Journal of Elasticity (2024) 156:255–278
https://doi.org/10.1007/s10659-024-10049-w

RESEARCH

Internally Balanced Elasticity Tensor in Terms of Principal Stretches

Ashraf Hadoush[1]

Received: 11 May 2023 / Accepted: 21 January 2024 / Published online: 5 February 2024

Abstract
A new scheme for hyperelastic material is developed based on applying the argument of calculus variation to two-factor multiplicative decomposition of the deformation gradient. Then, Piola–Kirchhoff stress is coupled with internal balance equation. Strain energy function is expressed in terms of principal invariants of the deformation gradient decomposed counterparts. Recent work introduces a strain energy function in terms of principal stretches of the deformation gradient multiplicatively decomposed counterparts directly. Hence, a new reformulation of Piola–Kirchhoff stress and internal balance equation are provided. This work focuses on developing the mathematical framework to calculate the elasticity tensor for material model formulated in terms of decomposed principal stretches. This paves the way for future implementation of these classes of material model in FE formulation.

Keywords Principal stretches · Internal balance · Elasticity tensor

Mathematics Subject Classification 74B20

1 Introduction

Hyperelastic material develops its constitutive framework in terms of a strain energy function $W(\mathbf{F})$ where $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$ is the deformation gradient. Here $\mathbf{x} = \chi(\mathbf{X})$ is the mapping from the reference configuration $\mathbf{X}$ to the current configuration $\mathbf{x}$. More general treatment that models elastic and inelastic effects can be expressed in terms of deformation gradient $\mathbf{F}$ multiplicative decompositions [1–3] such as

$$\mathbf{F} = \hat{\mathbf{F}}\,\breve{\mathbf{F}}, \tag{1}$$

where $\hat{\mathbf{F}}$ models elastic response and it is associated to the rules of variational calculus while $\breve{\mathbf{F}}$ models inelastic response usually by means of a time dependent evolution law or balance principles. This treatment has been applied in plasticity [4–6], viscoelasticity [7–9], and shape memory alloys [10–12].

✉ A. Hadoush
a.hadoush@ptuk.edu.ps

[1] Mechanical Engineering, Palestine Technical University – Kadoorie, Tulkarm, Palestine

It is worth to mention that elastic and inelastic effects may prevail all aspects of a complex physical processes. Therefore, the decomposition sequences in which elastic and inelastic factors alternate with each other need to be considered [13]. This motivates the consideration of (1) in a context where both $\hat{\mathbf{F}}$ and $\breve{\mathbf{F}}$ are associated with a separate purely elastic type of effect. A theory of internally balanced incompressible elastic materials emerges under such considerations [14, 15] that is extended to model compressible hyperelastic materials in [16].

Now, strain energy function may depend on $\mathbf{F}$ and its decomposed counterparts $\hat{\mathbf{F}}$ and $\breve{\mathbf{F}}$. By virtue of (1), any one of $\mathbf{F}$, $\hat{\mathbf{F}}$ and $\breve{\mathbf{F}}$ can be eliminated in terms of the other two. Eliminating $\hat{\mathbf{F}}$ gives W as a function of $\mathbf{F}$ and $\breve{\mathbf{F}}$. To satisfy the requirement of material frame indifference [14], W is formulated in terms of $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ and $\breve{\mathbf{C}} = \breve{\mathbf{F}}^{\mathrm{T}}\breve{\mathbf{F}}$. The internal energy of the system is obtained by integrating W over the whole body.

The decomposition of $\mathbf{F}$ can be determined by a variety of ways. This work applies energy minimization to obtain the decomposition of $\mathbf{F}$. Energy minimization with respect to $\breve{\mathbf{C}}$ generates the internal balance equation which distinguishes the internally balanced material theory from the conventional hyperelasticity. This additional principle can be interpreted as requiring that the internal variable itself gives rise to a balanced internal configuration at all time. Keep in mind that energy minimization with respect to $\mathbf{C}$ generates stress equation of equilibrium. Clearly, this internally balanced hyperelastic procedure has connections with similar approaches found in literature considering for example the work presented in [17–19].

Special internally balanced Blatz–Ko material models have been introduced in [20–22]. These models are formulated in terms of principal invariant of $\hat{\mathbf{C}} = \hat{\mathbf{F}}^{\mathrm{T}}\hat{\mathbf{F}}$ and $\breve{\mathbf{C}}$. Nothing that tensors $\breve{\mathbf{C}}$ and $\hat{\mathbf{C}}$ are second order symmetric tensors that are defined in analogy with $\mathbf{C}$. Then, Second Piola–Kirchhoff stress and internal balance tensor become functions of $\mathbf{C}$ and $\breve{\mathbf{C}}$. A recent work demonstrates reformulation of this internally balanced compressible scheme in terms of principal stretches of tensors $\hat{\mathbf{C}}$ and $\breve{\mathbf{C}}$ [23]. This provides significant simplification for the representation of second Piola Kirchhoff stress and internal balance tensor.

It essential to calculate elasticity tensor to perform finite element analysis. The internal balance treatment requires to calculate four tensors in order to determine a condensed form of elasticity tensor [24]. For principal invariant formulation, these tensors are fourth order tensors and functions of $\mathbf{C}$ and $\breve{\mathbf{C}}$. The focus of this work is to illustrate the mathematical procedure that is needed to achieve condensed elasticity tensor for strain energy function based on principal stretches of $\hat{\mathbf{C}}$ and $\breve{\mathbf{C}}$.

2 Continuum Mechanics

Strain energy function W for isotropic hyperelastic material can be expressed using principal invariants I_1, I_2, I_3 of tensor $\mathbf{C}$. Second Piola–Kirchhoff PK2 stress $\mathbf{S}$ can be written as

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} = 2\Big(\frac{\partial W}{\partial I_1}\mathbf{I} + \frac{\partial W}{\partial I_2}\big(I_1\mathbf{I} - \mathbf{C}\big) + \frac{\partial W}{\partial I_3}\, I_3\mathbf{C}^{-1}\Big). \tag{2}$$

where $\mathbf{E}$ is the Green–Lagrange strain that is defined as $2\mathbf{E} = \mathbf{C} - \mathbf{I}$ and $\mathbf{I}$ is the second order identity tensor. The fourth order material elasticity tensor $\boldsymbol{\mathcal{C}}$ is obtained by

$$\boldsymbol{\mathcal{C}} = \frac{\partial \mathbf{S}}{\partial \mathbf{E}} = \frac{\partial^2 W}{\partial \mathbf{E}\partial \mathbf{E}}, \tag{3}$$

a detailed definition for $\boldsymbol{\mathcal{C}}$ in terms of principal invariants can be found in [25].

Also, W can be written in term of principal stretches λ_a which the square root of the eigenvalues λ_a^2 of tensor $\mathbf{C}$. The subscript a will always set to $a = 1, 2, 3$ in this manuscript. The superscript $^{\mathrm{P}}$ indicates that the tensor is expressed in principal directions. Now, $\mathbf{E}^{\mathrm{P}} = E_a \mathbf{N}_a \otimes \mathbf{N}_a$ where $E_a = (\lambda_a^2 - 1)/2$ is the eigenvalues of $\mathbf{E}$ and $\mathbf{N}_a$ is eigenvectors of tensor $\mathbf{C}$. The PK2 stress in principal directions $\mathbf{S}^{\mathrm{P}}$ becomes

$$\mathbf{S}^{\mathrm{P}} = \sum_{a=1}^{3} \frac{\partial W}{\partial \lambda_a} \frac{\partial \lambda_a}{\partial \mathbf{E}^{\mathrm{P}}} = \sum_{a=1}^{3} S_a \mathbf{N}_a \otimes \mathbf{N}_a, \tag{4}$$

with S_a defined as principal PK2 stress components

$$S_a = \frac{1}{\lambda_a} \frac{\partial W}{\partial \lambda_a}. \tag{5}$$

The derivative of λ_a with respect to $\mathbf{E}^{\mathrm{P}}$ is given as

$$\frac{\partial \lambda_a}{\partial \mathbf{E}^{\mathrm{P}}} = 2 \frac{\partial \lambda_a}{\partial \mathbf{C}^{\mathrm{P}}} = \lambda_a^{-1} \mathbf{N}_a \otimes \mathbf{N}_a, \tag{6}$$

further details about the derivatives of λ_a can be found in [25, 26].

In this work, the procedure of Crisfield [27] to determine the elasticity tensor is used because it simplifies the demonstration of internal balance treatment in the following sections. The reader is recommended to review appendix A to get familiar with the notation. Briefly, Green–Lagrange strain and PK2 stress are coaxial i.e. both tensors have the same principal directions. They are expressed in principal directions as

$$\mathbf{E}^{\mathrm{P}} = \mathbf{Q}\,\mathbf{Diag}(E_a)\,\mathbf{Q}^{\mathrm{T}}, \tag{7a}$$

$$\mathbf{S}^{\mathrm{P}} = \mathbf{Q}\,\mathbf{Diag}(S_a)\,\mathbf{Q}^{\mathrm{T}}, \tag{7b}$$

where $\mathbf{Q}$ consists of eigenvectors $\mathbf{N}_a$ such as $[\mathbf{Q}] = [\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3]$. The derivative of Green–Lagrange strain and PK2 stress w.r.t. time in base frame are given

$$\dot{\mathbf{E}} = \mathbf{Q}^{\mathrm{T}}\dot{\mathbf{E}}^{\mathrm{P}}\mathbf{Q} = \dot{\mathbf{Diag}}(E_a) + \boldsymbol{\Omega}\,\mathbf{Diag}(E_a) - \mathbf{Diag}(E_a)\,\boldsymbol{\Omega}^{\mathrm{T}}, \tag{8a}$$

$$\dot{\mathbf{S}} = \mathbf{Q}^{\mathrm{T}}\dot{\mathbf{S}}^{\mathrm{P}}\mathbf{Q} = \dot{\mathbf{Diag}}(S_a) + \boldsymbol{\Omega}\,\mathbf{Diag}(S_a) - \mathbf{Diag}(S_a)\,\boldsymbol{\Omega}^{\mathrm{T}}, \tag{8b}$$

here $\boldsymbol{\Omega}$ is an antisymmetric tensor as defined in (A.15). In Voigt notation, they are written as

$$\dot{\mathbf{E}}_{\mathrm{V}} = \begin{bmatrix} \dot{E}_{\mathrm{D}} \\ \dot{E}_{\mathrm{O}} \end{bmatrix} \quad \text{and} \quad \dot{\mathbf{S}}_{\mathrm{V}} = \begin{bmatrix} \dot{S}_{\mathrm{D}} \\ \dot{S}_{\mathrm{O}} \end{bmatrix}, \tag{9}$$

where the diagonal components are

$$[\dot{E}_{\mathrm{D}}] = \begin{bmatrix} \dot{E}_{11} & \dot{E}_{22} & \dot{E}_{33} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \dot{E}_1 & \dot{E}_2 & \dot{E}_3 \end{bmatrix}^{\mathrm{T}}, \tag{10a}$$

$$[\dot{S}_{\mathrm{D}}] = \begin{bmatrix} \dot{S}_{11} & \dot{S}_{22} & \dot{S}_{33} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \dot{S}_1 & \dot{S}_2 & \dot{S}_3 \end{bmatrix}^{\mathrm{T}}, \tag{10b}$$

and the off–diagonal components are

$$[\dot{E}_{\mathrm{O}}] = \begin{bmatrix} 2\dot{E}_{12} & 2\dot{E}_{23} & 2\dot{E}_{13} \end{bmatrix}^{\mathrm{T}}, \tag{11a}$$

$$[\dot{S}_{\rm O}] = \begin{bmatrix} \dot{S}_{12} & \dot{S}_{23} & \dot{S}_{13} \end{bmatrix}^{\rm T}. \tag{11b}$$

Using chain rule, $[\dot{S}_{\rm D}]$ is related to $[\dot{E}_{\rm D}]$ via

$$[\dot{S}_{\rm D}] = [S_{\rm a,E_a}]\,[\dot{E}_{\rm D}]\,, \tag{12}$$

where $[S_{\rm a,E_a}]$ is the partial derivatives of $S_{\rm a}$ with respect to $E_{\rm a}$,

$$[S_{\rm a,E_a}] = \begin{bmatrix} \dfrac{\partial S_1}{\partial E_1} & \dfrac{\partial S_1}{\partial E_2} & \dfrac{\partial S_1}{\partial E_3} \\ \dfrac{\partial S_2}{\partial E_1} & \dfrac{\partial S_2}{\partial E_2} & \dfrac{\partial S_2}{\partial E_3} \\ \dfrac{\partial S_3}{\partial E_1} & \dfrac{\partial S_3}{\partial E_2} & \dfrac{\partial S_3}{\partial E_3} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\lambda_1}\dfrac{\partial S_1}{\partial \lambda_1} & \dfrac{1}{\lambda_2}\dfrac{\partial S_1}{\partial \lambda_2} & \dfrac{1}{\lambda_3}\dfrac{\partial S_1}{\partial \lambda_3} \\ \dfrac{1}{\lambda_1}\dfrac{\partial S_2}{\partial \lambda_1} & \dfrac{1}{\lambda_2}\dfrac{\partial S_2}{\partial \lambda_2} & \dfrac{1}{\lambda_3}\dfrac{\partial S_2}{\partial \lambda_3} \\ \dfrac{1}{\lambda_1}\dfrac{\partial S_3}{\partial \lambda_1} & \dfrac{1}{\lambda_2}\dfrac{\partial S_3}{\partial \lambda_2} & \dfrac{1}{\lambda_3}\dfrac{\partial S_3}{\partial \lambda_3} \end{bmatrix}. \tag{13}$$

Note that the above equation makes use of $\partial\lambda_{\rm a}/\partial E_{\rm a} = 1/\lambda_{\rm a}$. The off–diagonal components of $\dot{\mathbf{S}}$ can be defined based on $E_{\rm a}$, $S_{\rm a}$ and $E_{\rm O}$ as

$$[\dot{S}_{\rm O}] = [\Delta S_{\rm a}/\Delta E_{\rm a}]\,[\dot{E}_{\rm O}]\,, \tag{14}$$

where

$$[\Delta S_{\rm a}/\Delta E_{\rm a}] = \begin{bmatrix} \dfrac{S_1 - S_2}{2(E_1 - E_2)} & 0 & 0 \\ 0 & \dfrac{S_2 - S_3}{2(E_2 - E_3)} & 0 \\ 0 & 0 & \dfrac{S_1 - S_3}{2(E_1 - E_3)} \end{bmatrix}. \tag{15}$$

The material elasticity tensor in principal directions using Voigt notation $\boldsymbol{\mathcal{C}}^{\rm P}_{\rm V}$ can be formed by

$$\dot{\mathbf{S}}_{\rm V} = \boldsymbol{\mathcal{C}}^{\rm P}_{\rm V}\,\dot{\mathbf{E}}_{\rm V} \Rightarrow [\boldsymbol{\mathcal{C}}^{\rm P}_{\rm V}] = \begin{bmatrix} [S_{\rm a,E_a}] & 0 \\ 0 & [\Delta S_{\rm a}/\Delta E_{\rm a}] \end{bmatrix}. \tag{16}$$

