see, e.g. [12, Chap. 20], and [13, p. 52])

$$\begin{aligned} &e_x = \frac{1-\nu_{31}\nu_{13}}{E_1}\sigma_x - \frac{\nu_{21}+\nu_{31}\nu_{23}}{E_2}\sigma_y, \quad e_y = -\frac{\nu_{12}+\nu_{32}\nu_{13}}{E_1}\sigma_x + \frac{1-\nu_{32}\nu_{23}}{E_2}\sigma_y, \\ &e_{xy} = \frac{\sigma_{xy}}{G_{12}}; \quad \frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}, \quad \frac{\nu_{32}\nu_{13}}{E_1} = \frac{\nu_{31}\nu_{23}}{E_2}, \end{aligned} \tag{3}$$

and the kinematic relations are given by

$$e_x = \frac{\partial u}{\partial x}, \quad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad e_y = \frac{\partial v}{\partial y}, \tag{4}$$

where σ_x is the normal-stress component, e_x, e_y and e_{xy} are the strain components, u is the horizontal displacement, ν_{ij} are Poisson's ratios, E_1 and E_2 are Young's moduli and G_{12} is the shear modulus. The relations (3)–(4) are accompanied by the equilibrium equations,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \tag{5}$$

and compatibility equation,

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x\, \partial y}. \tag{6}$$

3 Reduction to an integral equation

The normal and shear components of the stress may be expressed in terms of the Airy function $U(x, y)$ as

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}. \tag{7}$$

Assuming that the equilibrium conditions are satisfied, we insert formulae (7) into (3). Next, we insert the obtained relations into Eq. 6, arriving at a partial differential equation for the Airy function. It is

$$\frac{\partial^4 U}{\partial y^4} + 2A\frac{\partial^4 U}{\partial x^2\, \partial y^2} + B\frac{\partial^4 U}{\partial x^4} = 0, \tag{8}$$

where the coefficients A and B are defined by

$$A = \frac{E_1[E_2 - G_{12}(\nu_{21} + \nu_{31}\nu_{23})]}{G_{12}E_2(1 - \nu_{31}\nu_{13})}, \quad B = \frac{E_1(1 - \nu_{32}\nu_{23})]}{E_2(1 - \nu_{31}\nu_{13})}. \tag{9}$$

In the sequel we shall assume that $A > 0$ and $B > 0$.

Consider the following auxiliary problem with boundary conditions,

$$v(x, 0) = 0,\ \sigma_{xy}(x, 0) = 0,\ \sigma_{xy}(x, h) = 0,\ \sigma_y(x, h) = -\tilde{q}(x),\ |x| < \infty, \tag{10}$$

where

$$\tilde{q}(x) = \begin{cases} q(x), & |x| \le a, \\ 0, & |x| > a, \end{cases} \tag{11}$$

with $q(x)$ denoting the sought contact pressure. First, we present the Airy function in the form of a Fourier integral setting

$$U(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U^*(y, \zeta)\mathrm{e}^{-\mathrm{i}\zeta x}\, \mathrm{d}\zeta. \tag{12}$$

On using (8), we obtain the Fourier transform

$$U^*(y, \zeta) = c_1 \sinh(m_1\zeta y) + c_2 \cosh(m_1\zeta y) + c_3 \sinh(m_2\zeta y) + c_4 \cosh(m_2\zeta y), \tag{13}$$

with

$$m_{1,2} = \sqrt{A \pm \sqrt{A^2 - B}}; \tag{14}$$

$c_1, \ldots, c_4$ are functions of the Fourier-transform parameter ζ, which have to be determined from the boundary conditions (10) and the principal value of the square root is chosen. After finding these coefficients, we obtain the displacement $v(x, y)$ on the face $y = h$ from the solution of the auxiliary problem (10)–(11) which is given by

$$v(x, h) = -\frac{1}{2\pi\gamma}\int_{-\infty}^{\infty} Q(\zeta)\frac{L(\zeta h)}{\zeta}\mathrm{e}^{-\mathrm{i}\zeta x}\, \mathrm{d}\zeta, \quad Q(\zeta) = \int_{-a}^{a} q(\xi)\mathrm{e}^{\mathrm{i}\zeta\xi}\, \mathrm{d}\xi, \tag{15}$$

$$L(u) = \frac{m_2^2 - m_1^2}{m_1\, m_2(m_2 \coth(m_1 u) - m_1 \coth(m_2 u))}, \quad \gamma = \frac{E_2}{(1 - \nu_{32}\nu_{23})},$$

where $Q(\zeta)$ is the Fourier transform of the discontinues function $\tilde{q}(x)$ defined in (11).

