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RESEARCH

Classical Elastodynamics as a Linear Symmetric Hyperbolic System in Terms of $(\mathbf{u_x}, \mathbf{u}_t)$

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Abstract
Motivated from standard procedures in linear wave equations, we write the equations of classical elastodynamics as a linear symmetric hyperbolic system in terms of the displacement gradient $(\mathbf{u_x})$ and the velocity $(\mathbf{u}_t)$; this is in contrast with common practice, where the stress tensor and the velocity are used as the basic variables. We accomplish our goal by a judicious use of the compatibility equations. The approach using the stress tensor and the velocity requires use of the time differentiated constitutive law as a field equation; the present approach is devoid of this need. The symmetric form presented here is based on a Cartesian decomposition of the variables and the differential operators that does not alter the Hamiltonian structure of classical elastodynamics. We comment on the differences of our approach with that using the stress tensor in terms of the differentiability of the coefficients and the differentiability of the solution. Our analysis is confined to classical elastodynamics, namely geometrically and materially linear anisotropic elasticity which we treat as a linear theory per se and not as the linearization of the nonlinear theory. We, nevertheless, comment on the symmetrization processes of the nonlinear theories and the potential relation of them with the present approach.

Keywords Linear theory · Hyperbolicity · First order system · Elastodynamics · Hamiltonian structure

Mathematics Subject Classification 74B05 · 74B99 · 35L02

1 Motivation

When writing the system of classical elastodynamics as a linear symmetric hyperbolic system ([7]) it is a common practice to use the stress tensor and the velocity field as the basic variables. Such an approach requires the time differentiated constitutive law to be used as a field equation ([12, 21, 22]). We here present an alternative path for writing the system of classical elastodynamics as a linear symmetric hyperbolic system using the displacement

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gradient, $\mathbf{u}_{\mathbf{x}}$, and the velocity, $\mathbf{u}_t$, as the basic variables. Our motivation stems from standard procedures in linear wave equations: in order to accomplish our goal we judiciously use linear combinations of the compatibility equations. Our analysis is confined to classical elastodynamics, namely geometrically and materially linear anisotropic elasticity.

In order to present our motivation and our strategy clearly we start from the classical linear wave equation. When the unknown field u is scalar and we are limited to one space dimension the wave equation reads ([5] p. 402)[1]

$$u_{,tt} - a^{11} u_{,11} = 0. \tag{1}$$

Setting

$$v^1 = u_{,1}, \quad v^2 = u_{,t}, \tag{2}$$

we have from the differential equation

$$v^2_{,t} - a^{11} v^1_{,1} = 0, \tag{3}$$

and

$$v^1_{,t} - v^2_{,1} = 0, \tag{4}$$

from the compatibility relation. The compatibility Eq. (4) is multiplied by a^{11} and this new equation is used together with Eq. (3) to form a system of first order for $\mathbf{q} = (v^1, v^2)^T$ in the form

$$A^{\text{1d-w}} \frac{\partial \mathbf{q}}{\partial t} + B^{\text{1d-w}} \frac{\partial \mathbf{q}}{\partial x_1} = 0, \tag{5}$$

with

$$A^{\text{1d-w}} = \begin{pmatrix} a^{11} & 0 \\ 0 & 1 \end{pmatrix}, B^{\text{1d-w}} = \begin{pmatrix} 0 & -a^{11} \\ -a^{11} & 0 \end{pmatrix}. \tag{6}$$

By this judicious multiplication of the compatibility relation with a^{11} the system is brought into symmetric form ([5]): both matrices $A^{\text{1d-w}}$ and $B^{\text{1d-w}}$ are symmetric and $A^{\text{1d-w}}$ is positive definite when $a^{11} > 0$.

For a scalar unknown function u in a two dimensional space the situation is analogous. The differential equation is

$$\begin{aligned} &u_{,tt} - \sum_{i,j=1}^{2} a^{ij} u_{,ij} = 0 \rightarrow \\ &u_{,tt} - a^{11} u_{,11} - a^{12} u_{,12} - a^{21} u_{,21} - a^{22} u_{,22} = 0. \end{aligned} \tag{7}$$

Setting

$$v^1 = u_{,1}, \quad v^2 = u_{,2}, \quad v^3 = u_{,t}, \tag{8}$$

[1] In line with common practice, time and space derivatives of a quantity () are denoted by $()_{,t} = \frac{\partial ()}{\partial t}$ and $()_{,i} = \frac{\partial ()}{\partial x_i}$.

the differential equation render

$$v^3_{,t} - a^{11} v^1_{,1} - a^{12} v^1_{,2} - a^{21} v^2_{,1} - a^{22} v^2_{,2} = 0. \tag{9}$$

The compatibility relations are two in this case:

$$v^3_{,1} = v^1_{,t}, \quad v^3_{,2} = v^2_{,t}. \tag{10}$$

Multiplying the first by a^{11} and the second by a^{12} (using that $a^{12} = a^{21}$) and adding them one obtains

$$a^{11} v^1_{,t} + a^{12} v^2_{,t} - a^{11} v^3_{,1} - a^{21} v^3_{,2} = 0. \tag{11}$$

Starting from the same compatibility Eqs. (10) one can multiply the first by a^{12} and the second by a^{22} and add them to get

$$a^{12} v^1_{,t} + a^{22} v^2_{,t} - a^{12} v^3_{,1} - a^{22} v^3_{,2} = 0. \tag{12}$$

So, instead of working with Eqs. (9, 10) one may use Eqs. (9, 11, 12) and write them as a system in terms of $\mathbf{q} = (v^1, v^2, v^3)^T$ of the form ([5] p. 402)

$$A^{\text{2d-w}} \frac{\partial \mathbf{q}}{\partial t} + \sum_{i=1}^{2} B_i^{\text{2d-w}} \frac{\partial \mathbf{q}}{\partial x_i} = 0, \tag{13}$$

where

$$A^{\text{2d-w}} = \begin{pmatrix} a^{11} & a^{12} & 0 \\ a^{12} & a^{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, B_1^{\text{2d-w}} = \begin{pmatrix} 0 & 0 & -a^{11} \\ 0 & 0 & -a^{12} \\ -a^{11} & -a^{12} & 0 \end{pmatrix},$$
$$B_2^{\text{2d-w}} = \begin{pmatrix} 0 & 0 & -a^{21} \\ 0 & 0 & -a^{22} \\ -a^{21} & -a^{22} & 0 \end{pmatrix}. \tag{14}$$

With this writing matrices $A^{\text{2d-w}}$, $B_1^{\text{2d-w}}$, $B_2^{\text{2d-w}}$ are symmetric. Note that $a^{12} = a^{21}$ is important for the symmetry of matrix $A^{\text{2d-w}}$. Keep also in mind that matrix $B_1^{\text{2d-w}}$ is related with derivatives with respect to x_1 while matrix $B_2^{\text{2d-w}}$ with x_2. One can generalize this procedure straightforwardly to a scalar unknown function in R^n (see [5] p. 402) and a vector function in any space dimension with similar outcomes.

What can one infer from this analysis is that by judiciously substituting the compatibility relations with independent sum and/or subtractions of them, the system can be put into symmetric form. Certainly, for the examples above the used compatibility relations are independent as linear combinations of the initial compatibility relations.

Note that this analysis is confined to the linear case; when nonlinear differential equations are studied a similar goal is accomplished using entropy-flux pairs. Systems of conservation laws which possess entropy functions ([4, 5, 9]) are equations (commonly of mathematical physics) that can be written in a symmetric form which retains the conservation properties of the system. It seems that the existence of an entropy-flux pair for a system of conservation laws for the specific case of Euler fluids starts with the work of Godunov ([10]) who shows that a system which can be symmetrized has an entropy function (see also [1, 8]). Conversely, Mock ([16]) shows that when a system has an entropy function then it can be symmetrized. In the relativistic case the symmetrization procedure is done by Ruggeri and Sturmia ([19], see also [20]) and for the specific case of nonlinear elasticity by Boillat and Ruggeri ([2], see also, [3]). For the nonlinear elastic case, symmetrization is also accomplished in the work of Qin ([18]): the author starts from a polyconvex stored energy function and writes the system in a symmetric hyperbolic conservation law type.

In the next two sections we put the equations of classical linear elasticity in symmetric form. The strategy is very similar with the above described cases for the linear wave equation: we seek for the right substitutes of the initial compatibility relations in order to accomplish our goal. This is done in an inverse way: we start by writing our initial equations as a first order system of the form

$$A\frac{\partial \mathbf{q}}{\partial t}+\sum_{i=1}^{n}B_i\frac{\partial \mathbf{q}}{\partial x_i}=0, \tag{15}$$

with $n=2,3$ depending on whether we are in 2 or 3 dimensions. We then check what are the symmetric forms of matrices B_i and which combinations of the compatibility equations should be used in order the symmetric form of matrices B_i to appear into the system. This "symmetrization" process alters matrix A and if the resulting matrix A is symmetric our goal is accomplished. We only have to check whether the new compatibility relations are independent.

Compared to the classical wave equation (see e.g. [5]), the equations of elasticity are slightly different, since the strain appears in the momentum equation which in the linear case is related with the displacement gradient by the relation

$$e_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right). \tag{16}$$

Our analysis is similar with that of [12, 21, 22] in the sense that the ultimate purpose is to write the equations as a linear symmetric system. But there is a fundamental difference: these authors use the stress tensor and the velocity as the basic variables, while we use the displacement gradient and the velocity. The approach using the stress tensor instead of the displacement gradient requires the time differentiated constitutive law to be used as a field equation instead of the compatibility relations which are used in our approach.

Our starting point is the momentum equation which, in the absence of body forces and with a unit density, has the form

$$\ddot{u}_i=\sigma_{ij,j}, \tag{17}$$

where σ is the classical Cauchy stress tensor. Classical elasticity utilizes the energy

$$W=\frac{1}{2}e_{ij}\mathcal{C}_{ijkl}e_{kl}, \tag{18}$$

where $\mathcal{C}$ are the elasticities of a generically anisotropic material which satisfy minor and major symmetries $\mathcal{C}_{ijkl}=\mathcal{C}_{jikl}=\mathcal{C}_{ijlk}=\mathcal{C}_{klij}$. The stress tensor is then

$$\sigma_{ij}=\frac{\partial W}{\partial e_{ij}}=\mathcal{C}_{ijkl}e_{kl}, \tag{19}$$

so the momentum equation reads

$$\ddot{u}_i=\mathcal{C}_{ijkl}e_{kl,j}. \tag{20}$$

If one furthermore uses Eq. (16), then the momentum equation written with respect to the displacement has the form

$$u_{i,tt} = \frac{1}{2}\mathcal{C}_{ijkl}(u_{k,lj} + u_{l,kj}) \rightarrow$$
$$u_{i,tt} - \frac{1}{2}\mathcal{C}_{ijkl}u_{k,lj} - \frac{1}{2}\mathcal{C}_{ijkl}u_{l,kj} = 0. \tag{21}$$

In Sects. 2 and 3 we write Eqs. (21) as a linear symmetric hyperbolic system in terms of $\mathbf{u_x}$ and $\mathbf{u}_t$ in two and three space dimensions, respectively.