As a fourth order tensor, the component of $\boldsymbol{\mathcal{C}}^{\rm P}$ are

$$\mathcal{C}^{\rm P}_{\rm aabb} = \frac{\partial S_{\rm a}}{\partial E_{\rm b}} = \frac{1}{\lambda_{\rm b}}\frac{\partial S_{\rm a}}{\partial \lambda_{\rm b}}\,, \tag{17a}$$

$$\mathcal{C}^{\rm P}_{\rm abab} = \mathcal{C}^{\rm P}_{\rm abba} = \mathcal{C}^{\rm P}_{\rm baab} = \frac{S_{\rm a} - S_{\rm b}}{2(E_{\rm a} - E_{\rm b})} = \frac{S_{\rm a} - S_{\rm b}}{\lambda_{\rm a}^2 - \lambda_{\rm b}^2} \quad (a \neq b)\,. \tag{17b}$$

The elastcity tensor in base frame $\boldsymbol{\mathcal{C}}$ is related $\boldsymbol{\mathcal{C}}^{\rm P}$ via

$$\mathcal{C}_{\rm ijkl} = Q_{\rm ia}\,Q_{\rm jb}\,Q_{\rm kc}\,Q_{\rm ld}\,\mathcal{C}^{\rm P}_{\rm abcd}\,. \tag{18}$$

3 Internal Balance: Principal Invariants

A new scheme of viewing compressible hyperelastic material response is introduced in [16]. The arguments of variational calculus are applied to both portions of the deformation gradient decomposition. The decomposition itself is determined on the basis of an additional internal balance equation that emerges naturally from the variational treatment. The mathematical formulation is demonstrated using strain energy function based on principal invariant of tensors $\hat{\mathbf{C}}$ and $\breve{\mathbf{C}}$ which are $\hat{I}_1$, $\hat{I}_2$, $\hat{I}_3$ and $\breve{I}_1$, $\breve{I}_2$, $\breve{I}_3$, respectively. The PK2 stress $\mathbf{S}$ is obtained as consequence of applying the argument of variation with respect to $\mathbf{E}$ at fixed $\breve{\mathbf{E}}$. Tensor $\breve{\mathbf{E}}$ is defined in analogy with $\mathbf{E}$ such as $2\breve{\mathbf{E}} = \breve{\mathbf{C}} - \mathbf{I}$. Then, PK2 stress is written as

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} = 2\Big(\frac{\partial W}{\partial \hat{I}_1}\breve{\mathbf{C}}^{-1} + \frac{\partial W}{\partial \hat{I}_2}(\hat{I}_1\breve{\mathbf{C}}^{-1} - \mathbf{M}) + \frac{\partial W}{\partial \hat{I}_3}\hat{I}_3\,\mathbf{C}^{-1}\Big), \tag{19}$$

where $\mathbf{M} = \breve{\mathbf{C}}^{-1}\mathbf{C}\,\breve{\mathbf{C}}^{-1}$. The decomposition of the deformation gradient is found by solving an internal balance equation that arises from the variation with respect to $\breve{\mathbf{E}}$ at fixed $\mathbf{E}$. The internal balance equation is $\boldsymbol{\Psi} = 0$ where $\boldsymbol{\Psi}$ is an internal balance tensor

$$\boldsymbol{\Psi} = \frac{\partial W}{\partial \breve{\mathbf{E}}} = \hat{\boldsymbol{\Psi}} + \breve{\boldsymbol{\Psi}}, \tag{20}$$

with

$$\hat{\boldsymbol{\Psi}} = 2\Big(-\frac{\partial W}{\partial \hat{I}_1}\mathbf{M} + \frac{\partial W}{\partial \hat{I}_2}\big(\hat{I}_3\,\mathbf{C}^{-1} - \hat{I}_2\breve{\mathbf{C}}^{-1}\big) - \frac{\partial W}{\partial \hat{I}_3}\,\hat{I}_3\,\breve{\mathbf{C}}^{-1}\Big), \tag{21a}$$

$$\breve{\boldsymbol{\Psi}} = 2\Big(\frac{\partial W}{\partial \breve{I}_1}\mathbf{I} + \frac{\partial W}{\partial \breve{I}_2}\big(\breve{I}_1\mathbf{I} - \breve{\mathbf{C}}\big) + \frac{\partial W}{\partial \breve{I}_3}\,\breve{I}_3\,\breve{\mathbf{C}}^{-1}\Big). \tag{21b}$$

Further details about these individual derivatives can be found in [16, 21]. For conventional hyperelasticity treatment, the equations of equilibrium in absence of body and inertia forces is governed by $\mathrm{Div}\,\mathbf{P} = \mathbf{0}$ where $\mathbf{P}$ is the first Piola–Kirchhoff stress that is related to $\mathbf{S}$ via $\mathbf{P} = \mathbf{FS}$ where $\mathbf{S}$ is given as a function of $\mathbf{E}$ (2). In contrast, it is reformulated in the internal balance treatment as $\mathrm{Div}\,\mathbf{P} = \mathbf{0}$ coupled with $\boldsymbol{\Psi} = \mathbf{0}$ where $\mathbf{S}$ and $\boldsymbol{\Psi}$ are given as functions of $\mathbf{E}$ and $\breve{\mathbf{E}}$ as defined in (19) and (20), respectively. This requires to determine $\breve{\mathbf{E}}$ by solving $\boldsymbol{\Psi} = \mathbf{0}$ for given $\mathbf{E}$ then $\mathbf{S}$ can be obtained.

The finite element formulation of internally balanced compressible hyperelastic material is demonstrated in [24]. It is based on calculating a condensed fourth order elasticity tensor $\boldsymbol{\mathcal{C}}_{\mathrm{con}}$ that is defined as

$$\boldsymbol{\mathcal{C}}_{\mathrm{con}} = \boldsymbol{\mathcal{C}}_{\mathbf{E}} - \boldsymbol{\mathcal{C}}_{\breve{\mathbf{E}}} : \boldsymbol{\Psi}_{\breve{\mathbf{E}}}^{-1} : \boldsymbol{\Psi}_{\mathbf{E}}\,. \tag{22}$$

The tensors $\boldsymbol{\mathcal{C}}_{\mathbf{E}}$ and $\boldsymbol{\mathcal{C}}_{\breve{\mathbf{E}}}$ are obtained by differentiating PK2 stress (19) with respect to $\mathbf{E}$ and $\breve{\mathbf{E}}$ respectively

$$\boldsymbol{\mathcal{C}}_{\mathbf{E}} = \frac{\partial \mathbf{S}}{\partial \mathbf{E}} \quad \text{and} \quad \boldsymbol{\mathcal{C}}_{\breve{\mathbf{E}}} = \frac{\partial \mathbf{S}}{\partial \breve{\mathbf{E}}}\,. \tag{23}$$

The tensors $\boldsymbol{\Psi}_{\breve{\mathbf{E}}}$ and $\boldsymbol{\Psi}_{\mathbf{E}}$ are obtained by differentiating internal balance tensor (20) with respect to $\mathbf{E}$ and $\breve{\mathbf{E}}$ respectively

$$\boldsymbol{\Psi}_{\mathbf{E}} = \frac{\partial \boldsymbol{\Psi}}{\partial \mathbf{E}} \quad \text{and} \quad \boldsymbol{\Psi}_{\breve{\mathbf{E}}} = \frac{\partial \boldsymbol{\Psi}}{\partial \breve{\mathbf{E}}}\,. \tag{24}$$

Note that these tensors $\boldsymbol{\mathcal{C}}_{\mathbf{E}}$, $\boldsymbol{\mathcal{C}}_{\breve{\mathbf{E}}}$, $\boldsymbol{\Psi}_{\breve{\mathbf{E}}}$ and $\boldsymbol{\Psi}_{\mathbf{E}}$ are fourth-order tensors.

4 Internal Balance: Principal Stretches

Principal stretches of the deformation gradient can be multiplicatively decomposed as

$$\lambda_a = \hat{\lambda}_a \breve{\lambda}_a \,, \tag{25}$$

where $\hat{\lambda}_a$ and $\breve{\lambda}_a$ are the principal stretches of $\hat{\mathbf{F}}$ and $\breve{\mathbf{F}}$, respectively. Now, the strain energy can have the form of $W(\hat{\lambda}_a, \breve{\lambda}_a)$ [23, 28]. Then, PK2 stress using $(19)_1$ becomes

$$\mathbf{S}^P = \sum_{a=1}^{3} \frac{\partial W}{\partial \hat{\lambda}_a} \frac{\partial \hat{\lambda}_a}{\partial \mathbf{E}^P} = \sum_{a=1}^{3} S_a \mathbf{N}_a \otimes \mathbf{N}_a \,, \tag{26}$$

where

$$S_a = \frac{\hat{\lambda}_a}{\lambda_a^2} \frac{\partial W}{\partial \hat{\lambda}_a} \,, \tag{27}$$

and the derivative of $\hat{\lambda}_a$ with respect to $\mathbf{E}^P$ is given as

$$\frac{\partial \hat{\lambda}_a}{\partial \mathbf{E}^P} = \frac{\hat{\lambda}_a}{\lambda_a^2} \mathbf{N}_a \otimes \mathbf{N}_a \,. \tag{28}$$

The individual parts of the internal balance tensor defined in (20) now becomes

$$\hat{\boldsymbol{\Psi}}^P = \sum_{a=1}^{3} \frac{\partial W}{\partial \hat{\lambda}_a} \frac{\partial \hat{\lambda}_a}{\partial \breve{\mathbf{E}}^P} \quad \text{and} \quad \breve{\boldsymbol{\Psi}}^P = \sum_{a=1}^{3} \frac{\partial W}{\partial \breve{\lambda}_a} \frac{\partial \breve{\lambda}_a}{\partial \breve{\mathbf{E}}^P}; \tag{29}$$

here, $\breve{\mathbf{E}}^P = \breve{E}_a \breve{\mathbf{N}}_a \otimes \breve{\mathbf{N}}_a$ where $\breve{E}_a = (\breve{\lambda}_a^2 - 1)/2$ are the eigenvalues of $\breve{\mathbf{E}}$ and $\breve{\mathbf{N}}_a$ are the eigenvectors of $\breve{\mathbf{C}}$. The derivatives of principal stretches $\hat{\lambda}_a$ and $\breve{\lambda}_a$ with respect to $\breve{\mathbf{E}}^P$ are given as

$$\frac{\partial \hat{\lambda}_a}{\partial \breve{\mathbf{E}}^P} = -\frac{\hat{\lambda}_a}{\breve{\lambda}_a^2} \breve{\mathbf{N}}_a \otimes \breve{\mathbf{N}}_a \quad \text{and} \quad \frac{\partial \breve{\lambda}_a}{\partial \breve{\mathbf{E}}^P} = \frac{1}{\breve{\lambda}_a} \breve{\mathbf{N}}_a \otimes \breve{\mathbf{N}}_a \,. \tag{30}$$

Back substituting (30) into (29), respectively, and then substituting the results of that into (20) gives

$$\boldsymbol{\Psi}^P = \sum_{a=1}^{3} \Psi_a \breve{\mathbf{N}}_a \otimes \breve{\mathbf{N}}_a \,, \tag{31}$$

where Ψ_a is the principal internal balance scalar components

$$\Psi_a = \frac{1}{\breve{\lambda}_a} \frac{\partial W}{\partial \breve{\lambda}_a} - \frac{\hat{\lambda}_a}{\breve{\lambda}_a^2} \frac{\partial W}{\partial \hat{\lambda}_a} \,. \tag{32}$$

Further details about these formulations can be found in [23].

5 Classes of Material

This work focuses on particular classes of material where $\breve{\mathbf{E}}$ and Green–Lagrange strain $\mathbf{E}$ are coaxial i.e. $\mathbf{N}_\mathrm{a}$ coincides with $\breve{\mathbf{N}}_\mathrm{a}$. Two classes of material are presented to demonstrate this coaxiality. The analysis is carried out using principal invariants for simplicity. Note that $\breve{\mathbf{E}}$ and $\mathbf{E}$ are not coaxial in general. Then, it is required to calculate the derivative of $\breve{\mathbf{N}}_\mathrm{a}$ with respect to time for linearization procedure. This is not the scope of this paper.

5.1 $W(\hat{I}_1, \hat{I}_3, \breve{I}_1, \breve{I}_2, \breve{I}_3)$

The internal balance equation of this material model is achieved by (20) as

$$w_0\,\breve{\mathbf{C}}^{-1}\mathbf{C}\,\breve{\mathbf{C}}^{-1} + w_1\,\breve{\mathbf{C}}^{-1} + w_2\,\mathbf{I} + w_3\,\breve{\mathbf{C}} = \mathbf{0}\,, \tag{33}$$

where

$$\begin{aligned} w_0 &= \frac{\partial W}{\partial \hat{I}_1}, \quad w_1 = \frac{\partial W}{\partial \hat{I}_3}\,\hat{I}_3 - \frac{\partial W}{\partial \breve{I}_3}\,\breve{I}_3, \\ w_2 &= -\Big(\frac{\partial W}{\partial \breve{I}_1} + \frac{\partial W}{\partial \breve{I}_2}\,\breve{I}_1\Big), \quad w_3 = \frac{\partial W}{\partial \breve{I}_2}\,. \end{aligned} \tag{34}$$

Premultiplying and post multiplying (33) by $\breve{\mathbf{C}}$ gives

$$w_0\,\mathbf{C} + w_1\,\breve{\mathbf{C}} + w_2\,\breve{\mathbf{C}}^2 + w_3\,\breve{\mathbf{C}}^3 = \mathbf{0}. \tag{35}$$

Then multiplying the above equation with $\breve{\mathbf{N}}_\mathrm{a}$ and making use of (A.5) gives

$$\mathbf{C}\,\breve{\mathbf{N}}_\mathrm{a} = -\Big(\frac{w_1\,\hat{\lambda}_\mathrm{a}^2 + w_2\,\hat{\lambda}_\mathrm{a}^4 + w_3\,\hat{\lambda}_\mathrm{a}^6}{w_0}\Big)\,\breve{\mathbf{N}}_\mathrm{a}; \tag{36}$$

note that the term inside the bracket is scalar and, hence, that $\breve{\mathbf{N}}_\mathrm{a}$ is an eigenvector of $\mathbf{C}$ and $\mathbf{E}$.

