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# What makes the Peregrine soliton so special as a prototype of freak waves?

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**Abstract** The formation of breathers as prototypes of freak waves is studied within the framework of the classic 'focussing' nonlinear Schrödinger (NLS) equation. The analysis is confined to evolution of localised initial perturbations upon an otherwise uniform wave train. For a breather to emerge out of an initial hump, a certain integral over the hump, which we refer to as the "area", should exceed a certain critical value. It is shown that the breathers produced by the critical and slightly supercritical initial perturbations are described by the Peregrine soliton which represents a spatially localised breather with only one oscillation in time and thus captures the main feature of freak waves: a propensity to appear out of nowhere and disappear without trace. The maximal amplitude of the Peregrine soliton equals exactly three times the amplitude of the unperturbed uniform wave train. It is found that, independently of the proximity to criticality, all small-amplitude supercritical humps generate the Peregrine solitons to leading order. Since the criticality condition requires the spatial scale of the initially small perturbation to be very large (inversely proportional to the square root of the smallness of the hump magnitude), this allows one to predict a priori whether a freak wave could develop judging just by the presence/absence of the corresponding scales in the initial conditions. If a freak wave does develop, it will be most likely the Peregrine soliton with the peak amplitude close to three times the background level. Hence, within the framework of the one-dimensional NLS equation the Peregrine soliton describes the most likely freak-wave patterns. The prospects of applying the findings to real-world freak waves are also discussed.

**Keywords** Breathers · Nonlinear Schrödinger equation · Pulses in optical fibres · Rogue waves · Water waves

## 1 Introduction

Our civilisation increasingly depends on shipping and ever expanding offshore activities. However, despite the progress of engineering, freak waves in the ocean remain a serious danger for ships and offshore structures.

The key and the most unpleasant feature of freak waves making them such a serious threat for all human activities in the sea is their propensity to appear seemingly out of nowhere, i.e., in otherwise relatively benign conditions and

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without any visible precursors. There was a massive effort to study these rare, but unfortunately not negligibly rare, events. Considerable progress has been made, and, on the theoretical front, so far two main generic mechanisms have been identified in the absence of wave-current interaction: the Benjamin–Feir (BF) or modulational instability [1–5] and an essentially linear space-time focussing; see e.g. [6,7] for a recent review. A combination of both mechanisms should be regarded as the general case. For the purposes of the present paper we roughly define freak wave as a very short wave packet (could be one wave) having amplitude significantly larger than the surrounding waves.

Although observations suggest that a freak wave could be essentially nonlinear, we pursue the line of thought that linear and weakly nonlinear mechanisms play the key role in its formation, even if they cannot explain some strongly nonlinear features. The situation resembles that of wave breaking which in itself is an essentially nonlinear phenomenon, but is caused by the development of some linear and weakly nonlinear effects [8]

Howell Peregrine was one of the first who identified the key role of the modulational instability in the formation of patterns resembling freak waves, in particular, as early as 1983 he attracted attention to some simple solutions of the NLS which could serve as the freak-wave weakly nonlinear prototypes [3]. His insight was later supported by numerical experiments reported in [5] and [9] within the framework of exact 2D potential theory. The next important step was the extension of the analysis (so far confined to model equations) to random wave fields [10–12]. The criterion specifying when the modulational instability persists for narrow-band random wave fields was found and on its basis Janssen [13] introduced the *Benjamin–Feir Index* (BFI) to characterize the narrowness of the wind-wave spectra (more precisely the ratio of nonlinear and dispersion effects for narrow band spectra) and to identify *a priori* those situations, where the BF instability is more likely. The direct numerical simulations of random wind waves indeed show that the strong departure of wind-wave height statistics from Gaussianity occurs only for rare situations of narrow band spectra [14]. The recorded observations of freak waves also do gravitate to such situations. However, it is not possible to quantify the risk. The fact that the wave field is modulationally unstable does not imply that a freak wave necessarily emerges. Thus, at present the only way to minimize the risk of encountering a freak wave is to avoid the areas where operational wave models predict BFI $\sim$1.

In order to progress to better prediction of freak waves, it would be highly desirable to identify some other features of the sea state (apart from the BFI) or environment (e.g., wind field, bottom relief, currents) with causality relationships with freak waves in order to develop some elements of deterministic forecasting of freak waves. It would be even better if one could know in advance what kind of freak wave to expect. There exists a line of research where wave-field evolution is linked to an integrable system, say NLS, whose associated Inverse Scattering problem spectra are somehow extracted from field data; see e.g. [15–18]. The present paper is close to this line and makes a step aimed at addressing these needs.

The situations characterised by the case BFI $\sim$1 are not well understood theoretically since, in this range of parameters, standard kinetic description of random wave fields is not applicable and new approaches taking into account existence of coherent patterns have yet to be developed. Our strategy is to get a new insight by analysis of the simplest possible model in a deterministic setting and then by stretching the argument to try to project the model predictions into the real world. In discussing the applications it is also worth noting that freak waves have recently become a subject of intense interest in nonlinear optics (see e.g. [19] and references therein), where the underlying mathematics is often remarkably similar to that of water waves and application of simple models is more straightforward.