Writing the elasticity equations in the symmetric form of Eq. (15) essentially uses a Cartesian decompositions of the variables and the operators which does not alter the Hamiltonian structure of linear elasticity ([12, 14, 15]). We comment on the Hamiltonian structure of classical elastodynamics in Sect. 4. In Sect. 5 we compare our analysis with the one using the stress tensor in terms of the differentiability of the coefficients and the differentiability of the solution. In doing so, we use a standard theorem for linear symmetric hyperbolic systems ([6, 7]). In the same section we present the two alternative writings, i.e. the one using the displacement gradient and the other using the stress tensor, in the one dimensional setting for a non-homogeneous body, where one can see more clearly the differences that appear in the requirements on the coefficients. In Sect. 6 we put our per se linear theory under the perspective of the nonlinear approach by commenting on the potential relation of the linear theory with two prominent symmetrization processes of nonlinear elasticity: that of Boillat and Ruggeri ([2]) and that of Qin ([18]). The article ends up with some concluding remarks in Sect. 7.

The present framework for the 2d and 3d case is valid for homogeneous materials in the sense that the elasticity tensor, $\mathcal{C}$, does not depend on the spatial coordinate. When such a spatial dependence is introduced lower order terms should be added in the first order systems; these terms are related with the spatial gradient of the material parameters but do not alter the analysis (see [13] p. 48) since they remain lower order terms that have no effect on the principal part of the first order system. Nevertheless, they play an important role in the differentiability of the coefficients which can be better seen in the 1d case presented in Sect. 5.

The linear theory we work with stands on its own: we develop the linear theory on its own footing, rather as the linearization of the nonlinear theory. We do not examine the adopted linear approach as the limit of the nonlinear theory, but as a linear theory per se. To the best of our knowledge the symmetrization process established here is not reported in the literature of linear elasticity, when it is treated as a linear theory per se. This, we believe, constitutes the main novelty of our approach with respect to the existing literature of linear elasticity when treated as a linear theory on its own. In order to promptly infer to which nonlinear framework our linear approach corresponds one should carefully linearize the framework of Boillat and Ruggeri ([2]) and that of Qin ([18]) and then examine to which framework our approach corresponds.

2 Elasticity in 2d

In the two dimensional case there are two equations, one for each of the two components of the displacement vector. Summing the dummy indices in Eq. (21) for $i = 1$ we get the first equation which reads

$$
\begin{aligned}
& u_{1,tt} - \mathcal{C}_{1111} u_{1,11} - \mathcal{C}_{1211} u_{1,12} - \frac{1}{2}(\mathcal{C}_{1112} + \mathcal{C}_{1121}) u_{1,21} \\
& - \frac{1}{2}(\mathcal{C}_{1212} + \mathcal{C}_{1221}) u_{1,22} - \frac{1}{2}(\mathcal{C}_{1121} + \mathcal{C}_{1112}) u_{2,11} - \frac{1}{2}(\mathcal{C}_{1121} + \mathcal{C}_{1112}) u_{2,11} \\
& - \frac{1}{2}(\mathcal{C}_{1221} + \mathcal{C}_{1212}) u_{2,12} - \mathcal{C}_{1122} u_{2,21} - \mathcal{C}_{1222} u_{2,22} = 0.
\end{aligned} \tag{22}
$$

For $i = 2$ and summing all dummy indices we obtain the second equation

$$
\begin{aligned}
& u_{2,tt} - \mathcal{C}_{2111} u_{1,11} - \mathcal{C}_{2211} u_{1,12} - \frac{1}{2}(\mathcal{C}_{2112} + \mathcal{C}_{2121}) u_{1,21} \\
& - \frac{1}{2}(\mathcal{C}_{2212} + \mathcal{C}_{2221}) u_{1,22} - \frac{1}{2}(\mathcal{C}_{2121} + \mathcal{C}_{2112}) u_{2,11} - \frac{1}{2}(\mathcal{C}_{2121} + \mathcal{C}_{2112}) u_{2,11} \\
& - \frac{1}{2}(\mathcal{C}_{2221} + \mathcal{C}_{2212}) u_{2,12} - \mathcal{C}_{2122} u_{2,21} - \mathcal{C}_{2222} u_{2,22} = 0.
\end{aligned} \tag{23}
$$

While in the realm of classical elasticity $\mathcal{C}$ has the major and minor symmetries, we choose not to use these symmetries at this stage.

Setting

$$
\begin{aligned}
v^1 &= u_{1,1}, \quad v^2 = u_{1,2}, \quad v^3 = u_{1,t}, \\
v^4 &= u_{2,1}, \quad v^5 = u_{2,2}, \quad v^6 = u_{2,t},
\end{aligned} \tag{24}
$$

Equation (22) reads

$$
\begin{aligned}
& v^3_{,t} - \mathcal{C}_{1111} v^1_{,1} - \mathcal{C}_{1211} v^1_{,2} - \frac{1}{2}(\mathcal{C}_{1112} + \mathcal{C}_{1121}) v^2_{,1} \\
& - \frac{1}{2}(\mathcal{C}_{1212} + \mathcal{C}_{1221}) v^2_{,2} - \frac{1}{2}(\mathcal{C}_{1121} + \mathcal{C}_{1112}) v^4_{,1} \\
& - \frac{1}{2}(\mathcal{C}_{1221} + \mathcal{C}_{1212}) u^4_{,2} - \mathcal{C}_{1122} u^5_{,1} - \mathcal{C}_{1222} u^5_{,2} = 0,
\end{aligned} \tag{25}
$$

while Eq. (23) takes the form

$$
\begin{aligned}
& v^6_{,t} - \mathcal{C}_{2111} v^1_{,1} - \mathcal{C}_{2211} v^1_{,2} - \frac{1}{2}(\mathcal{C}_{2112} + \mathcal{C}_{2121}) v^2_{,1} \\
& - \frac{1}{2}(\mathcal{C}_{2212} + \mathcal{C}_{2221}) v^2_{,2} - \frac{1}{2}(\mathcal{C}_{2121} + \mathcal{C}_{2112}) v^4_{,1} \\
& - \frac{1}{2}(\mathcal{C}_{2221} + \mathcal{C}_{2212}) u^4_{,2} - \mathcal{C}_{2122} u^5_{,1} - \mathcal{C}_{2222} u_{5,2} = 0.
\end{aligned} \tag{26}
$$

The compatibility relations read

$$
\begin{aligned}
v^1_{,t} &= v^3_{,1} \quad v^2_{,t} = v^3_{,2}, \\
v^4_{,t} &= v^6_{,1} \quad v^5_{,t} = v^6_{,2}.
\end{aligned} \tag{27}
$$

So, as a system with respect to $\mathbf{q} = (v^1, v^2, v^3, v^4, v^5, v^6)^T$ Eqs. (25-27) are of the form

$$A^{2d}\frac{\partial \mathbf{q}}{\partial t} + \sum_{i=1}^{2} B_i^{2d}\frac{\partial \mathbf{q}}{\partial x_i} = 0, \tag{28}$$

with $A^{2d} = I$, the 6 by 6 unit tensor,

$$B_1^{2d} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathcal{C}_{1111} & -\frac{1}{2}(\mathcal{C}_{1112}+\mathcal{C}_{1121}) & 0 & -\frac{1}{2}(\mathcal{C}_{1121}+\mathcal{C}_{1112}) & -\mathcal{C}_{1122} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathcal{C}_{2111} & -\frac{1}{2}(\mathcal{C}_{2112}+\mathcal{C}_{2121}) & 0 & -\frac{1}{2}(\mathcal{C}_{2121}+\mathcal{C}_{2112}) & -\mathcal{C}_{2122} & 0 \end{bmatrix}, \tag{29}$$

and

$$B_2^{2d} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -\mathcal{C}_{1211} & -\frac{1}{2}(\mathcal{C}_{1212}+\mathcal{C}_{1221}) & 0 & -\frac{1}{2}(\mathcal{C}_{1221}+\mathcal{C}_{1212}) & -\mathcal{C}_{1222} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -\mathcal{C}_{2211} & -\frac{1}{2}(\mathcal{C}_{2212}+\mathcal{C}_{2221}) & 0 & -\frac{1}{2}(\mathcal{C}_{2221}+\mathcal{C}_{2212}) & -\mathcal{C}_{2222} & 0 \end{bmatrix}. \tag{30}$$

Keep in mind that matrices B_1^{2d} and B_2^{2d} are related with derivatives with respect to x_1 and x_2, respectively.

In line with the wave equation described in the motivation section, the idea is to built the symmetric form of the matrices B_1^{2d} and B_2^{2d} in the system by judiciously multiplying and adding the compatibility relations. To see how this can be done we start with matrix B_1^{2d}: its symmetric form is

$$B_1^{2d,\,sym} = \begin{bmatrix} 0 & 0 & -\mathcal{C}_{1111} & 0 & 0 & -\mathcal{C}_{2111} \\ 0 & 0 & -\frac{1}{2}(\mathcal{C}_{1112}+\mathcal{C}_{1121}) & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{2112}+\mathcal{C}_{2121}) \\ -\mathcal{C}_{1111} & -\frac{1}{2}(\mathcal{C}_{1112}+\mathcal{C}_{1121}) & 0 & -\frac{1}{2}(\mathcal{C}_{1121}+\mathcal{C}_{1112}) & -\mathcal{C}_{1122} & 0 \\ 0 & 0 & -\frac{1}{2}(\mathcal{C}_{1121}+\mathcal{C}_{1112}) & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{2121}+\mathcal{C}_{2112}) \\ 0 & 0 & -\mathcal{C}_{1122} & 0 & 0 & -\mathcal{C}_{2122} \\ -\mathcal{C}_{2111} & -\frac{1}{2}(\mathcal{C}_{2112}+\mathcal{C}_{2121}) & 0 & -\frac{1}{2}(\mathcal{C}_{2121}+\mathcal{C}_{2112}) & -\mathcal{C}_{2122} & 0 \end{bmatrix}. \tag{31}$$

In order to built the first line of $B_1^{2d,\,sym}$ into the system, we start from the compatibility

$$v^1_{,t} = v^3_{,1} \rightarrow \mathcal{C}_{1111}v^1_{,t} - \mathcal{C}_{1111}v^3_{,1} = 0, \tag{32}$$

which is added to

$$v^4_{,t} = v^6_{,1} \rightarrow \mathcal{C}_{2111}v^4_{,t} - \mathcal{C}_{2111}v^6_{,1} = 0, \tag{33}$$

in order to get

$$\mathcal{C}_{1111}v^1_{,t} + \mathcal{C}_{2111}v^4_{,t} - \mathcal{C}_{1111}v^3_{,1} - \mathcal{C}_{2111}v^6_{,1} = 0. \tag{34}$$

When Eq. (34) replaces the compatibility relation Eq. $(27)_1$, then the first line of B_1^{2d} in its symmetric form appears in the system. This change affects the matrix A^{2d} by changing its first line as

$$A^{2d} = \begin{bmatrix} \mathcal{C}_{1111} & 0 & 0 & \mathcal{C}_{2111} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{35}$$

