J Eng Math (2008) 61:357–369
DOI 10.1007/s10665-008-9214-6

The thermoelastic state of a thermosensitive sphere and space with a spherical cavity subject to complex heat exchange

R. M. Kushnir · V. S. Popovych · O. M. Vovk

Received: 4 December 2006 / Accepted: 26 January 2008 / Published online: 21 February 2008
© Springer Science+Business Media B.V. 2008

Abstract A solution is derived for the non-stationary heat-conduction problem involving a thermosensitive sphere and space with a spherical cavity under convective radial heat exchange with the environment. The influence of the material thermosensitivity on the temperature distribution and the stresses caused by it is analyzed for the cases when force loadings exist and when they are absent on the surface of the bodies considered.

Keywords Complex heat exchange · Space with a spherical cavity · Sphere · Thermoelasticity · Thermosensitivity of material

1 Introduction

Modern high requirements in engineering regarding the determination of the thermal stress state of design elements, operating at large temperature drops, can be satisfied only on the basis of mathematical models considering the temperature dependence of the thermal and mechanical characteristics of the material. Such models involve nonlinear boundary-value problems of mathematical physics. The corresponding models describing the stress–strain state make use of systems of differential equations with variable coefficients. Basically, numerical methods are used to construct the solutions to these problems. Reviews and analyses of mathematical methods for the solution of such heat-conduction and thermoelasticity problems are given, e.g., in [1–5].

Nowinski [6] considered the thermoelasticity problem for an incompressible isotropic sphere, and Parida and Das [7] analyzed the thermoelastic state of an incompressible inhomogeneous sphere in a periodic temperature field. Noda has given in [8] a general solution for the thermoelasticity problem for a sphere with a temperature-dependent coefficient of linear expansion only. For the temperature-dependent shear modulus also, such a problem was solved by a perturbation method. Stanišić and McKinley [9] studied the steady thermal stresses of an isotropic sphere for a temperature-independent Poisson ratio; Nyuko et al. [10] considered these problems for a composite hollow sphere. Kolyano and Mahorkin [11] and Mahorkin [12] constructed solutions for centrally symmetric thermoelasticity problems for thermosensitive continuous and hollow spheres, approximating the temperature dependence of the mechanical characteristics by unit functions. An analytical solution for the spherically symmetric thermal and

R. M. Kushnir (✉) · V. S. Popovych · O. M. Vovk
Pidstryhach Institute for Applied Problems of Mechanics and Mathematics, Ukrainian National Academy of Sciences, 3b Naukova Str., 79060 Lviv, Ukraine
e-mail: kushnir@iapmm.lviv.ua

mechanical stress in a thick hollow sphere made of functionally graded materials is presented in [13]. The material properties are assumed to be graded along the radial direction according to exponential functions in the radial direction.

Nowinski [14] determined the temperature field, and stress–strain state caused by it, for a thermosensitive space with a spherical cavity in the case when a constant temperature was prescribed on the cavity surface and the Poisson ratio was assumed to be constant.

In [15,16] the non-stationary temperature fields in a thermosensitive sphere and space with a spherical cavity with convective heat exchange with the environment were determined, and the influence of the material thermosensitivity on the temperature distribution, and stresses caused by it, for the case of an external load-free bounding surface was analyzed. Here, using the analytical-numerical procedure proposed in [17], we construct solutions for the analogous heat-conduction problems for a thermosensitive sphere and space with a spherical cavity under complex heat exchange with the environment; the thermoelastic state will also be determined. The temperature values, obtained on the basis of the given procedure, have been compared with those found by a purely numerical method under convective, radial, and convective-radial heat exchange. The influence of temperature dependence of the material characteristics on the value and character of the temperature distribution and the resulting stresses is studied for the cases of external load-free surfaces of bodies subject to constant-pressure conditions.

2 Statement of the heat-conduction problem for a sphere

Consider the problem of determining the non-stationary temperature field t and the stress–strain state (caused by it) in a sphere of radius r_0, the thermomechanical characteristics of which are functions of temperature. The sphere has a uniform temperature distribution t_p and its surface $r = r_0$ is under constant pressure p_0. At an initial moment $\tau = 0$ the sphere begins to heat through the surface $r = r_0$ by convective-radial heat exchange with the environment, the temperature of which is equal to t_c.

In this case the temperature field in the sphere is determined from the nonlinear heat-conduction equation [2, p. 91], [18, Chap. 1, Sect. 6]

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\lambda_t(t)\frac{\partial t}{\partial r}\right) = c_v(t)\frac{\partial t}{\partial \tau}, \tag{1}$$

subject to the boundary and initial conditions

$$\left[\lambda_t(t)\frac{\partial t}{\partial r} + \alpha(t - t_c) + \sigma\varepsilon(t^4 - t_c^4)\right]_{r=r_0} = 0, \quad t|_{r=0} < +\infty, \quad \left.\frac{\partial t}{\partial r}\right|_{r=0} = 0, \tag{2}$$

$$t|_{\tau=0} = t_p, \tag{3}$$

where α is the heat-transfer coefficient through the surface $r = r_0$; $c_v(t)$, $\lambda_t(t)$ are the temperature-dependent volumetric heat capacity and heat-conduction factor of the sphere material, respectively; σ is the Stefan–Boltzmann constant; ε is the degree of blackness.

Let t_0 be a reference temperature and the sphere radius r_0 be a characteristic size; then introduce the dimensionless temperature $T = t/t_0$, the coordinate $\rho = r/r_0$ and present the characteristics of the sphere material in the form $\chi(t) = \chi_0\chi^*(T)$. Here values with subscript zero have corresponding dimensions, and the values with an asterisk are functions of the dimensionless temperature, where $\chi(t_p) = \chi_0$, $\chi^*(t_p) = 1$ $(T_p = t_p/t_0)$. Then the problem (1)–(3) takes the form:

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\lambda_t^*(T)\frac{\partial T}{\partial \rho}\right) = c_v^*(T)\frac{\partial T}{\partial \mathrm{Fo}}, \tag{4}$$

$$\left[\lambda_t^*(T)\frac{\partial T}{\partial \rho} + \mathrm{Bi}(T - T_c) + \mathrm{Sk}(T^4 - T_c^4)\right]_{\rho=1} = 0, \quad T|_{\rho=0} < +\infty, \quad \left.\frac{\partial T}{\partial \rho}\right|_{\rho=0} = 0, \tag{5}$$

$$T|_{\mathrm{Fo}=0} = T_p. \tag{6}$$

Here $T_c = t_c/t_0$, $\mathrm{Bi} = \alpha r_0/\lambda_{t0}$, $\mathrm{Sk} = \sigma\varepsilon t_0^3 r_0/\lambda_{t0}$, $\mathrm{Fo} = a_0\tau/r_0^2$, $a_0 = \lambda_{t0}/c_{v0}$.