The basic model for a deterministic description of narrow-band wave fields is the classic 'focussing' nonlinear Schrödinger (NLS) equation (e.g., [20, Sect.14.8], [3], [21, Sect.4.3]),

$$\mathrm{i}u_t + u_{xx} + 2|u|^2u = 0, \tag{1}$$

where $u$ is the normalized envelope amplitude, $x = \varepsilon(X - C_gT)$ and $t = \varepsilon^2T$ are slow spatial and temporal variables, $X$ and $T$ are the spatial and temporal variables scaled by wave length and period, $C_g$ is the group velocity and $\varepsilon \ll 1$ is the nonlinearity parameter (the NLS in the original physical variables could be found e.g. in [3], [20, Sect.14.8]. The NLS is exactly solvable by the Inverse Scattering Technique (IST) [22–25] and some other methods, which enables one to find wide classes of exact solutions and (in principle) to solve an arbitrary initial-value problem. In the water-wave context it is natural to assume statistical spatial uniformity of the wave field. It is convenient to view the field as a sum of a basic state (strictly uniform wave train $|u| = \sqrt{\rho_0}$) and an initial (not necessarily small) perturbation. At present there is no technique enabling one to deal with the initial problem with random initial perturbation given on the infinite interval. We assume the initial perturbation of the wave train to be localized in the following sense: $|u| \to \text{const} = \sqrt{\rho_0}$ as $x \to \pm\infty$. Of course the localization assumption cannot be justified since a strictly uniform wave wave train $|u| = \sqrt{\rho_0}$ is unstable with respect to the modulational instability; we just assume that the perturbations arising from such an instability would not affect the processes in the domain under consideration. It is helpful first to review briefly the basic exact solutions which were discussed as weakly nonlinear prototypes of freak waves (for more detail see [26]).

There are two important families of simple solutions. The first one is often referred to as *Ma solitons* or *Ma breathers* [27]; we will, however, refer to it as *Kuznetsov–Ma* solitons, since these solutions were first found in [28]. The solutions have a simple explicit form

$$\frac{u}{\sqrt{\rho_0}} = \left[1 - 2s\,\frac{s\cos\Omega(t-t_0) + \mathrm{i}\sqrt{1+s^2}\sin\Omega(t-t_0)}{\sqrt{1+s^2}\cosh(2s(x-x_0)) - \cos\Omega(t-t_0)}\right]\mathrm{e}^{\mathrm{i}\sqrt{2}\,\rho_0 t}. \tag{2}$$

They describe spatially localized patterns ($|u|$ at $x \to \pm\infty$ is constant) oscillating in time with frequency $\Omega = 4s\sqrt{1+s^2}$, where $s$ is the amplitude parameter.

The second key family is the *Akhmediev breathers* which describe spatially periodic solutions which 'breathe' only once [29]. Here we do not provide the explicit expressions of the solutions which are linked via a simple transformation with the first family (see e.g. [26]), since we will not use them here. To complete the picture we mention that there are also more general solutions (see [19,29,30]) and a solution resulting from the interaction between the simple solutions. In our context it is important that, in the limit $s \to 0$, both the Kuznetsov–Ma and *Akhmediev breathers* tend to the same remarkably simple solution found by Peregrine [3] and called the *Peregrine soliton*

$$\frac{u}{\sqrt{\rho_0}} = \left[1 - \frac{4 + 16\mathrm{i}(t-t_0)}{1 + 4(x-x_0)^2 + 16(t-t_0)^2}\right]\mathrm{e}^{\mathrm{i}\sqrt{2}\,\rho_0 t}. \tag{3}$$

This soliton represents a single hump both in space and time centered at $x_0$ and $t_0$; it decays as $x^{-2}$ at $x \to \pm\infty$ (in contrast to the exponentially decaying non-degenerate Kuznetsov–Ma solitons (2)) and as $t^{-2}$ at $t \to \pm\infty$ (in contrast to the exponentially decaying non-degenerate Akhmediev breathers). The maximum amplitude in the Peregrine soliton (the maximal height of the elevation) is exactly three times the amplitude of the unperturbed uniform wave train, while both its characteristic length and the lifespan are of order one. In the context of freak waves, the distinguished character of the Peregrine soliton is immediately apparent: it is the simplest NLS solution localized both in space and time, which qualitatively reproduces the main feature of freak waves—their propensity to appear out of nowhere and then again disappear. Generalizations of (3) with the same fundamental property found in [19,30] are of great interest; however, as we will show below, only the Peregrine soliton represents asymptotics of a wide class of initial conditions. It was further noticed, on the basis of simulations within the framework of exact 2D potential equations [31–33], that the very steep short wave packets emerging as a result of the nonlinear stage of the BF instability are close to the above analytical solutions. The conjecture was later supported by the analysis of [26] (also within the framework of exact 2D potential theory). In the present work we discuss deeper reasons distinguishing this solution from the perspective of the initial-value problem within the framework of the NLS. Although the Cauchy problem for the NLS was studied in many papers, it has not be realized that the Peregrine soliton represents asymptotics of a wide class of initial conditions that are very interesting for applications. The present work shows it to be the case and specifies the conditions when the initial perturbation generates patterns close to the Peregrine soliton. Apparently this might shed a new light on the problem of freak-wave forecasting as well.