For building the second line of the symmetric form of matrix B_1^{2d} into the system we start from the compatibility

$$v^1_{,t} = v^3_{,1} \rightarrow \frac{1}{2}(\mathcal{C}_{1112} + \mathcal{C}_{1121})v^1_{,t} - \frac{1}{2}(\mathcal{C}_{1112} + \mathcal{C}_{1121})v^3_{,1} = 0, \tag{36}$$

which is added to

$$v^4_{,t} = v^6_{,1} \rightarrow \frac{1}{2}(\mathcal{C}_{2112} + \mathcal{C}_{2121})v^4_{,t} - \frac{1}{2}(\mathcal{C}_{2112} + \mathcal{C}_{2121})v^6_{,1} = 0. \tag{37}$$

When the last two equations are added we obtain

$$\begin{aligned} &\frac{1}{2}(\mathcal{C}_{1112} + \mathcal{C}_{1121})v^1_{,t} + \frac{1}{2}(\mathcal{C}_{2112} + \mathcal{C}_{2121})v^4_{,t} \\ &-\frac{1}{2}(\mathcal{C}_{1112} + \mathcal{C}_{1121})v^3_{,1} - \frac{1}{2}(\mathcal{C}_{2112} + \mathcal{C}_{2121})v^6_{,1} = 0, \end{aligned} \tag{38}$$

which when substitutes the compatibility relation Eq. $(27)_2$, the second line of the symmetric form of the matrix B_1^{2d} is built into the system and matrix A^{2d} is changed into the form

$$A^{2d} = \begin{bmatrix} \mathcal{C}_{1111} & 0 & 0 & \mathcal{C}_{2111} & 0 & 0 \\ \frac{1}{2}(\mathcal{C}_{1112} + \mathcal{C}_{1121}) & 0 & 0 & \frac{1}{2}(\mathcal{C}_{2112} + \mathcal{C}_{2121}) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{39}$$

Now, there is an important question that appears: we substitute the compatibility relation Eq. $(27)_2$ by an equation which does not contain the initial variables. So, the new compatibility relations are really independent? To answer this question, we finish the scheme of producing the symmetric matrices and then check under what conditions the newly used compatibility relations are independent.

Working in an analogous fashion for building each line of the symmetric form of B_1^{2d} we arrive at a matrix A^{2d} of the form

$$A^{2d} = \begin{bmatrix} \mathcal{C}_{1111} & 0 & 0 & \mathcal{C}_{2111} & 0 & 0 \\ \frac{1}{2}(\mathcal{C}_{1112} + \mathcal{C}_{1121}) & 0 & 0 & \frac{1}{2}(\mathcal{C}_{2112} + \mathcal{C}_{2121}) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2}(\mathcal{C}_{1121} + \mathcal{C}_{1112}) & 0 & 0 & \frac{1}{2}(\mathcal{C}_{2121} + \mathcal{C}_{2112}) & 0 & 0 \\ \mathcal{C}_{1122} & 0 & 0 & \mathcal{C}_{2122} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{40}$$

Matrix B_2^{2d} is related with derivatives with respect to x_2, so one should do an analogous procedure using the compatibility relations that contain derivatives with respect to x_2, namely

$$v_t^2 = v_{,2}^3, \quad v_t^5 = v_{,2}^6. \tag{41}$$

As an outcome of this building of the symmetric form of matrix B_2^{2d}, matrix A^{2d} has the form

$$A^{2d} = \begin{bmatrix} \mathcal{C}_{1111} & \mathcal{C}_{1211} & 0 & \mathcal{C}_{2111} & \mathcal{C}_{2211} & 0 \\ \frac{1}{2}(\mathcal{C}_{1112}+\mathcal{C}_{1121}) & \frac{1}{2}(\mathcal{C}_{1212}+\mathcal{C}_{1221}) & 0 & \frac{1}{2}(\mathcal{C}_{2112}+\mathcal{C}_{2121}) & \frac{1}{2}(\mathcal{C}_{2212}+\mathcal{C}_{2221}) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2}(\mathcal{C}_{1121}+\mathcal{C}_{1112}) & \frac{1}{2}(\mathcal{C}_{1221}+\mathcal{C}_{1212}) & 0 & \frac{1}{2}(\mathcal{C}_{2121}+\mathcal{C}_{2112}) & \frac{1}{2}(\mathcal{C}_{2221}+\mathcal{C}_{2212}) & 0 \\ \mathcal{C}_{1122} & \mathcal{C}_{1222} & 0 & \mathcal{C}_{2122} & \mathcal{C}_{2222} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{42}$$

When major and minor symmetries are enforced matrix A^{2d} has the symmetric form

$$A^{2d} = \begin{bmatrix} \mathcal{C}_{1111} & \mathcal{C}_{1211} & 0 & \mathcal{C}_{2111} & \mathcal{C}_{2211} & 0 \\ \mathcal{C}_{1112} & \mathcal{C}_{1221} & 0 & \mathcal{C}_{2121} & \mathcal{C}_{2221} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \mathcal{C}_{1112} & \mathcal{C}_{1212} & 0 & \mathcal{C}_{2112} & \mathcal{C}_{2212} & 0 \\ \mathcal{C}_{1122} & \mathcal{C}_{1222} & 0 & \mathcal{C}_{2122} & \mathcal{C}_{2222} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{43}$$

Additionally, when the matrix A^{2d} is positive definite the system is symmetric hyperbolic.

As we claimed, we need to show that the compatibility relations used in this procedure are independent. Since the third and sixth line of the system are the momentum equations and all other lines are the compatibility relations, independence of the compatibility equations used is equivalent to

$$\det \begin{bmatrix} \mathcal{C}_{1111} & \mathcal{C}_{1211} & \mathcal{C}_{2111} & \mathcal{C}_{2211} \\ \mathcal{C}_{1112} & \mathcal{C}_{1221} & \mathcal{C}_{2121} & \mathcal{C}_{2221} \\ \mathcal{C}_{1112} & \mathcal{C}_{1212} & \mathcal{C}_{2112} & \mathcal{C}_{2212} \\ \mathcal{C}_{1122} & \mathcal{C}_{1222} & \mathcal{C}_{2122} & \mathcal{C}_{2222} \end{bmatrix} \neq 0. \tag{44}$$

Essentially, these are the $2^4 = 16$ components of $\mathcal{C}$ for the two dimensional case, so, Eq. (44) equals to the invertibility of $\mathcal{C}$.

Note that an alternative symmetric writing of the equations of isotropic linear elastodynamics in 2d is presented by Morando and Serre ([17]).

3 Elasticity in 3d

In the three dimensional case, the situation is analogous; the difference lies in the fact that there are three equations, so the first order system is a 12 by 12 system at the end. The first of the momentum equations stems from Eq. (21) for $i = 1$ by summing all other indices:

$$u_{1,tt} - \mathcal{C}_{1111}u_{1,11} - \mathcal{C}_{1211}u_{1,12} - \mathcal{C}_{1311}u_{1,13} - \frac{1}{2}(\mathcal{C}_{1112} + \mathcal{C}_{1121})u_{1,21}$$

$$
\begin{aligned}
&-\frac{1}{2}(\mathcal{C}_{1212}+\mathcal{C}_{1221})u_{1,22}-\frac{1}{2}(\mathcal{C}_{1312}+\mathcal{C}_{1321})u_{1,23}-\frac{1}{2}(\mathcal{C}_{1113}+\mathcal{C}_{1131})u_{1,31}\\
&-\frac{1}{2}(\mathcal{C}_{1213}+\mathcal{C}_{1231})u_{1,32}-\frac{1}{2}(\mathcal{C}_{1313}+\mathcal{C}_{1331})u_{1,33}-\frac{1}{2}(\mathcal{C}_{1112}+\mathcal{C}_{1112})u_{2,11}\\
&-\frac{1}{2}(\mathcal{C}_{1221}+\mathcal{C}_{1212})u_{2,12}-\frac{1}{2}(\mathcal{C}_{1321}+\mathcal{C}_{1312})u_{2,13}-\mathcal{C}_{1122}u_{2,21}-\mathcal{C}_{1222}u_{2,22}\\
&-\mathcal{C}_{1322}u_{2,23}-\frac{1}{2}(\mathcal{C}_{1123}+\mathcal{C}_{1132})u_{2,31}-\frac{1}{2}(\mathcal{C}_{1223}+\mathcal{C}_{1232})u_{2,32}\\
&-\frac{1}{2}(\mathcal{C}_{1323}+\mathcal{C}_{1332})u_{2,33}-\frac{1}{2}(\mathcal{C}_{1131}+\mathcal{C}_{1113})u_{3,11}-\frac{1}{2}(\mathcal{C}_{1231}+\mathcal{C}_{1213})u_{3,12}\\
&-\frac{1}{2}(\mathcal{C}_{1331}+\mathcal{C}_{1313})u_{3,13}-\frac{1}{2}(\mathcal{C}_{1132}+\mathcal{C}_{1123})u_{3,21}-\frac{1}{2}(\mathcal{C}_{1232}+\mathcal{C}_{1223})u_{3,22}\\
&-\frac{1}{2}(\mathcal{C}_{1332}+\mathcal{C}_{1323})u_{3,23}-\mathcal{C}_{1233}u_{3,31}-\mathcal{C}_{1233}u_{3,32}-\mathcal{C}_{1333}u_{3,33}=0.
\end{aligned}
\tag{45}
$$

The second of the momentum equations stems from Eq. (21) for $i=2$ by summing all other indices:

$$
\begin{aligned}
&u_{2,tt}-\mathcal{C}_{2111}u_{1,11}-\mathcal{C}_{2211}u_{1,12}-\mathcal{C}_{2311}u_{1,13}-\frac{1}{2}(\mathcal{C}_{2112}+\mathcal{C}_{2121})u_{1,21}\\
&-\frac{1}{2}(\mathcal{C}_{2212}+\mathcal{C}_{2221})u_{1,22}-\frac{1}{2}(\mathcal{C}_{2312}+\mathcal{C}_{2321})u_{1,23}-\frac{1}{2}(\mathcal{C}_{2113}+\mathcal{C}_{2131})u_{1,31}\\
&-\frac{1}{2}(\mathcal{C}_{2213}+\mathcal{C}_{2231})u_{1,32}-\frac{1}{2}(\mathcal{C}_{2313}+\mathcal{C}_{2331})u_{1,33}-\frac{1}{2}(\mathcal{C}_{2112}+\mathcal{C}_{2112})u_{2,11}\\
&-\frac{1}{2}(\mathcal{C}_{2221}+\mathcal{C}_{2212})u_{2,12}-\frac{1}{2}(\mathcal{C}_{2321}+\mathcal{C}_{2312})u_{2,13}-\mathcal{C}_{2122}u_{2,21}-\mathcal{C}_{2222}u_{2,22}\\
&-\mathcal{C}_{2322}u_{2,23}-\frac{1}{2}(\mathcal{C}_{2123}+\mathcal{C}_{2132})u_{2,31}-\frac{1}{2}(\mathcal{C}_{2223}+\mathcal{C}_{2232})u_{2,32}\\
&-\frac{1}{2}(\mathcal{C}_{2323}+\mathcal{C}_{2332})u_{2,33}-\frac{1}{2}(\mathcal{C}_{2131}+\mathcal{C}_{2113})u_{3,11}-\frac{1}{2}(\mathcal{C}_{2231}+\mathcal{C}_{2213})u_{3,12}\\
&-\frac{1}{2}(\mathcal{C}_{2331}+\mathcal{C}_{2313})u_{3,13}-\frac{1}{2}(\mathcal{C}_{2132}+\mathcal{C}_{2123})u_{3,21}-\frac{1}{2}(\mathcal{C}_{2232}+\mathcal{C}_{2223})u_{3,22}\\
&-\frac{1}{2}(\mathcal{C}_{2332}+\mathcal{C}_{2323})u_{3,23}-\mathcal{C}_{2233}u_{3,31}-\mathcal{C}_{2233}u_{3,32}-\mathcal{C}_{2333}u_{3,33}=0.
\end{aligned}
\tag{46}
$$