Let us now incorporate the last boundary condition of (1) in (15) (all the other boundary conditions have already been satisfied when deriving (15)). As a result, we arrive at an integral equation for the contact pressure $q(x)$ in the original problem. The latter takes the form

$$\int_{-a}^{a} q(\xi) K\left(\frac{\xi - x}{h}\right) \mathrm{d}\xi = \pi \gamma g(x), \quad |x| \leq a, \tag{16}$$

with

$$K(t) = \int_{0}^{\infty} \frac{L(u)}{u} \cos ut \, \mathrm{d}u. \tag{17}$$

The kernel (17) in (16) has an important property given by

Lemma 1 *For all values of* $0 \leq |t| < \infty$*, the kernel* $K(t)$ *may be represented as*

$$K(t) = -\log|t| - F(t), \tag{18}$$

$$F(t) = \int_{0}^{\infty} \frac{[1 - L(u)] \cos ut - \mathrm{e}^{-u}}{u} \mathrm{d}u. \tag{19}$$

The function $F(t)$ *is an analytic function of the complex variable* $w = t + \mathrm{i}\tau$ *in the strip* $|t| < \infty$, $|\tau| < 2$.

For a detailed proof of Lemma 1 and the analyticity of the function $F(t)$ the reader is referred to the text [5, p. 220].

In view of Lemma 1, Eq. 16 can be rewritten as

$$-\int_{-a}^{a} q(\xi) \log\left|\frac{\xi - x}{h}\right| \mathrm{d}\xi = \pi \gamma g(x) + \int_{-a}^{a} q(\xi) F\left(\frac{\xi - x}{h}\right) \mathrm{d}\xi, \quad |x| \leq a, \tag{20}$$

by substituting expression (18) for the kernel $K(t)$ in (16).

4 Solution for a relatively thick strip

In this section we develop an iterative method for solving the integral equation (16). First we nondimensionalise the problem setting

$$\xi' = \frac{\xi}{a}, \; x' = \frac{x}{a}, \; \phi(x') = \frac{q(x)}{\gamma}, \; g'(x') = \frac{g(x)}{a}, \; \lambda = \frac{h}{a}, \tag{21}$$

where λ denotes the relative thickness of the strip which plays a role of the main problem parameter. Below we determine the bounds of this parameter corresponding to the validity range of the method.

By inserting the nondimensional quantities into (16) we get

$$\int_{-1}^{1} \phi(\xi) \log\left|\frac{\xi - x}{\lambda}\right| \mathrm{d}\xi = \pi g(x) + \int_{-1}^{1} \phi(\xi) F\left(\frac{\xi - x}{\lambda}\right) \mathrm{d}\xi, \quad |x| \leq 1. \tag{22}$$

Here and henceforth, the primes in the nondimensional variables will be dropped. Let us start from the lemma [5] stating

Lemma 2 *If $g'(x) \in L_p[-1,1],\ p > 3/4$, then every solution of the integral equations (16) or (22) from the class $L_p[-1,1],\ p > 1$ is also the solution of the integral equation*

$$\phi(x) = \frac{1}{\pi\sqrt{1-x^2}}\left[P_0 - \int_{-1}^{1}\frac{g'(t)\sqrt{1-t^2}}{t-x}\,\mathrm{d}t + \frac{1}{\pi\lambda}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\,\mathrm{d}t\int_{-1}^{1}\phi(\xi)F'\left(\frac{\xi-t}{\lambda}\right)\mathrm{d}\xi\right] \tag{23}$$

$$P_0 = \frac{1}{\log 2\lambda}\left[\int_{-1}^{1}\frac{g(t)}{\sqrt{1-t^2}}\,\mathrm{d}t - \frac{1}{\pi}\int_{-1}^{1}\frac{\mathrm{d}t}{\sqrt{1-t^2}}\int_{-1}^{1}\phi(\xi)F\left(\frac{\xi-t}{\lambda}\right)\mathrm{d}\xi\right], \quad |x| \le 1, \tag{24}$$

where the nondimensional indenting force P_0 is defined by

$$P_0 = \frac{P}{\gamma\, a}. \tag{25}$$

The inverse of this lemma is also true.