In Sect. 2 we briefly discuss the mathematical formulation of the problem and the underlying physics. We are interested in the evolution of initially small perturbations of a uniform train which attain large amplitudes at the moment of observation due to the modulational instability. We are interested in the basic questions: what are the likely maximum amplitudes of such perturbations and what initial conditions allow the perturbations to attain substantial amplitudes? We formulate the problem in terms of the Inverse Scattering Technique (IST) formalism for the NLS as a problem of finding discrete eigenvalues for the chosen class of initial conditions. The contribution of the continuous spectrum could result in essentially linear focussing which we do not consider in this paper. In Sect. 3 we show that $O(\epsilon)$, $(\epsilon \ll 1)$ small generic initial perturbations could generate a finite number of Kuznetsov–Ma breathers (2) with small $O(\epsilon^{1/2})$ values of the amplitude parameter $s$, which implies that their maximal amplitudes are approximately thrice the unperturbed wave amplitude. The Peregrine soliton (3) is shown to be a very good approximation of such solutions. For such breathers to emerge the initial perturbation must have positive elevation and be $O(\epsilon^{-1/2})$ long. More precisely, in order for at least one Peregrine soliton to emerge, a certain normalised integral (which we will refer to for simplicity as normalized "area") of the initial perturbation should exceed $\pi/4$; for $N$ solitons to emerge the "area" must be $\geq (N + 1/4)\pi$, but less than $(N + 5/4)\pi/2$, provided the unperturbed amplitude is normalized to be unity. In the concluding Sect. 4 we discuss the implications of the findings for real-world situations.

## 2 Formulation of the problem

Our starting point is the one-dimensional nonlinear Schrödinger equation (1). We confine our attention to the class of initial conditions which sufficiently rapidly tend to a finite constant at infinity: $|u| \to \sqrt{\rho_0}$ as $x \to \pm\infty$. The assumption of localised initial conditions is adopted to allow the use of the powerful Inverse Scattering Technique (IST) (e.g. [22–25]). It is not as restricting as it may seem, since in reality we are interested in what happens in a certain finite domain in the $x, t$-plane; this selects a finite interval on the $x$-axis where initial data might affect the outcome in the space–time window of interest. Since small perturbations are most common in nature we will focus on studying nonlinear evolution of initially small perturbations of the uniform wave train.

The IST is based on the remarkable fact that to find nonlinear evolution of the solutions to the NLS equation (1) satisfying a given localized initial condition $u(x, t)|_{t=0} \equiv u(x)$, it is sufficient to solve the following linear system of equations, often referred to as the *inverse scattering problem*, where the initial distribution $u(x)$ enters as a given function ($\bar{u}(x)$ is complex conjugate of $u(x)$) [22,23], [25, Chap. 1]:

$$\psi_x^{(1)} = -\frac{1}{2}\mathrm{i}\lambda\psi^{(1)} + \mathrm{i}\bar{u}\psi^{(2)}, \quad \psi_x^{(2)} = \frac{1}{2}\mathrm{i}\lambda\psi^{(2)} + \mathrm{i}u\psi^{(1)}. \tag{4}$$

Since the IST problem (4) represents a second-order linear problem with variable coefficients dependent on $u(x)$, there is no analytic solution for arbitrary $u(x)$. However, for many purposes it is often sufficient to analyse the spectral properties of the solutions of (4) that vanish as $x \to \pm\infty$.

A basic property of system (4) is that it has an involution, that is, if $\psi = \begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \end{pmatrix}$ is a solution corresponding to a value of the spectral parameter $\lambda$, then $\tilde{\psi} = \begin{pmatrix} -\bar{\psi}^{(2)} \\ \bar{\psi}^{(1)} \end{pmatrix}$ is also a solution of (4) with the same $\lambda$. It is easy to see that for $x \to \pm\infty$ (where $|u| \to \sqrt{\rho_0}$) the system has two linearly independent solutions, called the Jost functions,

$$\psi = \begin{pmatrix} 1 \\ \frac{\lambda - k}{2\sqrt{\rho_0}} \end{pmatrix} \mathrm{e}^{-\frac{\mathrm{i}}{2}kx} \quad \text{and} \quad \tilde{\psi} = \begin{pmatrix} -\frac{\lambda - k}{2\sqrt{\rho_0}} \\ 1 \end{pmatrix} \mathrm{e}^{\frac{\mathrm{i}}{2}\bar{k}x}.$$

The parameter $k$ we introduced here is expressed in terms of the spectral parameter $\lambda$ and the amplitude of the constant potential $\sqrt{\rho_0}$,

$$k = \sqrt{\lambda^2 + 4\rho_0}. \tag{5}$$

Note that $k = 0$ at $\lambda = 2\mathrm{i}\sqrt{\rho_0}$. The Jost functions describe exponentially decaying asymptotics of all localized solutions to (4), The properties of system (4) are accumulated in the so-called "scattering matrix" which links the two Jost functions and depends only on $\lambda$ [22,25,28].

We focus upon the analysis of the discrete spectrum, since the discrete eigenvalues of (4) correspond to solitons of the NLS (e.g. [25, Chap. 1], [28]). In particular, discrete imaginary values of the spectral parameter

$$\lambda = 2\mathrm{i}\sqrt{\rho_0(1+4s^2)} \tag{6}$$

correspond to the Kuznetsov–Ma solitons (2) with $s$ being the soliton-amplitude parameter [28]. The Peregrine soliton is the limiting case corresponding to $s \to 0$ or, in terms of $\lambda$, $\mathfrak{Im}\,\lambda \to 2\sqrt{\rho_0}$. Spectral points with non-zero $\mathfrak{Re}\,\lambda$ correspond to "moving solitons".