The third of the momentum equations stems from Eq. (21) for $i=3$ by summing all other indices:

$$
\begin{aligned}
&u_{3,tt}-\mathcal{C}_{3111}u_{1,11}-\mathcal{C}_{3211}u_{1,12}-\mathcal{C}_{3311}u_{1,13}-\frac{1}{2}(\mathcal{C}_{3112}+\mathcal{C}_{3121})u_{1,21}\\
&-\frac{1}{2}(\mathcal{C}_{3212}+\mathcal{C}_{3221})u_{1,22}-\frac{1}{2}(\mathcal{C}_{3312}+\mathcal{C}_{3321})u_{1,23}-\frac{1}{2}(\mathcal{C}_{3113}+\mathcal{C}_{3131})u_{1,31}\\
&-\frac{1}{2}(\mathcal{C}_{3213}+\mathcal{C}_{3231})u_{1,32}-\frac{1}{2}(\mathcal{C}_{3313}+\mathcal{C}_{3331})u_{1,33}-\frac{1}{2}(\mathcal{C}_{3112}+\mathcal{C}_{3112})u_{2,11}\\
&-\frac{1}{2}(\mathcal{C}_{3221}+\mathcal{C}_{3212})u_{2,12}-\frac{1}{2}(\mathcal{C}_{3321}+\mathcal{C}_{3312})u_{2,13}-\mathcal{C}_{3122}u_{2,21}-\mathcal{C}_{3222}u_{2,22}
\end{aligned}
$$

$$-\mathcal{C}_{3322}u_{2,23} - \frac{1}{2}(\mathcal{C}_{3123} + \mathcal{C}_{3132})u_{2,31} - \frac{1}{2}(\mathcal{C}_{3223} + \mathcal{C}_{3232})u_{2,32}$$
$$-\frac{1}{2}(\mathcal{C}_{3323} + \mathcal{C}_{3332})u_{2,33} - \frac{1}{2}(\mathcal{C}_{3131} + \mathcal{C}_{3113})u_{3,11} - \frac{1}{2}(\mathcal{C}_{3231} + \mathcal{C}_{3213})u_{3,12}$$
$$-\frac{1}{2}(\mathcal{C}_{3331} + \mathcal{C}_{3313})u_{3,13} - \frac{1}{2}(\mathcal{C}_{3132} + \mathcal{C}_{3123})u_{3,21} - \frac{1}{2}(\mathcal{C}_{3232} + \mathcal{C}_{3223})u_{3,22}$$
$$-\frac{1}{2}(\mathcal{C}_{3332} + \mathcal{C}_{3323})u_{3,23} - \mathcal{C}_{3233}u_{3,31} - \mathcal{C}_{3233}u_{3,32} - \mathcal{C}_{3333}u_{3,33} = 0. \tag{47}$$

We now set

$$\begin{aligned} &v^1 = u_{1,1}, \quad v^2 = u_{1,2}, \quad v^3 = u_{1,3}, \quad v^4 = u_{1,t}, \\ &v^5 = u_{2,1}, \quad v^6 = u_{2,2}, \quad v^7 = u_{2,3}, \quad v^8 = u_{2,t}, \\ &v^9 = u_{3,1}, \quad v^{10} = u_{3,2}, \quad v^{11} = u_{3,3}, \quad v^{12} = u_{3,t}. \end{aligned} \tag{48}$$

So, the first of the momentum equation reads

$$\begin{aligned} &v_{4,t} - \mathcal{C}_{1111}v_{1,1} - \mathcal{C}_{1211}v_{1,2} - \mathcal{C}_{1311}v_{1,3} - \frac{1}{2}(\mathcal{C}_{1112} + \mathcal{C}_{1121})v_{2,1} \\ &-\frac{1}{2}(\mathcal{C}_{1212} + \mathcal{C}_{1221})v_{2,2} - \frac{1}{2}(\mathcal{C}_{1312} + \mathcal{C}_{1321})v_{2,3} - \frac{1}{2}(\mathcal{C}_{1113} + \mathcal{C}_{1131})v_{3,1} \\ &-\frac{1}{2}(\mathcal{C}_{1213} + \mathcal{C}_{1231})v_{3,2} - \frac{1}{2}(\mathcal{C}_{1313} + \mathcal{C}_{1331})v_{3,3} - \frac{1}{2}(\mathcal{C}_{1112} + \mathcal{C}_{1112})u_{5,1} \\ &-\frac{1}{2}(\mathcal{C}_{1221} + \mathcal{C}_{1212})v_{5,2} - \frac{1}{2}(\mathcal{C}_{1321} + \mathcal{C}_{1312})v_{5,3} - \mathcal{C}_{1122}v_{6,1} - \mathcal{C}_{1222}v_{6,2} \\ &-\mathcal{C}_{1322}v_{6,3} - \frac{1}{2}(\mathcal{C}_{1123} + \mathcal{C}_{1132})u_{7,1} - \frac{1}{2}(\mathcal{C}_{1223} + \mathcal{C}_{1232})u_{7,2} \\ &-\frac{1}{2}(\mathcal{C}_{1323} + \mathcal{C}_{1332})v_{7,3} - \frac{1}{2}(\mathcal{C}_{1131} + \mathcal{C}_{1113})v_{9,1} - \frac{1}{2}(\mathcal{C}_{1231} + \mathcal{C}_{1213})v_{9,2} \\ &-\frac{1}{2}(\mathcal{C}_{1331} + \mathcal{C}_{1313})v_{9,3} - \frac{1}{2}(\mathcal{C}_{1132} + \mathcal{C}_{1123})v_{10,1} - \frac{1}{2}(\mathcal{C}_{1232} + \mathcal{C}_{1223})v_{10,2} \\ &-\frac{1}{2}(\mathcal{C}_{1332} + \mathcal{C}_{1323})v_{10,3} - \mathcal{C}_{1233}u_{11,1} - \mathcal{C}_{1233}u_{11,2} - \mathcal{C}_{1333}u_{11,3} = 0. \end{aligned} \tag{49}$$

The second momentum equation reads

$$\begin{aligned} &v_{8,t} - \mathcal{C}_{2111}v_{1,1} - \mathcal{C}_{2211}v_{1,2} - \mathcal{C}_{2311}v_{1,3} - \frac{1}{2}(\mathcal{C}_{2112} + \mathcal{C}_{2121})v_{2,1} \\ &-\frac{1}{2}(\mathcal{C}_{2212} + \mathcal{C}_{2221})v_{2,2} - \frac{1}{2}(\mathcal{C}_{2312} + \mathcal{C}_{2321})v_{2,3} - \frac{1}{2}(\mathcal{C}_{2113} + \mathcal{C}_{2131})v_{3,1} \\ &-\frac{1}{2}(\mathcal{C}_{2213} + \mathcal{C}_{2231})v_{3,2} - \frac{1}{2}(\mathcal{C}_{2313} + \mathcal{C}_{2331})v_{3,3} - \frac{1}{2}(\mathcal{C}_{2112} + \mathcal{C}_{2112})u_{5,1} \\ &-\frac{1}{2}(\mathcal{C}_{2221} + \mathcal{C}_{2212})v_{5,2} - \frac{1}{2}(\mathcal{C}_{2321} + \mathcal{C}_{2312})v_{5,3} - \mathcal{C}_{2122}v_{6,1} - \mathcal{C}_{2222}v_{6,2} \\ &-\mathcal{C}_{2322}v_{6,3} - \frac{1}{2}(\mathcal{C}_{2123} + \mathcal{C}_{2132})u_{7,1} - \frac{1}{2}(\mathcal{C}_{2223} + \mathcal{C}_{2232})u_{7,2} \end{aligned} \tag{50}$$

$$
\begin{aligned}
&-\frac{1}{2}(\mathcal{C}_{2323}+\mathcal{C}_{2332})v_{7,3}-\frac{1}{2}(\mathcal{C}_{2131}+\mathcal{C}_{2113})v_{9,1}-\frac{1}{2}(\mathcal{C}_{2231}+\mathcal{C}_{2213})v_{9,2}\\
&-\frac{1}{2}(\mathcal{C}_{2331}+\mathcal{C}_{2313})v_{9,3}-\frac{1}{2}(\mathcal{C}_{2132}+\mathcal{C}_{2123})v_{10,1}-\frac{1}{2}(\mathcal{C}_{2232}+\mathcal{C}_{2223})v_{10,2}\\
&-\frac{1}{2}(\mathcal{C}_{2332}+\mathcal{C}_{2323})v_{10,3}-\mathcal{C}_{2233}u_{11,1}-\mathcal{C}_{2233}u_{11,2}-\mathcal{C}_{2333}u_{11,3}=0.
\end{aligned}
$$

The third momentum equation reads

$$
\begin{aligned}
&v_{12,t}-\mathcal{C}_{3111}v_{1,1}-\mathcal{C}_{3211}v_{1,2}-\mathcal{C}_{3311}v_{1,3}-\frac{1}{2}(\mathcal{C}_{3112}+\mathcal{C}_{3121})v_{2,1}\\
&-\frac{1}{2}(\mathcal{C}_{3212}+\mathcal{C}_{3221})v_{2,2}-\frac{1}{2}(\mathcal{C}_{3312}+\mathcal{C}_{3321})v_{2,3}-\frac{1}{2}(\mathcal{C}_{3113}+\mathcal{C}_{3131})v_{3,1}\\
&-\frac{1}{2}(\mathcal{C}_{3213}+\mathcal{C}_{3231})v_{3,2}-\frac{1}{2}(\mathcal{C}_{3313}+\mathcal{C}_{3331})v_{3,3}-\frac{1}{2}(\mathcal{C}_{3112}+\mathcal{C}_{3112})u_{5,1}\\
&-\frac{1}{2}(\mathcal{C}_{3221}+\mathcal{C}_{3212})v_{5,2}-\frac{1}{2}(\mathcal{C}_{3321}+\mathcal{C}_{3312})v_{5,3}-\mathcal{C}_{3122}v_{6,1}-\mathcal{C}_{3222}v_{6,2}\\
&-\mathcal{C}_{3322}v_{6,3}-\frac{1}{2}(\mathcal{C}_{3123}+\mathcal{C}_{3132})u_{7,1}-\frac{1}{2}(\mathcal{C}_{3223}+\mathcal{C}_{3232})u_{7,2}\\
&-\frac{1}{2}(\mathcal{C}_{3323}+\mathcal{C}_{3332})v_{7,3}-\frac{1}{2}(\mathcal{C}_{3131}+\mathcal{C}_{3113})v_{9,1}-\frac{1}{2}(\mathcal{C}_{3231}+\mathcal{C}_{3213})v_{9,2}\\
&-\frac{1}{2}(\mathcal{C}_{3331}+\mathcal{C}_{3313})v_{9,3}-\frac{1}{2}(\mathcal{C}_{3132}+\mathcal{C}_{3123})v_{10,1}-\frac{1}{2}(\mathcal{C}_{3232}+\mathcal{C}_{3223})v_{10,2}\\
&-\frac{1}{2}(\mathcal{C}_{3332}+\mathcal{C}_{3323})v_{10,3}-\mathcal{C}_{3233}u_{11,1}-\mathcal{C}_{3233}u_{11,2}-\mathcal{C}_{3333}u_{11,3}=0.
\end{aligned}
\tag{51}
$$