5.2 $W(\hat{I}_2, \hat{I}_3, \breve{I}_1, \breve{I}_2, \breve{I}_3)$

Substituting the form of this material into (20) leads to

$$w_4\,\mathbf{C}^{-1} + w_5\,\breve{\mathbf{C}}^{-1} + w_6\,\mathbf{I} + w_7\,\breve{\mathbf{C}} = \mathbf{0}, \tag{37}$$

where

$$\begin{aligned} w_4 &= \frac{\partial W}{\partial \hat{I}_2}\,\hat{I}_3, \quad w_5 = -\Big(\frac{\partial W}{\partial \hat{I}_2}\,\hat{I}_2 + \frac{\partial W}{\partial \hat{I}_3}\,\hat{I}_3 + \frac{\partial W}{\partial \breve{I}_3}\,\breve{I}_3\Big), \\ w_6 &= \frac{\partial W}{\partial \breve{I}_1} + \frac{\partial W}{\partial \breve{I}_2}\,\breve{I}_1, \quad w_7 = -\frac{\partial W}{\partial \breve{I}_2}\,. \end{aligned} \tag{38}$$

Multiplying (38) by $\breve{\mathbf{N}}_\mathrm{a}$ and applying (A.5) gives

$$\mathbf{C}^{-1}\,\breve{\mathbf{N}}_\mathrm{a} = -\Big(\frac{w_5\,\hat{\lambda}_\mathrm{a}^{-2} + w_6 + w_7\,\hat{\lambda}_\mathrm{a}^2}{w_4}\Big)\,\breve{\mathbf{N}}_\mathrm{a}\,, \tag{39}$$

the term inside the bracket is a scalar and, hence, $\breve{\mathbf{N}}_\mathrm{a}$ is an eigenvector of $\mathbf{C}$ and $\mathbf{E}$ for this material model as well.

6 Linearization

Following the leads of Crisfield procedure [27] for conventional hyperelasticity, the internal balance formulation can be expressed in principal directions as

$$\mathbf{E}^{\mathrm{P}} = \mathbf{Q}\,\mathbf{Diag}(E_{\mathrm{a}})\,\mathbf{Q}^{\mathrm{T}}, \tag{40a}$$

$$\breve{\mathbf{E}}^{\mathrm{P}} = \mathbf{Q}\,\mathbf{Diag}(\breve{E}_{\mathrm{a}})\,\mathbf{Q}^{\mathrm{T}}, \tag{40b}$$

$$\mathbf{S}^{\mathrm{P}} = \mathbf{Q}\,\mathbf{Diag}(S_{\mathrm{a}})\,\mathbf{Q}^{\mathrm{T}}, \tag{40c}$$

$$\boldsymbol{\Psi}^{\mathrm{P}} = \mathbf{Q}\,\mathbf{Diag}(\Psi_{\mathrm{a}})\,\mathbf{Q}^{\mathrm{T}}. \tag{40d}$$

Tensors $\mathbf{E}$, $\breve{\mathbf{E}}$, $\mathbf{S}$ and $\boldsymbol{\Psi}$ are coaxial considering the material classes presented in Sect. 5. Their derivative w.r.t. time in base frame are given by

$$\dot{\mathbf{E}} = \mathbf{Q}^{\mathrm{T}}\dot{\mathbf{E}}^{\mathrm{P}}\mathbf{Q} = \dot{\mathbf{Diag}}(E_{\mathrm{a}}) + \boldsymbol{\Omega}\,\mathbf{Diag}(E_{\mathrm{a}}) - \mathbf{Diag}(E_{\mathrm{a}})\,\boldsymbol{\Omega}^{\mathrm{T}}, \tag{41a}$$

$$\dot{\breve{\mathbf{E}}} = \mathbf{Q}^{\mathrm{T}}\dot{\breve{\mathbf{E}}}^{\mathrm{P}}\mathbf{Q} = \dot{\mathbf{Diag}}(\breve{E}_{\mathrm{a}}) + \boldsymbol{\Omega}\,\mathbf{Diag}(\breve{E}_{\mathrm{a}}) - \mathbf{Diag}(\breve{E}_{\mathrm{a}})\,\boldsymbol{\Omega}^{\mathrm{T}}, \tag{41b}$$

$$\dot{\mathbf{S}} = \mathbf{Q}^{\mathrm{T}}\dot{\mathbf{S}}^{\mathrm{P}}\mathbf{Q} = \dot{\mathbf{Diag}}(S_{\mathrm{a}}) + \boldsymbol{\Omega}\,\mathbf{Diag}(S_{\mathrm{a}}) - \mathbf{Diag}(S_{\mathrm{a}})\,\boldsymbol{\Omega}^{\mathrm{T}}, \tag{41c}$$

$$\dot{\boldsymbol{\Psi}} = \mathbf{Q}^{\mathrm{T}}\dot{\boldsymbol{\Psi}}^{\mathrm{P}}\mathbf{Q} = \dot{\mathbf{Diag}}(\Psi_{\mathrm{a}}) + \boldsymbol{\Omega}\,\mathbf{Diag}(\Psi_{\mathrm{a}}) - \mathbf{Diag}(\Psi_{\mathrm{a}})\,\boldsymbol{\Omega}^{\mathrm{T}}, \tag{41d}$$

where $\boldsymbol{\Omega}$ is defined in (A.15). In Voigt notation, $\dot{\mathbf{E}}_{\mathrm{V}}$ and $\dot{\mathbf{S}}_{\mathrm{V}}$ have same representation as in (9). Similarly, $\dot{\breve{\mathbf{E}}}_{\mathrm{V}}$ and $\dot{\boldsymbol{\Psi}}_{\mathrm{V}}$ is written in Voigt notation as

$$\dot{\breve{\mathbf{E}}}_{\mathrm{V}} = \begin{bmatrix} \dot{\breve{E}}_{\mathrm{D}} \\ \dot{\breve{E}}_{\mathrm{O}} \end{bmatrix} \quad \text{and} \quad \dot{\boldsymbol{\Psi}}_{\mathrm{V}} = \begin{bmatrix} \dot{\Psi}_{\mathrm{D}} \\ \dot{\Psi}_{\mathrm{O}} \end{bmatrix}, \tag{42}$$

where the diagonal components are

$$[\dot{\breve{E}}_{\mathrm{D}}] = \begin{bmatrix} \dot{\breve{E}}_{11} & \dot{\breve{E}}_{22} & \dot{\breve{E}}_{33} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \dot{\breve{E}}_{1} & \dot{\breve{E}}_{2} & \dot{\breve{E}}_{3} \end{bmatrix}^{\mathrm{T}}, \tag{43a}$$

$$[\dot{\Psi}_{\mathrm{D}}] = \begin{bmatrix} \dot{\Psi}_{11} & \dot{\Psi}_{22} & \dot{\Psi}_{33} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \dot{\Psi}_{1} & \dot{\Psi}_{2} & \dot{\Psi}_{3} \end{bmatrix}^{\mathrm{T}}, \tag{43b}$$

and off–diagonal components are

$$[\dot{\breve{E}}_{\mathrm{O}}] = \begin{bmatrix} \dot{\breve{E}}_{12} & \dot{\breve{E}}_{23} & \dot{\breve{E}}_{13} \end{bmatrix}^{\mathrm{T}}, \tag{44a}$$

$$[\dot{\Psi}_{\mathrm{O}}] = \begin{bmatrix} \dot{\Psi}_{12} & \dot{\Psi}_{23} & \dot{\Psi}_{13} \end{bmatrix}^{\mathrm{T}}. \tag{44b}$$

6.1 Diagonal Components

In view of (27) and (41c), $[\dot{S}_{\mathrm{D}}]$ can be related to $[\dot{E}_{\mathrm{D}}]$ and $[\dot{\breve{E}}_{\mathrm{D}}]$ using the chain rule. This yields

$$[\dot{S}_{\mathrm{D}}] = [S_{\mathrm{a,E_a}}]\,[\dot{E}_{\mathrm{D}}] + [S_{\mathrm{a,\breve{E}_a}}]\,[\dot{\breve{E}}_{\mathrm{D}}], \tag{45}$$

where $[S_{\mathrm{a,E_a}}]$ is partial derivative of S_a w.r.t. E_a at fixed $\breve{E}_\mathrm{a}$

$$[S_{\mathrm{a,E_a}}] = \begin{bmatrix} \frac{\partial S_1}{\partial E_1} & \frac{\partial S_1}{\partial E_2} & \frac{\partial S_1}{\partial E_3} \\ \frac{\partial S_2}{\partial E_1} & \frac{\partial S_2}{\partial E_2} & \frac{\partial S_2}{\partial E_3} \\ \frac{\partial S_3}{\partial E_1} & \frac{\partial S_3}{\partial E_2} & \frac{\partial S_3}{\partial E_3} \end{bmatrix} \tag{46a}$$

$$= \begin{bmatrix} \frac{1}{\lambda_1}\frac{\partial S_1}{\partial \lambda_1} & \frac{1}{\lambda_2}\frac{\partial S_1}{\partial \lambda_2} & \frac{1}{\lambda_3}\frac{\partial S_1}{\partial \lambda_3} \\ \frac{1}{\lambda_1}\frac{\partial S_2}{\partial \lambda_1} & \frac{1}{\lambda_2}\frac{\partial S_2}{\partial \lambda_2} & \frac{1}{\lambda_3}\frac{\partial S_2}{\partial \lambda_3} \\ \frac{1}{\lambda_1}\frac{\partial S_3}{\partial \lambda_1} & \frac{1}{\lambda_2}\frac{\partial S_3}{\partial \lambda_2} & \frac{1}{\lambda_3}\frac{\partial S_3}{\partial \lambda_3} \end{bmatrix} + \begin{bmatrix} \frac{\hat{\lambda}_1}{\lambda_1^2}\frac{\partial S_1}{\partial \hat{\lambda}_1} & \frac{\hat{\lambda}_2}{\lambda_2^2}\frac{\partial S_1}{\partial \hat{\lambda}_2} & \frac{\hat{\lambda}_3}{\lambda_3^2}\frac{\partial S_1}{\partial \hat{\lambda}_3} \\ \frac{\hat{\lambda}_1}{\lambda_1^2}\frac{\partial S_2}{\partial \hat{\lambda}_1} & \frac{\hat{\lambda}_2}{\lambda_2^2}\frac{\partial S_2}{\partial \hat{\lambda}_2} & \frac{\hat{\lambda}_3}{\lambda_3^2}\frac{\partial S_2}{\partial \hat{\lambda}_3} \\ \frac{\hat{\lambda}_1}{\lambda_1^2}\frac{\partial S_3}{\partial \hat{\lambda}_1} & \frac{\hat{\lambda}_2}{\lambda_2^2}\frac{\partial S_3}{\partial \hat{\lambda}_2} & \frac{\hat{\lambda}_3}{\lambda_3^2}\frac{\partial S_3}{\partial \hat{\lambda}_3} \end{bmatrix}, \tag{46b}$$

and $[S_{\mathrm{a,\breve{E}_a}}]$ is partial derivative of S_a w.r.t. $\breve{E}_\mathrm{a}$ at fixed E_a

$$[S_{\mathrm{a,\breve{E}_a}}] = \begin{bmatrix} \frac{\partial S_1}{\partial \breve{E}_1} & \frac{\partial S_1}{\partial \breve{E}_2} & \frac{\partial S_1}{\partial \breve{E}_3} \\ \frac{\partial S_2}{\partial \breve{E}_1} & \frac{\partial S_2}{\partial \breve{E}_2} & \frac{\partial S_2}{\partial \breve{E}_3} \\ \frac{\partial S_3}{\partial \breve{E}_1} & \frac{\partial S_3}{\partial \breve{E}_2} & \frac{\partial S_3}{\partial \breve{E}_3} \end{bmatrix} \tag{47a}$$

$$= \begin{bmatrix} \frac{1}{\breve{\lambda}_1}\frac{\partial S_1}{\partial \breve{\lambda}_1} & \frac{1}{\breve{\lambda}_2}\frac{\partial S_1}{\partial \breve{\lambda}_2} & \frac{1}{\breve{\lambda}_3}\frac{\partial S_1}{\partial \breve{\lambda}_3} \\ \frac{1}{\breve{\lambda}_1}\frac{\partial S_2}{\partial \breve{\lambda}_1} & \frac{1}{\breve{\lambda}_2}\frac{\partial S_2}{\partial \breve{\lambda}_2} & \frac{1}{\breve{\lambda}_3}\frac{\partial S_2}{\partial \breve{\lambda}_3} \\ \frac{1}{\breve{\lambda}_1}\frac{\partial S_3}{\partial \breve{\lambda}_1} & \frac{1}{\breve{\lambda}_2}\frac{\partial S_3}{\partial \breve{\lambda}_2} & \frac{1}{\breve{\lambda}_3}\frac{\partial S_3}{\partial \breve{\lambda}_3} \end{bmatrix} - \begin{bmatrix} \frac{\hat{\lambda}_1}{\breve{\lambda}_1^2}\frac{\partial S_1}{\partial \hat{\lambda}_1} & \frac{\hat{\lambda}_2}{\breve{\lambda}_2^2}\frac{\partial S_1}{\partial \hat{\lambda}_2} & \frac{\hat{\lambda}_3}{\breve{\lambda}_3^2}\frac{\partial S_1}{\partial \hat{\lambda}_3} \\ \frac{\hat{\lambda}_1}{\breve{\lambda}_1^2}\frac{\partial S_2}{\partial \hat{\lambda}_1} & \frac{\hat{\lambda}_2}{\breve{\lambda}_2^2}\frac{\partial S_2}{\partial \hat{\lambda}_2} & \frac{\hat{\lambda}_3}{\breve{\lambda}_3^2}\frac{\partial S_2}{\partial \hat{\lambda}_3} \\ \frac{\hat{\lambda}_1}{\breve{\lambda}_1^2}\frac{\partial S_3}{\partial \hat{\lambda}_1} & \frac{\hat{\lambda}_2}{\breve{\lambda}_2^2}\frac{\partial S_3}{\partial \hat{\lambda}_2} & \frac{\hat{\lambda}_3}{\breve{\lambda}_3^2}\frac{\partial S_3}{\partial \hat{\lambda}_3} \end{bmatrix}. \tag{47b}$$

The above equations make use of

$$\frac{\partial \lambda_\mathrm{a}}{\partial E_\mathrm{a}} = \frac{1}{\lambda_\mathrm{a}}, \quad \frac{\partial \hat{\lambda}_\mathrm{a}}{\partial E_\mathrm{a}} = \frac{\hat{\lambda}_\mathrm{a}}{\lambda_\mathrm{a}^2}, \quad \frac{\partial \breve{\lambda}_\mathrm{a}}{\partial \breve{E}_\mathrm{a}} = \frac{1}{\breve{\lambda}_\mathrm{a}}, \quad \frac{\partial \hat{\lambda}_\mathrm{a}}{\partial \breve{E}_\mathrm{a}} = -\frac{\hat{\lambda}_\mathrm{a}}{\breve{\lambda}_\mathrm{a}^2}. \tag{48}$$