Genesis of the Kuznetsov–Ma solutions characterised by small values of the amplitude parameter $s$ is the prime subject of the present paper and, as we will show below, might be of great interest in the context of freak waves. Although the continuous-spectrum solutions could also contribute to the formation of freak-wave events via the well-understood mechanism of focussing [6,34], a study of the combined effect of discrete and the continuous spectrum goes beyond the scope of this work and the continuous spectrum will be ignored throughout the paper. We justify the neglect by arguing that a continuous spectrum leads to comparatively rare freak events with already well-understood mechanisms and it is relatively straightforward to find near-Gaussian statistical distributions. In terms of spectral problem (4) the central issue of the study can be formulated as finding properties of the discrete spectrum, primarily for small-amplitude initial perturbations. In plain words, we will attempt to understand what discrete eigenvalues are possible and under what conditions. Then important conclusions on the types and likelihood of possible solitary/breather freak waves will follow.

## 3 Origination of "small" solitons

In this section we first briefly consider a simple model of a rectangular hump in the initial envelope amplitude where the analysis of the required spectral properties of the inverse problem (4) could be carried out easily and without any additional assumptions. The explicit condition for spectral points to appear requires a certain integral property of the initial perturbation, which, for brevity, we will refer to as "area", to exceed $\pi/4$. We show that in case of small initial perturbations, as well as for perturbations whose "area" just exceeds a certain critical value, we found the resulting solitons are characterised by small values of the amplitude parameter $s$. Breathers with zero value of $s$ correspond to the Peregrine soliton while those with small $s$ are shown to be well approximated by this solution. In Sect. 3.2 we address the problem of generic small initial perturbations: we find how the eigenvalues of (4) and soliton amplitude parameters scale with the amplitude of initial perturbations of arbitrary shape. For small initial perturbations we arrive at bounds on the soliton amplitude parameter. Then in Sect. 3.3 for smooth generic small initial perturbations we show how the discrete eigenvalues of (4) could be found by virtue of the WKB method and confirm our preliminary conclusions on the special role of the Peregrine soliton based on the model of rectangular potentials.

### 3.1 Model with rectangular potential

First consider as a simple example $u(x)$ with a rectangular hump in $|u|$

$$u(x) = \begin{cases} \sqrt{\rho_0}\,\mathrm{e}^{\mathrm{i}\sqrt{2}\,\rho_0 t}, & x < 0 \\ \sqrt{\rho_1}\,\mathrm{e}^{\mathrm{i}\sqrt{2}\,\rho_1 t}, & 0 < x < x_0 \\ \sqrt{\rho_0}\,\mathrm{e}^{\mathrm{i}\sqrt{2}\,\rho_0 t}, & x > x_0, \end{cases}$$

where $\rho_0$, $\rho_1$ are positive constants, and the phase is assumed constant and without loss of generality is set to be zero. The behavior of the solution of (4) is in many aspects similar to the classic solution of Schrödinger's equation with rectangular potential (see e.g. [35, Chap. 3]). For a solution to be localized it has to decay for $x \to \pm\infty$, that is to behave as $\mathrm{e}^{-\frac{\mathrm{i}}{2}kx}$ or $\mathrm{e}^{\frac{\mathrm{i}}{2}\bar{k}x}$ in the corresponding limits. By virtue of (5) this requirement implies that $\mathfrak{Im}\,\lambda > 2\sqrt{\rho_0}$.

If $\mathfrak{Im}\,\lambda > 2\sqrt{\rho_1}$, the behavior of the solution is exponential for any $x$, which means that it cannot decay simultaneously for $x \to -\infty$ and $x \to +\infty$. Hence the spectrum of eigenvalues $\lambda$ for this potential is confined to the range $2\sqrt{\rho_0} < \mathfrak{Im}\,\lambda < 2\sqrt{\rho_1}$.

We begin by constructing a localised solution by choosing the decaying exponent for $x < 0$,

$$\psi = \begin{pmatrix} 1 \\ \frac{\lambda-k}{2\sqrt{\rho_0}} \end{pmatrix} e^{-\frac{i}{2}kx}, \quad (x < 0),$$

i.e., with $\mathfrak{Im}\,k < 0$, where $k = \sqrt{\lambda^2 + 4\rho_0}$ and, correspondingly, $\lambda^2 + 4\rho_0 < 0$. If also $\lambda^2 + 4\rho_1 > 0$, this solution could be continued into the interval $0 < x < x_0$ as an oscillating function

$$\psi = \begin{pmatrix} 1 \\ \frac{\lambda-k}{2\sqrt{\rho_0}} \end{pmatrix} \cos\frac{k_1 x}{2} + \begin{pmatrix} \frac{\lambda-k}{2\sqrt{\rho_0}}\frac{\lambda+k_1}{2\sqrt{\rho_1}} \\ \frac{-\lambda-k_1}{2\sqrt{\rho_1}} \end{pmatrix} \sin\frac{k_1 x}{2}, \quad k_1 = \sqrt{\lambda^2 + 4\rho_1}.$$

For $x > x_0$ the solution turns into a decaying exponent

$$\psi = C \begin{pmatrix} \frac{\lambda-k}{2\sqrt{\rho_0}} \\ 1 \end{pmatrix} e^{\frac{i}{2}kx}.$$

In order for $\lambda$ to be a spectral point, this solution should match the oscillating solution. The matching condition (in addition to the already utilised $\rho_1 > -\lambda^2 > 4\rho_0$) yields

$$\tan\frac{k_1 x_0}{2} = \sqrt{\frac{\rho_1}{\rho_0}}, \quad \text{where} \quad \frac{\rho_1}{\rho_0} > 1\,. \tag{7}$$

It is easy to see that (7) allows only real $k_1$ and, since $k_1 = \sqrt{\lambda^2 + 4\rho_1}$, all the spectral points in terms of $\lambda$ are purely imaginary. The physical interpretation of $k_1$ is the rate of the solution oscillation ("wavenumber") in the hump area. The solution could have vanishing exponents at both sides of the hump only if the phase shift over the length of the hump is approximately $\pi/2$. For such a shift to occur, the hump "area" has to exceed a certain critical value which we will specify below.