3 Construction of an analytical solution to the heat-conduction problem

Let us apply the Kirchhoff transform to the nonlinear problem (4–6):

$$\theta = \int_{T_p}^{T} \lambda_t^*(T)\mathrm{d}T. \tag{7}$$

As a result, we obtain the following boundary-value problem for the variable $\theta(\mathrm{Fo}, \rho)$,

$$\frac{\partial^2(\rho\theta)}{\partial\rho^2} = \frac{1}{a^*(\theta)}\frac{\partial(\rho\theta)}{\partial \mathrm{Fo}}, \tag{8}$$

$$\left[\frac{\partial\theta}{\partial\rho} + \mathrm{Bi}(T(\theta) - T_c) + \mathrm{Sk}(T^4(\theta) - T_c^4)\right]_{\rho=1} = 0, \quad \theta|_{\rho=0} < +\infty, \quad \left.\frac{\partial\theta}{\partial\rho}\right|_{\rho=0} = 0, \tag{9}$$

$$\theta|_{\mathrm{Fo}=0} = 0. \tag{10}$$

Here $T(\theta)$ is an expression for the dimensionless temperature in terms of the Kirchhoff variable θ, which is found from the integral equation (7) for a particular dependence of the heat-conduction factor on the temperature, $a^*(\theta) = \lambda_t^*[T(\theta)]/c_v^*[T(\theta)]$.

To find the solution of the boundary-value problem for the variable θ, (8–10), we use the method of successive approximations [17]. As the mth ($m = 1, 2, \ldots$) approximation of the solution to the problem we take the analytical solution of the following linear problem:

$$\frac{\partial^2(\rho\theta_m)}{\partial\rho^2} = \frac{\partial(\rho\theta_m)}{\partial \mathrm{Fo_m}}, \tag{11}$$

$$\left[\frac{\partial\theta_m}{\partial\rho} + \mathrm{Bi}_m(\theta_m - \theta_c)\right]_{\rho=1} = 0, \quad \theta_m|_{\rho=0} < +\infty, \quad \left.\frac{\partial\theta_m}{\partial\rho}\right|_{\rho=0} = 0, \tag{12}$$

$$\theta_\mathrm{m}|_{\mathrm{Fo_m}=0} = 0, \tag{13}$$

where $\theta_c = \int_{T_p}^{T_c} \lambda_t^*(T)\mathrm{d}T$, $\mathrm{Fo}_1 = \mathrm{Fo}$, $\mathrm{Bi}_1 = \mathrm{Bi}$, $\mathrm{Fo}_m = a^*(\theta_{m-1}(\mathrm{Fo}^*, 1))\mathrm{Fo}$, $\mathrm{Bi}_m = [\theta_{m-1}(\mathrm{Fo}^*, 1) - \theta_c]^{-1} \times \{\mathrm{Bi}[T(\theta_{m-1}(\mathrm{Fo}^*, 1)) - T_c] + \mathrm{Sk}[(T(\theta_{m-1}(\mathrm{Fo}^*, 1)))^4 - T_c^4]\}$, $(m \geq 2)$; Fo^* is the moment for which the temperature is calculated.

It may be noted that the condition $|\theta_m - \theta_{m-1}| < \varepsilon$ (where ε is the prescribed numerical accuracy) is the convergence criterion of the iteration process. Although convergence of the proposed variant of the iteration process is not proved theoretically, it is confirmed by numeral experiments and comparison of the solutions found on its basis with solutions obtained by an entirely numerical method. On the other hand, it is not difficult to see that, when the number of iterations increases, θ_m differs only sligthtly (on ε) from θ_{m-1} and the first condition (12) becomes a nonlinear condition (the first condition (9)), which corresponds to the exact statement of the problem. Thus, the first iteration takes into account only the convective heat-transfer constituent, which allows to obtain the analytical solution of the corresponding problem.

To find the solution of the problem (11–13), we apply the Laplace integral transform by the variable Fo_m [19,20]. As a result we obtain a boundary-value problem for the transform of the mth Kirchhoff approximation variable:

$$\frac{\mathrm{d}^2(\rho\tilde{\theta}_m)}{\mathrm{d}\rho^2} - s\rho\tilde{\theta}_m = 0, \tag{14}$$

$$\left[\frac{\mathrm{d}\tilde{\theta}_m}{\mathrm{d}\rho} + \mathrm{Bi}_m\left(\tilde{\theta}_m - \frac{\theta_c}{s}\right)\right]_{\rho=1} = 0, \quad \left.\tilde{\theta}_m\right|_{\rho=0} < +\infty, \quad \left.\frac{\partial\tilde{\theta}_m}{\partial\rho}\right|_{\rho=0} = 0, \tag{15}$$

where $\tilde{\theta}_m = \int_0^\infty \theta_m \mathrm{e}^{-s\mathrm{Fo}_m}\mathrm{dFo}_m$; s is the Laplace transform parameter.

From the boundary-value problem (14–15) we find the following Laplace transform of the Kirchhoff variable $\tilde{\theta}_m$

$$\tilde{\theta}_m = \frac{1}{\rho}\left(\frac{\phi(s)}{s\varphi(s)}\right), \tag{16}$$

where $\phi(s) = \mathrm{Bi}_m\theta_c \frac{\sin \mathrm{h}\sqrt{s}\rho}{\sqrt{s}}$, $\varphi(s) = \cos \mathrm{h}\sqrt{s} + (\mathrm{Bi}_m - 1)\frac{\sin \mathrm{h}\sqrt{s}}{\sqrt{s}}$.

Since the solution (16) is a relation of generalized polynomials with respect to the parameter s, the polynomial of the denominator being such that it does not contain a free term, then for the Laplace inverse transform we can use the expansion theorem [4, pp. 490–496]. The denominator has a simple root $s = 0$ and an infinite number of simple roots $s_n = -\mu_{nm}^2(\mu_{nm} = \mathrm{i}\sqrt{s})$, where μ_{nm} are the roots of characteristic equation

$$\mu \cot\mu = 1 - \mathrm{Bi}_m \tag{17}$$

obtained from the equation $\varphi(s) = 0$.

Taking into account the above, we find

$$\theta_m = \theta_c\left[1 - \sum_{n=1}^{\infty} A_{nm}\frac{\sin(\mu_{nm}\rho)}{\mu_{nm}\rho}\mathrm{e}^{-\mu_{nm}^2\mathrm{Fo_m}}\right], \quad \text{where } A_{nm} = \frac{2(\sin\mu_{nm} - \mu_{nm}\cos\mu_{nm})}{\mu_{nm} - \sin\mu_{nm}\cos\mu_{nm}}.$$

For numerical calculations it is convenient to use an expression for A_{nm}, in which the trigonometrical functions are expressed in terms of μ_{nm} and Bi_m by the characteristic equation (17) [4, p. 228], namely

$$A_{nm} = (-1)^{n+1}\frac{2\mathrm{Bi}_m\sqrt{\mu_{nm}^2 + (\mathrm{Bi}_m - 1)^2}}{\mu_{nm}^2 + \mathrm{Bi}_m^2 - \mathrm{Bi}_m}.$$

To calculate θ_m for values of ρ close to zero, we can use the asymptotic expression

$$\theta_m \approx \theta_c\left[1 - \sum_{n=1}^{\infty} A_{nm}\mathrm{e}^{-\mu_{nm}^2\mathrm{Fo_m}}\right].$$

If, for example, the heat-conduction factor is the linear temperature function $\lambda_t^*(T) = 1 + k(T - T_p)(k = \mathrm{const})$, then the mth temperature approximation in the sphere follows from

$$T_m = k^{-1}\left(\sqrt{1 + 2k\theta_m} - 1\right) + T_p.$$

4 Numerical solution of the heat-conduction problem

For comparison, we solve the boundary-value problem for the Kirchhoff variable, (8–10), by the method of lines, which involves two operations: spatial discretization and time integration. During spatial discretization the equation with partial derivatives is transformed into a system of ordinary differential equations, where the time variable remains continuous [21].