The compatibility relations are

$$
\begin{aligned}
&v^1_{,t}=v^4_{,1},\quad v^2_{,t}=v^4_{,2},\quad v^3_{,t}=v^4_{,3},\\
&v^5_{,t}=v^8_{,1},\quad v^6_{,t}=v^8_{,2},\quad v^7_{,t}=v^8_{,3},\\
&v^9_{,t}=v^{12}_{,1},\quad v^{10}_{,t}=v^{12}_{,2},\quad v^{11}_{,t}=v^{12}_{,3}.
\end{aligned}
\tag{52}
$$

Equations (49-52) is a 12 by 12 first order system in terms of $\mathbf{q}=(v^1, v^2, v^3, v^4, v^5, v^6, v^7, v^8, v^9, v^{10}, v^{11}, v^{12})^T$ which can be written in the form

$$
A^{3\mathrm{d}}\frac{\partial\mathbf{q}}{\partial t}+\sum_{i=1}^{3}B_i^{3\mathrm{d}}\frac{\partial\mathbf{q}}{\partial x_i}=0,
\tag{53}
$$

where matrix $A^{3\mathrm{d}}$ is the identity matrix. Matrix $B_1^{3\mathrm{d}}$ is related with the derivatives with respect to x_1 and has the form

$$B_1^{3\mathrm{d}} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathcal{C}_{1111} & -\frac{1}{2}(\mathcal{C}_{1112}+\mathcal{C}_{1121}) & -\frac{1}{2}(\mathcal{C}_{1113}+\mathcal{C}_{1131}) & 0 & -\frac{1}{2}(\mathcal{C}_{1121}+\mathcal{C}_{1112}) & -\mathcal{C}_{1122} & -\frac{1}{2}(\mathcal{C}_{1123}+\mathcal{C}_{1132}) & 0 & -\frac{1}{2}(\mathcal{C}_{1131}+\mathcal{C}_{1113}) & -\frac{1}{2}(\mathcal{C}_{1123}+\mathcal{C}_{1132}) & -\mathcal{C}_{1133} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathcal{C}_{2111} & -\frac{1}{2}(\mathcal{C}_{2112}+\mathcal{C}_{2121}) & -\frac{1}{2}(\mathcal{C}_{2113}+\mathcal{C}_{2131}) & 0 & -\frac{1}{2}(\mathcal{C}_{2121}+\mathcal{C}_{2112}) & -\mathcal{C}_{2122} & -\frac{1}{2}(\mathcal{C}_{2123}+\mathcal{C}_{2132}) & 0 & -\frac{1}{2}(\mathcal{C}_{2131}+\mathcal{C}_{2113}) & -\frac{1}{2}(\mathcal{C}_{2123}+\mathcal{C}_{2132}) & -\mathcal{C}_{2133} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathcal{C}_{3111} & -\frac{1}{2}(\mathcal{C}_{3112}+\mathcal{C}_{3121}) & -\frac{1}{2}(\mathcal{C}_{3113}+\mathcal{C}_{3131}) & 0 & -\frac{1}{2}(\mathcal{C}_{3121}+\mathcal{C}_{3112}) & -\mathcal{C}_{3122} & -\frac{1}{2}(\mathcal{C}_{3123}+\mathcal{C}_{3132}) & 0 & -\frac{1}{2}(\mathcal{C}_{3131}+\mathcal{C}_{3113}) & -\frac{1}{2}(\mathcal{C}_{3123}+\mathcal{C}_{3132}) & -\mathcal{C}_{3133} & 0 \end{bmatrix}. \tag{54}$$

Matrix $B_2^{3\mathrm{d}}$ is related with the derivatives with respect to x_2 and has the form

$$B_2^{3\mathrm{d}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathcal{C}_{1211} & -\frac{1}{2}(\mathcal{C}_{1212}+\mathcal{C}_{1221}) & -\frac{1}{2}(\mathcal{C}_{1213}+\mathcal{C}_{1231}) & 0 & -\frac{1}{2}(\mathcal{C}_{1221}+\mathcal{C}_{1212}) & -\mathcal{C}_{1222} & -\frac{1}{2}(\mathcal{C}_{1223}+\mathcal{C}_{1232}) & 0 & -\frac{1}{2}(\mathcal{C}_{1231}+\mathcal{C}_{1213}) & -\frac{1}{2}(\mathcal{C}_{1223}+\mathcal{C}_{1232}) & -\mathcal{C}_{1233} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathcal{C}_{2211} & -\frac{1}{2}(\mathcal{C}_{2212}+\mathcal{C}_{2221}) & -\frac{1}{2}(\mathcal{C}_{2213}+\mathcal{C}_{2231}) & 0 & -\frac{1}{2}(\mathcal{C}_{2221}+\mathcal{C}_{2212}) & -\mathcal{C}_{2222} & -\frac{1}{2}(\mathcal{C}_{2223}+\mathcal{C}_{2232}) & 0 & -\frac{1}{2}(\mathcal{C}_{2231}+\mathcal{C}_{2213}) & -\frac{1}{2}(\mathcal{C}_{2223}+\mathcal{C}_{2232}) & -\mathcal{C}_{2233} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathcal{C}_{3211} & -\frac{1}{2}(\mathcal{C}_{3212}+\mathcal{C}_{3221}) & -\frac{1}{2}(\mathcal{C}_{3213}+\mathcal{C}_{3231}) & 0 & -\frac{1}{2}(\mathcal{C}_{3221}+\mathcal{C}_{3212}) & -\mathcal{C}_{3222} & -\frac{1}{2}(\mathcal{C}_{3223}+\mathcal{C}_{3232}) & 0 & -\frac{1}{2}(\mathcal{C}_{3231}+\mathcal{C}_{3213}) & -\frac{1}{2}(\mathcal{C}_{3223}+\mathcal{C}_{3232}) & -\mathcal{C}_{3233} & 0 \end{bmatrix}. \tag{55}$$

Matrix $B_3^{3\mathrm{d}}$ is related with the derivatives with respect to x_3 and has the form

$$B_3^{3\mathrm{d}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathcal{C}_{1311} & -\frac{1}{2}(\mathcal{C}_{1312}+\mathcal{C}_{1321}) & -\frac{1}{2}(\mathcal{C}_{1313}+\mathcal{C}_{1331}) & 0 & -\frac{1}{2}(\mathcal{C}_{1321}+\mathcal{C}_{1312}) & -\mathcal{C}_{1322} & -\frac{1}{2}(\mathcal{C}_{1323}+\mathcal{C}_{1332}) & 0 & -\frac{1}{2}(\mathcal{C}_{1331}+\mathcal{C}_{1313}) & -\frac{1}{2}(\mathcal{C}_{1323}+\mathcal{C}_{1332}) & -\mathcal{C}_{1333} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -\mathcal{C}_{2311} & -\frac{1}{2}(\mathcal{C}_{2312}+\mathcal{C}_{2321}) & -\frac{1}{2}(\mathcal{C}_{2313}+\mathcal{C}_{2331}) & 0 & -\frac{1}{2}(\mathcal{C}_{2321}+\mathcal{C}_{2312}) & -\mathcal{C}_{2322} & -\frac{1}{2}(\mathcal{C}_{2323}+\mathcal{C}_{2332}) & 0 & -\frac{1}{2}(\mathcal{C}_{2331}+\mathcal{C}_{2313}) & -\frac{1}{2}(\mathcal{C}_{2323}+\mathcal{C}_{2332}) & -\mathcal{C}_{2333} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -\mathcal{C}_{3311} & -\frac{1}{2}(\mathcal{C}_{3312}+\mathcal{C}_{3321}) & -\frac{1}{2}(\mathcal{C}_{3313}+\mathcal{C}_{3331}) & 0 & -\frac{1}{2}(\mathcal{C}_{3321}+\mathcal{C}_{3312}) & -\mathcal{C}_{3322} & -\frac{1}{2}(\mathcal{C}_{3323}+\mathcal{C}_{3332}) & 0 & -\frac{1}{2}(\mathcal{C}_{3331}+\mathcal{C}_{3313}) & -\frac{1}{2}(\mathcal{C}_{3323}+\mathcal{C}_{3332}) & -\mathcal{C}_{3333} & 0 \end{bmatrix}. \tag{56}$$

The symmetric form of the matrix $B_1^{3\mathrm{d}}$ is

$$
B_1^{3\mathrm{d,\,sym}} = \begin{bmatrix}
0 & 0 & 0 & -\mathcal{C}_{1111} & 0 & 0 & 0 & -\mathcal{C}_{2111} & 0 & 0 & 0 & -\mathcal{C}_{3111} \\
0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{1112}+\mathcal{C}_{1121}) & 0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{2112}+\mathcal{C}_{2121}) & 0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{3112}+\mathcal{C}_{3121}) \\
0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{1113}+\mathcal{C}_{1131}) & 0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{2113}+\mathcal{C}_{2131}) & 0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{1113}+\mathcal{C}_{1131}) \\
-\mathcal{C}_{1111} & -\frac{1}{2}(\mathcal{C}_{1112}+\mathcal{C}_{1121}) & -\frac{1}{2}(\mathcal{C}_{1113}+\mathcal{C}_{1131}) & 0 & -\frac{1}{2}(\mathcal{C}_{1121}+\mathcal{C}_{1112}) & -\mathcal{C}_{1122} & -\frac{1}{2}(\mathcal{C}_{1123}+\mathcal{C}_{1132}) & 0 & -\frac{1}{2}(\mathcal{C}_{1131}+\mathcal{C}_{1113}) & -\frac{1}{2}(\mathcal{C}_{1123}+\mathcal{C}_{1132}) & -\mathcal{C}_{1133} & 0 \\
0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{1121}+\mathcal{C}_{1112}) & 0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{2121}+\mathcal{C}_{2112}) & 0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{3121}+\mathcal{C}_{3112}) \\
0 & 0 & 0 & -\mathcal{C}_{1122} & 0 & 0 & 0 & -\mathcal{C}_{2122} & 0 & 0 & 0 & -\mathcal{C}_{2122} \\
0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{1123}+\mathcal{C}_{1132}) & 0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{2123}+\mathcal{C}_{2132}) & 0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{3123}+\mathcal{C}_{3132}) \\
-\mathcal{C}_{2111} & -\frac{1}{2}(\mathcal{C}_{2112}+\mathcal{C}_{2121}) & -\frac{1}{2}(\mathcal{C}_{2113}+\mathcal{C}_{2131}) & 0 & -\frac{1}{2}(\mathcal{C}_{2121}+\mathcal{C}_{2112}) & -\mathcal{C}_{2122} & -\frac{1}{2}(\mathcal{C}_{2123}+\mathcal{C}_{2132}) & 0 & -\frac{1}{2}(\mathcal{C}_{2131}+\mathcal{C}_{2113}) & -\frac{1}{2}(\mathcal{C}_{2123}+\mathcal{C}_{2132}) & -\mathcal{C}_{2133} & 0 \\
0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{1131}+\mathcal{C}_{1113}) & 0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{2131}+\mathcal{C}_{2113}) & 0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{3131}+\mathcal{C}_{3113}) \\
0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{1132}+\mathcal{C}_{1123}) & 0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{2132}+\mathcal{C}_{2123}) & 0 & 0 & 0 & -\frac{1}{2}(\mathcal{C}_{3132}+\mathcal{C}_{3123}) \\
0 & 0 & 0 & -\mathcal{C}_{1133} & 0 & 0 & 0 & -\mathcal{C}_{2133} & 0 & 0 & 0 & -\mathcal{C}_{1133} \\
-\mathcal{C}_{3111} & -\frac{1}{2}(\mathcal{C}_{3112}+\mathcal{C}_{3121}) & -\frac{1}{2}(\mathcal{C}_{3113}+\mathcal{C}_{3131}) & 0 & -\frac{1}{2}(\mathcal{C}_{3121}+\mathcal{C}_{3112}) & -\mathcal{C}_{3122} & -\frac{1}{2}(\mathcal{C}_{3123}+\mathcal{C}_{3132}) & 0 & -\frac{1}{2}(\mathcal{C}_{3131}+\mathcal{C}_{3113}) & -\frac{1}{2}(\mathcal{C}_{3123}+\mathcal{C}_{3132}) & -\mathcal{C}_{3133} & 0
\end{bmatrix}. \tag{57}
$$