By definition $k_1 = \sqrt{\lambda^2 + 4\rho_1}$; it is convenient by virtue of (5) to express it as $k_1 = \sqrt{k^2 + 4(\rho_1 - \rho_0)}$. Since, for a localized mode to exist, $k^2$ must be negative, the maximal value of $k_1$ corresponding to the critical (threshold) situation is $2\sqrt{\rho_1 - \rho_0}$. Thus, for a discrete eigenvalue, and, correspondingly, a soliton to appear, the length of the positive perturbation should exceed a threshold:

$$x_0 \geq \frac{\arctan\sqrt{\frac{\rho_1}{\rho_0}}}{\sqrt{\rho_1 - \rho_0}}\,. \tag{8}$$

For simplicity we refer to the quantity $x_0\sqrt{\rho_1 - \rho_0}$ as the "*hump area*", which for a small hump is equal to the product of its true length $x_0$ and square root of "height" defined as $\rho_1 - \rho_0$. The critical (threshold) eigenvalue in terms of $\lambda$ is $2i\sqrt{\rho_0}$ which by virtue (6) implies that the critical solutions have $s = 0$ and therefore are the Peregrine solitons (3).

Since we are primarily interested in small perturbations to the uniform wave train, assume $\rho_1 - \rho_0 \sim \rho_0\epsilon$, $(\epsilon \ll 1)$. It is easy to see from condition (7) that to leading order

$$k_1 x_0 \simeq 2\pi(1/4 + N), \quad (N \text{ is an arbitrary integer}). \tag{9}$$

Thus, the criticality condition (8) implies that for small $\epsilon$, in order for a discrete eigenevalue, and, correspondingly, a soliton to appear, the length of the positive perturbation should exceed a threshold inversely proportional to the square root of the hump height:

$$x_0 \geq \frac{\pi}{4}\frac{1}{\sqrt{\rho_1 - \rho_0}} \simeq \frac{\pi}{4\sqrt{\rho_0\epsilon}} \sim \frac{1}{\sqrt{\epsilon}}\,. \tag{10}$$

Similarly, for a hump to produce $N$ solitons:

$$x_0 \geq \pi(1/4 + N)\frac{1}{\sqrt{\rho_1 - \rho_0}}\,. \tag{11}$$

Obviously, as in the general case, the critical solutions for small humps are the Peregrine solitons. Now consider the range of allowed $s$ for small-amplitude humps. Recall that, whatever the number of the solitons, the range of allowed $\lambda$ is restricted by the height of the hump:

$$2\sqrt{\rho_0} < \mathfrak{Im}\,\lambda < 2\sqrt{\rho_1} = 2\sqrt{\rho_0(1+\epsilon)}\ . \tag{12}$$

Because of (6), the soliton amplitude parameter $s$ is confined to the range $0 < s < 2\sqrt{\epsilon}$. Note that (12) does not require $\epsilon \ll 1$, however; obviously, if $\epsilon \ll 1$, the amplitude parameter $s$ cannot exceed $2\sqrt{\epsilon}$.

Consider the Kuznetsov–Ma solitons (2) with small $s$ in more detail. Expansion of the exponentials and trigonometric functions in (2) by assuming smallness of their arguments yields the Peregrine soliton. This expansion is valid as long as $s|x-x_0| \ll 1,\ \ s|t-t_0| \ll 1$. Thus, for small but finite $s$, the Kuznetsov–Ma solitons have a "hybrid" character: the core where the solution attains significant amplitude is described by the Peregrine solution, while the tails for $|x-x_0| \sim 1/s$ become exponential and the motion maintains its periodic character, which becomes apparent only for $O(s^{-1})$ large times. The amplitudes of such hybrid creatures being $\sqrt{\rho_0}(1+2\sqrt{1+s^2}) \simeq \sqrt{\rho_0}(3+s^2)$ are slightly larger than the amplitudes of the true Peregrine solutions (3) which are equal to $3\sqrt{\rho_0}$: the difference is $s^2\sqrt{\rho_0}$ in terms of $s$ and in terms of $\epsilon$ is confined by $4\epsilon\sqrt{\rho_0}$. It is therefore expected that in all experimental and numerical observations the Kuznetsov–Ma solitons with small $s$ will be practically indistinguishable from the Peregrine soliton.

### 3.2 Scaling of the eigenvalues for generic shallow potential

So far we have described the behaviour of the eigenvalues for simple model potentials corresponding to initial rectangular humps of height $\epsilon$. We found that the soliton amplitude parameter $s$ is bounded from above by $2\sqrt{\epsilon}$. Here we attempt to address the question on whether these findings are robust enough to hold for shallow potentials of arbitrary shape. To this end we consider $O(\epsilon)$ small generic localised initial perturbations, i.e., $|u(x)-\sqrt{\rho_0}| \sim \epsilon\sqrt{\rho_0}$.