We omit the proof of Lemma 2 for brevity; however, the reader is referred to the famous texts by Gakhov [6] and Muskhelishvili [7]. Let $B_k^{\alpha}(D)$ be the class of functions with derivatives up to the order k inclusively existing in the closed domain $\bar{D}$, and let the k-th derivative satisfy Hölder's condition with Hölder constant α. If the function $g(x)$ is from the class $B_1^{\alpha}[-1,1]$, $0 < \alpha \le 1$, then the solution of the integral equation (23) takes the form

$$\phi(x) = \frac{\Phi(x)}{(1-x^2)^{1/2}}, \tag{26}$$

where $\Phi(x) \in C[-1,1]$. Under these assumptions Eq. 23 may be presented in the form

$$\Phi(x) = \mathbf{A}(\Phi) + \Phi_0(x), \quad \mathbf{A}(\Phi) = \int_{-1}^{1}\Phi(\xi)m(\xi,x)\,\mathrm{d}\xi, \tag{27}$$

where

$$\Phi_0(x) = \frac{1}{\pi}\left[P_0 - \int_{-1}^{1}\frac{g'(t)\sqrt{1-t^2}}{t-x}\,\mathrm{d}t\right], \tag{28}$$

and

$$m = \frac{M^*(\xi,x)}{\pi^2\lambda\sqrt{1-\xi^2}}, \tag{29}$$

with

$$M^*(\xi,x) = \int_{-1}^{1}F'\left(\frac{\xi-t}{\lambda}\right)\frac{\sqrt{1-t^2}}{t-x}\,\mathrm{d}t. \tag{30}$$

The operator $\mathbf{A}$, defined by (27), (29), acts in the space of continuous functions $C[-1,1]$ see [11, Chap. 2, Sect. 8]. Thus the analysed contact problem is reduced to the integral equation (27). Its solution follows from the theorem (see [13])

Theorem 1 *Let $g(x) \in B_1^{\alpha}[-1,1],\ 0 < \alpha \le 1$, and let the inequality*

$$\lambda > \lambda_\infty = \frac{D_2}{-D_1 + \sqrt{D_1^2 + 2D_2}} \tag{31}$$

be valid, where

$$D_1 = \max\left|F'(t)\right|, \quad D_2 = \max\left|F''(t)\right|, \quad t \in [0,\infty). \tag{32}$$

Then the solution of the integral equation (27) exists and is unique in the class $C[-1,1]$, and may be obtained by the iterative formula

$$\Phi_n(x) = \mathbf{A}(\Phi_{n-1}) + \Phi_0(x). \tag{33}$$

Theorem 1 not only provides the conditions for the existence and uniqueness of the solution, but also suggests an efficient iterative approach for calculating this solution. In addition, this theorem determines the lower bounds of the nondimensional parameter λ corresponding to the relative thickness of the strip.

The next theorem results in a simple formula for the nth iterative function $\Phi_n(x)$ defined by the formulae (28)–(33).

Theorem 2 *Let $g(x) \in B_1^{\alpha}[-1,1]$, $0 < \alpha \le 1$. Then the function $\Phi_n(x)$, defined by (28) and (33), may be written as*

$$\Phi_n(x) = \frac{P_0}{\pi}\left\{1 + \sum_{i=1}^{n}\int_{-1}^{1} K_i(\tau,x)\,\mathrm{d}\tau\right\} - \frac{1}{\pi}\int_{-1}^{1} g'(t)\sqrt{1-t^2}\left\{\frac{1}{t-x} + \sum_{i=1}^{n}\int_{-1}^{1}\frac{K_i(\tau,x)}{t-\tau}\,\mathrm{d}\tau\right\}\mathrm{d}t, \tag{34}$$

where $K_j(\tau,x)$ are iterated kernels (see [8]) defined by

$$K_1(\tau,x) = m(\tau,x), \tag{35}$$

$$K_j(\tau,x) = \int_{-1}^{1} K_1(\tau,s)K_{j-1}(s,x)\,\mathrm{d}s, \quad j = 2,3,\ldots,n. \tag{36}$$

We are now in a position to calculate the solution $q(x)$ of the contact problem. However, the form of the function $F(t)$ given by (19) is not optimal for the numerical evaluation. We may nevertheless rewrite $F(t)$ as an absolutely convergent power series in terms of the variable t, $|t| < 2$, which immediately follows from Lemma 1. Since $F(t)$ is an even function, the power series has the form

$$F(t) = \sum_{i=0}^{\infty} b_i t^{2i}, \tag{37}$$

where the coefficients of the series are given by

$$b_0 = \int_0^{\infty}\frac{1 - L(u) - \mathrm{e}^{-u}}{u}\,\mathrm{d}u, \tag{38}$$

and

$$b_i = \frac{(-1)^i}{(2i)!}\int_0^{\infty}(1 - L(u))\,u^{2i-1}\,\mathrm{d}u, \quad i = 1,2,\ldots. \tag{39}$$

