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Equilibrium of elastic hollow inhomogeneous cylinders of corrugated elliptic cross-section

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Abstract. The approach to the solution of a three-dimensional boundary-value stress problem for elastic hollow inhomogeneous cylinders of corrugated elliptic cross-section is proposed. The boundary conditions make it possible to separate variables along the length at the cylinder ends. It is proposed to include additional functions into the resolving system of differential equations. These functions enable the variables to be separated along a directrix using discrete Fourier series. The boundary-value problem derived for the system of ordinary differential equations is solved by the stable numerical method of discrete orthogonalization over the cylinder thickness. Results in the form of plots and tables are presented.

Key words: corrugated ellipse, discrete orthogonalization method, inhomogeneous noncircular cylinder

1. Introduction

Problems on the stress state of circular cylinders under mechanical and thermal loads have been addressed before by Timoshenko [1, Chapter 11]. In a review by Soldatos [2, pp. 237–238] it is pointed out, that noncircular cylindrical shells with small or large eccentricity are widely used in aerospace and mechanical engineering applications. It was also noted that the number of studies devoted to noncircular cylinders is rather scanty in comparison with the extended literature on circular cylinders. Among these investigations there are studies on the solution of problems for hollow noncircular cylinders with arbitrary cross-section, including those with corrugated elliptical cross-section being considered in the present paper. However, in contrast to circular cylinders, where in solving the boundary-value problems the dimensionality may be reduced by representing resolving functions in the form of Fourier series along a circumferential coordinate, the problem solution in the case of noncircular cylinders is complicated, making it necessary to solve a three-dimensional boundary-value problem, as was demonstrated by Grigorenko and Rozhok [3]. Some semi-analytical approaches to the solution of problems of this class are given in [2] and [4, Chapter 9]. There the dimensionality over the cylinder thickness is reduced by the finite-difference method. Thus, in [3] the problem for an elliptical isotropic cylinder under certain boundary conditions imposed on the ends after separating the variables along the longitudinal coordinate is reduced to a two-dimensional one which is solved using a discrete Fourier series (see [5], [6, Chapter 6], [7, Chapter 19.7]) and the stable numerical discrete orthogonalization method proposed by Godunov [8], Grigorenko [9, pp. 80–84, 90–94], Bellman and Kalaba [10, Chapter 4.2].

This paper considers an approach to the solution of a stress problem for inhomogeneous orthotropic cylinders with corrugated elliptic cross-section under specified conditions at the

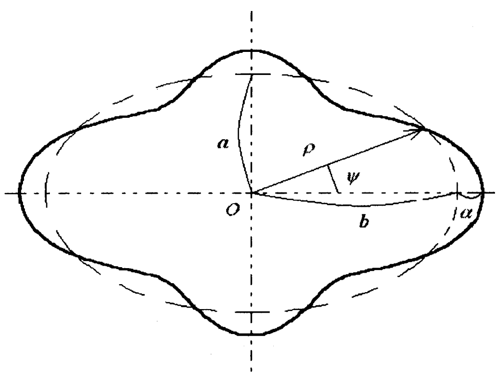

Figure 1. Cross-section of the reference surface.

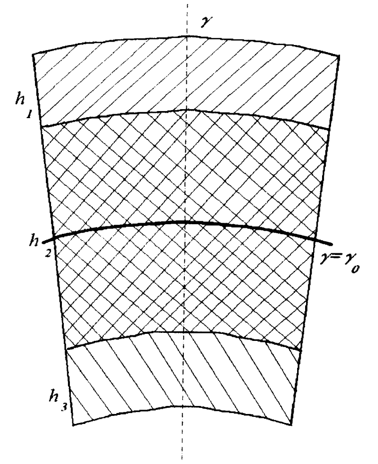

Figure 2. Structure of the cylinder cross-section.

ends. The problem is solved by using a discrete Fourier series and a numerical discrete-orthogonalization method.

2. Statement of the problem

Let us consider a stress problem for elastic hollow layered orthotropic noncircular constant-thickness cylinders with the cross-section at each point being corrugated and elliptical. The problem is described by the elasticity equations for an orthotropic body [4, Chapters 9–10], [11, pp. 59–64], [12, pp. 22–36]. In this case the first quadratic form can be written as

$$\mathrm{d}S^2 = \mathrm{d}s^2 + A_2^2(\psi,\gamma)\mathrm{d}\psi^2 + \mathrm{d}\gamma^2, \tag{1}$$

where s, ψ, γ are the orthogonal curvilinear coordinates, s is the arc length along a generatrix, ψ is the polar angle in the cross-section, γ is the normal coordinate to the reference surface $\gamma = \gamma_0$.

The directrix of the reference surface at the cross-section (Figure 1) is specified in polar coordinates in the form

$$\rho(\psi) = \frac{a}{(1 - e^2\cos^2\psi)^{1/2}} + \alpha\cos m\psi \quad (0 \le \psi \le 2\pi), \tag{2}$$

$$e = \sqrt{1-(a/b)^2} = 2\sqrt{\Delta}/(1+\Delta), \quad \Delta = (b-a)/(b+a),$$

where α is the amplitude, m is the corrugation frequency, a, b, and e are the semi-axes and eccentricity of the ellipse ($b > a$). The point O ($\rho = 0$) lies at the intersection of the ellipse axes. When $\alpha = 0$, the curve (2) describes an ellipse and at $a = b$ a corrugated circumference. The element of the cylinder cross-section is shown in Figure 2.

Then, we have:

$$A_2(\psi,\gamma) = H_2(\psi,\gamma)\omega(\psi), \tag{3}$$

where

$$H_2(\psi,\gamma)=1+\gamma/R(\psi),\quad R(\psi)=\frac{[\rho^2+(\rho')^2]^{3/2}}{\rho^2+2(\rho')^2-\rho\rho''},\quad \omega(\psi)=[\rho^2+(\rho')^2]^{1/2},$$
$$\rho'=-\left[\frac{ae^2\sin 2\psi}{2(1-e^2\cos^2\psi)^{3/2}}+\alpha m\sin m\psi\right],$$
$$\rho''=-\left\{\frac{ae^2}{2}\left[\frac{2\cos 2\psi}{(1-e^2\cos^2\psi)^{3/2}}-\frac{3e^2\sin^2\psi}{2(1-e^2\cos^2\psi)^{5/2}}\right]+\alpha m^2\cos m\psi\right\},$$

where $R(\psi)$ is the radius of the reference surface curvature.