Viewing (31) and (41d), $[\dot{\Psi}_\mathrm{D}]$ can be related to $[\dot{E}_\mathrm{D}]$ and $[\dot{\breve{E}}_\mathrm{D}]$ using the chain rule and (48). For instance,

$$[\dot{\Psi}_\mathrm{D}] = [\Psi_{\mathrm{a,E_a}}]\,[\dot{E}_\mathrm{D}] + [\Psi_{\mathrm{a,\breve{E}_a}}]\,[\dot{\breve{E}}_\mathrm{D}], \tag{49}$$

where $[\Psi_{\mathrm{a,E_a}}]$ is partial derivative of Ψ_a w.r.t. E_a at fixed $\breve{E}_\mathrm{a}$,

$$[\Psi_{\mathrm{a,E_a}}] = \begin{bmatrix} \frac{\partial \Psi_1}{\partial E_1} & \frac{\partial \Psi_1}{\partial E_2} & \frac{\partial \Psi_1}{\partial E_3} \\ \frac{\partial \Psi_2}{\partial E_1} & \frac{\partial \Psi_2}{\partial E_2} & \frac{\partial \Psi_2}{\partial E_3} \\ \frac{\partial \Psi_3}{\partial E_1} & \frac{\partial \Psi_3}{\partial E_2} & \frac{\partial \Psi_3}{\partial E_3} \end{bmatrix} \tag{50a}$$

$$= \begin{bmatrix} \frac{1}{\lambda_1}\frac{\partial\Psi_1}{\partial\lambda_1} & \frac{1}{\lambda_2}\frac{\partial\Psi_1}{\partial\lambda_2} & \frac{1}{\lambda_3}\frac{\partial\Psi_1}{\partial\lambda_3} \\ \frac{1}{\lambda_1}\frac{\partial\Psi_2}{\partial\lambda_1} & \frac{1}{\lambda_2}\frac{\partial\Psi_2}{\partial\lambda_2} & \frac{1}{\lambda_3}\frac{\partial\Psi_2}{\partial\lambda_3} \\ \frac{1}{\lambda_1}\frac{\partial\Psi_3}{\partial\lambda_1} & \frac{1}{\lambda_2}\frac{\partial\Psi_3}{\partial\lambda_2} & \frac{1}{\lambda_3}\frac{\partial\Psi_3}{\partial\lambda_3} \end{bmatrix} + \begin{bmatrix} \frac{\hat{\lambda}_1}{\lambda_1^2}\frac{\partial\Psi_1}{\partial\hat{\lambda}_1} & \frac{\hat{\lambda}_2}{\lambda_2^2}\frac{\partial\Psi_1}{\partial\hat{\lambda}_2} & \frac{\hat{\lambda}_3}{\lambda_3^2}\frac{\partial\Psi_1}{\partial\hat{\lambda}_3} \\ \frac{\hat{\lambda}_1}{\lambda_1^2}\frac{\partial\Psi_2}{\partial\hat{\lambda}_1} & \frac{\hat{\lambda}_2}{\lambda_2^2}\frac{\partial\Psi_2}{\partial\hat{\lambda}_2} & \frac{\hat{\lambda}_3}{\lambda_3^2}\frac{\partial\Psi_2}{\partial\hat{\lambda}_3} \\ \frac{\hat{\lambda}_1}{\lambda_1^2}\frac{\partial\Psi_3}{\partial\hat{\lambda}_1} & \frac{\hat{\lambda}_2}{\lambda_2^2}\frac{\partial\Psi_3}{\partial\hat{\lambda}_2} & \frac{\hat{\lambda}_3}{\lambda_3^2}\frac{\partial\Psi_3}{\partial\hat{\lambda}_3} \end{bmatrix}, \tag{50b}$$

and $[\Psi_{\mathrm{a},\breve{E}_\mathrm{a}}]$ is partial derivative of Ψ_a w.r.t. $\breve{E}_\mathrm{a}$ at fixed E_a,

$$[\Psi_{\mathrm{a},\breve{E}_\mathrm{a}}] = \begin{bmatrix} \frac{\partial\Psi_1}{\partial\breve{E}_1} & \frac{\partial\Psi_1}{\partial\breve{E}_2} & \frac{\partial\Psi_1}{\partial\breve{E}_3} \\ \frac{\partial\Psi_2}{\partial\breve{E}_1} & \frac{\partial\Psi_2}{\partial\breve{E}_2} & \frac{\partial\Psi_2}{\partial\breve{E}_3} \\ \frac{\partial\Psi_3}{\partial\breve{E}_1} & \frac{\partial\Psi_3}{\partial\breve{E}_2} & \frac{\partial\Psi_3}{\partial\breve{E}_3} \end{bmatrix} \tag{51a}$$

$$= \begin{bmatrix} \frac{1}{\breve{\lambda}_1}\frac{\partial\Psi_1}{\partial\breve{\lambda}_1} & \frac{1}{\breve{\lambda}_2}\frac{\partial\Psi_1}{\partial\breve{\lambda}_2} & \frac{1}{\breve{\lambda}_3}\frac{\partial\Psi_1}{\partial\breve{\lambda}_3} \\ \frac{1}{\breve{\lambda}_1}\frac{\partial\Psi_2}{\partial\breve{\lambda}_1} & \frac{1}{\breve{\lambda}_2}\frac{\partial\Psi_2}{\partial\breve{\lambda}_2} & \frac{1}{\breve{\lambda}_3}\frac{\partial\Psi_2}{\partial\breve{\lambda}_3} \\ \frac{1}{\breve{\lambda}_1}\frac{\partial\Psi_3}{\partial\breve{\lambda}_1} & \frac{1}{\breve{\lambda}_2}\frac{\partial\Psi_3}{\partial\breve{\lambda}_2} & \frac{1}{\breve{\lambda}_3}\frac{\partial\Psi_3}{\partial\breve{\lambda}_3} \end{bmatrix} - \begin{bmatrix} \frac{\hat{\lambda}_1}{\breve{\lambda}_1^2}\frac{\partial\Psi_1}{\partial\hat{\lambda}_1} & \frac{\hat{\lambda}_2}{\breve{\lambda}_2^2}\frac{\partial\Psi_1}{\partial\hat{\lambda}_2} & \frac{\hat{\lambda}_3}{\breve{\lambda}_3^2}\frac{\partial\Psi_1}{\partial\hat{\lambda}_3} \\ \frac{\hat{\lambda}_1}{\breve{\lambda}_1^2}\frac{\partial\Psi_2}{\partial\hat{\lambda}_1} & \frac{\hat{\lambda}_2}{\breve{\lambda}_2^2}\frac{\partial\Psi_2}{\partial\hat{\lambda}_2} & \frac{\hat{\lambda}_3}{\breve{\lambda}_3^2}\frac{\partial\Psi_2}{\partial\hat{\lambda}_3} \\ \frac{\hat{\lambda}_1}{\breve{\lambda}_1^2}\frac{\partial\Psi_3}{\partial\hat{\lambda}_1} & \frac{\hat{\lambda}_2}{\breve{\lambda}_2^2}\frac{\partial\Psi_3}{\partial\hat{\lambda}_2} & \frac{\hat{\lambda}_3}{\breve{\lambda}_3^2}\frac{\partial\Psi_3}{\partial\hat{\lambda}_3} \end{bmatrix}. \tag{51b}$$

6.2 Off–Diagonal Components

The off–diagonal components $\dot{S}_\mathrm{O}$ and $\dot{\Psi}_\mathrm{O}$ are related to $\dot{E}_\mathrm{O}$ only because both $\mathbf{S}$ and $\boldsymbol{\Psi}$ are coaxial with $\mathbf{E}$. Then, they are expressed as

$$[\dot{S}_\mathrm{O}] = [\Delta S_\mathrm{a}/\Delta E_\mathrm{a}]\,[\dot{E}_\mathrm{O}], \tag{52a}$$

$$[\dot{\Psi}_\mathrm{O}] = [\Delta \Psi_\mathrm{a}/\Delta E_\mathrm{a}]\,[\dot{E}_\mathrm{O}], \tag{52b}$$

where

$$[\Delta S_\mathrm{a}/\Delta E_\mathrm{a}] = \begin{bmatrix} \frac{S_1 - S_2}{2(E_1 - E_2)} & 0 & 0 \\ 0 & \frac{S_2 - S_3}{2(E_2 - E_3)} & 0 \\ 0 & 0 & \frac{S_1 - S_3}{2(E_1 - E_3)} \end{bmatrix}, \tag{53a}$$

$$[\Delta \Psi_\mathrm{a}/\Delta E_\mathrm{a}] = \begin{bmatrix} \frac{\Psi_1 - \Psi_2}{2(E_1 - E_2)} & 0 & 0 \\ 0 & \frac{\Psi_2 - \Psi_3}{2(E_2 - E_3)} & 0 \\ 0 & 0 & \frac{\Psi_1 - \Psi_3}{2(E_1 - E_3)} \end{bmatrix}. \tag{53b}$$

Note that the representation of $[\Delta S_\mathrm{a}/\Delta E_\mathrm{a}]$ for the internal balance treatment (53a) is similar to its representation in conventional hyperelasticty (13) because this term is related to time derivative of eigenvectors $\mathbf{N}_\mathrm{a}$ and it has nothing to do with decomposition of the deformation gradient but S_a is defined by internal balance treatment (27). Care has to be taken during the calculation of $[\Delta S_\mathrm{a}/\Delta E_\mathrm{a}]$ and $[\Delta \Psi_\mathrm{a}/\Delta E_\mathrm{a}]$ to avoid dividing zero by zero which can be achieved by applying l'Hôpital rule.

6.3 Condensed Elasticity

The main task now is to determine elasticity tensor in similar manner to (16). Up to this end, $\dot{\mathbf{S}}$ expressed in (41c) can be reformed by the virtue of (9), $(41\mathrm{d})_1$, (45) and (52a) such as

$$\dot{\mathbf{S}}_\mathrm{V} = \begin{bmatrix} \dot{S}_\mathrm{D} \\ \dot{S}_\mathrm{O} \end{bmatrix} = \begin{bmatrix} [S_{\mathrm{a},\mathrm{E}_\mathrm{a}}] & 0 \\ 0 & [\Delta S_\mathrm{a}/\Delta E_\mathrm{a}] \end{bmatrix} \begin{bmatrix} \dot{E}_\mathrm{D} \\ \dot{E}_\mathrm{O} \end{bmatrix} + \begin{bmatrix} [S_{\mathrm{a},\breve{\mathrm{E}}_\mathrm{a}}] & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\breve{E}}_\mathrm{D} \\ \dot{\breve{E}}_\mathrm{O} \end{bmatrix}, \tag{54}$$

therefore it is required to eliminate the dependency of $\dot{\mathbf{S}}_\mathrm{V}$ upon $[\dot{\breve{E}}_\mathrm{D}]$. Starting from (41d), the internal balance condition $\dot{\boldsymbol{\Psi}} = \mathbf{0}$ gives

$$\dot{\mathbf{Diag}}(\Psi_\mathrm{a}) = -\boldsymbol{\Omega}\,\mathbf{Diag}(\Psi_\mathrm{a}) + \mathbf{Diag}(\Psi_\mathrm{a})\,\boldsymbol{\Omega}^\mathrm{T}. \tag{55}$$

It is worth to notice the equivalency of $\dot{\mathbf{Diag}}(\Psi_\mathrm{a})$ and $[\dot{\Psi}_\mathrm{D}]$, the components $\dot{\Psi}_\mathrm{a}$ are formed in matrix representation using $\dot{\mathbf{Diag}}(\Psi_\mathrm{a})$ as in (A.6) and in vector representing using $[\dot{\Psi}_\mathrm{D}]$ as in (43b). By the virtue of (49), (52b) and (55), the term $[\dot{\breve{E}}_\mathrm{D}]$ can be expressed as

$$\begin{aligned} &[\Psi_{\mathrm{a},\mathrm{E}_\mathrm{a}}]\,[\dot{E}_\mathrm{D}] + [\Psi_{\mathrm{a},\breve{\mathrm{E}}_\mathrm{a}}]\,[\dot{\breve{E}}_\mathrm{D}] = -[\Delta\Psi_\mathrm{a}/\Delta E_\mathrm{a}]\,[\dot{E}_\mathrm{O}] \Rightarrow \\ &[\dot{\breve{E}}_\mathrm{D}] = -[\Psi_{\mathrm{a},\breve{\mathrm{E}}_\mathrm{a}}]^{-1}\Big([\Psi_{\mathrm{a},\mathrm{E}_\mathrm{a}}]\,[\dot{E}_\mathrm{D}] + [\Delta\Psi_\mathrm{a}/\Delta E_\mathrm{a}]\,[\dot{E}_\mathrm{O}]\Big), \end{aligned} \tag{56}$$

where $[\Psi_{\mathrm{a},\breve{\mathrm{E}}_\mathrm{a}}]^{-1}$ is the inverse of matrix $[\Psi_{\mathrm{a},\breve{\mathrm{E}}_\mathrm{a}}]$. Substituting $(56)_2$ into (54) gives in Voigt notation

$$\dot{\mathbf{S}}_\mathrm{V} = \boldsymbol{\mathcal{C}}^\mathrm{P}_\mathrm{Con,V}\,\dot{\mathbf{E}}_\mathrm{V} \Rightarrow [\boldsymbol{\mathcal{C}}^\mathrm{P}_\mathrm{Con,V}] = \begin{bmatrix} \mathcal{C}^\mathrm{P}_\mathrm{Con,D} & 0 \\ 0 & \mathcal{C}^\mathrm{P}_\mathrm{Con,O} \end{bmatrix}, \tag{57}$$

where

$$\mathcal{C}^\mathrm{P}_\mathrm{Con,D} = [S_{\mathrm{a},\mathrm{E}_\mathrm{a}}] - [S_{\mathrm{a},\breve{\mathrm{E}}_\mathrm{a}}][\Psi_{\mathrm{a},\breve{\mathrm{E}}_\mathrm{a}}]^{-1}[\Psi_{\mathrm{a},\mathrm{E}_\mathrm{a}}], \tag{58a}$$

$$\mathcal{C}^\mathrm{P}_\mathrm{Con,O} = [\Delta S_\mathrm{a}/\Delta E_\mathrm{a}] - [S_{\mathrm{a},\breve{\mathrm{E}}_\mathrm{a}}][\Psi_{\mathrm{a},\breve{\mathrm{E}}_\mathrm{a}}]^{-1}[\Delta\Psi_\mathrm{a}/\Delta E_\mathrm{a}]. \tag{58b}$$

The condensed fourth order elasticity tensor in principal direction $\boldsymbol{\mathcal{C}}^\mathrm{P}_\mathrm{Con}$ is related to condensed elasticity in base frame $\boldsymbol{\mathcal{C}}_\mathrm{Con}$ via (18). Note that $\boldsymbol{\mathcal{C}}_\mathrm{Con}$ is introduced previously in terms of principal invariants [24] and it is presented again here in (22).