Finally, we obtain the solution $\phi(x)$ of the integral equation (23) by employing (34) and using the iterated kernels defined by (36) together with (37)–(39). The series expressed in terms of the negative powers of the parameter λ (with property (31)) can be written as

$$\begin{aligned}\phi(x) = {} & \frac{P_0}{\pi\sqrt{1-x^2}}\left[1 - \frac{2b_1}{\lambda^2}\left(\frac{1}{2}-x^2\right) - \frac{4b_2}{\lambda^4}\left(\frac{7}{8}-x^2-x^4\right) - \frac{3b_1b_2}{\lambda^6}\left(\frac{1}{2}-x^2\right) - \frac{6b_3}{\lambda^6}\left(\frac{13}{8}+\frac{3}{4}x^2-\frac{9}{2}x^4-x^6\right)\right] \\ & - \frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^{1}\sqrt{1-t^2}\,g'(t)\left(\frac{1}{t-x} + \frac{2b_1x}{\lambda^2} + \frac{2}{\lambda^4}(-b_1^2x + b_2(3t - 2x + 2xt^2 - 6x^2t + 6x^3)\right) \\ & + \frac{1}{\lambda^6}\left(2b_1^3x - 2b_1b_2(x + 2t^2x + 12x^3) + 3b_3\left(5t + 5t^3 - \frac{x}{2}(11 + 18t^2 - 4t^4) + x^2(5t - 10t^3)\right.\right. \\ & \left.\left. + x^3(5 + 20t^2) - 20x^4t + 10x^5\right)\right)\mathrm{d}t + \mathcal{O}\left(\frac{1}{\lambda^8}\right),\end{aligned} \tag{40}$$

The indenting force P_0 appearing in (40) has to be determined from the relation (24). By substituting the solution $\phi(x)$ given by (40) in (24) we get, for values of λ satisfying inequality (31), the asymptotic formula

$$P_0 = \left[\log 2\lambda - b_0 - \frac{b_1}{\lambda^2} - \frac{b_1^2}{4\lambda^4} - \frac{9b_2}{4\lambda^4} - \frac{2b_1 b_2}{\lambda^6} - \frac{25b_3}{4\lambda^6} + \mathcal{O}\left(\frac{1}{\lambda^8}\right)\right]^{-1}$$
$$\times \left\{\int_{-1}^{1} \frac{g(t)\,\mathrm{d}t}{\sqrt{1-t^2}} + \int_{-1}^{1} \sqrt{1-t^2}g'(t)\,t\left[\frac{b_1}{\lambda^2} + \frac{b_2}{\lambda^4}\left(\frac{7}{2}+t^2\right)\right.\right.$$
$$\left.\left. + \frac{b_3}{\lambda^6}\left(\frac{39}{4}+8t^2+t^4\right) - \frac{3}{2\lambda^6}b_1 b_2 t\right]\mathrm{d}t + \mathcal{O}\left(\frac{1}{\lambda^8}\right)\right\}. \tag{41}$$

The moment acting on the punch can be calculated in a similar manner. Let us introduce the notation

$$M_0 = \frac{P_0 e}{a} = \int_{-1}^{1} \xi\phi(\xi)\,\mathrm{d}\xi, \tag{42}$$

and substitute (40) in (42). The result is

$$M_0 = \int_{-1}^{1} \sqrt{1-t^2}g'(t)\left[1 - \frac{b_1}{\lambda^2} + \frac{b_1^2}{\lambda^4} - \frac{b_2}{\lambda^4}\left(\frac{5}{2}+2t^2\right) - \frac{b_1^3}{\lambda^6}\right.$$
$$\left. + \frac{b_1 b_2}{\lambda^6}\left(\frac{11}{2}+2t^2\right) - \frac{3b_3}{\lambda^6}\left(\frac{9}{4}+3t^2+t^4\right)\right]\mathrm{d}t + \mathcal{O}\left(\frac{1}{\lambda^8}\right). \tag{43}$$

Formulae (41) and (43) will be used in Sect. 6 when computing the force and the moment acting on the punch for various values of λ.

5 Asymptotic analysis of a thin strip

Here we develop an approach based on the asymptotic integration of the differential equations in plane elasticity (e.g., [11]) in the case of a strip subject to a surface transverse load. This approach allows direct evaluation of the contact pressure without making use of the integral-equations theory.