Taking into account (1–3), we can write the initial equations and relations, which describe the equilibrium of cylinders for the given class, for the i-th layer as follows:
– expressions for strains in terms of displacements

$$\begin{aligned}
&e_s^i=\frac{\partial u_s^i}{\partial s},\quad e_\psi^i=\frac{1}{H_2\omega}\frac{\partial u_\psi^i}{\partial\psi}+\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}u_\gamma^i,\quad e_\gamma^i=\frac{\partial u_\gamma^i}{\partial\gamma},\\
&e_{s\psi}^i=\frac{1}{H_2\omega}\frac{\partial u_s^i}{\partial\psi}+\frac{\partial u_\psi^i}{\partial s},\quad e_{s\gamma}^i=\frac{\partial u_\gamma^i}{\partial s}+\frac{\partial u_s^i}{\partial\gamma},\\
&e_{\psi\gamma}^i=H_2\frac{\partial}{\partial\gamma}\left(\frac{u_\psi^i}{H_2}\right)+\frac{1}{H_2\omega}\frac{\partial u_\gamma^i}{\partial\psi};
\end{aligned}\tag{4}$$

– equilibrium equations

$$\begin{aligned}
&H_2\frac{\partial\sigma_s^i}{\partial s}+\frac{1}{\omega}\frac{\partial\tau_{s\psi}^i}{\partial\psi}+\frac{\partial}{\partial\gamma}\left(H_2\tau_{s\gamma}^i\right)=0,\\
&\frac{1}{\omega}\frac{\partial\sigma_\psi^i}{\partial\psi}+\frac{\partial}{\partial\gamma}\left(H_2\tau_{\psi\gamma}^i\right)+H_2\frac{\partial\tau_{s\psi}^i}{\partial s}+\frac{\partial H_2}{\partial\gamma}\tau_{\psi\gamma}^i=0,\\
&\frac{\partial}{\partial\gamma}\left(H_2\sigma_\gamma^i\right)+H_2\frac{\partial\tau_{s\gamma}^i}{\partial s}+\frac{1}{\omega}\frac{\partial\tau_{\psi\gamma}^i}{\partial\psi}-\frac{\partial H_2}{\partial\gamma}\sigma_\psi^i=0;
\end{aligned}\tag{5}$$

– relations of generalized Hooke's law for an orthotropic body

$$\begin{aligned}
&e_s^i=a_{11}^i\sigma_s^i+a_{12}^i\sigma_\psi^i+a_{13}^i\sigma_\gamma^i;\quad e_{\psi\gamma}^i=a_{44}^i\tau_{\psi\gamma}^i,\\
&e_\psi^i=a_{12}^i\sigma_s^i+a_{22}^i\sigma_\psi^i+a_{23}^i\sigma_\gamma^i;\quad e_{s\gamma}^i=a_{55}^i\tau_{s\gamma}^i,\\
&e_\gamma^i=a_{13}^i\sigma_s^i+a_{23}^i\sigma_\psi^i+a_{33}^i\sigma_\gamma^i;\quad e_{s\psi}^i=a_{66}^i\tau_{s\psi}^i,
\end{aligned}\tag{6}$$

where

$$\begin{aligned}
&a_{11}^i=\frac{1}{E_s^i},\quad a_{12}^i=-\frac{\nu_{s\psi}^i}{E_\psi^i}=-\frac{\nu_{\psi s}^i}{E_s^i},\quad a_{13}^i=-\frac{\nu_{s\gamma}^i}{E_\gamma^i}=-\frac{\nu_{\gamma s}^i}{E_s^i},\quad a_{22}^i=\frac{1}{E_\psi^i},\\
&a_{23}^i=-\frac{\nu_{\gamma\psi}^i}{E_\psi^i}=-\frac{\nu_{\psi\gamma}^i}{E_\gamma^i},\quad a_{33}^i=\frac{1}{E_\gamma^i},\quad a_{44}^i=\frac{1}{G_{\psi\gamma}^i},\quad a_{55}^i=\frac{1}{G_{s\gamma}^i},\quad a_{66}^i=\frac{1}{G_{\psi s}^i},\\
&\qquad(0\le s\le l,\quad 0\le\psi\le 2\pi,\quad \gamma_p\le\gamma\le\gamma_q)\quad (i=\overline{1,T}).
\end{aligned}\tag{7}$$

In the case of an isotropic body, relations (6) become:

$$
\begin{aligned}
e_s^i &= \frac{1}{E^i}(\sigma_s^i - \nu^i(\sigma_\psi^i + \sigma_\gamma^i)), \quad e_{\psi\gamma}^i = \frac{1}{G^i}\tau_{\psi\gamma}^i, \\
e_\psi^i &= \frac{1}{E^i}(\sigma_\psi^i - \nu^i(\sigma_s^i + \sigma_\gamma^i)), \quad e_{s\gamma}^i = \frac{1}{G^i}\tau_{s\gamma}^i, \\
e_\gamma^i &= \frac{1}{E^i}(\sigma_\gamma^i - \nu^i(\sigma_s^i + \sigma_\psi^i)), \quad e_{s\psi}^i = \frac{1}{G^i}\tau_{s\psi}^i.
\end{aligned} \tag{8}
$$

In (4–6) u_s^i, u_ψ^i, u_γ^i are the displacements along the generatrix, directrix, and over the thickness, e_s^i, e_ψ^i, e_γ^i, $e_{s\psi}^i$, $e_{\psi\gamma}^i$, $e_{s\gamma}^i$, $\sigma_s^i, \sigma_\psi^i, \sigma_\gamma^i, \tau_{s\psi}^i, \tau_{\psi\gamma}^i, \tau_{s\gamma}^i$ are the strains and stresses in the i-th layer, $E_s^i, E_\psi^i, E_\gamma^i, G_{s\gamma}^i, G_{s\psi}^i, G_{\psi\gamma}^i, \nu_{s\gamma}^i, \nu_{s\psi}^i, \nu_{\psi\gamma}^i$ are the corresponding elastic moduli, shear moduli, and Poisson's ratios.

Adding to Equations (4–6) the boundary conditions at the ends $s=0$, $s=l$ and the lateral surfaces $\gamma=\gamma_p$; $\gamma=\gamma_q$ of the cylinder, we arrive at a three-dimensional boundary-value problem.

Let us consider conditions for cylinders with simply supported [2] ends:

$$
\sigma_s^i = 0, \quad u_\psi^i = 0, \quad u_\gamma^i = 0 \quad \text{at} \quad s=0, \quad s=l. \tag{9}
$$

These conditions correspond to the presence at the ends of a diaphragm that is perfectly rigid in its plane and flexible out of it.