To construct a semi-discrete model of the problem (8–10) we introduce a uniform grid $w_h = \{\rho_i = ih, i = 0, \ldots, n, h = 1/n\}$ on the segment [0, 1]. We replace the partial derivatives with respect to the variable ρ by the difference derivatives with the second-order approximation with respect to h. Then functions $\theta_i = \theta(\rho_i, \mathrm{Fo})$ are approximated by the functions $\theta_i(\mathrm{Fo})$, $i = 0, \ldots, n$, satisfying the following system of ordinary differential equations

$$\frac{\mathrm{d}\theta_i}{\mathrm{dFo}} = \frac{a^*(\theta_i)}{h^2}\left\{\frac{1}{i}[(1+i)\theta_{i+1} - (1-i)\theta_{i-1}] - 2\theta_i\right\}, \quad i = 1, \ldots, n-1, \tag{18}$$

where

$$a^*(\theta_i) = a^*(T(\theta_i)) = \lambda_t^*(T(\theta_i))/c_v^*(T(\theta_i)), \quad T(\theta_i) = k^{-1}\left(\sqrt{1 + 2k\theta_i} - 1\right) + T_p. \tag{19}$$

Note that (8) has a singularity at $\rho = 0$. Using the condition $\left.\frac{\partial\theta}{\partial\rho}\right|_{\rho=0} = 0$ and de L'Hôpital rule, we may present the left part of (8), divided by ρ, i.e. $\frac{1}{\rho}\frac{\partial^2(\rho\theta)}{\partial\rho^2}$, in the form

$$\lim_{\rho\to 0}\left[\frac{1}{\rho}\frac{\partial^2(\rho\theta)}{\partial\rho^2}\right] = \lim_{\rho\to 0}\left[\frac{\partial^2\theta}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial\theta}{\partial\rho}\right] = \left[\frac{\partial^2\theta}{\partial\rho^2} + 2\frac{\partial^2\theta}{\partial\rho^2}\right]_{\rho=0} = 3\left.\frac{\partial^2\theta}{\partial\rho^2}\right|_{\rho=0}. \tag{20}$$

Utilizing the representation (20) and relation $\theta_{-1} = \theta_1$ (it follows from condition $\left.\frac{\partial\theta}{\partial\rho}\right|_{\rho=0} = 0$, $\frac{\theta_1-\theta_{-1}}{2h} = 0$), we obtain an ordinary differential equation in the sphere center,

$$\frac{\mathrm{d}\theta_0}{\mathrm{dFo}} = \frac{a^*(\theta_n)}{h^2}\left\{\frac{1}{n}\left[(1+n)\theta_{n+1} - (1-n)\theta_{n-1}\right] - 2\theta_n\right\},$$

where θ_{n+1} is determined from the boundary condition (9), replacing the derivative with respect to the variable ρ by the central-difference derivative ($\left.\frac{\partial\theta}{\partial\rho}\right|_{\rho=1} \approx \frac{\theta_{n+1}-\theta_{n-1}}{2h}$).

As a result we obtain

$$\theta_{n+1} = 2h[-\mathrm{Bi}(T(\theta_n) - T_c) - \mathrm{Sk}(T^4(\theta_n) - T_c^4)] + \theta_{n-1}. \tag{21}$$

Substituting (21) in (20), we obtain the final form in terms of ordinary differential equations on the sphere surface.

Thus, having realized the difference approximation of the boundary conditions (9) and taking into account the initial condition (10), we obtain the following semi-discrete model, which is a difference analogue of the boundary-value problem (8)–(10):

$$\begin{cases} \dfrac{\mathrm{d}\theta_0}{\mathrm{dFo}} = \dfrac{6a^*(\theta_0)}{h^2}(\theta_1 - \theta_0), \\ \dfrac{\mathrm{d}\theta_i}{\mathrm{dFo}} = \dfrac{a^*(\theta_i)}{h^2}\left\{\frac{1}{i}[(1+i)\theta_{i+1} - (1-i)\theta_{i-1}] - 2\theta_i\right\}, i = 1, \ldots, n-1, \\ \dfrac{\mathrm{d}\theta_n}{\mathrm{dFo}} = \left\{\dfrac{1}{n}[(1+n)\{2h[-\mathrm{Bi}(T_n - T_c) - \mathrm{Sk}(T_n^4 - T_c^4)] + \theta_{n-1}\} + 2n\theta_{n-1}] - 2\theta_n\right\} \times \dfrac{a^*(\theta_n)}{h^2}, \\ \theta_i(0) = 0, \end{cases} \tag{22}$$

The solution to the initial-value problem for a system of ordinary differential equations (22) can be found using the formulas of backward differentiation (the Hier method with a strip structure of the Jacobi matrix calculated by numerical differentiation). When the values of the Kirchhoff variable θ_i are found, the temperature in the sphere is calculated by (19).

5 Stress–strain state of a sphere

The stress–strain state of a sphere in a centrally symmetric temperature field under constant pressure p on the surface $r = r_0$, is determined [16] by the dimensionless radial displacement $\bar{u} = u/r_0\alpha_0 t_0$, radial $\sigma_\rho = \sigma_r/2G_0\alpha_0 t_0$ and circumferential $\sigma_\varphi = \sigma_{\varphi\varphi}/2G_0\alpha_0 t_0$ stresses, radial e_ρ and circumferential e_φ strains ($G_0 = G(t_p), \alpha_{t0} = \alpha_t(t_p)$), where

$$\{\bar{u}, \sigma_\rho, \sigma_\varphi, e_\rho, e_\varphi\} = \sum_{k=0}^{\infty}\{\bar{u}_k, \sigma_{\rho k}, \sigma_{\varphi k}, e_{\rho k}, e_{\varphi k}\}, \tag{23}$$

and the terms of the series (23) are calculated by

$$\bar{u}_0 = c_0\rho + \rho^{-2}[H(\rho) - \frac{1}{3}H_3(\rho)] + \frac{\rho}{3}H_0(\rho),$$

$$\sigma_{\rho 0} = G^*(T)[\bar{\nu}(T)c_0 - 2\rho^{-3}H(\rho) + \bar{\nu}(T)H_0(\rho)/3 + 2\rho^{-3}H_3(\rho)/3],$$