Let us scale the original variables in (3)–(6) setting

$$x = a\eta, \quad y = h\zeta, \tag{44}$$

and express strains through displacements. We get

$$\lambda\frac{\partial\sigma_x}{\partial\eta} + \frac{\partial\sigma_{xy}}{\partial\zeta} = 0, \quad \lambda\frac{\partial\sigma_{xy}}{\partial\eta} + \frac{\partial\sigma_y}{\partial\zeta} = 0, \tag{45}$$

and

$$\frac{G_{12}}{a}\frac{\partial u}{\partial\eta} = k_x\sigma_x - k_{xy}\sigma_y, \quad \frac{G_{12}}{h}\frac{\partial v}{\partial\zeta} = -k_{xy}\sigma_x + k_y\sigma_y, \quad \frac{G_{12}}{2h}\left(\frac{\partial u}{\partial\zeta} + \lambda\frac{\partial v}{\partial\eta}\right) = \sigma_{xy}, \tag{46}$$

where

$$k_x = \frac{(1-\nu_{31}\nu_{13})G_{12}}{E_1}, \quad k_y = \frac{(1-\nu_{32}\nu_{23})G_{12}}{E_2}, \quad k_{xy} = \frac{(\nu_{21}+\nu_{31}\nu_{23})G_{12}}{E_2} \tag{47}$$

The boundary conditions for the base of the punch become ($|\eta| \le 1$)

$$\sigma_{xy}(\eta, 0) = \sigma_{xy}(\eta, 1) = v(\eta, 0) = 0, \quad v(\eta, 1) = -ag(\eta). \tag{48}$$

As above in the paper, here we operate with the dimensionless quantity g normalized by a. The required solution can be expanded in an asymptotic series in terms of the small parameter λ. They are given by

$$u(\eta,\zeta)=a\lambda^{-1}\left(u^{(0)}(\eta,\zeta)+\lambda^2u^{(2)}(\eta,\zeta)+\ldots\right),\quad v(\eta,\zeta)=a\left(v^{(0)}(\eta,\zeta)+\lambda^2v^{(2)}(\eta,\zeta)+\ldots\right),$$
$$\sigma_x(\eta,\zeta)=G_{12}\lambda^{-1}\left(\sigma_x^{(0)}(\eta,\zeta)+\lambda^2\sigma_x^{(2)}(\eta,\zeta)+\ldots\right),\ \sigma_y(\eta,\zeta)=G_{12}\lambda^{-1}\left(\sigma_y^{(0)}(\eta,\zeta)+\lambda^2\sigma_y^{(2)}(\eta,\zeta)+\ldots\right),$$
$$\sigma_{xy}(\eta,\zeta)=G_{12}\lambda^{-1}\left(\sigma_{xy}^{(0)}(\eta,\zeta)+\lambda^2\sigma_{xy}^{(2)}(\eta,\zeta)+\ldots\right). \tag{49}$$

By substituting these relations in the formulae (44)–(46) we have, at leading order,

$$\frac{\partial\sigma_x^{(0)}}{\partial\eta}=0,\quad \frac{\partial\sigma_y^{(0)}}{\partial\zeta}=0,\quad \frac{\partial u^{(0)}}{\partial\eta}=k_x\sigma_x^{(0)}-k_{xy}\sigma_y^{(0)},\quad \frac{\partial v^{(0)}}{\partial\zeta}=k_y\sigma_y^{(0)}-k_{xy}\sigma_x^{(0)},\quad \frac{\partial u^{(0)}}{\partial\zeta}=0 \tag{50}$$

and

$$v^{(0)}(\eta,0)=0,\quad v^{(0)}(\eta,1)=-g(\eta). \tag{51}$$

The solution of the problem (50)–(51) takes the form of a power series in the transverse variable ζ

$$u^{(0)}(\eta,\zeta)=u^{(0,0)}(\eta),\quad v^{(0)}(\eta,\zeta)=\zeta v^{(0,1)}(\eta),\quad \sigma_x^{(0)}(\eta,\zeta)=\sigma_x^{(0,0)}(\eta),\quad \sigma_y^{(0)}(\eta,\zeta)=\sigma_y^{(0,0)}(\eta), \tag{52}$$

where

$$\frac{\partial\sigma_x^{(0,0)}}{\partial\eta}=0,\quad \frac{\partial u^{(0,0)}}{\partial\eta}k_x\sigma_x^{(0,0)}-k_{xy}\sigma_y^{(0,0)},\quad v^{(0,1)}=k_y\sigma_y^{(0,0)}-k_{xy}\sigma_x^{(0,0)}=-g(\eta). \tag{53}$$

The equilibrium equation $(53)_1$ results in a uniform solution $\sigma_x^{(0,0)}(\eta)=C$, where C is a constant. Moreover, $C\equiv 0$, since the strip is not subject to external longitudinal loads. Then, we derive from $(53)_3$ that $\sigma_y^{(0,0)}=-g(\eta)/k_y$, and finally have at leading order

$$\mathrm{q}(x)=\frac{\gamma}{\lambda}g(x), \tag{54}$$

generalising the consideration in [10] to a stamp of arbitrary profile.