The boundary conditions at the lateral surfaces of the cylinder are specified in the form:

$$
\begin{aligned}
\sigma_\gamma &= q_\gamma^-, \quad \tau_{s\gamma} = q_s^-, \quad \tau_{\psi\gamma} = q_\psi^- \quad \text{at} \quad \gamma=\gamma_p, \\
\sigma_\gamma &= q_\gamma^+, \quad \tau_{s\gamma} = q_s^+, \quad \tau_{\psi\gamma} = q_\psi^+ \quad \text{at} \quad \gamma=\gamma_q.
\end{aligned} \tag{10}
$$

When the adjacent layers contact without slip and separation, conditions of continuity should be met:

$$
\begin{aligned}
\sigma_\gamma^i &= \sigma_\gamma^{i+1}, \quad \tau_{s\gamma}^i = \tau_{s\gamma}^{i+1}, \quad \tau_{\psi\gamma}^i = \tau_{\psi\gamma}^{i+1}, \\
u_\gamma^i &= u_\gamma^{i+1}, \quad u_s^i = u_s^{i+1}, \quad u_\psi^i = u_\psi^{i+1}, \qquad (i=\overline{1, T-1}).
\end{aligned} \tag{11}
$$

As resolving functions, with allowance for the conjugation conditions, we choose the stress and displacement components $\sigma_\gamma, \tau_{s\gamma}, \tau_{\psi\gamma}, u_\gamma, u_s, u_\psi$. Upon performing some transformations, from (4–6) we obtain the resolving system of partial differential equations with sixth-order variable coefficients

$$
\begin{aligned}
\frac{\partial\sigma_\gamma^i}{\partial\gamma} &= (c_2^i - 1)\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}\sigma_\gamma^i - \frac{\partial\tau_{s\gamma}^i}{\partial s} - \frac{1}{H_2\omega}\frac{\partial\tau_{\psi\gamma}^i}{\partial\psi} \\
&\quad + b_{22}^i\left(\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}\right)^2 u_\gamma^i + b_{12}^i\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}\frac{\partial u_s^i}{\partial s} + b_{22}^i\frac{1}{H_2^2}\frac{\partial H_2}{\partial\gamma}\frac{1}{\omega}\frac{\partial u_\psi^i}{\partial\psi}, \\
\frac{\partial\tau_{s\gamma}^i}{\partial\gamma} &= -c_1^i\frac{\partial\sigma_\gamma^i}{\partial s} - \frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}\tau_{s\gamma}^i - b_{12}^i\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}\frac{\partial u_\gamma^i}{\partial s} - b_{11}^i\frac{\partial^2 u_s^i}{\partial s^2} \\
&\quad - b_{66}^i\frac{1}{H_2\omega}\frac{\partial}{\partial\psi}\left(\frac{1}{H_2\omega}\frac{\partial u_s^i}{\partial\psi}\right) - (b_{12}^i + b_{66}^i)\frac{1}{H_2\omega}\frac{\partial^2 u_\psi^i}{\partial s\partial\psi}, \\
\frac{\partial\tau_{\psi\gamma}^i}{\partial\gamma} &= -c_2^i\frac{1}{H_2\omega}\frac{\partial\sigma_\gamma^i}{\partial\psi} - \frac{2}{H_2}\frac{\partial H_2}{\partial\gamma}\tau_{\psi\gamma}^i - b_{22}^i\frac{1}{H_2\omega}\frac{\partial}{\partial\psi}\left(\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}u_\gamma^i\right) \\
&\quad - (b_{12}^i + b_{66}^i)\frac{1}{H_2\omega}\frac{\partial^2 u_s^i}{\partial s\partial\psi} - b_{22}^i\frac{1}{H_2\omega}\frac{\partial}{\partial_\psi}\left(\frac{1}{H_2\omega}\frac{\partial u_\psi^i}{\partial\psi}\right) - b_{66}^i\frac{\partial^2 u_\psi^i}{\partial s^2},
\end{aligned} \tag{12}
$$

$$\frac{\partial u_\gamma^i}{\partial\gamma}=c_4^i\sigma_\gamma^i-c_2^i\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}u_\gamma^i-c_1^i\frac{\partial u_s^i}{\partial s}-c_2^i\frac{1}{H_2\omega}\frac{\partial u_\psi^i}{\partial\psi},$$
$$\frac{\partial u_s^i}{\partial\gamma}=a_{55}^i\tau_{s\gamma}^i-\frac{\partial u_\gamma^i}{\partial s},\quad \frac{\partial u_\psi^i}{\partial\gamma}=a_{44}^i\tau_{\psi\gamma}^i-\frac{1}{H_2\omega}\frac{\partial u_\gamma^i}{\partial\psi}+\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}u_\psi^i,$$

where

$$\begin{aligned}&b_{11}^i=a_{22}^ia_{66}^i/\Omega^i,\quad b_{12}^i=-a_{12}^ia_{66}^i/\Omega^i,\quad b_{22}^i=a_{11}^ia_{66}^i/\Omega^i,\\&b_{66}^i=1/a_{66}^i,\quad \Omega^i=(a_{11}^ia_{22}^i-a_{12}^{i2})a_{66}^i,\\&c_1^i=-(b_{11}^ia_{13}^i+b_{12}^ia_{23}^i),\quad c_2^i=-(b_{12}^ia_{13}^i+b_{22}^ia_{23}^i),\quad c_4^i=a_{33}^i+c_1^ia_{13}^i+c_2^ia_{23}^i.\end{aligned}\tag{13}$$

3. Method for solving the problem

The boundary conditions (9) specified at ends allow us to separate in system (12) the variables in s and to reduce the problem to a two-dimensional one. Let us present the resolving functions and load components in the form of a Fourier series along a cylinder generatrix as

$$\begin{aligned}&X(s,\psi,\gamma)=\sum_{n=1}^{N}X_n(\psi,\gamma)\sin\lambda_n s,\quad Y(s,\psi,\gamma)=\sum_{n=0}^{N}Y_n(\psi,\gamma)\cos\lambda_n s,\\&X=\{\sigma_\gamma^i;\tau_{\psi\gamma}^i;u_\gamma^i;u_\psi^i;q_\gamma^i;q_\psi^i\},\quad Y=\{\tau_{s\gamma}^i;u_s^i;q_s^i\},\quad \lambda_n=\pi n/l,\quad (0\le s\le l;\, i=\overline{1,T}).\end{aligned}\tag{14}$$