$$\sigma_{\varphi 0} = G^*(T)[\bar{\nu}(T)(c_0 + H_0(\rho)/3) + \rho^{-3}(H(\rho) - H_3(\rho)/3) - \Phi^*(T)],$$
$$e_{\rho 0} = \frac{\partial \bar{u}_0}{\partial \rho} = c_0 + \rho^{-3}[-2H(\rho) + \frac{2}{3}H_3(\rho)] + \frac{\rho}{3}H_0(\rho) + \Phi^*(T),\ e_{\varphi 0} = \frac{\bar{u}_0}{\rho},$$
$$\bar{u}_k = c_k\rho - \frac{1}{3}\left(\rho H_0^{(k-1)}(\rho) - \rho^{-2}H_3^{(k-1)}(\rho)\right),$$
$$\sigma_{\rho k} = G^*(T)[\bar{\nu}(T)(c_k - H_0^{(k-1)}(\rho)/3) - 2\rho^{-3}H_3^{(k-1)}(\rho)/3],$$
$$\sigma_{\varphi k} = G^*(T)[\bar{\nu}(T)(c_k - H_0^{(k-1)}(\rho)/3) + \rho^{-3}H_3^{(k-1)}(\rho)/3)],$$
$$e_{\rho k} = \frac{\partial \bar{u}_k}{\partial \rho} = c_k - \frac{1}{3}\left(H_0^{(k-1)}(\rho) - 2\rho^{-3}H_3^{(k-1)}(\rho)\right),\ e_{\varphi k} = \frac{\bar{u}_k}{\rho},\ (k \geq 1), \tag{24}$$
$$\bar{\nu}(T) = \frac{1+\nu(T)}{1-2\nu(T)},\quad H(\rho) = \int_0^\rho \xi^2\Phi^*(\xi,\mathrm{Fo})\mathrm{d}\xi,\quad H_m(\rho) = \int_0^\rho \xi^m\bar{\Phi}(\xi,\mathrm{Fo})\,\mathrm{d}\xi,\quad \bar{\Phi}(T) = \psi(T)\Phi^*(T),$$
$$\Phi^*(T) = \frac{1+\nu(T)}{1-\nu(T)}\int_{T_p}^T \alpha_t^*(T)\mathrm{d}T,\ H_m^{(k-1)}(\rho) = \int_0^\rho \xi^m f_{k-1}(\xi,\mathrm{Fo})\,\mathrm{d}\xi,\ \psi(T) = \frac{\partial}{\partial\rho}\left(\log\left(G^*(T)\frac{1-\nu(T)}{1-2\nu(T)}\right)\right),$$
$$f_{k-1}(\rho,\mathrm{Fo}) = \psi(T)\left(e_{\rho k-1} + 2m(T)e_{\varphi k-1}\right),\quad m(T) = \frac{\partial}{\partial\rho}\left(G^*(T)\frac{\nu(T)}{1-2\nu(T)}\right)\Big/\frac{\partial}{\partial\rho}\left(G^*(T)\frac{1-\nu(T)}{1-2\nu(T)}\right),$$
$$c_0 = \frac{B_0}{\bar{\nu}_0},\quad c_k = \frac{B_{k-1}}{\bar{\nu}_0},\quad B_0 = 2H(\rho_0) - \frac{1}{3}\bar{\nu}_0H_0(\rho_0) - \frac{2}{3}H_3(\rho_0) - \frac{p_0}{G_0^*},\quad \bar{\nu}_0 = \bar{\nu}(T)|_{\rho=\rho_0=1},$$
$$G_0^* = G^*(T)\big|_{\rho=\rho_0=1},\quad B_{k-1} = \frac{1}{3}\left(\bar{\nu}_0H_0^{(k-1)}(\rho_0) + 2H_3^{(k-1)}(\rho_0)\right).$$

Details of the derivation of the displacements and stresses in the sphere center and in its vicinity (for small ρ) are given in [16].

6 Temperature-field determination for a space with a spherical cavity

Now consider a space with a spherical cavity of radius r_0, all thermal and mechanical characteristics depending on temperature. The space has a constant initial temperature t_p and from $\tau = 0$ it is exchanging heat with the environment of constant temperature t_c through the cavity surface.

In this case the temperature field is determined from the heat-conduction equation (1) subject to the boundary condition

$$\left[\lambda_t(t)\frac{\partial t}{\partial r} - \alpha(t - t_c) - \sigma\varepsilon(t^4 - t_c^4)\right]\Bigg|_{r=r_0} = 0,\quad \lim_{r\to\infty} t = t_p, \tag{25}$$

and the initial condition (3).

After writing down condition (25) in dimensionless coordinates and for the values introduced above and having applied the Kirchhoff transform of (7), we obtain

$$\left[\frac{\partial\theta}{\partial\rho} - \mathrm{Bi}(T(\theta) - T_c) - \mathrm{Sk}(T^4(\theta) - T_c^4)\right]\Bigg|_{\rho=1} = 0,\quad \lim_{\rho\to\infty}\theta = 0. \tag{26}$$

Thus, we have the boundary-value problem (8), (10), (26) to determine the Kirchhoff variable. Its solution is found by the method of successive approximations where the mth approximation is the solution to Eq. (11) with initial (13) and boundary conditions:

$$\left[\frac{\partial\theta_m}{\partial\rho} - \mathrm{Bi}_m(\theta_m - \theta_c)\right]\Bigg|_{\rho=1} = 0,\quad \lim_{\rho\to\infty}\theta_m = 0. \tag{27}$$

It is found with the help of the Laplace integral transform and is of the following form

$$\theta_m = \frac{\mathrm{Bi}_m\theta_r}{(1+\mathrm{Bi}_m)\rho}\left[\mathrm{erfc}\frac{\rho-1}{2\sqrt{\mathrm{Fo}_m}} - \mathrm{e}^{(1+\mathrm{Bi}_m)(\rho-1+(1+\mathrm{Bi}_m)\mathrm{Fo}_m)}\mathrm{erfc}\left((1+\mathrm{Bi}_m)\sqrt{\mathrm{Fo}_m} + \frac{\rho-1}{2\sqrt{\mathrm{Fo}_m}}\right)\right], \tag{28}$$

where $\mathrm{erfc}\xi = 1 - \mathrm{erf}\xi$; $\mathrm{erf}\xi$ is the probability integral.

7 Numerical solution of the heat-conduction problem for a space with a spherical cavity

A semi-discrete model of the problem (8), (10), (26), constructed by the integro-interpolation method, is the Cauchy problem for a system of ordinary differential equations

$$\begin{aligned}
\frac{\mathrm{d}\theta_0}{\mathrm{dFo}} &= \frac{2a^*(\theta_0)}{h}\left[\frac{1}{h}(\theta_1-\theta_0)\rho^2_{\frac{1}{2}} - \mathrm{Bi}(T(\theta_0)-T_r) - \mathrm{Sk}(T^4(\theta_0)-T_r^4)\right],\\
\frac{\mathrm{d}\theta_i}{\mathrm{dFo}} &= \frac{a^*(\theta_i)}{h^2(1+ih)^2}[(\theta_{i+1}-\theta_i)\rho^2_{i+\frac{1}{2}} - (\theta_i-\theta_{i-1})\rho^2_{i-\frac{1}{2}}],\ i=1,\ldots,n-1,\ \theta_i(0)=0,\\
\frac{\mathrm{d}\theta_n}{\mathrm{dFo}} &= -\frac{2a^*(\theta_n)}{h^2(1+nh)^2}(\theta_n-\theta_{n-1})\rho^2_{n-\frac{1}{2}}.
\end{aligned} \tag{29}$$

8 Stress–strain state of a space with a spherical cavity

The stress–strain state of a space with a spherical cavity, that is in a centrally symmetric temperature field under constant pressure p_1 on the surface $\rho = 1$, will be determined by the dimensionless radial displacement component $\bar{u}$ different from 0, in terms of which the dimensionless radial σ_ρ and circumferential σ_φ stresses are expressed:

$$\begin{aligned}
\sigma_\rho &= \bar{G}(T)[(1-\nu(T))\partial\bar{u}/\partial\rho + 2\nu(T)\bar{u}/\rho - (1-\nu(T))\Phi^*(T)],\\
\sigma_\varphi &= \bar{G}(T)[\nu(T)\partial\bar{u}/\partial\rho + \bar{u}/\rho - (1-\nu(T))\Phi^*(T)],
\end{aligned} \tag{30}$$

satisfying the equilibrium equation

$$\partial\sigma_\rho/\partial\rho + 2(\sigma_\rho - \sigma_\varphi)/\rho = 0, \tag{31}$$

where $\bar{G}(T) = G^*(T)/(1-2\nu(T))$.