7 Blatz–Ko Model

A special case of the generalized Blatz–Ko [29, 30] in conventional hyperelasticity has the form of

$$W(I_1, I_3) = \mu(I_1 + 2I_3^{-1/2} - 5)/2, \tag{59}$$

and it can be reformulated in term of principal stretches by the use of (A.4) such as

$$W(\lambda_1,\lambda_2,\lambda_3)=\frac{\mu}{2}\big(\lambda_1^2+\lambda_2^2+\lambda_3^2+\frac{2}{\lambda_1\lambda_2\lambda_3}-5\big)\,, \tag{60}$$

where μ is the shear modulus. Substituting (60) into (5) gives the principal PK2 stress component

$$S_1=\mu\Big(1-\frac{1}{\lambda_1^3\lambda_2\lambda_3}\Big),\ S_2=\mu\Big(1-\frac{1}{\lambda_1\lambda_2^3\lambda_3}\Big),\ S_3=\mu\Big(1-\frac{1}{\lambda_1\lambda_2\lambda_3^3}\Big). \tag{61}$$

In analogy with (59), an internal balance Blatz–Ko material model is defined in [16] as

$$W(\hat{I}_1,\hat{I}_3,\breve{I}_1,\breve{I}_3)=\frac{\bar{\mu}}{2}(\hat{I}_1+2\hat{I}_3^{-1/2}-5)+\frac{\bar{\mu}}{2\beta}(\breve{I}_1+2\breve{I}_3^{-1/2}-5)\,. \tag{62}$$

where $\bar{\mu}=\mu(\beta+1)$. The positive material parameter β quantifies the contribution of two–factor multiplicative decomposition (1). The limits $\beta\to 0$ and $\beta\to\infty$ retrieve the hyperelastic behavior in full nonlinear strain range [16]. Note that (62) has the form of $W(\hat{I}_1,\ \hat{I}_3,\ \breve{I}_1,\ \breve{I}_2=0,\ \breve{I}_3)$, therefore tensors $\mathbf{E}$ and $\breve{\mathbf{E}}$ are coaxial as concluded by (35). Using principal stretches of $\hat{\mathbf{C}}$ and $\breve{\mathbf{C}}$, equation (62) is re-written as

$$\begin{aligned}W(\hat{\lambda}_1,\hat{\lambda}_2,\hat{\lambda}_3,\breve{\lambda}_1,\breve{\lambda}_2,\breve{\lambda}_3)=&\frac{\bar{\mu}}{2}\big(\hat{\lambda}_1^2+\hat{\lambda}_2^2+\hat{\lambda}_3^2+\frac{2}{\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3}-5\big)\\&+\frac{\bar{\mu}}{2\beta}\big(\breve{\lambda}_1^2+\breve{\lambda}_2^2+\breve{\lambda}_3^2+\frac{2}{\breve{\lambda}_1\breve{\lambda}_2\breve{\lambda}_3}-5\big).\end{aligned} \tag{63}$$

The principal $PK2$ stress components are obtained by substituting (63) into (27)

$$S_1=\frac{\bar{\mu}\,\hat{\mathcal{L}}_1}{\lambda_1^2}\,,\quad S_2=\frac{\bar{\mu}\,\hat{\mathcal{L}}_2}{\lambda_2^2}\,,\quad S_3=\frac{\bar{\mu}\,\hat{\mathcal{L}}_3}{\lambda_3^2}\,. \tag{64}$$

Substituting (63) into (32) gives the principal internal balance components

$$\Psi_1=\bar{\mu}\Big(\frac{\breve{\mathcal{M}}_1}{\beta}-\frac{\hat{\mathcal{L}}_1}{\breve{\lambda}_1^2}\Big),\quad \Psi_2=\bar{\mu}\Big(\frac{\breve{\mathcal{M}}_2}{\beta}-\frac{\hat{\mathcal{L}}_2}{\breve{\lambda}_2^2}\Big),\quad \Psi_3=\bar{\mu}\Big(\frac{\breve{\mathcal{M}}_3}{\beta}-\frac{\hat{\mathcal{L}}_3}{\breve{\lambda}_3^2}\Big), \tag{65}$$

where

$$\begin{aligned}&\hat{\mathcal{L}}_1=\hat{\lambda}_1^2-1/\hat{J},\ \hat{\mathcal{L}}_2=\hat{\lambda}_2^2-1/\hat{J},\ \hat{\mathcal{L}}_3=\hat{\lambda}_3^2-1/\hat{J},\ \hat{J}=\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3\,,\\&\breve{\mathcal{M}}_1=1-\frac{1}{\breve{\lambda}_1^3\breve{\lambda}_2\breve{\lambda}_3},\ \breve{\mathcal{M}}_2=1-\frac{1}{\breve{\lambda}_1\breve{\lambda}_2^3\breve{\lambda}_3},\ \breve{\mathcal{M}}_3=1-\frac{1}{\breve{\lambda}_1\breve{\lambda}_2\breve{\lambda}_3^3}\,.\end{aligned} \tag{66}$$

The condensed elasticity tensor in principal direction $\boldsymbol{\mathcal{C}}^{\mathrm{P}}_{\mathrm{Con,V}}$ as expressed in (57) depends on several 3×3 matrices (58a)–(58b) that are determined by the following procedure. Substituting (64) into (46b) and (47b), respectively, gives

$$[S_{\mathrm{a,E_a}}]=\frac{\bar{\mu}}{\hat{J}}\begin{bmatrix}3/\lambda_1^4 & 1/(\lambda_1^2\lambda_2^2) & 1/(\lambda_1^2\lambda_3^2)\\ 1/(\lambda_1^2\lambda_2^2) & 3/\lambda_2^4 & 1/(\lambda_2^2\lambda_3^2)\\ 1/(\lambda_1^2\lambda_3^2) & 1/(\lambda_2^2\lambda_3^2) & 3/\lambda_3^4\end{bmatrix}, \tag{67}$$

$$[S_{\mathrm{a},\breve{\mathrm{E}}_\mathrm{a}}] = -\bar{\mu}\begin{bmatrix} \hat{\mathcal{N}}_1/(\lambda_1^2\breve{\lambda}_1^2) & 1/(\lambda_1^2\breve{\lambda}_2^2\hat{J}) & 1/(\lambda_1^2\breve{\lambda}_3^2\hat{J}) \\ 1/(\lambda_2^2\breve{\lambda}_1^2\hat{J}) & \hat{\mathcal{N}}_2/(\lambda_2^2\breve{\lambda}_2^2) & 1/(\lambda_2^2\breve{\lambda}_3^2\hat{J}) \\ 1/(\lambda_3^2\breve{\lambda}_1^2\hat{J}) & 1/(\lambda_3^2\breve{\lambda}_2^2\hat{J}) & \hat{\mathcal{N}}_3/(\lambda_3^2\breve{\lambda}_3^2) \end{bmatrix}, \tag{68}$$

where

$$\hat{\mathcal{N}}_1 = 2\hat{\lambda}_1^2 + 1/\hat{J}\,, \quad \hat{\mathcal{N}}_2 = 2\hat{\lambda}_2^2 + 1/\hat{J}\,, \quad \hat{\mathcal{N}}_3 = 2\hat{\lambda}_3^2 + 1/\hat{J}\,. \tag{69}$$

Substituting (65) into (50b) and (51b), respectively, gives

$$[\Psi_{\mathrm{a},\mathrm{E}_\mathrm{a}}] = -\bar{\mu}\begin{bmatrix} \hat{\mathcal{N}}_1/(\lambda_1^2\breve{\lambda}_1^2) & 1/(\lambda_2^2\breve{\lambda}_1^2\hat{J}) & 1/(\lambda_3^2\breve{\lambda}_1^2\hat{J}) \\ 1/(\lambda_1^2\breve{\lambda}_2^2\hat{J}) & \hat{\mathcal{N}}_2/(\lambda_2^2\breve{\lambda}_2^2) & 1/(\lambda_3^2\breve{\lambda}_2^2\hat{J}) \\ 1/(\lambda_1^2\breve{\lambda}_3^2\hat{J}) & 1/(\lambda_2^2\breve{\lambda}_3^2\hat{J}) & \hat{\mathcal{N}}_3/(\lambda_3^2\breve{\lambda}_3^2) \end{bmatrix}, \tag{70}$$

$$[\Psi_{\mathrm{a},\breve{\mathrm{E}}_\mathrm{a}}] = \bar{\mu}\begin{bmatrix} \dfrac{2\hat{\mathcal{L}}_1}{\breve{\lambda}_1^4} + \dfrac{3}{\beta\breve{\lambda}_1^5\breve{\lambda}_2\breve{\lambda}_3} & \dfrac{1}{\beta\breve{\lambda}_1^3\breve{\lambda}_2^3\breve{\lambda}_3} & \dfrac{1}{\beta\breve{\lambda}_1^3\breve{\lambda}_2\breve{\lambda}_3^3} \\ \dfrac{1}{\beta\breve{\lambda}_1^3\breve{\lambda}_2^3\breve{\lambda}_3} & \dfrac{2\hat{\mathcal{L}}_2}{\breve{\lambda}_2^4} + \dfrac{3}{\beta\breve{\lambda}_1\breve{\lambda}_2^5\breve{\lambda}_3} & \dfrac{1}{\beta\breve{\lambda}_1\breve{\lambda}_2^3\breve{\lambda}_3^3} \\ \dfrac{1}{\beta\breve{\lambda}_1^3\breve{\lambda}_2\breve{\lambda}_3^3} & \dfrac{1}{\beta\breve{\lambda}_1\breve{\lambda}_2^3\breve{\lambda}_3^3} & \dfrac{2\hat{\mathcal{L}}_3}{\breve{\lambda}_3^4} + \dfrac{3}{\beta\breve{\lambda}_1\breve{\lambda}_2\breve{\lambda}_3^5} \end{bmatrix}$$
$$+\,\bar{\mu}\begin{bmatrix} \hat{\mathcal{N}}_1/\breve{\lambda}_1^4 & 1/(\breve{\lambda}_1^2\breve{\lambda}_2^2\hat{J}) & 1/(\breve{\lambda}_1^2\breve{\lambda}_3^2\hat{J}) \\ 1/(\breve{\lambda}_1^2\breve{\lambda}_2^2\hat{J}) & \hat{\mathcal{N}}_2/\breve{\lambda}_2^4 & 1/(\breve{\lambda}_2^2\breve{\lambda}_3^2\hat{J}) \\ 1/(\breve{\lambda}_1^2\breve{\lambda}_3^2\hat{J}) & 1/(\breve{\lambda}_2^2\breve{\lambda}_3^2\hat{J}) & \hat{\mathcal{N}}_3/\breve{\lambda}_3^4 \end{bmatrix}. \tag{71}$$

Next, $[\Psi_{\mathrm{a},\breve{\mathrm{E}}_\mathrm{a}}]^{-1}$ has to be calculated in order to determine $\mathcal{C}^{\mathrm{P}}_{\mathrm{Con,D}}$ as in (58a). Finally, $\mathcal{C}^{\mathrm{P}}_{\mathrm{Con,O}}$ as in (58b) is determined after calculating $[\Delta S_\mathrm{a}/\Delta E_\mathrm{a}]$ and $[\Delta\Psi_\mathrm{a}/\Delta E_\mathrm{a}]$ in (53a) and (53b).

8 Verification I: Uniaxial Loading

The elasticity tensor for uniaxial loading is caculated using principal stretches formulation as expressed in $(57)_2$. It is essential to perform this verification task because the eigenvectors of $\mathbf{C}$ and $\breve{\mathbf{C}}$ coincide with the directions of the base frame unit vectors during uniaxial loading. The deformation gradient for uniaxail loading is given as $\mathbf{F} = \lambda_1\,\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2\,\mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3\,\mathbf{e}_3 \otimes \mathbf{e}_3$. The principal component S_a in conventional hyperelasticity treatment is obtained by (61). Applying lateral traction free condition $S_2 = S_3 = 0$ gives $\lambda_2 = \lambda_3 = \lambda_1^{-1/4}$. This leads to $S_1 = \mu(1 - \lambda^{-5/2})$.

For internal balance treatment, S_a and Ψ_a are coupled with each other and are obtained by (64) and (65), respectively. Applying lateral traction free condition $S_2 = S_3 = 0$ gives $\hat{\lambda}_2 = \hat{\lambda}_3 = \hat{\lambda}_1^{-1/4}$. The internal balance scalar equations $\Psi_2 = \Psi_3 = 0$ provide that $\breve{\lambda}_2 = \breve{\lambda}_3 = \breve{\lambda}_1^{-1/4}$. By the virtue of (25), it can be shown that $\lambda_2 = \lambda_3 = \lambda_1^{-1/4}$. Now, S_1 and $\Psi_1 = 0$ are simplified such as

$$S_1 = \frac{\mu(1+\beta)\,\breve{\mathcal{L}}_1}{\lambda_1^2} = \frac{\mu(1+1/\beta)\,\breve{\mathcal{M}}_1\breve{\lambda}_1^2}{\lambda_1^2}, \tag{72a}$$

$$\breve{\mathcal{M}}_1 - \frac{\beta\breve{\mathcal{L}}_1}{\breve{\lambda}_1^2} = \frac{\breve{\mathcal{M}}_1}{\beta} - \frac{\breve{\mathcal{L}}_1}{\breve{\lambda}_1^2} = 0, \tag{72b}$$

where $\breve{\mathcal{L}} = \lambda_1^2 \breve{\lambda}_1^{-2} - \lambda_1^{-1/2} \breve{\lambda}_1^2$ and $\breve{\mathcal{M}} = 1 - \breve{\lambda}_1^{-5/2}$. The value of $\breve{\lambda}_1$ has to be determined by solving (72b) for given value of β and λ_1 in order to calculate S_1 as expressed in (72a). If $\beta \to 0$ then $(72b)_1$ gives $\breve{\mathcal{M}}_1 \to 0$ i.e. $\breve{\lambda}_1 \to 1$ then $(72a)_1$ becomes $S_1 = \mu(1 - \lambda^{-5/2})$. If $\beta \to \infty$ then $(72b)_2$ gives $\breve{\mathcal{L}}_1 \to 0$ i.e. $\breve{\lambda}_1 \to \lambda_1$ then $(72a)_2$ becomes $S_1 = \mu(1 - \lambda^{-5/2})$. Note that the calculated stress S_1 by internal balance treatment for both special cases $\beta \to 0$ and $\beta \to \infty$ retrieve the conventional hyperelastic response. If $\beta = 1$, an analytical solution can be achieved for (72b) such as $\hat{\lambda}_a = \breve{\lambda}_a = \lambda_a^{1/2}$. This special case represents an equal contribution of multiplicative decomposition counterparts to the stored energy function. Now, the uniaxial stress (72a) become $S_1 = 2\mu(\lambda_1^{-1} - \lambda_1^{-9/4})$. If β is not one of these special cases then $\breve{\lambda}_1$ has to determined numerically for given β and λ_1. The Reader can find more detail about uniaxial loading using internal balance material models in [16, 22].