Substituting (14) in system (12) and boundary conditions (10), we obtain for the n-th term in expansions (14) the two-dimensional boundary-value problem (for simplicity the index i is omitted):

$$\begin{aligned}\frac{\partial\sigma_{\gamma,n}}{\partial\gamma}&=(c_2-1)\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}\sigma_{\gamma,n}+\lambda_n\tau_{s\gamma,n}-\frac{1}{H_2\omega}\frac{\partial\tau_{\psi\gamma,n}}{\partial\psi}\\&\quad+b_{22}\left(\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}\right)^2u_{\gamma,n}-b_{12}\lambda_n\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}u_{s,n}+b_{22}\frac{1}{H_2^2}\frac{\partial H_2}{\partial\gamma}\frac{1}{\omega}\frac{\partial u_{\psi,n}}{\partial\psi},\\\frac{\partial\tau_{s\gamma,n}}{\partial\gamma}&=-c_1\lambda_n\sigma_{\gamma,n}-\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}\tau_{s\gamma,n}-b_{12}\lambda_n\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}u_{\gamma,n}+b_{11}\lambda_n^2u_{s,n}\\&\quad-b_{66}^i\frac{1}{H_2\omega}\frac{\partial}{\partial\psi}\left(\frac{1}{H_2\omega}\frac{\partial u_{s,n}}{\partial\psi}\right)-(b_{12}+b_{66})\lambda_n\frac{1}{H_2\omega}\frac{\partial u_{\psi,n}}{\partial\psi},\\\frac{\partial\tau_{\psi\gamma,n}}{\partial\gamma}&=-c_2\frac{1}{H_2\omega}\frac{\partial\sigma_{\gamma,n}}{\partial\psi}-\frac{2}{H_2}\frac{\partial H_2}{\partial\gamma}\tau_{\psi\gamma,n}-b_{22}\frac{1}{H_2\omega}\frac{\partial}{\partial\psi}\left(\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}u_{\gamma,n}\right)\\&\quad+(b_{12}+b_{66})\lambda_n\frac{1}{H_2\omega}\frac{\partial u_{s,n}}{\partial\psi}-b_{22}\frac{1}{H_2\omega}\frac{\partial}{\partial_\psi}\left(\frac{1}{H_2\omega}\frac{\partial u_{\psi,n}}{\partial\psi}\right)+b_{66}\lambda_n^2u_{\psi,n},\\\frac{\partial u_{\gamma,n}}{\partial\gamma}&=c_4\sigma_{\gamma,n}-c_2\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}u_{\gamma,n}+c_1\lambda_nu_{s,n}-c_2\frac{1}{H_2\omega}\frac{\partial u_{\psi,n}}{\partial\psi},\quad\frac{\partial u_{s,n}}{\partial\gamma}=a_{55}\tau_{s\gamma,n}^i-\lambda_nu_{\gamma,n},\\\frac{\partial u_\psi,n}{\partial\gamma}&=a_{44}\tau_{\psi\gamma,n}-\frac{1}{H_2\omega}\frac{\partial u_{\gamma,n}}{\partial\psi}+\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}u_{\psi,n}(n=\overline{0,N})\end{aligned}\tag{15}$$

with boundary conditions

$$\begin{aligned}&\sigma_{\gamma,n}=q_{\gamma,n}^-,\quad\tau_{s\gamma,n}=q_{s,n}^-,\quad\tau_{\psi\gamma,n}=q_{\psi,n}^-\quad\text{at }\gamma=\gamma_p,\\&\sigma_{\gamma,n}=q_{\gamma,n}^+,\quad\tau_{s\gamma,n}=q_{s,n}^+,\quad\tau_{\psi\gamma,n}=q_{\psi,n}^+\quad\text{at }\gamma=\gamma_q.\end{aligned}\tag{16}$$

To reduce the two-dimensional boundary-value problem (15) and (16) to a one-dimensional one, we will use an approach [3], [13] based on the change of the terms in the resolving system (15), which interfere with the separation of variables along the cylinder directrix, with new additional functions. These functions, expressed in terms of resolving functions, include geometrical parameters and have the form

$$\begin{aligned}
&\varphi_1^j = \frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}\left\{\sigma_{\gamma,n};\,\tau_{s\gamma,n};\,u_{\gamma,n};\,u_{s,n};\,\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}u_{\gamma,n}\right\},\quad (j=\overline{1,5}),\\
&\varphi_2^j = \frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}\{\tau_{\psi\gamma,n};\,u_{\psi,n}\},\quad (j=\overline{1,2}),\\
&\varphi_3^j = \frac{1}{H_2\omega}\frac{\partial}{\partial\psi}\{\sigma_{\gamma,n};\,u_{\gamma,n};\,u_{s,n}\},\quad (j=\overline{1,3}),\\
&\varphi_4^j = \frac{1}{H_2\omega}\left\{\frac{\partial\tau_{\psi\gamma,n}}{\partial\psi};\,\frac{\partial u_{\psi,n}}{\partial\psi};\,\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}\frac{\partial u_{\psi,n}}{\partial\psi}\right\},\quad (j=\overline{1,3})(n=\overline{0,N}),\\
&\varphi_5 = \frac{1}{H_2\omega}\frac{\partial\varphi_1^3}{\partial\psi},\quad \varphi_6 = \frac{1}{H_2\omega}\frac{\partial\varphi_3^3}{\partial\psi},\quad \varphi_7 = \frac{1}{H_2\omega}\frac{\partial\varphi_4^2}{\partial\psi}.
\end{aligned}\tag{17}$$

After substituting the additional functions (17) in the system (15), we arrive at the following system of equations:

$$\begin{aligned}
&\frac{\partial\sigma_\gamma}{\partial\gamma} = (c_2-1)\varphi_1^1 + \lambda_n\tau_{s\gamma} - \varphi_4^1 + b_{22}\varphi_1^5 - b_{12}\lambda_n\varphi_1^4 + b_{22}\varphi_4^3,\\
&\frac{\partial\tau_{s\gamma}}{\partial\gamma} = -c_1\lambda_n\sigma_\gamma - \varphi_1^2 - b_{12}\lambda_n\varphi_1^3 + b_{11}\lambda_n^2 u_s - b_{66}\varphi_6 - (b_{12}+b_{66})\lambda_n\varphi_4^2,\\
&\frac{\partial\tau_{\psi\gamma}}{\partial\gamma} = -c_2\varphi_3^1 - 2\varphi_2^1 - b_{22}\varphi_5 + (b_{12}+b_{66})\lambda_n\varphi_3^3 - b_{22}\varphi_7 + b_{66}\lambda_n^2 u_\psi,\\
&\frac{\partial u_\gamma}{\partial\gamma} = c_4\sigma_\gamma - c_2\varphi_4^2 + c_1\lambda_n u_s - c_2\varphi_1^3,\\
&\frac{\partial u_s}{\partial\gamma} = a_{55}\tau_{s\gamma} - \lambda_n u_\gamma,\quad \frac{\partial u_\psi}{\partial\gamma} = a_{44}\tau_{\psi\gamma} - \varphi_3^2 + \varphi_2^2
\end{aligned}\tag{18}$$

with the boundary conditions (16). Here the index n, in the notations of resolving and additional functions is omitted.