Substituting the relation (30) in Eq. (31), we obtain a differential equation for the determination of $\bar{u}$ [16]:

$$\frac{\partial}{\partial\rho}\left(\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\bar{u})\right) = \frac{\partial\Phi^*}{\partial\rho} - \psi(T)\left(\frac{\partial\bar{u}}{\partial\rho} + 2m(T)\frac{\bar{u}}{\rho} - \Phi^*(T)\right), \tag{32}$$

where

$$\psi(T) = \frac{\partial}{\partial\rho}\left(\log\left(G^*(T)\frac{1-\nu(T)}{1-2\nu(T)}\right)\right),\quad m(T) = \frac{\partial}{\partial\rho}\left(G^*(T)\frac{\nu(T)}{1-2\nu(T)}\right)\Big/\frac{\partial}{\partial\rho}\left(G^*(T)\frac{1-\nu(T)}{1-2\nu(T)}\right).$$

The solution to Eq. (32) will be constructed by a perturbation method. Contrary to [16], we shall rearrange the terms in the right part of (32) somewhat differently and along with it we shall consider the following differential equation with variable coefficients

$$\frac{\partial}{\partial\rho}\left(\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\bar{u})\right) = \frac{\partial\Phi^*(T)}{\partial\rho} + \psi(T)\Phi^*(T) - \varepsilon\psi(T)\left(\frac{\partial\bar{u}}{\partial\rho} + 2m(T)\frac{\bar{u}}{\rho}\right), \tag{33}$$

which coincides with (32) when $\varepsilon = 1$. The solution to Eq. (33) is given in the form of an expansion in powers of the parameter ε:

$$\bar{u} = \sum_{k=0}^{\infty}\varepsilon^k\bar{u}_k(\rho, \mathrm{Fo}). \tag{34}$$

Substituting (34) in Eq. (33) and equating terms having equal powers ε, we obtain a differential equation relative to the zero component $\bar{u}_0$:

$$\frac{\partial}{\partial\rho}\left(\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\bar{u}_0)\right) = \frac{\partial\Phi^*(T)}{\partial\rho} + \psi(T)\Phi^*(T) \tag{35}$$

and a recursion sequence of differential equations relative to the kth $\bar{u}_k$ $(k \geq 1)$ component

$$\frac{\partial}{\partial \rho}\left(\frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 \bar{u}_k)\right) = -f_{k-1}(\rho, \text{Fo}), \tag{36}$$

of displacement $\bar{u}$, where $f_{k-1}(\rho, \text{Fo}) = \psi(T)\left(\frac{\partial \bar{u}_{k-1}}{\partial \rho} + 2m(T)\frac{\bar{u}_{k-1}}{\rho}\right)$.

Taking into account the above, the solution to Eq. (32) reads

$$\bar{u} = \sum_{k=0}^{\infty} \bar{u}_k(\rho, \text{Fo}), \tag{37}$$

where $\bar{u}_0$, $\bar{u}_k$, are the solutions to (35), (36), which, respectively, are of the form:

$$\bar{u}_0 = c_{10}\rho + \rho^{-2}(c_{20} + H(\rho) - H_3(\rho)/3) + \rho H_0(\rho)/3, \tag{38}$$

$$\bar{u}_k = c_{1k}\rho + \rho^{-2}(c_{2k} + H_3^{k-1}(\rho)/3) - \rho H_0^{k-1}(\rho)/3. \tag{39}$$

Here $c_{ik}(i = 1, 2)$ are the integration constants, and $H(\rho) = \int_1^{\rho} \xi^2 \Phi^*(\xi, \text{Fo}) d\xi$, $H_m(\rho) = \int_1^{\rho} \xi^m \psi(T)\Phi^*(\xi, \text{Fo})$ $d\xi$, $H_m^{(k-1)}(\rho) = \int_1^{\rho} \xi^m f_{k-1}(\xi, \text{Fo}) d\xi$.

Taking into account the representation (34), we may calculate the thermal stresses by (23), where the terms of the corresponding series are of the form

$$\sigma_{\rho 0} = G^*(T)[\bar{\nu}(T)(c_{10} + H_0(\rho)/3) - 2\rho^{-3}(c_{20} + H(\rho) - H_3(\rho)/3)], \tag{40}$$

$$\sigma_{\varphi 0} = G^*(T)[\bar{\nu}(T)(c_{10} + H_0(\rho)/3) + \rho^{-3}(c_{20} + H(\rho) - H_3(\rho)/3) - \Phi^*(T)], \tag{41}$$

$$\sigma_{\rho k} = G^*(T)[\bar{\nu}(T)(c_{1k} - H_0^{(k-1)}(\rho)/3) - 2\rho^{-3}(c_{2k} - H_3^{(k-1)}(\rho)/3)], \tag{42}$$

$$\sigma_{\varphi k} = G^*(T)[\bar{\nu}(T)(c_{1k} - H_0^{(k-1)}(\rho)/3) + \rho^{-3}(c_{2k} + H_3^{(k-1)}(\rho)/3)]. \tag{43}$$

The integration constants $c_{ik}(i = 1, 2)$ are determined from the conditions that there is a pressure $\bar{p}_1$ on the cavity surface $\rho = 1$ and that the stresses vanish at infinity, i.e.

$$\sigma_{\rho k}\big|_{\rho=\rho_1} = -\bar{p}_1\delta_{0k}, \ \lim_{\rho\to\infty}\{\sigma_{\rho k}, \sigma_{\Phi k}\} = 0, \tag{44}$$

where $\bar{p}_1 = p_1/(2G_0\alpha_{t0}t_0)$; δ_{0k} is the Kronecker symbol.

Demanding that the stress components (40)–(43) should satisfy the condition (44), while assuming $\lim_{\rho\to\infty}\Phi^*(T) = 0$, we find:

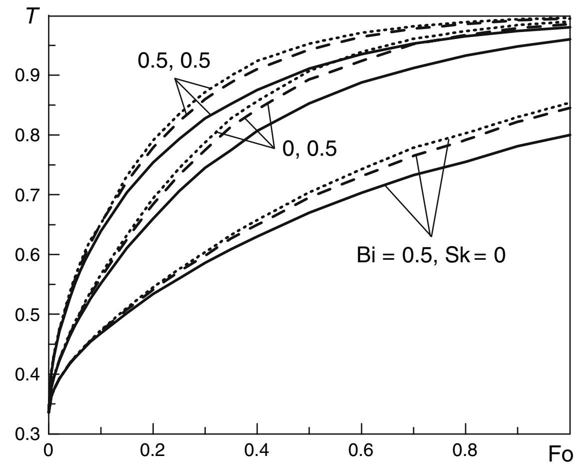
$$c_{10} = -\frac{1}{3}\left. H_0(\rho)\right|_{\rho=\infty}, \quad c_{1k} = \frac{1}{3}\left. H_0^{k-1}(\rho)\right|_{\rho=\infty},$$

$$c_{20} = \frac{\bar{p}_1}{2G^*(T)|_{\rho=1}} + \frac{\bar{\nu}(T)|_{\rho=1}\, c_{10}}{2}, \quad c_{2k} = \frac{1}{2}c_{1k}\left.\bar{\nu}(T)\right|_{\rho=1}.$$