In order to built the first line of $B_1^{\text{3d, sym}}$ into the system, we have to substitute the first compatibility equation (Eq. (52)$_1$) with a judicious combination of some of the compatibility relations. We start with

$$v^1_{,t} = v^4_{,1} \rightarrow \mathcal{C}_{1111} v^1_{,t} - \mathcal{C}_{1111} v^4_{,1} = 0, \tag{58}$$

and add it to the two equations

$$v^5_{,t} = v^8_{,1} \rightarrow \mathcal{C}_{2111} v^5_{,t} - \mathcal{C}_{2111} v^8_{,1} = 0, \tag{59}$$

$$v^9_{,t} = v^{12}_{,1} \rightarrow \mathcal{C}_{3111} v^9_{,t} - \mathcal{C}_{3111} v^{12}_{,1} = 0, \tag{60}$$

to get

$$\begin{aligned} &\mathcal{C}_{1111} v^1_{,t} + \mathcal{C}_{2111} v^5_{,t} + \mathcal{C}_{3111} v^9_{,t} \\ &-\mathcal{C}_{1111} v^4_{,1} - \mathcal{C}_{2111} v^8_{,1} - \mathcal{C}_{3111} v^{12}_{,1} = 0. \end{aligned} \tag{61}$$

This equation substitutes the compatibility Eq. (52)$_1$ and affects the first line of the matrix A^{3d}: instead of only 1 in the first slot, now matrix A^{3d} has three non null components. These are $\mathcal{C}_{1111}$ in the first slot, $\mathcal{C}_{2111}$ in the fifth slot and $\mathcal{C}_{3111}$ in the ninth slot.

Continuing in this vein, building each term of the symmetric form of matrices $B_i^{3\mathrm{d}}$, $i = 1, 2, 3$, we arrive finally at matrix $A^{3\mathrm{d}}$ of the form

$$A^{3\mathrm{d}} = \begin{bmatrix}
C_{1111} & C_{1121} & C_{1131} & 0 & C_{2111} & C_{2211} & C_{2311} & 0 & C_{3111} & C_{3211} & C_{3311} & 0 \\
\frac{1}{2}(C_{1112}+C_{1121}) & \frac{1}{2}(C_{1212}+C_{1221}) & \frac{1}{2}(C_{1312}+C_{1321}) & 0 & \frac{1}{2}(C_{2112}+C_{2121}) & \frac{1}{2}(C_{2212}+C_{2221}) & \frac{1}{2}(C_{2312}+C_{2321}) & 0 & \frac{1}{2}(C_{3112}+C_{3121}) & \frac{1}{2}(C_{3212}+C_{3221}) & \frac{1}{2}(C_{3312}+C_{3321}) & 0 \\
\frac{1}{2}(C_{1113}+C_{1131}) & \frac{1}{2}(C_{1213}+C_{1231}) & \frac{1}{2}(C_{1313}+C_{1331}) & 0 & \frac{1}{2}(C_{2113}+C_{2131}) & \frac{1}{2}(C_{2213}+C_{2231}) & \frac{1}{2}(C_{2313}+C_{2331}) & 0 & \frac{1}{2}(C_{3113}+C_{3131}) & \frac{1}{2}(C_{3213}+C_{3231}) & \frac{1}{2}(C_{3313}+C_{3331}) & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}(C_{1121}+C_{1112}) & \frac{1}{2}(C_{1221}+C_{1212}) & \frac{1}{2}(C_{1321}+C_{1312}) & 0 & \frac{1}{2}(C_{2121}+C_{2112}) & \frac{1}{2}(C_{2221}+C_{2212}) & \frac{1}{2}(C_{2321}+C_{2312}) & 0 & \frac{1}{2}(C_{3121}+C_{3112}) & \frac{1}{2}(C_{3221}+C_{3212}) & \frac{1}{2}(C_{3321}+C_{3312}) & 0 \\
C_{1122} & C_{1222} & C_{1322} & 0 & C_{2122} & C_{2222} & C_{2322} & 0 & C_{3122} & C_{3222} & C_{3322} & 0 \\
\frac{1}{2}(C_{1123}+C_{1132}) & \frac{1}{2}(C_{1223}+C_{1232}) & \frac{1}{2}(C_{1323}+C_{1332}) & 0 & \frac{1}{2}(C_{2123}+C_{2132}) & \frac{1}{2}(C_{2223}+C_{2232}) & \frac{1}{2}(C_{2323}+C_{2332}) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2}(C_{1131}+C_{1113}) & \frac{1}{2}(C_{1231}+C_{1213}) & \frac{1}{2}(C_{1331}+C_{1313}) & 0 & \frac{1}{2}(C_{2131}+C_{2113}) & \frac{1}{2}(C_{2231}+C_{2213}) & \frac{1}{2}(C_{2331}+C_{2313}) & 0 & \frac{1}{2}(C_{3131}+C_{3113}) & \frac{1}{2}(C_{3231}+C_{3213}) & \frac{1}{2}(C_{3331}+C_{3313}) & 0 \\
\frac{1}{2}(C_{1132}+C_{1123}) & \frac{1}{2}(C_{1232}+C_{1223}) & \frac{1}{2}(C_{1332}+C_{1323}) & 0 & \frac{1}{2}(C_{2132}+C_{2123}) & \frac{1}{2}(C_{2232}+C_{2223}) & \frac{1}{2}(C_{2332}+C_{2323}) & 0 & \frac{1}{2}(C_{3132}+C_{3123}) & \frac{1}{2}(C_{3232}+C_{3223}) & \frac{1}{2}(C_{3332}+C_{3323}) & 0 \\
C_{1133} & C_{1233} & C_{1333} & 0 & C_{2133} & C_{2233} & C_{2333} & 0 & C_{3133} & C_{3233} & C_{3333} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}. \tag{62}$$

When major and minor symmetries of $\mathcal{C}$ are enforced, matrix A^{3d} takes the symmetric form

$$A^{3d} = \begin{bmatrix}
\mathcal{C}_{1111} & \mathcal{C}_{1121} & \mathcal{C}_{1311} & 0 & \mathcal{C}_{2111} & \mathcal{C}_{2211} & \mathcal{C}_{2311} & 0 & \mathcal{C}_{3111} & \mathcal{C}_{3211} & \mathcal{C}_{3311} & 0 \\
\mathcal{C}_{1112} & \mathcal{C}_{1212} & \mathcal{C}_{1312} & 0 & \mathcal{C}_{2112} & \mathcal{C}_{2212} & \mathcal{C}_{2312} & 0 & \mathcal{C}_{3112} & \mathcal{C}_{3212} & \mathcal{C}_{3312} & 0 \\
\mathcal{C}_{1113} & \mathcal{C}_{1213} & \mathcal{C}_{1313} & 0 & \mathcal{C}_{2113} & \mathcal{C}_{2213} & \mathcal{C}_{2312} & 0 & \mathcal{C}_{3113} & \mathcal{C}_{3213} & \mathcal{C}_{3313} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathcal{C}_{1121} & \mathcal{C}_{1221} & \mathcal{C}_{1321} & 0 & \mathcal{C}_{2121} & \mathcal{C}_{2221} & \mathcal{C}_{2321} & 0 & \mathcal{C}_{3121} & \mathcal{C}_{3221} & \mathcal{C}_{3321} & 0 \\
\mathcal{C}_{1122} & \mathcal{C}_{1222} & \mathcal{C}_{1322} & 0 & \mathcal{C}_{2122} & \mathcal{C}_{2222} & \mathcal{C}_{2322} & 0 & \mathcal{C}_{3122} & \mathcal{C}_{3222} & \mathcal{C}_{3322} & 0 \\
\mathcal{C}_{1123} & \mathcal{C}_{1223} & \mathcal{C}_{1323} & 0 & \mathcal{C}_{2123} & \mathcal{C}_{2223} & \mathcal{C}_{2323} & 0 & \mathcal{C}_{3123} & \mathcal{C}_{3223} & \mathcal{C}_{3323} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\mathcal{C}_{1131} & \mathcal{C}_{1231} & \mathcal{C}_{1331} & 0 & \mathcal{C}_{2131} & \mathcal{C}_{2231} & \mathcal{C}_{2331} & 0 & \mathcal{C}_{3131} & \mathcal{C}_{3231} & \mathcal{C}_{3331} & 0 \\
\mathcal{C}_{1132} & \mathcal{C}_{1232} & \mathcal{C}_{1332} & 0 & \mathcal{C}_{2132} & \mathcal{C}_{2232} & \mathcal{C}_{2332} & 0 & \mathcal{C}_{3132} & \mathcal{C}_{3232} & \mathcal{C}_{3332} & 0 \\
\mathcal{C}_{1133} & \mathcal{C}_{1233} & \mathcal{C}_{1333} & 0 & \mathcal{C}_{2133} & \mathcal{C}_{2233} & \mathcal{C}_{2333} & 0 & \mathcal{C}_{3133} & \mathcal{C}_{3233} & \mathcal{C}_{3333} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}. \tag{63}$$

When matrix A^{3d} is also positive definite we speak about a symmetric hyperbolic system.