Now represent all the functions entering into the system (18) and boundary conditions (16) as the expansions into the Fourier series along the coordinate ψ:

$$\begin{aligned}
&X(\psi,\gamma) = \sum_{k=0}^{K} X_k(\gamma)\cos k\psi,\quad Y(\psi,\gamma) = \sum_{k=1}^{K} Y_k(\gamma)\sin k\psi,\\
&X = \left\{\sigma_\gamma;\,\tau_{s\gamma};\,u_\gamma;\,u_s;\,\varphi_1^j;\,\varphi_4^j;\,\varphi_6;\,q_\gamma;\,q_s\right\},\quad Y = \left\{\tau_{\psi\gamma};\,u_\psi;\,\varphi_2^j;\,\varphi_3^j;\,\varphi_5;\,\varphi_7;\,q_\psi\right\}.
\end{aligned}\tag{19}$$

In this case the number of series terms K should be not less than the number of terms in the series for the functions q_γ, q_s, q_ψ.

Expansions (19) are based on the fact that functions $X = \{\sigma_\gamma;\,\tau_{s\gamma};\,u_\gamma;\,u_s;\,\varphi_1^j;\,\varphi_4^j;\,\varphi_6;\,q_\gamma;\,q_s\}$ entering into Equations (18) are even and $Y = \{\tau_{\psi\gamma};\,u_\psi;\,\varphi_2^j;\,\varphi_3^j;\,\varphi_5;\,\varphi_7;\,q_\psi\}$ are odd. Because of this, we can restrict ourselves in the Fourier series to one term only, namely cosine or sine. This is supported also by the situation that, after substitution of (19) in (18), the variables are separated. If the functions X and Y do not have the property of being even, the Fourier series have to include both the sine and cosine. This takes place in the case of anisotropic layers.

Having substituted the series (19) in the equations of system (18) and boundary conditions (16) and having separated the variables, we arrive at the following system of ordinary differential equations with respect to the amplitude values of the functions entering into system (18):

$$
\begin{aligned}
&\frac{d\sigma_{\gamma,k}}{d\gamma} = (c_2-1)\varphi^1_{1,k} + \lambda_n\tau_{s\gamma,k} - \varphi^1_{4,k} + b_{22}\varphi^5_{1,k} - b_{12}\lambda_n\varphi^4_{1,k} + b_{22}\varphi^3_{4,k},\\
&\frac{d\tau_{s\gamma,k}}{d\gamma} = -c_1\lambda_n\sigma_{\gamma,k} - \varphi^2_{1,k} - b_{12}\lambda_n\varphi^3_{1,k} + b_{11}\lambda_n^2 u_{s,k} - b_{66}\varphi_{6,k} - (b_{12}+b_{66})\lambda_n\varphi^2_{4,k},\\
&\frac{d\tau_{\psi\gamma,k}}{d\gamma} = -c_2\varphi^1_{3,k} - 2\varphi^1_{2,k} - b_{22}\varphi_{5,k} + (b_{12}+b_{66})\lambda_n\varphi^3_{3,k} - b_{22}\varphi_{7,k} + b_{66}\lambda_n^2 u_{\psi,k},\\
&\frac{du_{\gamma,k}}{d\gamma} = c_4\sigma_{\gamma,k} - c_2\varphi^2_{4,k} + c_1\lambda_n u_{s,k} - c_2\varphi^3_{1,k},\\
&\frac{du_{s,k}}{d\gamma} = a_{55}\tau_{s\gamma,k} - \lambda_n u_{\gamma,k}, \qquad \frac{du_{\psi,k}}{d\gamma} = a_{44}\tau_{\psi\gamma,k} - \varphi^2_{3,k} + \varphi^2_{2,k}
\end{aligned}
\tag{20}
$$

with the boundary conditions

$$
\begin{aligned}
&\sigma_{\gamma,k} = q^-_{\gamma,k}, \quad \tau_{s\gamma} = q^-_{s,k}, \quad \tau_{\psi\gamma,k} = q^-_{\psi,k} \qquad \text{at } \gamma = \gamma_p;\\
&\sigma_{\gamma,k} = q^+_{\gamma,k}, \quad \tau_{s\gamma,k} = q^+_{s,k}, \quad \tau_{\psi\gamma,k} = q^+_{\psi,k} \qquad \text{at } \gamma = \gamma_q \quad (k = \overline{0,K}).
\end{aligned}
\tag{21}
$$

In view of (17) for each k in Equations (20) we have:

$$
\begin{aligned}
&\varphi^j_{1,k} = \varphi^j_{1,k}(\gamma; \sigma_{\gamma,l}; \tau_{s\gamma,l}; u_{s,l}), && (j=\overline{1,5}),\\
&\varphi^j_{2,k} = \varphi^j_{2,k}(\gamma; \tau_{\psi\gamma,l}; u_{\psi,l}), && (j=\overline{1,2}),\\
&\varphi^j_{3,k} = \varphi^j_{3,k}(\gamma; \sigma_{\gamma,l}; u_{\gamma,l}; u_{s,l}), && (j=\overline{1,3}),\\
&\varphi^j_{4,k} = \varphi^j_{4,k}(\gamma; \tau_{\psi\gamma,l}; u_{\psi,l}), && (j=\overline{1,3}),\\
&\varphi_{5,k} = \varphi_{5,k}(\gamma; u_{\gamma,l}); \varphi_{6,k} = \varphi_{6,k}(\gamma; u_{s,l}), &&\\
&\varphi_{7,k} = \varphi_{7,k}(\gamma; u_{\psi,l}), && (l=\overline{0,K}).
\end{aligned}
\tag{22}
$$