We exploit a scaling transformation of the IST system (4) which expands the potential along the $x$-axis simultaneously contracting its magnitude. To this end we re-write (4) excluding $\psi^{(2)}$ as a single second-order equation

$$\psi^{(1)}_{xx} - \frac{\bar{u}_x}{\bar{u}}\,\psi^{(1)}_x + \frac{\lambda}{2}\,\frac{\bar{u}_x}{\bar{u}}\,\psi^{(1)} + \frac{\lambda^2}{4}\,\psi^{(1)} + |u|^2\psi^{(1)} = 0.$$

We change the dependent variable by substituting

$$\psi^{(1)} = \exp\left(\mathrm{i}\int\left(\chi + \frac{\mathrm{i}}{2}\frac{\bar{u}_x}{\bar{u}}\right)\mathrm{d}x\right),$$

which yields the following Riccati equation for the new dependent variable $\chi$

$$\mathrm{i}\chi_x = \chi^2 + \frac{1}{4}\lambda^2 - |u|^2 + \frac{1}{4}\left(\frac{\bar{u}_x}{\bar{u}}\right)^2 + \frac{\mathrm{i}}{2}\partial_x\left(\frac{\bar{u}_x}{\bar{u}}\right). \tag{13}$$

We re-write (13) employing auxiliary variables $v = \bar{u}_x/\bar{u}$ and $h = |u|^2 - \rho_0$; on using relation (5) for $k(\lambda, \rho_0)$ we get

$$\mathrm{i}\chi_x = \chi^2 + \frac{1}{4}k^2 - h + \frac{1}{4}v^2 + \frac{\mathrm{i}}{2}v_x.$$

The spectrum of this equation is more convenient to analyse in terms of $k$ instead of $\lambda$. The equation allows the scaling transformation

$$h \to \epsilon h,\quad v \to \sqrt{\epsilon}v,\quad \chi \to \sqrt{\epsilon}\chi,\quad x \to 1/\sqrt{\epsilon}x,\quad k \to \epsilon k. \tag{14}$$

For any given shape of the potential $u(x)$ transformation (14) shows the following: if the "depth" of the potential becomes smaller, say, $\sim\epsilon$ (i.e., the height of the hump decreases to become $O(\epsilon)$), the eigenvalues in terms of $k$ also scale as $\epsilon$. Therefore, the $O(\epsilon)$ shallow potentials allow only $O(\epsilon^{1/2})$ "small" solitons in terms of the soliton amplitude parameter $s$. Since, according to (14), the length scales as $\epsilon^{-1/2}$, the condition on the length scaling of the potentials containing the discrete spectrum of the perturbation (10) derived from an analysis of rectangular potentials, also holds for arbitrary ones; indeed, the length of such a hump should be $\sim\epsilon^{-1/2}$.

### 3.3 Eigenvalues for smooth generic shallow potentials

Although we established the scaling of the eigenvalues for generic shallow potentials, explicit expressions for the eigenvalues were derived only for rectangular potentials. Here we will show that the eigenvalues could be found by the WKB method for generic shallow potentials under the additional assumption of smoothness.

Consider solutions of (4) for $O(\epsilon)$ shallow potentials with discrete eigenvalues. By virtue of (14) $|\lambda - 2\mathrm{i}\sqrt{\rho_0}|$ is $O(\epsilon^{1/2})$. Recall that if $|u| \equiv \sqrt{\rho} = \text{const} = \sqrt{\rho_0}$ and $\lambda^2/4 = \rho_0$, two linearly independent solutions of (4) can be presented as

$$\psi = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \mathrm{e}^{-\frac{\mathrm{i}}{2}kx}, \quad \tilde{\psi} = \begin{pmatrix} -\mathrm{i} \\ 1 \end{pmatrix} \mathrm{e}^{\frac{\mathrm{i}}{2}\bar{k}x}, \quad \text{where } k = \sqrt{\lambda^2 + 4\rho_0}\,.$$

Our analysis of rectangular potentials as well as (14) suggests that if the initial deviation from $\sqrt{\rho_0}$ is $O(\epsilon)$ small, this requires the initial perturbations to be $\epsilon^{-1/2}$ long in order for discrete eigenvalues to exist. Assume that in the generic case of a shallow potential $u$ varies slowly around $\sqrt{\rho_0}$, which requires an additional but not restrictive assumption on the smoothness of the potential. Then it is natural to apply the WKB method to seek solutions to (4) in the standard WKB form

$$\psi = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \mathrm{e}^{-\frac{\mathrm{i}}{2}\int k\,\mathrm{d}x}, \qquad \tilde{\psi} = \begin{pmatrix} -\mathrm{i} \\ 1 \end{pmatrix} \mathrm{e}^{\frac{\mathrm{i}}{2}\int \bar{k}\,\mathrm{d}x}, \qquad k = \sqrt{\lambda^2 + 4\rho}.$$

As is standard for the WKB (see e.g. [35, Chap.3]), the condition of having a discrete eigenvalue is

$$\int_{x_-}^{x_+} k\,\mathrm{d}x = \frac{\pi}{2} + 2\pi N, \tag{15}$$

where the integration is taken between the two turning points.