Consider the special case of $\beta = 1$, the individual matrices in (58a) for uniaxial loadining become

$$[S_{\mathrm{a,E_a}}] = 2\mu \begin{bmatrix} 3\lambda_1^{-17/4} & \lambda_1^{-7/4} & \lambda_1^{-7/4} \\ \lambda_1^{-7/4} & 3\lambda_1^{3/4} & \lambda_1^{3/4} \\ \lambda_1^{-7/4} & \lambda_1^{3/4} & 3\lambda_1^{3/4} \end{bmatrix}, \tag{73a}$$

$$[S_{\mathrm{a,\breve{E}_a}}] = -2\mu \begin{bmatrix} c_1 & \lambda_1^{-2} & \lambda_1^{-2} \\ \lambda_1^{-3/4} & 3\lambda_1^{1/2} & \lambda_1^{1/2} \\ \lambda_1^{-3/4} & \lambda_1^{1/2} & 3\lambda_1^{1/2} \end{bmatrix}, \tag{73b}$$

$$[\Psi_{\mathrm{a,E_a}}] = -2\mu \begin{bmatrix} c_1 & \lambda_1^{-3/4} & \lambda_1^{-3/4} \\ \lambda_1^{-2} & 3\lambda_1^{1/2} & \lambda_1^{1/2} \\ \lambda_1^{-2} & \lambda_1^{1/2} & 3\lambda_1^{1/2} \end{bmatrix}, \tag{73c}$$

$$[\Psi_{\mathrm{a,\breve{E}_a}}] = 4\mu \begin{bmatrix} c_2 & \lambda_1^{-1} & \lambda_1^{-1} \\ \lambda_1^{-1} & 3\lambda_1^{1/4} & \lambda_1^{1/4} \\ \lambda_1^{-1} & \lambda_1^{1/4} & 3\lambda_1^{1/4} \end{bmatrix}, \tag{73d}$$

$$[\Psi_{\mathrm{a,\breve{E}_a}}]^{-1} = \frac{1}{8c_3\mu} \begin{bmatrix} 4\lambda_1^{9/4} & -\lambda_1 & -\lambda_1 \\ -\lambda_1 & c_4\lambda_1^{-1/4} & -\lambda_1 \\ -\lambda_1 & -\lambda_1 & c_4\lambda_1^{-1/4} \end{bmatrix}, \tag{73e}$$

now $\mathcal{C}^{\mathrm{P}}_{\mathrm{Con,D}}$ becomes

$$\mathcal{C}^{\mathrm{P}}_{\mathrm{Con,D}} = \begin{bmatrix} c_5 & 1/\lambda_1^{7/4} & 1/\lambda_1^{7/4} \\ 1/\lambda_1^{7/4} & 3\lambda_1^{3/4} & \lambda_1^{3/4} \\ 1/\lambda_1^{7/4} & \lambda_1^{3/4} & 3\lambda_1^{3/4} \end{bmatrix}, \tag{74}$$

where

$$\begin{aligned} & c_1 = 2\lambda_1^{-2} + \lambda_1^{-13/4}, \quad c_2 = 2\lambda_1^{-1} + \lambda_1^{-9/4}, \quad c3 = 4\lambda_1^{5/4} + 1, \\ & c_4 = 3\lambda_1^{5/4} + 1, \quad c_5 = (5 - 2\lambda_1^{5/4})/\lambda_1^{17/4}. \end{aligned} \tag{75}$$

The calculation of $[\Delta S_a/\Delta E_a]$ and $[\Delta\Psi_a/\Delta E_a]$ require the use of l'Hôpital rule to avoid the case of 0/0. This is observed in two cases because of $S_2 = S_3 = 0$, $\Psi_2 = \Psi_3 = 0$ and $E_2 = E_3$. Applying l'Hôpital rule for both cases gives

$$\frac{S_2 - S_3}{2(E_2 - E_3)} = \frac{1}{2}\left(\frac{\partial S_2}{\partial E_2} - \frac{\partial S_3}{\partial E_2}\right) = 2\mu\lambda_1^{3/4}, \tag{76a}$$

$$\frac{\Psi_2-\Psi_3}{2(E_2-E_3)}=\frac{1}{2}\left(\frac{\partial\Psi_2}{\partial E_2}-\frac{\partial\Psi_3}{\partial E_2}\right)=-2\mu\lambda_1^{1/2}. \tag{76b}$$

The remaining components $[\Delta S_{\mathrm{a}}/\Delta E_{\mathrm{a}}]$ and $[\Delta\Psi_{\mathrm{a}}/\Delta E_{\mathrm{a}}]$ are calculated by direct substitution such as

$$\begin{aligned}\frac{S_1-S_2}{2(E_1-E_2)}&=\frac{S_1-S_3}{2(E_1-E_3)}=\frac{2\mu(\lambda_1-\lambda_1^{-1/4})}{\lambda_1^2(\lambda_1^2-\lambda_1^{-1/2})}\\&=\frac{2\mu(\lambda_1-\lambda_1^{-1/4})}{\lambda_1^2(\lambda_1-\lambda_1^{-1/4})(\lambda_1+\lambda_1^{-1/4})}=\frac{2\mu}{\lambda_1^3+\lambda_1^{7/4}},\end{aligned} \tag{77a}$$

$$\frac{\Psi_1-\Psi_2}{2(E_1-E_2)}=\frac{\Psi_1-\Psi_3}{2(E_1-E_3)}=0. \tag{77b}$$

The final achieved results are

$$[\Delta S_{\mathrm{a}}/\Delta E_{\mathrm{a}}]=\begin{bmatrix}\dfrac{2\mu}{\lambda_1^3+\lambda_1^{7/4}}&0&0\\0&2\mu\lambda_1^{3/4}&0\\0&0&\dfrac{2\mu}{\lambda_1^3+\lambda_1^{7/4}}\end{bmatrix}, \tag{78a}$$

$$[\Delta\Psi_{\mathrm{a}}/\Delta E_{\mathrm{a}}]=\begin{bmatrix}0&0&0\\0&-2\mu\lambda_1^{1/2}&0\\0&0&0\end{bmatrix}. \tag{78b}$$

Now, $\mathcal{C}^{\mathrm{P}}_{\mathrm{Con,O}}$ is achieved by substituting (78a)–(78b), (73b) and (73e) into (58b) to give

$$\mathcal{C}^{\mathrm{P}}_{\mathrm{Con,O}}=\mu\begin{bmatrix}\dfrac{2}{\lambda_1^3+\lambda_1^{7/4}}&0&0\\0&\lambda_1^{3/4}&0\\0&0&\dfrac{2}{\lambda_1^3+\lambda_1^{7/4}}\end{bmatrix}; \tag{79}$$

then, $\boldsymbol{\mathcal{C}}^{\mathrm{P}}_{\mathrm{Con,V}}$ is obtained by substituting (74) and (79) into $(57)_2$ to give

$$[\boldsymbol{\mathcal{C}}^{\mathrm{P}}_{\mathrm{Con,V}}]=\begin{bmatrix}\mathcal{C}^{\mathrm{P}}_{\mathrm{Con,D}}&0\\0&\mathcal{C}^{\mathrm{P}}_{\mathrm{Con,O}}\end{bmatrix}=\mu\begin{bmatrix}c_5&1/\lambda_1^{7/4}&1/\lambda_1^{7/4}&0&0&0\\1/\lambda_1^{7/4}&3\lambda_1^{3/4}&\lambda_1^{3/4}&0&0&0\\1/\lambda_1^{7/4}&\lambda_1^{3/4}&3\lambda_1^{3/4}&0&0&0\\0&0&0&c_6&0&0\\0&0&0&0&\lambda_1^{3/4}&0\\0&0&0&0&0&c_6\end{bmatrix}, \tag{80}$$

where $c_6=2/(\lambda_1^3+\lambda_1^{7/4})$. Note that $\boldsymbol{\mathcal{C}}^{\mathrm{P}}_{\mathrm{Con,V}}$ as calculated in (80) is identical to $\boldsymbol{\mathcal{C}}_{\mathrm{Con,V}}$ (B.2) that is calculated using principal invariants because principal directions of $\mathbf{C}$ and $\breve{\mathbf{C}}$ coincide with the direction of the base frame unit vectors.

9 Verification II: Numerical Example

A general deformation is presented in this verification to ensure the correctness of the new procedure. It is given as

$$\mathbf{F} = \begin{bmatrix} 2.0 & 0 & 0 \\ 1.5 & 1.0 & 2.0 \\ 3.1 & 3.4 & 2.5 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 15.86 & 12.04 & 10.75 \\ 12.04 & 12.56 & 10.50 \\ 10.75 & 10.50 & 10.25 \end{bmatrix}. \tag{81}$$

The eigenvalues and eignenvectors of $\mathbf{C}$ are

$$\lambda_1^2 = 0.84136, \quad \lambda_2^2 = 2.4873, \quad \lambda_3^2 = 35.341, \tag{82a}$$

$$\mathbf{N}_1 = -1.0592 \times 10^{-2}\,\mathbf{e}_1 + 0.67330\,\mathbf{e}_2 - 0.73930\,\mathbf{e}_3, \tag{82b}$$

$$\mathbf{N}_2 = 0.76997\,\mathbf{e}_1 - 0.46623\,\mathbf{e}_2 - 0.43564\,\mathbf{e}_3, \tag{82c}$$

$$\mathbf{N}_3 = 0.63800\,\mathbf{e}_1 + 0.57385\,\mathbf{e}_2 + 0.51348\,\mathbf{e}_3\,. \tag{82d}$$

The material parameter β is set to one to get an analytical solution of the internal balance equation $\boldsymbol{\Psi} = \mathbf{0}$ such as $\breve{\mathbf{C}} = \sqrt{\mathbf{C}}$ leading to

$$\begin{aligned} \hat{\lambda}_1 &= \breve{\lambda}_1 = \lambda_1^{1/2} = 0.95774, \\ \hat{\lambda}_2 &= \breve{\lambda}_2 = \lambda_2^{1/2} = 1.2558, \\ \hat{\lambda}_3 &= \breve{\lambda}_3 = \lambda_3^{1/2} = 2.4382. \end{aligned} \tag{83}$$

Setting $\mu = 1$, then the individual matrices in (58a) become

$$[S_{\mathrm{a,E_a}}] = 10^{-3} \begin{bmatrix} 2890.2 & 325.89 & 22.936 \\ 325.89 & 330.71 & 7.7583 \\ 22.936 & 7.7583 & 1.6381 \end{bmatrix}, \tag{84a}$$

$$[S_{\mathrm{a,\breve{E}_a}}] = 10^{-2} \begin{bmatrix} -563.79 & -51.396 & -13.635 \\ -29.892 & -178.20 & -4.6122 \\ -2.1038 & -1.2236 & -11.643 \end{bmatrix}, \tag{84b}$$

$$[\Psi_{\mathrm{a,E_a}}] = 10^{-2} \begin{bmatrix} -563.79 & -29.892 & -2.1038 \\ -51.396 & -178.20 & -1.2236 \\ -13.635 & -4.6122 & -11.643 \end{bmatrix}, \tag{84c}$$

$$[\Psi_{\mathrm{a,\breve{E}_a}}] = \begin{bmatrix} 10.343 & 0.94287 & 0.25014 \\ 0.94287 & 5.6209 & 0.14548 \\ 0.25014 & 0.14548 & 1.3843 \end{bmatrix}, \tag{84d}$$

$$[\Psi_{\mathrm{a,\breve{E}_a}}]^{-1} = 10^{-2} \begin{bmatrix} 9.8544 & -1.6113 & -1.6113 \\ -1.6113 & 18.103 & -1.6113 \\ -1.6113 & -1.6113 & 72.699 \end{bmatrix}, \tag{84e}$$

then, $\mathcal{C}^{\mathrm{P}}_{\mathrm{Con,D}}$ is calculated

$$\mathcal{C}^{\mathrm{P}}_{\mathrm{Con,D}} = 10^{-3} \begin{bmatrix} -182.98 & 162.94 & 11.468 \\ 162.94 & -234.25 & 3.8792 \\ 11.468 & 3.8792 & -8.1542 \end{bmatrix}. \tag{85}$$

The component S_a and E_a are

$$S_1 = 1.3698, \quad S_2 = 0.99394, \quad S_3 = 0.31713, \tag{86}$$

$$E_1 = -7.9318 \times 10^{-2}, \quad E_2 = 0.74365, \quad E_3 = 17.171, \tag{87}$$

while the internal balance equation $\boldsymbol{\Psi} = \mathbf{0} \to \Psi_a = 0$ hence $[\Delta\Psi_a/\Delta E_a] = 0$. Consequently, (58b) is simplified to $\mathcal{C}^{P}_{Con,O} = [\Delta S_a/\Delta E_a]$ and it is given as:

$$\mathcal{C}^{P}_{Con,O} = [\Delta S_a/\Delta E_a] = 10^{-2} \begin{bmatrix} -22.837 & 0 & 0 \\ 0 & 2.0601 & 0 \\ 0 & 0 & -3.0513 \end{bmatrix}; \tag{88}$$

$\boldsymbol{\mathcal{C}}^{P}_{Con,V}$ as presented in (57) is assembled using (85) and (89):

$$\boldsymbol{\mathcal{C}}^{P}_{Con,V} = 10^{-3} \begin{bmatrix} -182.98 & 162.94 & 11.468 & 0 & 0 & 0 \\ 162.94 & -234.25 & 3.8792 & 0 & 0 & 0 \\ 11.468 & 3.8792 & -8.1542 & 0 & 0 & 0 \\ 0 & 0 & 0 & -228.37 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20.601 & 0 \\ 0 & 0 & 0 & 0 & 0 & -30.513 \end{bmatrix}. \tag{89}$$

Now, $\boldsymbol{\mathcal{C}}_{Con}$ as expressed in base frame is achieved by transforming $\boldsymbol{\mathcal{C}}^{P}_{Con}$ from principal directions via (18) such as

$$[\boldsymbol{\mathcal{C}}_{Con,V}] = 10^{-2} \begin{bmatrix} -10.174 & 2.4533 & 4.0158 & 4.7388 & -6.8516 & 4.1830 \\ 2.4533 & -12.760 & 8.6890 & 5.9487 & 3.1511 & -7.5777 \\ 4.0158 & 8.6890 & -14.263 & -7.9974 & 2.9696 & 5.8854 \\ 4.7388 & 5.9487 & -7.9974 & -9.9987 & 3.8526 & 4.3129 \\ -6.8516 & 3.1511 & 2.9696 & 3.8526 & -9.7281 & 4.8045 \\ 4.1830 & -7.5777 & 5.8854 & 4.3129 & 4.8045 & -10.892 \end{bmatrix}. \tag{90}$$

Note that $[\boldsymbol{\mathcal{C}}_{Con,V}]$ that is calculated by the new approach (90) is identical to $[\boldsymbol{\mathcal{C}}_{Con,V}]$ as presented in (C.5) that is calculated based on the principal invariants formulation [24].