The values φ^j_{ik} which enter into the coefficients of the Fourier series, are not expressed explicitly in terms of the Fourier-series coefficients of the resolving functions and are calculated by integrating system (20) by the Runge scheme, using the discrete Fourier series at each step $\gamma = \text{const}$. Relations (22) demonstrate the dependency of these coefficients on the amplitude values of the determined resolving functions and connectivity of all $6K+4$ equations of system (20). The equations of this system are integrated simultaneously for all harmonics by a stable discrete numerical orthogonalization method. To find the amplitude values of the functions (22) during integration by the current values of the resolving functions for a fixed magnitude of γ, we will calculate the values of the functions (22) at a number of points ψ_r $(r = \overline{1,R})$ on the interval $[0, 2\pi]$ and construct a discrete Fourier series, *i.e.*, the Fourier series for the discretely specified function. As the number of points, where values of additional functions are calculated, increases, the discrete Fourier series becomes progressively less distinguished from the exact Fourier series. Earlier on, while integrating, we have used the boundary conditions. The amplitude values found for the functions (22) are substituted in the system (20) with integration being continued in γ. In this case the Runge-Kutta method with orthogonalization at separate points of the interval $(\gamma_p \le \gamma \le \gamma_q)$ is applied. When solving the ill-conditioned boundary-value problems, this method is stable [9, pp. 80–84].

The applied approach, based on the use of the discrete orthogonalization method and discrete Fourier-series method, has been described in detail for solving the problem of the bending of rectangular plates and illustrated by the examples in [14]. Here too the convergence to the problem solutions was shown. These solutions were obtained by the proposed approach, depending on the choice of a varying number of orthogonalization points and points for which the tabular values of additional functions (17) are calculated, as well as on the number of terms that are retained in expansions (19).

4. Numerical results and discussion

Based on the above approach, we will analyze the stress state of three-layered cylinders of a corrugated elliptical cross-section under internal pressure $q=q_0\sin(\pi s/l)$, $q_0=\text{const}$. The problem is solved for $h_1=h_3=2$, $h_2=4$, where h_1 and h_3 are the thicknesses of the external and internal layers, respectively, h_2 is the thickness of the middle layer. The other parameters are equal to: $r_0=40$, $l=60$, $m=4$, $\alpha=2,3$, $\nu=0{\cdot}3$, $\Delta=0,0{\cdot}1,0{\cdot}2$. It was assumed that the perimeter of the surface-reduction cross-section for an ellipse is equal to the length of a circumference of the radius r_0.

At first, we will consider the problem involving an inhomogeneous three-layered cylinder with isotropic layers, when the elastic moduli are $E_1=E_3=E_0$, the middle layer has the elastic modulus $E_2=dE_0$, where $d=1,0{\cdot}1$. For all the layers we used Poisson's ratio $\nu=0{\cdot}3$. Results of the problem solution are presented in Figures 3–6 and Tables 1 and 2 for the internal cylinder surface.

In Figures 3–6, where $\alpha=2$, the solid lines correspond to $d=1$, dashed ones to $d=0{\cdot}1$.

Figure 3 shows how the inhomogeneity of the structure and ellipticity of the cylinder affect the distribution of the displacements along the directrix. So, for $\Delta=0{\cdot}2$ the maximal displacements u_γ at $d=0{\cdot}1$ increase 1·9 times in comparison with a homogeneous cylinder. The maximal displacements u_γ (Figure 3) hold near the apex of the corrugation valley, that is, in the most pliable region of a corrugation. This point explains the above-mentioned distinction.

Figures 4–6 show how the stresses σ_ψ vary along the directrix and over the thickness, depending on the structure inhomogeneity and ellipticity level of the cylinder. It follows from Figure 4 that the maximal magnitudes of the stress σ_ψ hold at the corrugation apex for $\psi=0$

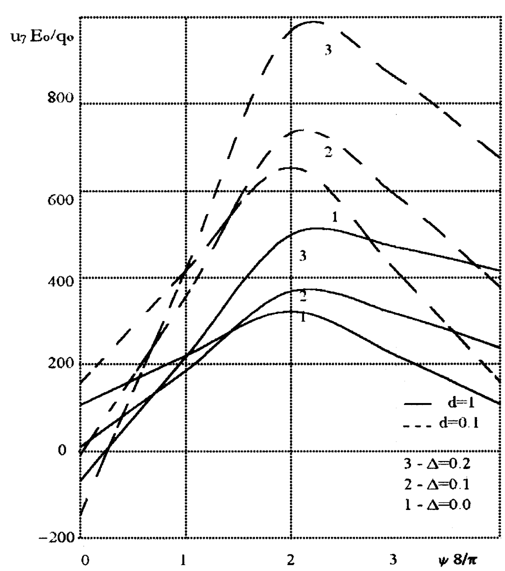

Figure 3. Distribution of displacements $u_\gamma E_0/q_0$ on the interval $0\le\psi\le\pi/2$ for various values of d.

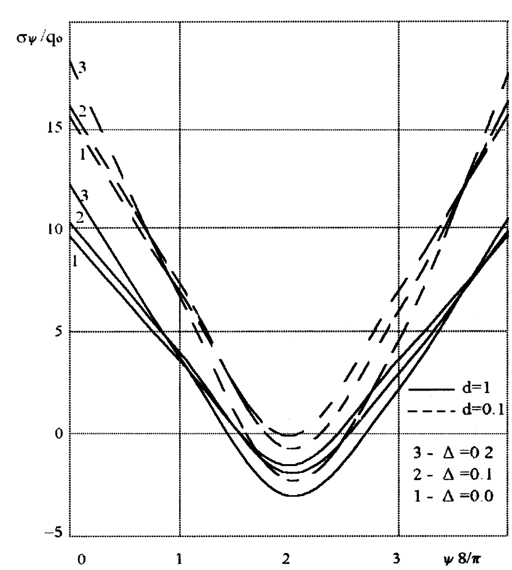

Figure 4. Distribution of stresses σ_ψ/q_0 on the interval $0\le\psi\le\pi/2$ for various values of d.

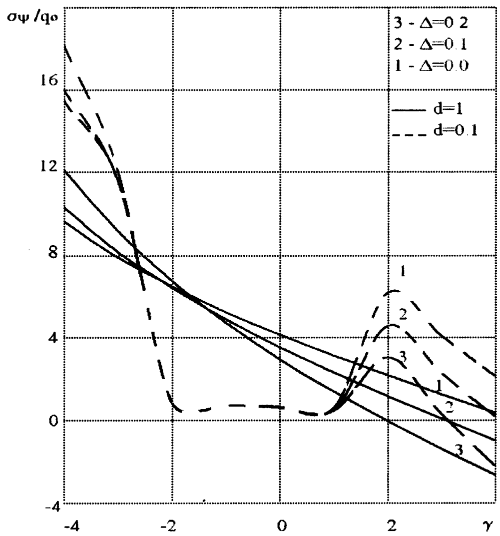

Figure 5. Distribution of stresses σ_ψ/q_0 over the cylinder thickness at $\psi=0$ for various values of d.