9 Numerical analysis

Numerical investigations have been carried out for the temperature field and stress–strain state (caused by it) in a sphere and space with a spherical cavity with characteristics in the temperature range 293–873 K (20–600 °C) for the cases of convective, radial, and convective-radial heat exchange, when there is force loading on their bounding surfaces and also when this is absent. The thermophysical characteristics vary linearly with temperature, and the mechanical ones vary according to a quadratic law and are of the form $\lambda_t(t) = 42.31 - 0.0087t$ [Ws/(m K)], $c(t) = 403.5608 + 0.54594t$[J/(Kg K)], $\rho = 7841$ [Kg/m^3], $\alpha_t(t) = 10.6429 + 0.00724t - 3.4102 \times 10^{-6}t^2$ [1/K],

Fig. 1 Distribution of the dimensionless temperature T versus Fo



$E(t) = (2.17573 - 0.00052t - 4.6271 \times 10^{-7}t^2)10^{11}$ [Pa], $\nu(t) = 0.31863 + 0.00003t + 7.126 \times 10^{-8}t^2$. The temperature of environment $t_c = 873$ K has been taken as a reference temperature, t_0 and the initial temperature t_p have been assumed to be equal to 293 K. Then $T_c = 1$, $T_p = 0.34$. For the above reference and initial temperatures the representations of the temperature dependence for the physical and mechanical characteristics in the form $\chi(t) = \chi_0\chi^*(T)$, where $\chi^*(T_p) = 1$, are as follows: $\lambda_t(t) = 39.76(1 - 0.19(T - T_p))$, $c(t) = 563.52(1 + 0.85(T - T_p))$, $\alpha_t(t) = 12.47(1 + 0.37(T - T_p)) - 0.2(T - T_p)^2) \times 10^{-6}$, $E(t) = 1.98(1 - 0.35(T - T_p) - 0.18(T - T_p)^2) \times 10^{11}$, $\nu(t) = 0.33(1 + 0.19(T - T_p) + 0.16(T - T_p)^2)$.

The calculations have been carried out for dimensionless values. The discrepancy between the temperature values of a sphere, obtained using the method of successive approximations and the numerical method, does not exceed 2%.

In Fig. 1 the graphs are given for the temperature change versus time Fo on the surface of the sphere in the cases of convective ($\text{Bi} = 0.5,\ \text{Sk} = 0$), radial ($\text{Bi} = 0,\ \text{Sk} = 0.5$), and convective-radial ($\text{Bi} = 0.5,\ \text{Sk} = 0.5$) heat exchange when the material of the sphere is thermosensitive (solid line), non-thermosensitive (material characteristics are equal to the reference values χ_0 – dashed line) and when the temperature dependence of thermal-conductivity factor is neglected ($a^* = 1$ – dotted line).

Figure 2 shows the results of an investigation into the stress–strain state of a sphere under convective-radial heating ($\text{Bi} = 0.5,\ \text{Sk} = 0.5$) and force loading $p = 0.035$ on the surface for various values of $\text{Fo} = 0.3, 0.4, 0.5$. Figure 2a presents graphs of the sphere-radius-displacement distributions, Fig. 2(b,c) presents graphs of radial and circumferential stresses, Fig. 2(d,e) presents graphs of radial and circumferential strains, respectively.

When time increases, the displacements increase from zero in the sphere center to their maximums on its surface, the level of which is lower for a thermosensitive sphere than for a non-thermosensitive sphere. The maximal discrepancy between them is 15%. The maximal discrepancy between the distribution of radial and circumferential strains does not exceed 15% either.

The value of the discrepancy between distributions of components of the stress–strain state in a thermosensitive and a non-thermosensitive sphere depends on the pressure applied to its surface. As can be seen from the graphs, these discrepancies are different for different moments when there is pressure.

A calculation of distributions of the dimensionless temperature field T, displacement $\bar{u}$ and stress-tensor components σ_ρ, σ_φ for a space with a spherical cavity has been carried out for convective, radial, and convective-radial heat exchange (Figs. 3, 4, respectively). Figure 5 illustrates the distribution of displacements and stresses when there is force loading $\bar{p}_1$ on the cavity surface and also when it is absent for convective-radial heat exchange. The dashed lines in these figures present the corresponding distributions in the case of non-thermosensitive material, the characteristics of which are equal to the reference values χ_0.

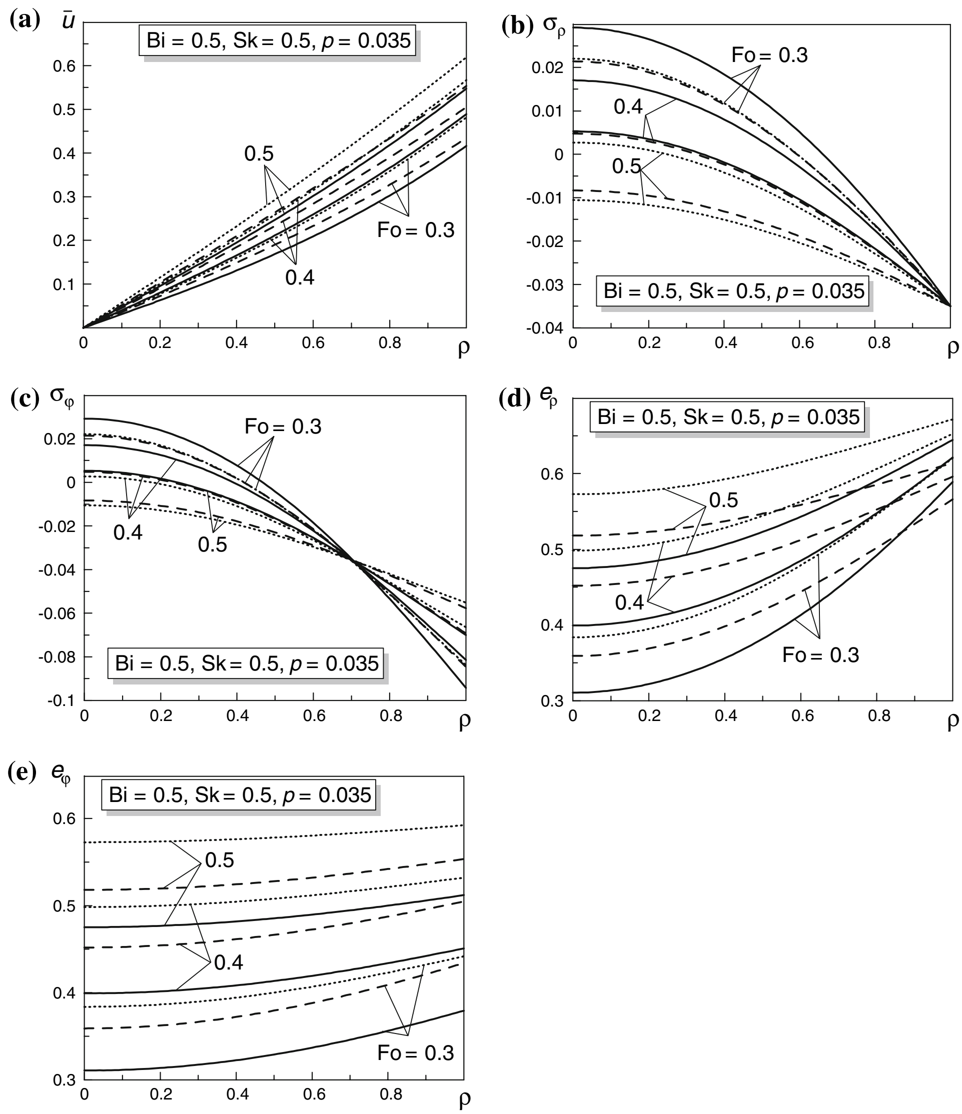
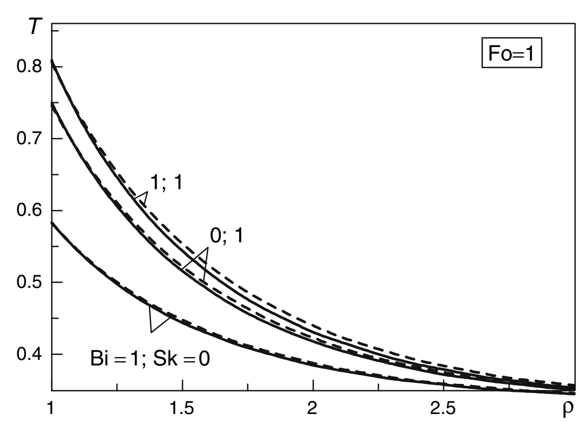