10 Conclusions

A new scheme for hyperelastic material is developed based on applying the argument of calculus variation to two-factor multiplicative decomposition of the deformation gradient. This work develops the procedure to calculate the elasticity tensor for material model formulated in terms of decomposed principal stretches. Focus is given for special classes of material model that grant the coaxiality of $\mathbf{S}$, $\boldsymbol{\Psi}$, $\mathbf{E}$ and $\check{\mathbf{E}}$. The new procedure is demonstrated using two loading scenarios namely uniaxial loading and general deformation. The calculated elasticity tensor by the new procedure is verified by the elasticity tensor that is calculated based on principal invariant formulations that has been introduced in [24].

Appendix A: Essentials and Notation

Let $\mathbf{A}$ be a second order symmetric tensor that is defined in the undeformed configuration using base frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The principal invariants of $\mathbf{A}$ are

$$I_1 = \text{tr}(\mathbf{A}), \quad I_2 = \frac{1}{2}\big[(\text{tr}(\mathbf{A}))^2 - \text{tr}(\mathbf{A}^2)\big], \quad I_3 = \det(\mathbf{A})\,, \tag{A.1}$$

and their partial derivatives with respect to $\mathbf{A}$

$$\frac{\partial I_1}{\partial \mathbf{A}} = \mathbf{I}, \quad \frac{\partial I_2}{\partial \mathbf{A}} = I_1\mathbf{I} - \mathbf{A}, \quad \frac{\partial I_3}{\partial \mathbf{A}} = I_3\mathbf{A}^{-1}\,. \tag{A.2}$$

Tensor $\mathbf{A}$ can be expressed in principal directions using its eigenvalues A_a and unit eigenvectors $\mathbf{N}_\text{a}$ pairs such as

$$\mathbf{A}^\text{P} = A_\text{a}\mathbf{N}_\text{a} \otimes \mathbf{N}_\text{a}\,, \tag{A.3}$$

where $a = 1, 2, 3$. Principal invariants and eigenvalues are related via

$$I_1 = A_1 + A_2 + A_3, \quad I_2 = A_1A_2 + A_2A_3 + A_1A_3, \quad I_3 = A_1A_2A_3. \tag{A.4}$$

It is important to know that

$$\mathbf{A}^\alpha\,\mathbf{N}_\text{a} = A_\text{a}^\alpha\,\mathbf{N}_\text{a}\,, \tag{A.5}$$

where $\alpha = \pm1, \pm2, \ldots$.

Following the notation of Crisfield in [31], $\mathbf{A}^\text{P}$ is expressed such as

$$\mathbf{A}^\text{P} = \mathbf{Q}\,\mathbf{Diag}(A_\text{a})\,\mathbf{Q}^\text{T} = A_1\,\mathbf{N}_1 \otimes \mathbf{N}_1 + A_2\,\mathbf{N}_2 \otimes \mathbf{N}_2 + A_3\,\mathbf{N}_3 \otimes \mathbf{N}_3\,, \tag{A.6}$$

where tensor $\mathbf{Q}$ is orthogonal tensor $\mathbf{Q}\mathbf{Q}^\text{T} = \mathbf{Q}^\text{T}\mathbf{Q} = \mathbf{I}$. The matrix representation of $\mathbf{Q}$ and $\mathbf{Diag}(A_\text{a})$ are

$$[\mathbf{Q}] = [\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3]_{1\times3}\,, \tag{A.7}$$

$$[\mathbf{Diag}(A_\text{a})] = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}_{3\times3}\,. \tag{A.8}$$

In order to perform (A.6), $[\mathbf{Q}]$ has the size of 1×3 and $[\mathbf{Diag}(A_\text{a})]$ has the size of 3×3. Tensors $\mathbf{A}^\text{P}$ and $\mathbf{A}$ are related via

$$\mathbf{A} = \mathbf{Q}^\text{T}\,\mathbf{A}^\text{P}\,\mathbf{Q}\,. \tag{A.9}$$

To calculate (A.9), $\mathbf{Q}$ must have the following order

$$[\mathbf{Q}] = \begin{bmatrix} \mathbf{N}_1(1) & \mathbf{N}_1(2) & \mathbf{N}_1(3) \\ \mathbf{N}_2(1) & \mathbf{N}_2(2) & \mathbf{N}_2(3) \\ \mathbf{N}_3(1) & \mathbf{N}_3(2) & \mathbf{N}_3(3) \end{bmatrix}_{3\times3}\,, \tag{A.10}$$

where each eigenvector $\mathbf{N}_\text{a}$ has three components $\{\mathbf{N}_\text{a}(1), \mathbf{N}_\text{a}(2), \mathbf{N}_\text{a}(3)\}$ e.g. the first eignvector $\mathbf{N}_1$ is written as $\mathbf{N}_1 = \mathbf{N}_1(1)\,\mathbf{e}_1 + \mathbf{N}_1(2)\,\mathbf{e}_2 + \mathbf{N}_1(3)\,\mathbf{e}_3$. Keep in mind that $\mathbf{Q}$ is often expressed by sorting each eigenvector as a column (not as a row) then $\mathbf{A} = \mathbf{Q}\,\mathbf{A}^{\mathrm{P}}\,\mathbf{Q}^{\mathrm{T}}$ however this notation is not used to avoid confusion with the representation of Crisfield in [31].

Tensor $\mathbf{A}$ is considered here as function of Green–Lagrange strain $\mathbf{E}$. Also, $\mathbf{A}$ and $\mathbf{E}$ are coaxial i.e. have same eigenvectors $\mathbf{N}_{\mathrm{a}}$. The derivative of $\mathbf{A}^{\mathrm{P}}$ with respect to time $\dot{\mathbf{A}}^{\mathrm{P}}$ is given as

$$\dot{\mathbf{A}}^{\mathrm{P}} = \mathbf{Q}\,\dot{\mathbf{Diag}}(A_{\mathrm{a}})\,\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\,\mathbf{Diag}(A_{\mathrm{a}})\,\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\,\mathbf{Diag}(A_{\mathrm{a}})\,\dot{\mathbf{Q}}^{\mathrm{T}}\,, \tag{A.11}$$

where the matrix representation of $\dot{\mathbf{Diag}}(A_{\mathrm{a}})$ is

$$[\dot{\mathbf{Diag}}(A_{\mathrm{a}})] = \begin{bmatrix} \dot{A}_1 & 0 & 0 \\ 0 & \dot{A}_2 & 0 \\ 0 & 0 & \dot{A}_3 \end{bmatrix}. \tag{A.12}$$

Subsituting (A.11) into (A.9) gives $\dot{\mathbf{A}}$ as

$$\dot{\mathbf{A}} = \dot{\mathbf{Diag}}(A_{\mathrm{a}}) + \Omega\,\mathbf{Diag}(A_{\mathrm{a}}) - \mathbf{Diag}(A_{\mathrm{a}})\,\mathbf{\Omega}\,, \tag{A.13}$$

where the derivative of $\mathbf{Q}$ w.r.t. time is given by antisymmetric $\mathbf{\Omega}$ tensor

$$\mathbf{\Omega}^{\mathrm{P}} = \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}\,, \quad \mathbf{\Omega} = \mathbf{Q}^{\mathrm{T}}\dot{\mathbf{Q}} = -\dot{\mathbf{Q}}^{\mathrm{T}}\mathbf{Q}\,, \tag{A.14}$$

that can be defined by mixed combination of components related Green–Lagrange strain and its derivative w.r.t. to time such as

$$[\mathbf{\Omega}] = \begin{bmatrix} 0 & \dfrac{\dot{E}_{12}}{E_2 - E_1} & \dfrac{\dot{E}_{13}}{E_3 - E_1} \\ \dfrac{\dot{E}_{12}}{E_1 - E_2} & 0 & \dfrac{\dot{E}_{23}}{E_3 - E_2} \\ \dfrac{\dot{E}_{13}}{E_1 - E_3} & \dfrac{\dot{E}_{23}}{E_2 - E_3} & 0 \end{bmatrix}. \tag{A.15}$$

Now, $\dot{\mathbf{A}}$ can be written as

$$[\dot{\mathbf{A}}] = \begin{bmatrix} \dot{A}_{11} & \dot{A}_{11} & \dot{A}_{13} \\ \dot{A}_{21} & \dot{A}_{2} & \dot{A}_{23} \\ \dot{A}_{31} & \dot{A}_{32} & \dot{A}_{33} \end{bmatrix} = \begin{bmatrix} \dot{A}_1 & \dfrac{(A_1 - A_2)\dot{E}_{12}}{E_1 - E_2} & \dfrac{(A_1 - A_3)\dot{E}_{13}}{E_1 - E_3} \\ \dfrac{(A_1 - A_2)\dot{E}_{12}}{E_1 - E_2} & \dot{A}_2 & \dfrac{(A_2 - A_3)\dot{E}_{23}}{E_2 - E_3} \\ \dfrac{(A_1 - A_3)\dot{E}_{13}}{E_1 - E_3} & \dfrac{(A_2 - A_3)\dot{E}_{23}}{E_2 - E_3} & \dot{A}_3 \end{bmatrix}. \tag{A.16}$$

The above equation make use of (A.8), (A.12) and (A.15). The Voigt notation can be used to reform $\dot{\mathbf{A}}$ into $\dot{\mathbf{A}}_{\mathrm{V}}$

$$[\dot{\mathbf{A}}_{\mathrm{V}}] = \begin{bmatrix} \dot{A}_{11} & \dot{A}_{22} & \dot{A}_{33} & \dot{A}_{12} & \dot{A}_{23} & \dot{A}_{13} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \dot{A}_{\mathrm{D}} \\ \dot{A}_{\mathrm{O}} \end{bmatrix}, \tag{A.17}$$

here $\dot{A}_{\mathrm{D}}$ is the diagonal components of $\dot{\mathbf{A}}$

$$[\dot{A}_{\mathrm{D}}] = \begin{bmatrix} \dot{A}_{11} & \dot{A}_{22} & \dot{A}_{33} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \dot{A}_1 & \dot{A}_2 & \dot{A}_3 \end{bmatrix}^{\mathrm{T}}, \tag{A.18}$$

and $\dot{A}_{\mathrm{O}}$ is the off–diagonal components of $\dot{\mathbf{A}}$

$$[\dot{A}_{\mathrm{O}}] = \begin{bmatrix} \dot{A}_{12} & \dot{A}_{23} & \dot{A}_{13} \end{bmatrix}^{\mathrm{T}} . \tag{A.19}$$

Using chain rule, $\dot{A}_{\mathrm{D}}$ can be expressed as

$$[\dot{A}_{\mathrm{D}}] = [A_{\mathrm{a,E_a}}]\,[\dot{E}_{\mathrm{D}}] , \tag{A.20}$$

where

$$[\dot{E}_{\mathrm{D}}] = \begin{bmatrix} \dot{E}_{11} & \dot{E}_{22} & \dot{E}_{33} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \dot{E}_{1} & \dot{E}_{2} & \dot{E}_{3} \end{bmatrix}^{\mathrm{T}} , \tag{A.21}$$

$$[A_{\mathrm{a,E_a}}] = \begin{bmatrix} \dfrac{\partial A_1}{\partial E_1} & \dfrac{\partial A_1}{\partial E_2} & \dfrac{\partial A_1}{\partial E_3} \\ \dfrac{\partial A_2}{\partial E_1} & \dfrac{\partial A_2}{\partial E_2} & \dfrac{\partial A_2}{\partial E_3} \\ \dfrac{\partial A_3}{\partial E_1} & \dfrac{\partial A_3}{\partial E_2} & \dfrac{\partial A_3}{\partial E_3} \end{bmatrix} . \tag{A.22}$$

The off–diagonal components of $\dot{\mathbf{A}}$ can be written as

$$[\dot{A}_{\mathrm{O}}] = [\Delta A_a/\Delta E_a][\dot{E}_{\mathrm{O}}] , \tag{A.23}$$

where

$$[\dot{E}_{\mathrm{O}}] = \begin{bmatrix} 2\dot{E}_{12} & 2\dot{E}_{23} & 2\dot{E}_{13} \end{bmatrix}^{\mathrm{T}} , \tag{A.24}$$

$$[\Delta A_a/\Delta E_a] = \begin{bmatrix} \dfrac{A_1 - A_2}{2(E_1 - E_2)} & 0 & 0 \\ 0 & \dfrac{A_2 - A_3}{2(E_2 - E_3)} & 0 \\ 0 & 0 & \dfrac{A_1 - A_3}{2(E_1 - E_3)} \end{bmatrix} . \tag{A.25}$$

In case $E_{\mathrm{a}} = E_{\mathrm{b}}$ and $A_{\mathrm{a}} = A_{\mathrm{b}}$, then l'Hôpital rule is applied to obtained

$$\lim_{E_{\mathrm{a}} \to E_{\mathrm{b}}} \frac{A_{\mathrm{a}} - A_{\mathrm{b}}}{E_{\mathrm{a}} - E_{\mathrm{b}}} = \frac{\partial A_a}{\partial E_a} - \frac{\partial A_b}{\partial E_a} . \tag{A.26}$$

Extensive explanations of these derivatives can be found in [27, 31].