Further analysis demonstrates that $\sigma_y^{(2)}=0$. Thus, the relative error of formula (54) is $\mathcal{O}(\lambda^4)$. The associated formulae for the contact force and moment become (see (41) and (43)):

$$P_0=\frac{1}{\lambda}\int_{-1}^{1}g(t)\,\mathrm{d}t, \tag{55}$$

$$M_0=\frac{1}{\lambda}\int_{-1}^{1}tg(t)\,\mathrm{d}t. \tag{56}$$

It is worth noting that the asymptotic methodology above allows only the leading-order approximations for the integral-contact parameters like the indenting force and moment. The point is that initial scaling does not take into consideration the boundary layers localized at the ends of the punch. At the same time, such a limitation does not affect the contact pressure $q(x)$ outside the ends of the punch.

6 Numerical results

Consider the indentation of a plane and parabolic punch to illustrate the efficiency of the developed methodology. For a plane inclined punch we have

$$g(x)=\delta^*+\alpha x,\quad \delta^*=\delta/a. \tag{57}$$

Here and below, in this section, x is again a nondimensionless variable. On substituting (57) in formulas (40), (41) and (43) we get for the contact pressure:

$$\phi(x) = \frac{P_0}{\sqrt{1-x^2}}\left\{1 - \frac{2b_1}{\lambda^2}\left(\frac{1}{2}-x^2\right) - \frac{4b_2}{\lambda^4}\left(\frac{7}{8}-x^2-x^4\right) - \frac{3b_1b_2}{\lambda^6}\left(\frac{1}{2}-x^2\right) - \frac{6b_3}{\lambda^6}\left(\frac{13}{8}+\frac{3}{4}x^2-\frac{9}{2}x^4-x^6\right)\right\}$$
$$+\frac{\alpha x}{\sqrt{1-x^2}}\left\{1 - \frac{b_1}{\lambda^2} + \frac{1}{\lambda^4}\left(b_1^2+\frac{3}{2}b_2\right) - \frac{1}{\lambda^6}\left(b_1^3 - 6b_1b_2 - \frac{45}{4}b_3\right) - x^2\left(\frac{6b_2}{\lambda^4} - \frac{12b_1b_2}{\lambda^6}\right) + \frac{5x^4}{\lambda^6} + \mathcal{O}\left(\frac{1}{\lambda^8}\right)\right\}; \tag{58}$$

for the force:

$$P_0 = \pi\delta^*\left[\log 2\lambda - b_0 - \frac{b_1}{\lambda^2} - \frac{b_1^2}{4\lambda^4} - \frac{9b_2}{4\lambda^4} - \frac{2b_1b_2}{\lambda^6} - \frac{25b_3}{4\lambda^6} + \mathcal{O}\left(\frac{1}{\lambda^8}\right)\right]^{-1}; \tag{59}$$

and for the moment

$$M_0 = \frac{\pi\alpha}{2}\left(1 - \frac{b_1}{\lambda^2} + \frac{b_1^2}{\lambda^4} - \frac{3b_2}{\lambda^4} - \frac{b_1^3}{\lambda^6} + \frac{6b_1b_2}{\lambda^6} - \frac{75b_3}{8\lambda^6} + \mathcal{O}\left(\frac{1}{\lambda^8}\right)\right). \tag{60}$$

It is now possible to evaluate the force and the moment for various values of λ using formulae (59)–(60) above, along with formulae (55) and (56) in the preceding section adapted for a plane punch. The numerical results computed for three cases ($A^2 > B$, $A^2 = B$, and $A^2 < B$, see (9)) are evaluated using a unique scheme in contrast to the consideration [10] separating the computations into three parts. Although the results for a plane punch in the cited and present paper are identical, the numerical data in these two papers differ due to the normalizing factor γ appearing in (15). Table 1 corresponds to the value $B = 1$. The numerics for $\lambda \geq 2$ and $\lambda \leq 2$ were computed using formulae (59)–(60) and (55)–(56), respectively. Note that the data for relatively small λ do not depend upon the material constant A as it follows from the direct asymptotic derivation presented in Sect. 3. Theorem 1 formulated in Sect. 4 predicts the lower bound for the validity range of the developed iterative approach, expressed in terms of the parameter λ (see (31)). This bound, given by the maximum values of both $|F'(t)|$ and $|F''(t)|$, can be calculated on using (19). The result is

$$A = 2, \quad B = 1, \quad \lambda_\infty = 2.33519, \tag{61}$$
$$A = 1, \quad B = 1, \quad \lambda_\infty = 1.75975, \tag{62}$$
$$A = 1/2, \quad B = 1, \quad \lambda_\infty = 1.37621. \tag{63}$$