Now find the eigenvalues corresponding to the eigenfunctions with minimal number of oscillations. As in the example with a rectangular potential, here the lowest discrete "energy levels" are very close to the bottom of the well, i.e. $\lambda \simeq 2\mathrm{i}\sqrt{\rho_0}$, whence

$$\lambda^2 + 4\rho_0 \ll \Delta\rho \equiv \rho - \rho_0\,.$$

In terms of $k$ this leads to a straightforward estimate:

$$k = \sqrt{\lambda^2 + 4\rho} = \sqrt{\lambda^2 + 4\rho_0 + 4(\rho - \rho_0)} \approx 2\sqrt{\Delta\rho} \tag{16}$$

By plugging the above estimate into (15), we get the condition for the existence of at least one spectral level

$$\int_{x_-}^{x_+} \sqrt{\Delta\rho}\,\mathrm{d}x \geq \frac{\pi}{4},$$

which is a straightforward generalization of condition (10). The generalized condition means that the discrete level and, correspondingly a soliton, appear only if the integral over the initial hump exceeds $\pi/4$. If the characteristic height of the hump is $\sim\epsilon$, then the minimal characteristic length of the hump is $1/\sqrt{\epsilon}$. At the critical value of the hump the eigenvalue in terms of $\lambda$ appears at $\lambda = 2\mathrm{i}\sqrt{\rho_0}$, which corresponds to the Kuznetsov–Ma solitons (2) with $s = 0$, i.e., the Peregrine soliton. Overall, in this generic case the picture is qualitatively the same as we discussed above for the particular case of the rectangular hump.

Note that the condition for the existence of a second discrete eigenvalue (second soliton) is

$$\int_{x_-}^{x_+} \sqrt{\Delta\rho}\,\mathrm{d}x \geq \frac{5\pi}{4},$$

which requires the hump to be five times longer.

### 3.4 Applicability of the WKB

Let us roughly estimate the gradient and the characteristic spatial scale of the inhomogeneity $L_{\text{inh}}$ created by the smooth potentials we are interested in. If the characteristic length of the potential well is $l$ and its characteristic (non-dimensional) depth is $\epsilon$, then the scale in question will be

$$L_{\text{inh}} = l/\epsilon$$

The wavenumber $k$ in the WKB solution scales with $\epsilon$ by virtue of (16) as $k \approx 2\sqrt{\Delta\rho} \sim \sqrt{\epsilon}$.

The minimal scale $l$ of the hump is given by the condition

$$\int_{x_-}^{x_+} \sqrt{\Delta\rho}\, \mathrm{d}x = \frac{\pi}{2},$$

which yields

$$l \sim O(1)/\sqrt{\varepsilon}.$$

The standard condition of the WKB validity requires $kL_{\text{inh}} \gg 1$. On gathering our estimates it is easy to see that

$$L_{\text{inh}} = l/\epsilon \sim \epsilon^{-3/2}, \qquad k \sim \epsilon^{1/2}$$

and, thus, $kL_{\text{inh}} \sim \epsilon^{-1} \gg 1$, which is satisfied as long as $\epsilon \ll 1$.

## 4 Discussion

First we briefly summarise our conclusions within the framework of the one-dimensional NLS and then discuss their possible implications for freak waves in nature.

It has long been known that Kuznetsov–Ma breathers describe the asymptotics of an initial problem and the Peregrine soliton is a member of this family corresponding to the limit of zero-amplitude parameter. Probably there was an understanding that these breathers could emerge out of an initial hump upon an otherwise uniform envelope amplitude, provided the hump "area" exceeds a certain critical value, although we cannot indicate any specific reference. In this paper we noticed that at the critical value of the hump the resulting breather is the Peregrine soliton. Our key finding is that for all initial "supercritical" humps of small amplitude the resulting breathers are very close to the Peregrine soliton. The second straightforward but nevertheless very important implication from the analysis of the condition for emergence of at least one breather is that it provides an explicit minimal spatial scale $l$ for the initial perturbations to be supercritical: if the characteristic height of the initial hump is $\epsilon$ then $l$ should be $\sim\epsilon^{-1/2}$. Hence, if there is an initial small-amplitude noise which does not contain such scales, then freak waves will be very rare and entirely due to the continuous spectrum or, in plain language, due to random focussing. If in the initial noise the relevant scales are present, even with very small amplitudes, then one could be guaranteed of having freak-wave events. Moreover, all such events will be described by the Peregrine soliton and, to leading order, have the same amplitude—three times the background uniform amplitude. Thus, within the NLS framework, once the appropriate large-scale perturbations have been identified, the occurrence of freak waves with *a priori* known parameters could be predicted.

We have considered well-separated continuous humps which assumes the distance between the humps to be much greater than the minimal spatial scale $l \sim \epsilon^{-1/2}$ required for the initial perturbations to produce at least one breather. If there is a number of humps with the distance between them $\sim l$ or smaller, such a group should be treated as a single potential in the IST boundary-value problem (4); such an analysis remains to be carried out. Here we would like to stress two points: first, for a soliton to emerge it is not necessary to have a *continuous positive perturbation* of length $l$, a sequence of perturbations of smaller length of both signs would also do, although a specific criterion has to be worked out for each case; second, by virtue of the scaling reasoning of Sect. 3.2 which does not depend on whether the hump is continuous or not, if the hump is $\sim\epsilon$, then according to (14), the eigenvalues are $l \sim \epsilon^{1/2}$. Hence the emerging patterns will be described well by the Peregrine soliton.