Appendix B: Uniaxial Loading: Principal Invariants

Consider the uniaxial loading of internally balanced material model (59) with $\beta = 1$. Then, $\mathbf{C} = \lambda_1^2\,\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_1^{-1/2}\,\mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_1^{-1/2}\,\mathbf{e}_3 \otimes \mathbf{e}_3$ and $\check{\mathbf{C}} = \lambda_1\,\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_1^{-1/4}\,\mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_1^{-1/4}\,\mathbf{e}_3 \otimes \mathbf{e}_3$. The individual fourth order tensors in (22) are determined by equations (59a)–(59d) in

[24]. In Voigt notation, they are expressed as

$$\mathcal{C}_{\mathrm{E,V}} = 2\mu \begin{bmatrix} 3\lambda_1^{-17/4} & \lambda_1^{-7/4} & \lambda_1^{-7/4} & 0 & 0 & 0 \\ \lambda_1^{-7/4} & 3\lambda_1^{3/4} & \lambda_1^{3/4} & 0 & 0 & 0 \\ \lambda_1^{-7/4} & \lambda_1^{3/4} & 3\lambda_1^{3/4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^{-7/4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^{3/4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^{-7/4} \end{bmatrix}, \tag{B.1a}$$

$$\mathcal{C}_{\breve{\mathrm{E}},\mathrm{V}} = -2\mu \begin{bmatrix} c_1 & \lambda_1^{-2} & \lambda_1^{-2} & 0 & 0 & 0 \\ \lambda_1^{-3/4} & 3\lambda_1^{1/2} & \lambda_1^{1/2} & 0 & 0 & 0 \\ \lambda_1^{-3/4} & \lambda_1^{1/2} & 3\lambda_1^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^{-3/4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^{1/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^{-3/4} \end{bmatrix}, \tag{B.1b}$$

$$\boldsymbol{\psi}_{\mathrm{E,V}} = -2\mu \begin{bmatrix} c_1 & \lambda_1^{-3/4} & \lambda_1^{-3/4} & 0 & 0 & 0 \\ \lambda_1^{-2} & 3\lambda_1^{1/2} & \lambda_1^{1/2} & 0 & 0 & 0 \\ \lambda_1^{-2} & \lambda_1^{1/2} & 3\lambda_1^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^{-3/4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^{1/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^{-3/4} \end{bmatrix}, \tag{B.1c}$$

$$\boldsymbol{\psi}_{\breve{\mathrm{E}},\mathrm{V}} = 4\mu \begin{bmatrix} c_2 & \lambda_1^{-1} & \lambda_1^{-1} & 0 & 0 & 0 \\ \lambda_1^{-1} & 3\lambda_1^{1/4} & \lambda_1^{1/4} & 0 & 0 & 0 \\ \lambda_1^{-1} & \lambda_1^{1/4} & 3\lambda_1^{1/4} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_7\lambda_1^{-1}/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^{1/4} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_7\lambda_1^{-1}/2 \end{bmatrix}, \tag{B.1d}$$

$$\boldsymbol{\psi}^{-1}_{\breve{\mathrm{E}},\mathrm{V}} = \frac{1}{8c_3\mu} \begin{bmatrix} 4\lambda_1^{9/4} & -\lambda_1 & -\lambda_1 & 0 & 0 & 0 \\ -\lambda_1 & c_4\lambda_1^{-1/4} & -\lambda_1 & 0 & 0 & 0 \\ -\lambda_1 & -\lambda_1 & c_4\lambda_1^{-1/4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 4c_3\lambda_1/c & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_3\lambda_1^{-1/4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 4c_3\lambda_1/c \end{bmatrix}. \tag{B.1e}$$

Now, $\mathcal{C}^{\mathrm{con}}$ as in (22) becomes

$$\mathcal{C}_{\mathrm{Con,V}} = \mu \begin{bmatrix} c_5 & 1/\lambda_1^{7/4} & 1/\lambda_1^{7/4} & 0 & 0 & 0 \\ 1/\lambda_1^{7/4} & 3\lambda_1^{3/4} & \lambda_1^{3/4} & 0 & 0 & 0 \\ 1/\lambda_1^{7/4} & \lambda_1^{3/4} & 3\lambda_1^{3/4} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^{3/4} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_6 \end{bmatrix}, \tag{B.2}$$

where $c_6 = 2/(\lambda_1^3 + \lambda_1^{7/4})$ and $c_7 = \lambda_1^{5/4} + 1$ while c_1, c_2, c_3, c_4 and c_5 are defined in (75).

Appendix C: Numerical Example: Principal Invariant

Consider the general deformation given in (81) then $I_3 = J^2 = 73.96$. Setting $\beta = 1$ for internal balance material model (59) to obtain an analytical solution of $\mathbf{\Psi} = \mathbf{0}$ such as $\check{\mathbf{C}} = \sqrt{\mathbf{C}}$

$$\check{\mathbf{C}} = \begin{bmatrix} 3.3549 & 1.6038 & 1.4257 \\ 1.6038 & 2.7163 & 1.6154 \\ 1.4257 & 1.6154 & 2.3681 \end{bmatrix}, \tag{C.1}$$

with $\check{I}_3 = \check{J}^2 = I_3^{1/2} = 8.6$. The inverse of tensors $\mathbf{C}$ and $\check{\mathbf{C}}$ is

$$\mathbf{C}^{-1} = \begin{bmatrix} 0.25000 & -0.14244 & -0.11628 \\ -0.14244 & 0.63551 & -0.50162 \\ -0.11628 & -0.50162 & 0.73337 \end{bmatrix}, \tag{C.2}$$

$$\check{\mathbf{C}}^{-1} = \begin{bmatrix} 0.44450 & -0.17381 & -0.14904 \\ -0.17381 & 0.68744 & -0.36432 \\ -0.14904 & -0.36432 & 0.76055 \end{bmatrix}. \tag{C.3}$$

Set $\mu = 1$, then the individual fourth order tensors in (22) are determined by equations (59a)–(59d) in [24]. In Voigt notation, they are expressed as

$$\mathcal{C}_{\mathbf{E},\mathrm{V}} = 10^{-3} \begin{bmatrix} 127.87 & 136.03 & 143.48 & -72.858 & -62.934 & -59.476 \\ 136.03 & 826.32 & 661.07 & -185.21 & -652.23 & 47.063 \\ 143.48 & 661.07 & 1100.4 & 8.3162 & -752.67 & -174.47 \\ -72.858 & -185.21 & 8.3162 & 136.03 & 47.063 & -62.934 \\ -62.934 & -652.23 & -752.67 & 47.063 & 661.07 & 8.3162 \\ -59.476 & 47.063 & -174.47 & -62.934 & 8.3162 & 143.48 \end{bmatrix}, \tag{C.4a}$$

$$\mathcal{C}_{\check{\mathbf{E}},\mathrm{V}} = 10^{-2} \begin{bmatrix} -86.610 & -23.804 & -21.852 & 33.866 & -4.1502 & 29.040 \\ -31.349 & -218.83 & -86.054 & 55.326 & 115.97 & -18.869 \\ -31.117 & -87.474 & -269.41 & -13.026 & 129.05 & 52.795 \\ 35.221 & 54.471 & -14.331 & -68.843 & 4.2880 & 25.759 \\ 4.8446 & 123.70 & 136.85 & 1.8811 & -143.57 & 10.479 \\ 30.024 & -19.877 & 51.372 & 25.828 & 12.689 & -73.237 \end{bmatrix}, \tag{C.4b}$$

$$\boldsymbol{\psi}_{\mathbf{E},\mathrm{V}} = 10^{-2} \begin{bmatrix} -86.610 & -31.349 & -31.117 & 35.221 & 4.8446 & 30.024 \\ -23.804 & -218.83 & -87.474 & 54.471 & 123.70 & -19.877 \\ -21.852 & -86.054 & -269.41 & -14.331 & 136.85 & 51.372 \\ 33.866 & 55.326 & -13.026 & -68.843 & 1.8811 & 25.828 \\ -4.1502 & 115.97 & 129.05 & 4.2880 & -143.57 & 12.689 \\ 29.040 & -18.869 & 52.795 & 25.759 & 10.479 & -73.237 \end{bmatrix}, \tag{C.4c}$$

$$\boldsymbol{\psi}_{\breve{\mathrm{E}},\mathrm{V}} = \begin{bmatrix} 3.8255 & 0.41679 & 0.46111 & -0.80061 & -0.22088 & -0.68652 \\ 0.41679 & 6.1441 & 0.71313 & -0.85821 & -1.7989 & -0.13975 \\ 0.46111 & 0.71313 & 6.8733 & -0.18030 & -1.8352 & -0.75077 \\ -0.80061 & -0.85821 & -0.18030 & 2.3051 & -0.21171 & -0.69330 \\ -0.22088 & -1.7989 & -1.8352 & -0.21171 & 3.0770 & -0.27355 \\ -0.68652 & -0.13975 & -0.75077 & -0.69330 & -0.27355 & 2.4404 \end{bmatrix}, \tag{C.4d}$$

$$\boldsymbol{\psi}^{-1}_{\breve{\mathrm{E}},\mathrm{V}} = 10^{-2} \begin{bmatrix} 32.672 & 2.6699 & 1.7599 & 17.940 & 7.5979 & 15.834 \\ 2.6699 & 22.984 & 3.4653 & 14.097 & 17.474 & 9.0970 \\ 1.7599 & 3.4653 & 19.542 & 8.0582 & 15.314 & 10.711 \\ 17.940 & 14.097 & 8.0582 & 66.374 & 21.534 & 29.603 \\ 7.5979 & 17.474 & 15.314 & 21.534 & 55.672 & 20.208 \\ 15.834 & 9.0970 & 10.711 & 29.603 & 20.208 & 59.923 \end{bmatrix}. \tag{C.4e}$$

Now, $\boldsymbol{\mathcal{C}}^{\text{con}}$ as in (22) becomes

$$\boldsymbol{\mathcal{C}}_{\text{Con},\mathrm{V}} = 10^{-2} \begin{bmatrix} -10.174 & 2.4533 & 4.0158 & 4.7388 & -6.8516 & 4.1830 \\ 2.4533 & -12.760 & 8.6890 & 5.9487 & 3.1511 & -7.5777 \\ 4.0158 & 8.6890 & -14.263 & -7.9974 & 2.9696 & 5.8854 \\ 4.7388 & 5.9487 & -7.9974 & -9.9987 & 3.8526 & 4.3129 \\ -6.8516 & 3.1511 & 2.9696 & 3.8526 & -9.7281 & 4.8045 \\ 4.1830 & -7.5777 & 5.8854 & 4.3129 & 4.8045 & -10.892 \end{bmatrix}. \tag{C.5}$$

Acknowledgements The author would like to acknowledges Palestine Technical University – Kadoorie for supporting this research.

Author contributions This is a manuscript prepared by a single author. The author have no relevant financial or non-financial interests to disclose

Declarations

Competing interests The authors declare no competing interests.

References

1. Bilby, B.A., Gardner, L.R.T., Stroh, A.N.: Continuous distributions of dislocations and theory of plasticity. In: Actes du Xième Congrès International de Mécanique Appliquée, vol. VIII, pp. 35–44. Université de Bruxelles, Belgium (1957)
2. Kröner, E.: Allgemeine kontinuumstheorie der versetzungen und eigenspannungen. Arch. Ration. Mech. Anal. **4**, 273–334 (1959)
3. Lee, E.H.: Elastic plastic deformation at finite strain. J. Appl. Mech. **36**, 1–6 (1969)
4. Simo, J.C.: A framework for finite strain elastoplasticity based on maximum plastic dissipation and the multiplicative decomposition. Part I: continuum formulation. Comput. Methods Appl. Mech. Eng. **66**, 199–219 (1988)
5. Piero, G.D.: On the decomposition of the deformation gradient in plasticity. J. Elast. **131**, 111–124 (2018)
6. Bruhns, O.T.: Large deformation plasticity. Acta Mech. Sin. **36**, 472–492 (2020)
7. Shutov, A.V., Landgraf, R., Ihlemann, J.: An explicit solution for implicit time stepping in multiplicative finite strain viscoelasticity. Comput. Methods Appl. Mech. Eng. **265**, 213–225 (2013)

8. Lengger, M., Possart, G., Steinmann, P.: A viscoelastic mooney–Rivlin model for adhesive curing and first steps toward its calibration based on photoelasticity measurements. Arch. Appl. Mech. (2022). https://doi.org/10.1007/s00419-022-02273-4
9. Bahreman, M., Darijani, H., Narooei, K.: Investigation of multiplicative decompositions in the form of fefv and fvfe to extend viscoelasticity laws from small to finite deformations. Mech. Mater. (2022). https://doi.org/10.1016/j.mechmat.2022.104235
10. Dunić, V., Slavković, R.: Implicit stress integration procedure for large strains of the reformulated shape memory alloys material model. Contin. Mech. Thermodyn. **32**, 1287–1309 (2020)
11. Zhao, W., Liu, L., Leng, J., Liu, Y.: Thermo-mechanical behavior prediction of shape memory polymer based on the multiplicative decomposition of the deformation gradient. Mech. Mater. (2019). https://doi.org/10.1016/j.mechmat.2019.103263
12. Wang, J., Gu, X., Xu, Y., Zhu, J., Zhang, W.: Thermomechanical modeling of nonlinear internal hysteresis due to incomplete phase transformation in pseudoelastic shape memory alloys. Nonlinear Dyn. **103**, 1393–1414 (2021)
13. Goriely, A., Amar, M.B.: On the definition and modeling of incremental, cumulative, and continuous growth laws in morphoelasticity. Biomech. Model. Mechanobiol. **6**, 289–296 (2007)
14. Demirkoparan, H., Pence, T.J., Tsai, H.: Hyperelastic internal balance by multiplicative decomposition of the deformation gradient. Arch. Ration. Mech. Anal. **214**, 923–970 (2014)
15. Zamani, V., Demirkoparan, H., Pence, T.J.: Material swelling with partial confinement in the internally balanced generalization of hyperelasticity. Math. Mech. Solids (2022). https://doi.org/10.1177/10812865221092377
16. Hadoush, A., Demirkoparan, H., Pence, T.J.: A constitutive model for an internally balanced compressible hyperelastic material. Math. Mech. Solids **22**, 372–400 (2015)
17. Neff, P., Forest, S.: A geometrically exact micromorphic model for elastic metallic foams accounting for affine microstructure. Modelling, existence of minimizers, identification of moduli and computational results. J. Elast. **87**, 239–276 (2007)
18. Owen, D.R.: Elasticity with gradient-disarrangements: a multiscale perspective for strain-gradient theories of elasticity and of plasticity. J. Elast. **127**, 115–150 (2017)
19. Zdunek, A.: On purely mechanical simple kinematic internal constraints. J. Elast. **139**, 123–152 (2020)
20. Hadoush, A., Demirkoparan, H., Pence, T.J.: Simple shearing and azimuthal shearing of an internally balanced compressible elastic material. Int. J. Non-Linear Mech. **79**, 99–114 (2016)
21. Hadoush, A.: Finite element formulation of internally balanced blatz–ko material model. Jordan J. Mech. Ind. Eng. **14**(2), 215–221 (2020)
22. Hadoush, A.: Effect of Poisson's ratio on internally balanced blatz–ko material model. Acta Mech. Sin. (2023). https://doi.org/10.1007/s10409-022-22350-x
23. Hadoush, A.: Internally balanced hyperelastic constitutive model in terms of principal stretches. Mech. Res. Commun. (2023). https://doi.org/10.1016/j.mechrescom.2023.104057
24. Hadoush, A., Demirkoparan, H., Pence, T.J.: Finite element analysis of internally balanced elastic materials. Comput. Methods Appl. Mech. Eng. **322**, 373–395 (2017)
25. Holzapfel, G.A.: Nonlinear Solid Mechanics. Wiley, England (2005)
26. Kuhl, E.: Continuum Mechanics, lecture notes edn. University of Stanford, Stanford (2008)
27. Crisfield, M.A.: Non-linear Finite Element Analysis of Solids and Structures: Advanced Topics. Wiley, Chichester (1991)
28. Hadoush, A., Demirkoparan, H., Pence, T.J.: Modeling of soft materials via multiplicative decomposition of deformation gradient. In: USNCTAM (2014)
29. Blatz, P.J., Ko, W.L.: Application of finite elasticity to the deformation of rubbery materials. Trans. Soc. Rheol. **6**, 223–251 (1962)
30. Beatty, M.F.: Topics in finite elasticity: hyperelasticity of rubber, elastomers and biological tissues–with examples. Appl. Mech. Rev. **40**, 1699–1734 (1987)
31. Crisfield, M.A.: Non-linear Finite Element Analysis of Solids and Structures: Essentials. Wiley, Chichester (1991)