Since the fourth, eighth and twelveth line of the system are the momentum equations, independence of the compatibility relations used is equivalent to the condition

$$\det \begin{bmatrix}
\mathcal{C}_{1111} & \mathcal{C}_{1121} & \mathcal{C}_{1311} & \mathcal{C}_{2111} & \mathcal{C}_{2211} & \mathcal{C}_{2311} & \mathcal{C}_{3111} & \mathcal{C}_{3211} & \mathcal{C}_{3311} \\
\mathcal{C}_{1112} & \mathcal{C}_{1212} & \mathcal{C}_{1312} & \mathcal{C}_{2112} & \mathcal{C}_{2212} & \mathcal{C}_{2312} & \mathcal{C}_{3112} & \mathcal{C}_{3212} & \mathcal{C}_{3312} \\
\mathcal{C}_{1113} & \mathcal{C}_{1213} & \mathcal{C}_{1313} & \mathcal{C}_{2113} & \mathcal{C}_{2213} & \mathcal{C}_{2312} & \mathcal{C}_{3113} & \mathcal{C}_{3213} & \mathcal{C}_{3313} \\
\mathcal{C}_{1121} & \mathcal{C}_{1221} & \mathcal{C}_{1321} & \mathcal{C}_{2121} & \mathcal{C}_{2221} & \mathcal{C}_{2321} & \mathcal{C}_{3121} & \mathcal{C}_{3221} & \mathcal{C}_{3321} \\
\mathcal{C}_{1122} & \mathcal{C}_{1222} & \mathcal{C}_{1322} & \mathcal{C}_{2122} & \mathcal{C}_{2222} & \mathcal{C}_{2322} & \mathcal{C}_{3122} & \mathcal{C}_{3222} & \mathcal{C}_{3322} \\
\mathcal{C}_{1123} & \mathcal{C}_{1223} & \mathcal{C}_{1323} & \mathcal{C}_{2123} & \mathcal{C}_{2223} & \mathcal{C}_{2323} & \mathcal{C}_{3123} & \mathcal{C}_{3223} & \mathcal{C}_{3323} \\
\mathcal{C}_{1131} & \mathcal{C}_{1231} & \mathcal{C}_{1331} & \mathcal{C}_{2131} & \mathcal{C}_{2231} & \mathcal{C}_{2331} & \mathcal{C}_{3131} & \mathcal{C}_{3231} & \mathcal{C}_{3331} \\
\mathcal{C}_{1132} & \mathcal{C}_{1232} & \mathcal{C}_{1332} & \mathcal{C}_{2132} & \mathcal{C}_{2232} & \mathcal{C}_{2332} & \mathcal{C}_{3132} & \mathcal{C}_{3232} & \mathcal{C}_{3332} \\
\mathcal{C}_{1133} & \mathcal{C}_{1233} & \mathcal{C}_{1333} & \mathcal{C}_{2133} & \mathcal{C}_{2233} & \mathcal{C}_{2333} & \mathcal{C}_{3133} & \mathcal{C}_{3233} & \mathcal{C}_{3333}
\end{bmatrix} \neq 0. \tag{64}$$

This essentially equals to the invertibility of $\mathcal{C}$.

4 Hamiltonian Structure

In this section we highlight the Hamiltonian structure of linear elastodynamics; our approach relies heavily on the fundamental works of Marsden and Hughes ([14, 15]). The classical elastodynamics problem can be phrased as a Hamiltonian system when the linear momentum $\mathbf{p} = \rho \frac{\partial \mathbf{u}}{\partial t}$ and the strain $\mathbf{e} = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u})$ are used as independent variables with the Hamiltonian

$$H = \frac{1}{2} \int_\Omega \left(\frac{1}{\rho} \|\mathbf{p}\|^2 + \mathbf{e}\mathcal{C}\mathbf{e} \right) d\Omega, \quad \Omega \subset \mathcal{R}, \quad d = \{1, 2, 3\}. \tag{65}$$

The elastodynamic problem can be rewritten as a Hamiltonian system

$$\frac{\partial}{\partial t} \begin{pmatrix} \mathbf{p} \\ \mathbf{e} \end{pmatrix} = \begin{pmatrix} 0 & \text{div} \\ \text{Grad} & 0 \end{pmatrix} \begin{pmatrix} \delta_\mathbf{p} H \\ \delta_\mathbf{e} H \end{pmatrix} = \begin{pmatrix} 0 & \text{div} \\ \text{Grad} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \end{pmatrix}, \tag{66}$$

$\mathbf{v}$ being the velocity, $\text{Grad} = \frac{1}{2}(\nabla + \nabla^T)$ the symmetrized gradient and $\delta_{\boldsymbol{\alpha}}$ the variational derivative with respect to α.

The operator

$$J = \begin{pmatrix} 0 & \text{div} \\ \text{Grad} & 0 \end{pmatrix} \tag{67}$$

defines a Poisson bracket being a formally skew-adjoint operator. The system can be re-written in terms of the velocity and stress as

$$\begin{pmatrix} \rho & 0 \\ 0 & \mathcal{C}^{-1} \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \end{pmatrix} = \begin{pmatrix} 0 & \text{div} \\ \text{Grad} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \end{pmatrix}. \tag{68}$$

The Hamiltonian structure is encoded in the skew symmetry of the matrix J.

For the classical case, using the Voigt notation, for the velocity and the stress tensor, the operator J is rewritten in the 2D case as

$$J = \begin{bmatrix} 0 & 0 & \partial_x & 0 & \partial_y \\ 0 & 0 & 0 & \partial_y & \partial_x \\ \partial_x & 0 & 0 & 0 & 0 \\ 0 & \partial_y & 0 & 0 & 0 \\ \partial_y & \partial_x & 0 & 0 & 0 \end{bmatrix}. \tag{69}$$

The symmetry of this matrix implies the skew symmetry of the operator. To see this, consider the case of the simple 1D wave equation setting all constants equal to one; then the matrix J has the form

$$J = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}. \tag{70}$$

The operator ∂_x is formally skew adjoint: given two smooth functions with compact support $f, g \in C_c^\infty$ for any interval $I \subset \mathcal{R}$ of the real line it holds

$$\int_I (\partial_x f) g dx = -\int_I f(\partial_x g) dx. \tag{71}$$

In terms of the L^2 inner product $(\partial_x f, g)_I = -(f, \partial_x g)_I$, so the operator J is a formally skew-adjoint operator and the same is true for linear elasticity. In the 2d case given above the operator can be written in a skew symmetric form as

$$J = \begin{pmatrix} 0 & \mathcal{D} \\ -\mathcal{D}^* & 0 \end{pmatrix}, \mathcal{D} = \begin{pmatrix} \partial_x & 0 & \partial_y \\ 0 & \partial_y & \partial_x \end{pmatrix}, \tag{72}$$

where $\mathcal{D}^*$ denotes the adjoint operator defined via integration by parts as illustrated above for ∂_x.

The calculations presented above are for the classical approach using the velocity and the stress tensor when writing the system in a first order form. When $(\mathbf{u}_x, \mathbf{u}_t)$ are used the corresponding operator J is skew symmetric from its construction. For the 2d case, it has the form

$$J = B_1^{\text{2d, sym}} \frac{\partial}{\partial x_1} + B_2^{\text{2d, sym}} \frac{\partial}{\partial x_2}, \tag{73}$$

where $B_1^{\text{2d, sym}}$ is given by Eq. (31) and $B_2^{\text{2d, sym}}$ is the symmetrized form of the matrix Eq. (30). A similar analysis holds for the 3d case, where again, by construction

$$J = B_1^{\text{3d, sym}} \frac{\partial}{\partial x_1} + B_2^{\text{3d, sym}} \frac{\partial}{\partial x_2} + B_3^{\text{3d, sym}} \frac{\partial}{\partial x_3}. \tag{74}$$

The symmetry of the matrices $B_i^{\text{3d, sym}}, i = 1, 2, 3$ implies the skew symmetry of the operator, so all in all, the Hamiltonian structure of the system is not affected when this Cartesian decomposition of the operator J is used.

5 Comparison of the Approach Using the Displacement Gradient Versus the One Using the Stress Tensor

In this section we compare the classical approach using the stress tensor versus the approach presented here where the displacement gradient is used. The comparison is with respect to the differentiability of the coefficients and the differentiability of the solutions and is based on a theorem by Fischer and Marsden ([6]) attributed to Friedrichs ([7]). Before presenting this comparison we give the one dimensional analogues of both approaches in the inhomogeneous case where the coefficients involved are presented more clearly.

5.1 Elasticity in 1d with Respect to (u_t, u_x)

The starting point is the elastic energy which for the one dimensional case has the form

$$W(u_x) = \frac{1}{2}\alpha u_x^2. \tag{75}$$

The momentum equation in the absence of body forces with unit density gives

$$u_{tt} = \sigma_x = \frac{\partial W}{\partial u_x} = \alpha u_{xx} + \alpha_x u_x, \tag{76}$$

α being the material parameter of the model. Setting $u_t = v$, $u_x = e$ the momentum equation renders

$$v_t = \alpha e_x + \alpha_x e, \tag{77}$$

while the compatibility equation renders

$$v_x = e_t \to v_x - e_t = 0. \tag{78}$$

We multiply the compatibility relation by α to obtain

$$\alpha v_x - \alpha e_t = 0. \tag{79}$$

So, as a system Eqs. (77, 79) are

$$v_t - \alpha e_x = \alpha_x e, \tag{80}$$

$$\alpha e_t - \alpha v_x = 0. \tag{81}$$

This system can be written in the form

$$A^{(u_t,u_x)}\frac{\partial \mathbf{q}}{\partial t}+B^{(u_t,u_x)}\frac{\partial \mathbf{q}}{\partial x}=f^{(u_t,u_x)}, \tag{82}$$

with respect to $\mathbf{q}=(v,e)^T$ with

$$A^{(u_t,u_x)}=\begin{pmatrix}1 & 0\\ 0 & \alpha\end{pmatrix}, B^{(u_t,u_x)}=\begin{pmatrix}0 & -\alpha\\ -\alpha & 0\end{pmatrix}, f^{(u_t,u_x)}=\begin{pmatrix}\alpha_x e\\ 0\end{pmatrix}. \tag{83}$$

5.2 Elasticity in 1d with Respect to (u_t, σ)

Using again as a starting point the energy of Eq. (75), stress is

$$\sigma=\frac{\partial W}{\partial u_x}=\alpha u_x. \tag{84}$$

Now, we need to time differentiate the constitutive law. If this is done directly in the last equation one obtains

$$\dot{\sigma}=\dot{\alpha}u_x+\alpha\dot{u}_x. \tag{85}$$

If we set $u_t=v$ then the first term in the right hand side of the last equation cannot be transformed to the new system of variables, (u_t,σ). To by-pass this issue we write the constitutive law of Eq. (84) in the form

$$\alpha^{-1}\sigma=u_x, \tag{86}$$

which requires that the material parameter α can be inverted. By time differentiating the last equation we obtain

$$\alpha_t^{-1}\sigma+\alpha^{-1}\sigma_t=\dot{u}_x. \tag{87}$$

When we set $u_t=v$ this equation can be written in the new system as

$$\alpha_t^{-1}\sigma+\alpha^{-1}\sigma_t=v_x. \tag{88}$$

If to this last equation we add the momentum equation we are essentially working with the system

$$v_t-\sigma_x=0, \tag{89}$$

$$\alpha^{-1}\sigma_t-v_x=-\dot{\alpha}^{-1}\sigma. \tag{90}$$

This system can be written in the form

$$A^{(u_t,\sigma)}\frac{\partial \mathbf{q}}{\partial t}+B^{(u_t,\sigma)}\frac{\partial \mathbf{q}}{\partial x}=f^{(u_t,\sigma)}, \tag{91}$$

with respect to $\mathbf{q}=(v,\sigma)^T$ with

$$A^{(u_t,\sigma)}=\begin{pmatrix}1 & 0\\ 0 & \alpha^{-1}\end{pmatrix}, B^{(u_t,\sigma)}=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}, f^{(u_t,\sigma)}=\begin{pmatrix}0\\ -\dot{\alpha}^{-1}\sigma\end{pmatrix}. \tag{92}$$