Fig. 2 Distribution of the components of the stress–strain state in a sphere. (**a**) Displacements $\bar{u}$, (**b**) radial stresses σ_{p}, (**c**) circumferential stresses σ_φ, (**d**) radial strains e_{p}, and (**e**) circumferential strains e_φ

Fig. 3 Distribution of the dimensionless temperature T versus ρ in a space with a spherical cavity



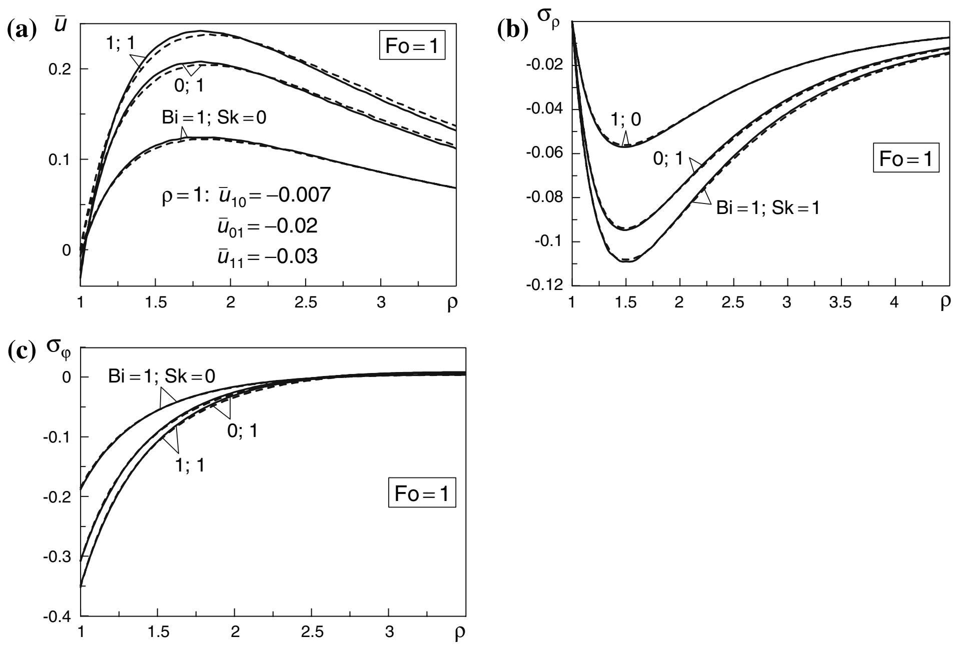


Fig. 4 Distributions of components of the stress–strain state for convective, radial, and convective-radial heat exchange. (**a**) Displacements $\bar{u}$ (**b**) radial stresses σ_ρ and (**c**) circumferential stresses σ_φ

The discrepancy between the temperature values obtained by the method of successive approximations and the numerical method does not exceed 2%.

From the graphs of Fig. 3, it is seen that by moving away from the cavity along the radius, the temperature tends to the initial value T_p. The calculations show that the maximal discrepancy between the temperature values in a thermosensitive and non-thermosensitive space is observed for convective-radial heat exchange and is approximately equal to 2%. The thermophysical characteristics in the given temperature range vary, respectively, $\lambda_t(t)$ by 11%, $c(t)$ by 48%.

The components of the displacement vector, $\bar{u}$ and the stress tensor, $\sigma_\rho, \sigma_\varphi$, have been calculated by Eqs. (38), (39), (40–43). The first few terms of the series (23) are important for practical calculations. Numerical investigations have shown that rearranging the terms in the right part of (32) (a more successful choice of the zero approximation) improves the convergence of the series (23). Approximately 90–95% of the total values of the displacements and stresses have given the zero and first approximation out of five approximations found in this case.

The graphs in Fig. 4a show that, in the case of thermosensitive material, the displacement $\bar{u}$ assumes a negative value on the cavity surface and when moving away from it along the radius it changes sign; at approximately $\rho = 1.7$ (for Fo $= 1$) it reaches its maximum and then decreases monotonically. For a non-thermosensitive material the displacements assume zero values on the cavity surface.

Figure 5a presents the displacement distributions in thermosensitive and non-thermosensitive spaces under convective-radial heating when there is ($\bar{p}_1 = 0.1$) pressure on the surface of a spherical cavity and also when it is absent ($\bar{p}_1 = 0$). The presence of pressure changes qualitatively and quantitatively the situation concerning the displacement distribution.

As can be seen from the graphs in Fig. 4b, the radial stresses σ_ρ increase in absolute value when moving away from the cavity along the radius, reach a maximal value for $\rho = 1.5$ and then decrease monotonically tending to zero. The stresses σ_ρ reach their maximum closer ($\rho = 1.3$) to the cavity surface when there is pressure (Fig. 5b).