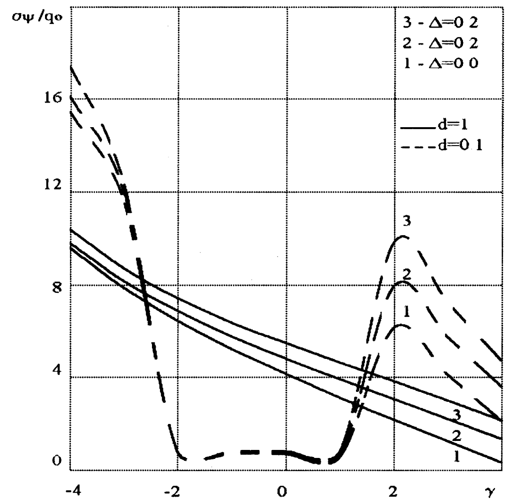

Figure 6. Distribution of stresses σ_ψ/q_0 over the cylinder thickness at $\psi=\pi/2$ for various values of d.

and $\psi=\pi/2$, that is, in the domains of maximal rigidity. In this case the maximal stress $\sigma_\psi(\psi=0)$ at $d=0{\cdot}1$ increases by a factor 1·7 in comparison with a homogeneous cylinder at $\Delta=0{\cdot}2$. The influence of ellipticity on the value of the displacements u_γ for $\Delta\neq 0$ is stronger for $\psi=0$ and $\psi=\pi/2$ than that of the stresses σ_ψ.

It can be noted regarding the change in the stress σ_ψ over the cylinder thickness (Figures 5 and 6) that this stress takes a maximum value on the internal surface of the cylinder, that

Table 1. Values of the displacements $u_\gamma E_0/q_0$.

Δ	H	$d=1{\cdot}0$			$d=0{\cdot}1$		
		$\psi=0$	$\psi=\pi/4$	$\psi=\pi/2$	$\psi=0$	$\psi=\pi/4$	$\psi=\pi/2$
0·0	−4	96·6	414·6	96·6	134·9	848·2	134·9
	−2	89·3	417·5	89·3	126·5	850·4	126·5
	0·0	85·8	416·7	85·8	124·5	839·0	124·5
	2	84·4	412·6	84·4	119·6	830·9	119·6
	4	84·6	404·6	84·6	117·5	820·7	117·5
0·1	−4	7·1	457·9	219·1	−17·4	927·5	343·4
	−2	−0·1	461·3	211·5	−25·3	930·5	334·5
	0·0	−2·8	460·8	207·2	−24·3	919·6	331·0
	2	−3·0	456·6	204·6	−25·7	912·0	232·0
	4	−1·3	448·2	203·2	−25·6	901·4	318·2
0·2	−4	−62·0	583·2	389·9	−138·6	1150·7	635·3
	−2	−69·4	588·0	381·9	−146·2	1155·7	626·1
	0·0	−71·3	588·3	376·9	−139·9	1146·6	621·6
	2	−70·0	584·0	373·0	−136·7	1140·6	610·0
	4	−66·7	574·5	369·8	−134·3	1109·0	602·0

Table 2. Values of the stresses σ_ψ/q_0.

Δ	H	$d=1{\cdot}0$			$d=0{\cdot}1$		
		$\psi=0$	$\psi=\pi/4$	$\psi=\pi/2$	$\psi=0$	$\psi=\pi/4$	$\psi=\pi/2$
0·0	−4	12·21	−4·15	12·21	18·84	−4·07	18·84
	−2	7·44	−0·26	7·44	0·92	0·16	0·92
	0·0	4·03	3·40	4·03	0·71	0·43	0·71
	2	1·23	7·16	1·23	5·57	7·34	5·57
	4	−1·33	11·44	−1·33	−0·02	17·25	−0·02
0·1	−4	12·54	−4·54	12·83	18·97	−4·61	19·99
	−2	7·27	−0·45	7·94	0·94	0·15	0·90
	0·0	3·51	3·44	4·56	0·66	0·44	0·77
	2	0·47	7·45	1·78	4·19	7·55	7·01
	4	−2·24	12·02	−0·82	−1·40	17·97	0·81
0·2	−4	14·08	−5·56	13·92	20·81	−6·01	21·67
	−2	7·47	−0·93	8·48	0·97	0·10	0·83
	0·0	2·96	3·54	4·98	0·64	0·45	0·80
	2	−0·57	8·21	2·14	2·82	8·04	8·38
	4	−3·63	13·56	−0·58	−3·41	19·80	1·26

is, at the same place where a load is applied. The stresses in the vicinity of a middle layer ($d=0{\cdot}1$) approach zero with distance from the internal surface; in the vicinity of the external layer they assume values that are almost three times smaller than the maximum values. In this case ellipticity has an insignificant effect.

The variation in numerical values of the displacements u_γ and stress σ_ψ over the thickness of the inhomogeneous cylinder depending on Δ and d at $\alpha=3$ can be seen from Tables 1 and 2. The given values of the displacements u_γ and the stresses σ_ψ characterize changes in the similar values for $\alpha=3$ as compared with those for $\alpha=2$ (Figures 3–6). Thus, among other things for $d=0{\cdot}1$, $\Delta=0{\cdot}2$ the maximal displacement increases by a factor of 1·15 (Table 1, Figure 3) and the maximal stress σ_ψ by 1·2. (Table 2).

The problem was also solved for a three-layered cylinder, when the external layers are isotropic and load-bearing, for $E_1=E_3=E_0$ and $\nu=0{\cdot}3$, and the middle layer is orthotropic, for $E_s=3{\cdot}68E_0$, $E_\psi=2{\cdot}68E_0$, $E_\gamma 1{\cdot}1E_0$, $\nu_{s\psi}=0{\cdot}105$, $\nu_{s\gamma}=0{\cdot}405$, $\nu_{\psi\gamma}=0{\cdot}431$, $G_{s\psi}=0{\cdot}5E_0$, $G_{s\gamma}=0{\cdot}45E_0$, $G_{\psi\gamma}=0{\cdot}41E_0$ [12, p. 64]. Corresponding results are given in Tables 3–4 which show how the displacements u_γ and stresses σ_s and σ_ψ vary over the cylinder thickness depending on Δ, ψ, and α. As this takes place, it can be noted (Table 3, Figure 3) that the values of the maximal displacements u_γ at $\alpha=2$, $\Delta=0{\cdot}2$ for a cylinder with a homogeneous orthotropic layer and with a soft middle layer are in the ratio $1:1{\cdot}54:3{\cdot}02$, that is, the cylinders become more pliable in the same sequence. As to the stresses (Tables 4, 5), it can be noted that the character of their distribution depends strongly on the orthotropy parameters.