5.3 Comparison with Respect to the Differentiability of the Coefficients

We compare the present approach with the classical one using the stress tensor instead of the displacement gradient by using a theorem of Fischer and Marsden ([6], Theorem 1.1 p. 3). These authors work with a linear symmetric hyperbolic system of the form

$$A^{0,\text{F-M}}(x,t)\frac{\partial u}{\partial t} = \sum_{i=1}^{n} A^{i,\text{F-M}}(x,t)\frac{\partial u}{\partial x^i} + B^{\text{F-M}}(x,t)u + C^{\text{F-M}}(x,t), \tag{93}$$

and they assume that

(i) $s > n/2 + 1$,

(ii) $A^{0,\text{F-M}}$ is symmetric positive definite and $t \mapsto A_t^{0,\text{F-M}}(x) - I \equiv A^{0,\text{F-M}}(x,t) - I$ is a C^1 map from $\mathcal{R}$ to $H^s(\mathcal{R}^n, \mathcal{R}^{m^2})$ and a C^0 map from $\mathcal{R}$ to $H^{s+1}(\mathcal{R}^n, \mathcal{R}^{m^2})$, I being the identity matrix,

(iii) $A^{i,\text{F-M}}$ is symmetric, $i = 1, 2, \ldots, n$ and $t \mapsto A_t^{i,\text{F-M}}(x) = A^{i,\text{F-M}}(x,t)$ is a C^0 map from $\mathcal{R}$ to $H^s(\mathcal{R}^n, \mathcal{R}^{m^2})$,

(iv) $t \mapsto B_t^{\text{F-M}}(x) = B^{\text{F-M}}(x,t)$ is a continuous curve from $\mathcal{R}$ to $H^s(\mathcal{R}^n, \mathcal{R}^{m^2})$,

(v) $t \mapsto C_t^{\text{F-M}}(x) = C^{\text{F-M}}(x,t)$ is a continuous curve from $\mathcal{R}$ to $H^s(\mathcal{R}^n, \mathcal{R}^m)$.

Under these assumptions, for any $u_0 \in H^s$ there is a unique continuous curve $t \mapsto H^s$ which is differentiable as a curve in H^{s-1} and equals u_0 at $t = 0$ and is a solution of Eq. (93). We adopt the assumptions of this theorem and examine what effect do they have in the differentiability of the coefficients when $(\mathbf{u}_t, \mathbf{u}_\mathbf{x})$ and $(\mathbf{u}_t, \boldsymbol{\sigma})$ are used.

For the one dimensional case, we see that the assumptions on the differentiability of the coefficients should be placed on terms α, α_x; namely, on the material parameter and its spatial gradient, when (u_t, u_x) are used as the new variables. When (u_t, σ) are used, one has to make the assumptions of differentiability on α^{-1}, $\dot{\alpha}^{-1}$. The main difference lies in the fact that use of the writing with respect to (u_t, σ) places differentiability requirements in the inverse of the material parameter α^{-1} and its temporal gradient $\dot{\alpha}^{-1}$. When (u_t, u_x) are used, the differentiability requirement should be placed on α and its spatial gradient.

So, when we are dealing with discontinuous in space material coefficients the writing using (u_t, σ) should be preferable since it avoids problems related with discontinuous material coefficients. On the other hand, when discontinuous in time material coefficients are involved (e.g. aging phenomena) the writing using (u_t, u_x) seems to be preferrable.

In the 3d case, the differentiability requirement for the coefficients should be placed again on $\mathcal{C}^{-1}$ when the stress is used as the new variable; to see this let us consider the approach of Yakhno and Akmaz ([22]): in the homogeneous case, they write the system in the form

$$A^{0,\text{Y-A}}(x,t)\frac{\partial u}{\partial t} + \sum_{i=1}^{3} A^{i,\text{Y-A}}(x,t)\frac{\partial u}{\partial x^i} = 0, \tag{94}$$

where

$$A^{0,\text{Y-A}} = \begin{pmatrix} I_3 & 0_{3\times 6} \\ 0_{6\times 3} & \mathcal{C}^{-1} \end{pmatrix}, A^{i,\text{Y-A}} = \begin{pmatrix} 0_{3\times 3} & \bar{A}_i^1 \\ (\bar{A}_i^1)^* & 0_{6\times 6} \end{pmatrix}. \tag{95}$$

$(\bar{A}_i^1)$, $(\bar{A}_i^1)^*$ being hard numbers, any requirement on differentiability should be placed on $\mathcal{C}^{-1}$. In contrast to that when (u_t, u_x) are used the requirement of differentiability should be placed on $\mathcal{C}$ and not on its inverse. Nevertheless the existence of $\mathcal{C}^{-1}$ is required from the condition for the independence of the compatibility relations used (Eq. (64) for the 3d case).

When the material is no longer homogeneous, in line with the 1d case, terms of the form $\dot{\mathcal{C}}^{-1}$ appear when the stress tensor is used, while terms of the form $\mathcal{C}_{\mathbf{x}}$ appear when $\mathbf{u_x}$ is used.

5.4 Comparison with Respect to the Differentiability of the Solution

Say that the assumptions (i-v) of the theorem of Fischer and Marsden described above are met; then for the solution one has $\mathbf{u}(\mathbf{x}, t) \in H^s$. When $(\mathbf{u}_t, \mathbf{u}_x)$, are used this means that $\mathbf{u}_t, \mathbf{u}_x \in H^s$. When $(\mathbf{u}_t, \boldsymbol{\sigma})$ are used it gives $\mathbf{u}_t, \boldsymbol{\sigma} \in H^s$. So, the first approach places restrictions in $\mathbf{u}_x$, while the second in $\boldsymbol{\sigma}$. Thus, the second framework can describe discontinuities in space since it does not involve $\mathbf{u}_x$ at all. In the linear regime, Hooke's law relates this two field in a linear way: for the one dimensional case, $\sigma = \alpha u_x$; so, when α is well behaved, σ and u_x belong essentially to the same space.

This is also true in the three dimensional case, where $\sigma_{ij} = \mathcal{C}_{ijkl} e_{kl}$, when $\mathcal{C}$ is well behaved. If we worked with inhomogeneous materials, we would arrive at the fact that in case of discontinuous in space material coefficients, use of the writing with the stress tensor avoids the problem of discontinuities in space material coefficients. On the other hand, use of the writing with the displacement gradient avoids the problem of discontinuous in time material coefficients.

6 Symmetrization in the Framework of Nonlinear Elasticity

In this section we put our per se linear theory under the perspective of the nonlinear framework, by commenting on two prominent symmetrization processes of the nonlinear theory. The first uses the notion of entropy-flux pairs ([1, 3, 8, 10]) for a system of conservation laws, while the second is the work of Qin ([18]) who starts from a polyconvex function.

6.1 Entropy-Flux Pairs in Nonlinear Elasticity

Systems of conservation laws which possess entropy functions ([4, 5, 9]) are equations (commonly of mathematical physics) that can be written in a symmetric form which retains the conservation properties of the system. It seems that the existence of an entropy-flux pair for a system of conservation laws for the specific case of Euler fluids starts with the work of Godunov ([10]) who shows that a system which can be symmetrized has an entropy function (see also [1, 8]). Conversely, Mock ([16]) shows that when a system has an entropy function then it can be symmetrized. In the relativistic case the symmetrization process is done by Ruggeri and Sturmia ([19], see also [20]) and for the specific case of nonlinear elasticity by Boillat and Ruggeri ([2], see also, [3]).

In their fundamental paper Boillat and Ruggeri ([2]) write the equations of nonlinear elasticity in symmetric form in two ways. The first one uses $(\mathbf{u}_t, \mathbf{F})$ ($\mathbf{F}$ being the deformation gradient) as the basic variables. The second way of writing utilizes a Legendre transformation and uses $(\mathbf{u}_t, -\mathbf{T})$ ($\mathbf{T}$ being the first Piola-Kirchhoff stress tensor) as the basic variables. This is for the nonlinear theory.

For the linear theory (as a theory per se) there are two approaches: the one presented here using $(\mathbf{u}_t, \mathbf{u_x})$ as the basic variables and the other one ([15, 21, 22]) using $(\mathbf{u}_t, \boldsymbol{\sigma})$ as the basic variables. In order to promptly infer to which linear way of writing the two nonlinear ways of writing by Boillat and Ruggeri ([2]) correspond one has to carefully linearize and examine the outcome. We, nevertheless, state that it seems our framework to be closer to the $(\mathbf{u}_t, \mathbf{F})$ of [2], while the $(\mathbf{u}_t, -\mathbf{T})$ seems to be closer to the approaches of ([15, 21, 22]). Since in general entropy-flux pairs are non-unique one has to carefully examine all possibilities when linearizing the nonlinear theory.

6.2 The Work of Qin ([18])

Qin ([18]), in his valuable paper, starts from a polyconvex stored energy function and writes the system in a symmetric conservation law type. He uses $(\mathbf{F}, \mathrm{cof}\mathbf{F}, \det\mathbf{F}, \mathbf{v})$ as the basic variables ($\mathbf{F}$ being the deformation gradient of the nonlinear theory, cof the cofactor matrix, det the determinant and $\mathbf{v}$ the velocity). For linearizing this framework one may use the formula $\det\mathbf{F} - 1 = \mathrm{div}\mathbf{u} + \mathcal{O}(\epsilon^2)$ (p.30 [11]) and replace $\det\mathbf{F}$ with $\mathrm{div}\mathbf{u}$ to within first order and $\mathbf{F}$ by $\nabla\mathbf{u}$.

It seems that, even if such an approximation is used the present framework is different from the one of Qin conceptually: Qin uses the compatibility relations without pre-multiplying them by a quantity and adding/subtracting them (see Eq. (3.12) in his paper); he uses as basic equations the time derivatives of the cofactor matrix and the determinant instead. We here, on the other hand, pre-multiply the compatibility equations judiciously and add/subtract them in order to accomplish our goal. Nevertheless, in order to give a precise answer one has to linearize Qin's framework systematically and then check exactly what it gives.

A comparison of Qin's work with that of Boillat and Ruggeri ([2]) would also help in the better understanding of the symmetric writing of the equations of nonlinear elasticity. From these possible ways of writing the equations of nonlinear elasticity one should linearize in order to examine under what conditions the present framework and that of [15, 21, 22] result and if these two approaches are the only possible one's for the linear elastic case.

7 Conclusion

The main achievement of the present approach is to offer an alternative path for writing the system of classical elastodynamics in a symmetric linear format. Instead of the stress tensor, the displacement gradient is considered to be the new dependent variable. Such a writing utilizes a Cartesian decomposition of the variables and operators that does not alter the Hamiltonian structure of classical elastodynamics. By means of the calculations, certainly, the present approach is more laborious than the one using the stress tensor. Mixed finite elements consider the velocity and the stress tensor as independent variables to avoid problems related with discontinuous in space material coefficients; in such cases the approach using the stress tensor seems superior to the present approach.

All in all, what the present approach teaches us about the subject is that an alternative way of writing the equations of classical elastodynamics exists using the displacement gradient instead of the stress tensor. While it is not perfectly clear what use of this approach enable us to do going forward in real calculations, it seems to be preferable when discontinuous in time material parameters are involved.

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Declarations

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