These computations justify making use of the iterative procedure above for the three left columns of Table 1. Next, we set

$$g(x) = \delta^* - \kappa^* x^2, \quad \kappa^* = \kappa/a \tag{64}$$

for a punch of the parabolic shape given by $f(x) = \alpha x + \kappa^* x^2$. We can once again insert (64) into (40), (41) and (43) to get

$$\phi(x) = \frac{P_0}{\sqrt{1-x^2}}\left\{1 - \frac{2b_1}{\lambda^2}\left(\frac{1}{2}-x^2\right) - \frac{4b_2}{\lambda^4}\left(\frac{7}{8}-x^2-x^4\right) - \frac{3b_1b_2}{\lambda^6}\left(\frac{1}{2}-x^2\right) - \frac{6b_3}{\lambda^6}\left(\frac{13}{8}+\frac{3}{4}x^2-\frac{9}{2}x^4-x^6\right)\right\}$$
$$+\frac{2\kappa^*}{\sqrt{1-x^2}}\left\{\left(\frac{1}{2}-x^2\right) - \frac{1}{\lambda^4}\left(\frac{3b_2}{4} - \frac{3b_2}{2}x^2\right) - \frac{1}{\lambda^6}\left(\frac{45b_3}{16} - \frac{5}{2}x^2\right)\right\}, \tag{65}$$

and

$$P_0 = \pi\delta^*\left[\log 2\lambda - b_0 - \frac{b_1}{\lambda^2} - \frac{b_1^2}{4\lambda^4} - \frac{9b_2}{4\lambda^4} - \frac{2b_1b_2}{\lambda^6} - \frac{25b_3}{4\lambda^6} + \mathcal{O}\left(\frac{1}{\lambda^8}\right)\right]^{-1} \times \left[\delta^* - \frac{\kappa^*}{2}\left(1 + \frac{b_1}{2\lambda^2} + \frac{2b_2}{\lambda^4} + \frac{225b_3}{32\lambda^6}\right)\right]. \tag{66}$$

We do not present here a formula for the moment M_0 since, for a parabolic punch, the integrand in (43) is an odd function and therefore the integral vanishes.

The numerical results for the quantity $P_0/(\delta^*\kappa^*)$ are given in Table 2. As before we take $B = 1$, and start from (66) and (55); for the latter we use the substitution (64). A similar table is presented in [5, Table 15] for the isotropic

Table 1 Numerical values of the pressure P_0/δ^* and moment M_0/α for different values of the relative thicknesses of the strip

A	$\lambda = 8$	$\lambda = 4$	$\lambda = 2$		$\lambda = 1$	$\lambda = 1/2$
P_0/δ^*						
2	0.583288	0.8299	1.28903	1	2	4
1	0.646694	0.893022	1.3692	1	2	4
1/2	0.69665	0.942544	1.41292	1	2	4
M_0/α						
2	0.650623	0.677439	0.774312	0.333	0.666	1.333
1	0.791767	0.810574	0.881693	0.333	0.666	1.333
1/2	0.911184	0.924126	0.976611	0.333	0.666	1.333

Table 2 Numerical values of the pressure $P_0/(\delta^*\kappa^*)$ for different base shapes of the punch and the relative strip thicknesses

A	δ^*/κ^*	$\lambda = 8$	$\lambda = 4$	$\lambda = 2$		$\lambda = 1$	$\lambda = 1/2$
2	1	0.293721	0.425686	0.70226	0.666667	1.333	2.66667
	2	0.877009	1.25559	1.99129	1.66667	3.333	6.66667
	3	1.4603	2.08549	3.28031	2.66667	5.333	10.6667
1	1	0.324642	0.453337	0.720242	0.666667	1.333	2.66667
	2	0.971336	1.34636	2.08945	1.66667	3.333	6.66667
	3	1.61803	2.23938	3.45865	2.66667	5.333	10.6667
1/2	1	0.349143	0.475646	0.731082	0.666667	1.333	2.66667
	2	1.04579	1.41819	2.144	1.66667	3.333	6.66667
	3	1.74244	2.36073	3.55692	2.66667	5.333	10.6667

case with no eccentricity. The data in Table 2 complement the numerics in both [5] and [10]. We also note that the explicit formulae for pressure and moment obtained in Sects. 4 and 5 (see (41), (55) and (43), (56)) may be combined to give composite formulae demonstrating the same asymptotic behaviour for small and large values of the parameter λ. In particular, a simple composite formula of this type for the force can be written as

$$P_0 = \frac{1}{\log 2\lambda - b_0} \int_{-1}^{1} \frac{g(t)}{\sqrt{1-t^2}}\,\mathrm{d}t + \frac{1}{\lambda}\int_{-1}^{1} g(t)\,\mathrm{d}t. \tag{67}$$