The circumferential stresses are maximal in the absolute value on the cavity surface, they decrease monotonically when moving away from it, changing their value on the opposite one (Fig. 4c). Loading the cavity surface by a

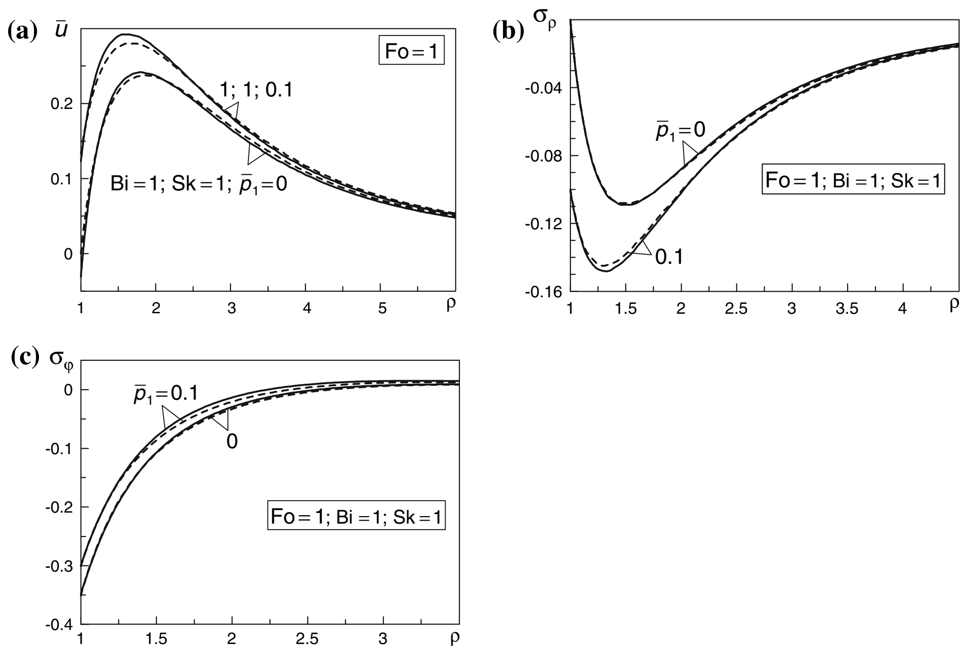


Fig. 5 Distributions of components of the stress–strain state under force loading $\bar{p}_1$ on the cavity surface in the absence of force loading. (**a**) Displacements $\bar{u}$, (**b**) radial stresses σ_ρ, and (**c**) circumferential stresses σ_φ

pressure $\bar{p}_1 = 0.1$ causes a qualitative and quantitative change of the distribution of the circumferential stresses (Fig. 5c).

10 Conclusions

The solutions to non-stationary heat-conduction problems for a sphere and space with a spherical cavity, all material characteristics of which are temperature-dependent, have been constructed. It has been assumed that these bodies are under convective-radial heat exchange through the bounding surface within a constant-temperature environment. We have used a method which employs partial linearization of the nonlinear heat-conduction problem by introducing the Kirchhoff variable with subsequent utilization of the proposed version of the method of successive approximations. The solutions of the corresponding thermoelasticity problems, when all mechanical characteristics are assumed temperature-dependent, have been constructed by a perturbation method. Using this method, the corresponding boundary-value problems (with equations containing variable coefficients) have been reduced to a sequence of boundary-value problems where the equations have constant coefficients. The solutions of the latter are written in quadratures. A numerical study has been carried out for the influence of material thermosensitivity on the distribution character and value of the stress–strain state characteristics of the bodies considered when the bounding surfaces were free or loaded by constant pressure. When solving the nonlinear heat-conduction problems for complex heat exchange, the numerical investigations show rapid convergence of the method of successive approximations (their number did not exceed 12). A numerical experiment has also shown rapid convergence of the perturbation method when solving thermoelasticity problems. The calculated first two terms of the expansion of the components of the stress–strain state give approximately 90–95% of the total values of the displacements and stresses. The influence of the temperature dependence of the linear-expansion coefficient is strongest when calculating the stresses. The difference between stresses in thermosensitive and non-thermosensitive solids depends also on the value of the given forces.

Investigations, carried out by the authors, and other work known to them, concerning the determination of the thermostressed state of thermosensitive solids under complex heating and simultaneous force loading have been realized mainly for one-dimensional mathematical models. Elaboration of methods for studying of two- and three-dimensional stress–strain states of thermosensitive solids under conditions of

- convective-radial heat exchange with the surroundings;
- presence of different heat sources in the bulk of the solids;
- force loading on the surfaces of the solid

is of practical interest.

References

1. Beliaiev NM, Riadno AA (1993) Mathematical methods for thermal conductivity. Vyshcha shk, Kyiv (in Russian)
2. Kolyano YuM (1992) Heat conduction and thermoelasticity method for inhomogeneous body. Naukova dumka, Eyiv (in Russian)
3. Lomakin VA (1976) Elasticity theory for inhomogeneous bodies. Izd-vo MGU, Moscow (in Russian)
4. Lykov AV (1967) Heat conduction theory. Vysshaja shkola, Moscow (in Russian)
5. Postol'nyk YuS, Ogurtzov AP (2002) Metallurgical thermomechanics. Systemni technolohii, Dnipropetrovs'k (in Ukrainian)
6. Nowinski J (1959) Thermoelastic problem for an isotropic sphere with temperature dependent properties. ZAMP 10(6):565–575
7. Parida J, Das AK (1970) Note on the thermal stresses in an incompressible non-homogeneous sphere in periodic temperature field. Bull Acad Polon Sci 18:1–6
8. Noda N (1986) Thermal stresses in materials with temperature-dependent properties. In: Hetnarski RB (ed) Thermal stresses I, Chap. 6. Elsevier Science, Amsterdam, pp 391–483
9. Stanišić MM, McKinley RM (1962) The steady-state thermal stress field in an isotropic sphere with temperature-dependent properties. Arch Appl Mech 31(4):241–249
10. Nyuko H, Takeuti Y, Noda N (1978) Stationary thermal stresses for a composite hollow sphere exhibiting temperature dependent properties. Trans Jpn Soc Mech Eng 44 (381):1454–1460
11. Kolyano YuM, Makhorkin IM (1984) Thermal stresses in a heat-sensitive sphere. J Eng Phys Thermophys 47(5):1373–1377
12. Makhorkin IN (1977) Thermal elasticity of a hollow sphere with temperature-dependent thermal conductivity. Strength Mater 9(12):1441–1442
13. Bahtui A, Poultangari R, Eslami MR (2007) Thermal and mechanical stresses in thick spheres with an extended FGM model. In Proceedings of the seventh international congress on thermal stresses, vol 2, Taiwan, pp 491–494
14. Nowinski J (1962) Transient thermoelastic problem for an infinite medium with a spherical cavity exhibiting temperature–dependent properties. J Appl Mech 29:399–407
15. Popovych VS, Harmatiy HYu (2004) Thermoelastic state of thermosensitive sphere under convective heat exchange with environment. Nauk notatky 15:252–264 (in Ukrainian)
16. Popovych VS, Sulym GT (2004) Centrally-symmetric quasi-static thermoelastisity problem for thermosensitive body. Mater Sci 3:365–375
17. Popovych VS (1997) Construction of solutions to thermoelastisity problems for thermosensitive bodies under convective-radial heat exchange. Dop NAN Ukraine 11:69–73 (in Ukrainian)
18. Carslaw HS, Jaeger JC (1959) Conduction of head in solids. Clarendon, Oxford
19. Prudnikov AV, Brychkov YuA, Marichev OI (1992) Direct Laplace transforms. Integrals and series, vol 4. Gordon and Breach, New York
20. Prudnikov AV, Brychkov YuA, Marichev OI (1992) Inverse Laplace transforms. Integrals and series, vol 5. Gordon and Breach, New York
21. Harmatiy HYu, Kutniv M, Popovych VS (2002) Numerical solution of non-stationary heat conduction problem for thermosensitive bodies under convective heat exchange. Mashynoznavstvo 1(55):21–25 (in Ukrainian)