Although we did not consider explicitly random initial conditions, we expect that for all fields characterised by initially small perturbations around a uniform background the Peregrine soliton solution will well describe the *most likely* pattern of freak waves. All other types of freak-wave events with magnitudes noticeably exceeding three are possible only as a result of either interaction (collision) between the elementary solutions or due to neglected effects of the continuous spectrum. Such events are much more rare. In the very recent work [36][1] devoted to freak waves within the framework of NLS generated by random initial conditions the attention was focussed upon the interaction between the elementary solitons. It was shown that the collision between two Akhmediev breathers is the main interaction resulting in events with amplitudes five times the background level. Their extensive numerical simulations indeed demonstrated a steep drop-off of the amplitude probability-density function at about five. At the same time their Fig. 14 also shows a significant drop-off of the probability-density function at about three; the drop-off is sharper for smaller amplitudes of initial noise. These simulations seem to support our conjecture that, within the framework of the one-dimensional NLS, the patterns approximately described by the Peregrine solitons emerge as the most likely freak events for wide classes of initial distributions.

The extremely challenging task of making sense of the predictions made within the simplest one-dimensional NLS model for real-world freak-wave events certainly requires a lot of further study. Nevertheless, it seems helpful to outline the possible implications as we see them now, just making sure that we clearly demarcated the speculations we inevitably have to make to fill the gaps.

The crucial question is how adequate the studied one-dimensional NLS model is in each particular context. First, we discuss the situations where one-dimensionality is not an obstacle. In nonlinear optics propagation of wave packets in optical fibers is naturally one-dimensional and wave evolution is usually described either by the NLS or its modifications. When the classic NLS is applicable, we can effectively control the occurrence of optical freak waves by playing with low-frequency perturbations. In particular, it might be of interest to use the described mechanism of breather formation as a way to select and amplify small-amplitude low-frequency signals of interest. Most of the physically relevant NLS modifications are not integrable and one has to carry out an analysis of the initial problem for each particular case, which would require heavy reliance on numerics. However, we expect the basic conclusions to hold in the following sense: similar breathers are likely to emerge for similar initial conditions, although the specific critical values might differ.

In the water-wave context the closest thing to the NLS model are numerical simulations of the full Euler equations for one-dimensional wave propagation with the initial conditions within or close to the range of the NLS validity. The similarity of "steep-wave events" with the Peregrine solution was indeed observed and mentioned in several papers (see [26] for further references), which was one of the initial motivations of the present study. Strongly nonlinear wave evolution outside the range of the NLS validity, although intensively studied numerically, has not been looked upon from this perspective. However, it is worth mentioning that breathers do emerge as asymptotic solutions of initial problems even in fully nonlinear settings [37]. The next step towards real-world applications are experiments in long and narrow laboratory tanks where wave propagation is to a good accuracy one-dimensional. The situation in such tanks is not much different from the "numerical tanks" discussed above and the NLS could be applied, at least as a first approximation for moderate wave amplitudes. However, to our knowledge, tank experiments have not been analyzed from this viewpoint; it could be that most of the tanks are just not long enough to exhibit the formation of breathers from small initial perturbations. Although there exist tank observations of steep-wave events carried out in the 330 m long facility with the maximal amplitudes of the events three times the amplitude of the unperturbed wavetrain [34], most likely these observations are not related to our findings and could be interpreted as manifestations of focussing. At the same time, we stress that, although in observations by [38] the Peregrine solitons have not been reported, the BF was found to play the key role in changing the waveheight probability distribution.

In the oceans wind-wave fields are never one-dimensional and often the wave spectra are not narrow enough for the NLS model to be applicable. When the spectra are broad, the modulational instability does not occur and freak waves are extremely rare [14]. Hence we confine our discussion to just the situations with narrow spectra.

[1] The paper appeared when the present paper was already under consideration.

First we attract attention to the interesting class of situations where the wave field is two-dimensional but could be described by the one-dimensional NLS or its generalisations: waves propagating in a wave guide created by bottom topography or currents. For example, waves propagating against the current on jet-like currents could be trapped and, if their interaction with the free modes is weak, the evolution of such trapped modes is governed by the one-dimensional NLS to leading order. There is vast although primarily anecdotal evidence that freak waves are much more frequent on the currents; the best known are those associated with the ill-famed Aguilhas current, but the mechanisms have not been understood so far. It is possible that atmospheric perturbations create large-scale "supercritical" inhomogeneities of the wave field which then develop into freak waves. At least for the Aguilhas current some documented freak-wave events for swell propagating against the current event have been tentatively linked to passage of atmospheric fronts [39].

For free waves it has been found that two-dimensionality of the spectra strongly inhibits freak-wave formation [40]. At the same time, we are not aware of any studies of the 2D NLS with the type of initial conditions required for the mechanism of freak-wave formation considered in this paper; it seems relatively straightforward to address analytically at least some aspects of this problem. Wind inhomogeneities translate very fast into wave-field inhomogeneities [41]. If we assume the wave length scale to be $\sim$200 m and the characteristic spatial scale of the perturbation to be $\sim$40 km, then the non-dimensional scale $l$ will be $O(10^2)$ which, according to the NLS-based estimates, implies that even a one percent increase in the mean wave amplitude within an area of increased wind would fall well into the "supercritical" zone. Of course, we cannot speak about any comparison with the data now. We do not know yet the criticality condition for two-dimensional fields, nor do we know the time it takes for breather-like patterns to emerge for slightly supercritical initial conditions. We can only speculate that, since freak waves in the ocean are rare, "supercritical" initial conditions are also rare, but slightly supercritical initial conditions are the most likely of those rare events. We can further assume that, although the criticality condition for two-dimensional fields might differ significantly from its one-dimensional analogue, the wave pattern corresponding to the slightly "supercritical" conditions might be robust and resemble the Peregrine soliton. At the moment this is just a hypothesis which, however, can be checked numerically.

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