Its counterpart for the moment is

$$M_0 = \int_{-1}^{1} \sqrt{1-t^2}\,g'(t)\,\mathrm{d}t + \frac{1}{\lambda}\int_{-1}^{1} t g(t)\,\mathrm{d}t. \tag{68}$$

Figure 2 illustrates the formula (67) for a plane punch of Example 1 for $A = 2,\ B = 1$. It is clear that the numerical values evaluated by the composite formulae tend to the asymptotic limits for small and large λ. We may expect that this formula also provides reasonable predictions for the intermediate values of the strip thickness.

As an example of an advanced orthotropic material we consider the glass-fibre plastic with the material constants

$$E_1 = 13, \quad E_2 = 19.8, \quad G_{12} = 4.05, \quad \nu_{12} = 0.176 \quad \nu_{13} = 0.25, \quad \nu_{23} = 0.06, \quad \nu_{32} = 0.4. \tag{69}$$

In doing so, the remaining constants can be found using relations (3). Inserting these values into the formula for A and B (see (9)) we get $A = 8.02956$ and $B = 1.75379$. Once again employing formulae (41) and (43) for a plane punch of Example 1, we plot in Fig. 3 the pressure and moment versus the parameter λ.

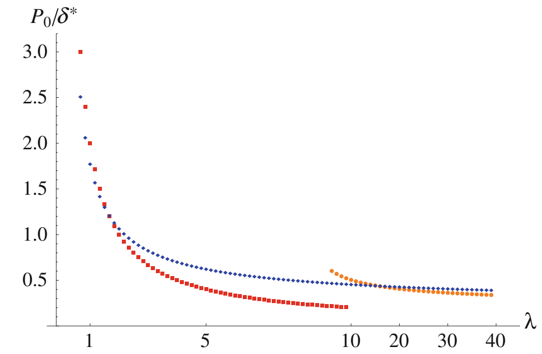

Fig. 2 The pressure versus the relative strip thickness λ (*filled circle*—formula (41) for large λ, *filled square*—formula (55) for small λ, *filled diamond*—composite formula (67))

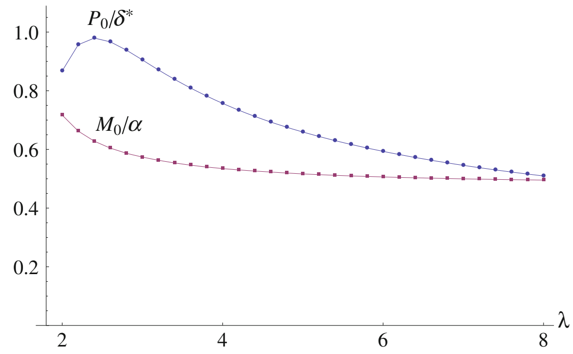

Fig. 3 The pressure and moment versus the relative strip thickness for the orthotropic glass-fibre plastic material

7 Conclusions

We have derived an integral equation for the contact problem involving an orthotropic strip that is unique for all types of orthotropy. The associated kernel corresponds to the normalisation of the contact stress by the parameter $\gamma = E_2/(1 - \nu_{23}\nu_{32})$. The latter seems to be crucial to characterise the contact interaction in question.

For a relatively thick strip we adapted an iterative approach developing the procedures established in [5,10] for isotropic and orthotropic formulations, respectively. For a thin orthotropic strip we developed a direct procedure, not assuming the reduction of the original mixed boundary-value problem to an integral equation. It originates in the asymptotic analysis of a plate subject to transverse surface loading; e.g. see [11]. The obtained explicit formulae for the contact force and moment may be extended to more-complicated geometries including arbitrary elastic shells.

Numerical results for plane and parabolic punches were presented for a number of combinations of the elastic constants. All the computed data for a plane punch were seen to coincide with those in [10] to within normalisation. We may expect that the composite formula constructed from the asymptotic expansions for a thin and a thick strip also provides a qualitative insight in the intermediate range of thicknesses.

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