In solving the above problems, to construct the discrete Fourier series for additional functions, we found their magnitudes at 80 points, taking into account the first 15 harmonics. In this case, to obtain a stable result by the numerical method used, we adopted 41 orthogonalization points.

Thus, variation in cylinder characteristics makes it possible to choose the rational parameters of similar structure elements.

Table 3. Values of the displacements $u_\gamma E_0/q_0$.

ψ	H	$\alpha=2$			$\alpha=3$		
		$\Delta=0$	$\Delta=0{\cdot}1$	$\Delta=0{\cdot}2$	$\Delta=0$	$\Delta=0{\cdot}1$	$\Delta=0{\cdot}2$
0·0	−4	40·2	−34·7	−94·9	32·7	−35·9	−88·7
	−2	34·9	−40·2	−100·8	26·4	−42·2	−95·2
	0·0	28·5	−45·4	−104·9	19·3	−47·9	−99·8
	2	25·8	−44·5	−102·3	17·8	−47·2	−96·6
	4	26·0	−44·1	−99·4	19·1	−44·9	−93·2
$\pi/4$	−4	239·4	275·0	380·3	325·1	360·3	461·2
	−2	239·7	275·7	382·5	327·3	362·9	465·1
	0·0	238·3	274·7	382·4	328·7	365·0	468·9
	2	229·9	265·9	372·4	319·9	356·0	459·5
	4	224·5	260·1	365·4	313·3	348·9	451·5
$\pi/2$	−4	40·2	139·2	276·7	32·7	127·3	261·4
	−2	34·9	133·8	271·2	26·4	120·7	254·4
	0·0	28·5	126·0	261·9	19·3	112·1	243·9
	2	25·8	120·7	253·5	17·8	108·1	237·1
	4	26·0	119·8	251·4	19·1	108·5	236·4

Table 4. Values of the stresses σ_s/q_0.

ψ	H	$\alpha=2$			$\alpha=3$		
		$\Delta=0$	$\Delta=0{\cdot}1$	$\Delta=0{\cdot}2$	$\Delta=0$	$\Delta=0{\cdot}1$	$\Delta=0{\cdot}2$
0·0	−4	7·18	7·58	8·93	9·49	9·60	10·27
	−2	11·00	10·75	11·34	13·19	12·71	13·03
	0·0	5·91	4·96	4·05	5·74	4·93	4·05
	2	0·61	−0·05	−0·90	−0·20	−0·67	−1·37
	4	−1·01	−1·84	−3·03	−2·46	−2·98	−3·92
$\pi/4$	−4	−2·35	−2·70	−3·64	−4·58	−4·91	−5·75
	−2	0·69	0·41	−0·32	−2·21	−2·50	−3·20
	0·0	5·68	5·82	6·17	4·99	5·06	5·26
	2	4·43	4·70	5·41	5·25	5·48	6·07
	4	6·78	7·28	8·66	9·01	9·50	10·75
$\pi/2$	−4	7·18	7·52	8·31	9·49	10·22	11·44
	−2	11·00	12·04	13·53	13·19	14·43	15·97
	0·0	5·91	6·99	8·13	5·75	6·57	7·26
	2	0·61	1·14	1·58	−0·20	0·08	0·19
	4	−1·01	−0·49	−0·18	−2·46	−2·32	−2·41

Table 5. Values of the stresses σ_ψ/q_0.

ψ	H	$\alpha=2$			$\alpha=3$		
		$\Delta=0$	$\Delta=0{\cdot}1$	$\Delta=0{\cdot}2$	$\Delta=0$	$\Delta=0{\cdot}1$	$\Delta=0{\cdot}2$
0·0	−4	1·04	1·55	2·13	1·78	2·14	2·62
	−2	−1·17	−1·05	−1·14	−0·46	−0·54	−0·63
	0·0	−0·34	−1·62	−2·85	0·10	−1·23	−2·30
	2	0·09	−0·81	−1·70	−0·10	−0·93	−1·71
	4	−0·25	−1·60	−2·93	−0·70	−1·90	−3·04
$\pi/4$	−4	−3·60	−4·10	−5·59	−5·36	−5·89	−7·44
	−2	−6·08	−6·88	−9·33	−8·75	−9·69	−12·52
	0·0	−0·69	−0·75	−1·06	−1·40	−1·61	−2·36
	2	2·18	2·42	3·05	2·66	2·84	3·30
	4	4·10	4·59	5·98	5·43	5·87	7·05
$\pi/2$	−4	1·04	0·51	−0·18	1·78	1·48	1·10
	−2	−1·17	−1·47	−1·95	−0·46	−0·34	−0·08
	0·0	−0·34	1·18	3·23	0·10	1·92	4·60
	2	0·09	1·12	2·45	−0·10	0·91	2·29
	4	−0·25	1·29	3·30	−0·70	0·76	2·77

5. Conclusions

In conclusion we note that the method proposed in this paper provides a tool for constructing solutions of the three-dimensional boundary-value problem concerning the equilibrium of hollow non-circular inhomogeneous cylinders of intricate cross-section. For this case we efficiently used the Fourier-series method for discretely specified functions and a stable numerical discrete orthogonalization method. This approach allowed us to solve the problems as applied to hollow cylinders of various cross-sectional shape.

Besides, the method being based on a continuous scheme makes it possible to obtain a relatively accurate approximate solution of the problem. This problem can not be solved by projective or variational methods. The method was found to be suitable for solving problems involving single-layered isotropic cylinders with a corrugated circular [13] and elliptical [15] cross-section. The possibility to realize the proposed approach to the solution of problems of the given class has been illustrated by various examples. The solution obtained falls in the category of exact solutions of the theory of elasticity which first of all serve as a basis for constructing and evaluation of the reliability of applied mathematical models and design schemes for both development and estimation of the accuracy of the approximate methods for calculating structural elements. The numerical results obtained and presented in Figures 3–6 and Tables 1–5 indicate that choosing rational parameters of similar structural elements can be performed by variation in ellipticity and corrugation characteristics.

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