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RESEARCH

A Second Gradient Theory of Thermoelasticity

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Abstract

This paper is concerned with a linear theory of thermoelasticity without energy dissipation, where the second gradient of displacement and the second gradient of the thermal displacement are included in the set of independent constitutive variables. In particular, in the case of rigid heat conductors the present theory leads to a fourth order equation for temperature. First, the basic equations of the second gradient theory of thermoelasticity are presented. The boundary conditions for thermal displacement are derived. The field equations for homogeneous and isotropic solids are established. Then, a uniqueness result for the basic boundary-initial-value problems is presented. An existence theorem is established for the first boundary value problem. The problem of a concentrated heat source is investigated using a solution of Cauchy-Kowalewski-Somigliana type.

Keywords Second gradient theory · Constitutive equations · Boundary conditions · Isotropic solids · Existence and uniqueness

Mathematics Subject Classification 35L57 · 74A15 · 74F05 · 74H20 · 74H25

1 Introduction

Green and Naghdi [1, 2] developed a thermomechanical theory of deformable continua that relies on an entropy balance law rather than an entropy inequality. The linear theory of thermoelasticity based on the new entropy balance law has been established in [3]. This theory does not sustain energy dissipation and permits the transmission of heat as thermal waves at finite speed. An account of the development of the subject as well as references to various contributions may be found in various works (see, e.g., [4–6] and references therein). The deformation of non-simple materials was extensively studied. The equations and the boundary conditions of the nonlinear strain gradient theory of elastic solids were first established by Toupin [7, 8]. The linear theory has been developed by Mindlin [9] and Mindlin and Eshel [10]. The interest in the gradient theory of elasticity is stimulated by the fact that this theory is adequate to investigate problems related to size effects and nanotechnology

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[11]. The gradient theories of thermomechanics have been studied in various papers (see, e.g., [4, 6, 12] and references therein). The motivations for introducing temperature gradient effects in thermomechanics were presented in [12].

This paper is concerned with a linear theory of thermoelasticity without energy dissipation where the second displacement gradient and the second thermal displacement gradient are included in the set of independent constitutive variables. In the first part of the paper we establish the basic equations of the theory. Following Toupin [8] we derive the boundary conditions for thermal displacement. In the case of homogeneous and isotropic solids we present the field equations and show that the present theory leads to a fourth order equation for temperature. Then, we establish a uniqueness result for the basic boundary-initial-value problems. In the case of the first boundary-initial-value problem, an existence result is established. The solution of the concentrated heat source problem is presented using a solution of Cauchy-Kowalewski-Somigliana type.

2 Basic Equations

In this section we establish the basic equations of the second gradient theory of thermoelasticity. We consider a body that at time t_0 occupies the region B of Euclidean three-dimensional space. The motion of the body is referred to the reference configuration B and to a fixed system of rectangular cartesian axes Ox_j, $(j = 1, 2, 3)$. We shall employ the usual summation and differentiation conventions: Latin subscripts, unless otherwise specified, are understood to range over the integers $(1, 2, 3)$, whereas Greek subscripts are confined to range $(1, 2)$; summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding cartesian coordinate. Boldface characters stand for tensors of an order $p \geq 1$ and if v has the order p, we write $v_{ij\ldots k}$ (p subscripts) for the components of v in the cartesian coordinate frame. We denote by ∂B the boundary of B. We assume that the boundary ∂B consists in the union of a finite number of smooth surfaces, smooth curves (edges) and points (corners). Let C be the union of edges. In what follows, we use a superposed dot to denote partial differentiation with respect to the time t.

Green and Naghdi [1, 3], presented a treatment of thermomechanical theory of deformable media which differs from the usual one and, in particular, makes use an entropy balance. Let $\mathcal{P}$ be an arbitrary material volume in the continuum, bounded by a surface $\partial\mathcal{P}$ at time t. We suppose that P is the corresponding region in the reference configuration B, bounded by a surface ∂P. In [1] the authors postulated an entropy balance in the form

$$\int_P \rho\dot{\eta}dv = \int_P \rho(s + \xi)dv + \int_{\partial P} \sigma da, \tag{1}$$

for all regions P of B and every time. Here, we have used the following notations: ρ is the reference mass density, η is the entropy per unit mass, σ is internal flux of entropy per unit area; s is the external rate of supply of entropy per unit mass; ξ is the internal rate of production of entropy per unit mass. Using the well-known method, from (1) we obtain

$$\sigma = \Lambda_j n_j, \tag{2}$$

and

$$\rho\dot{\eta} = \Lambda_{k,k} + \rho(s + \xi), \tag{3}$$

where n_j is the outward normal of ∂P and Λ_j is called entropy flux vector.

Following Green and Naghdi [1, 3] we introduce the thermal displacement α by

$$\dot{\alpha} = \theta, \tag{4}$$

where θ is the absolute temperature.

Following [1, 8], we postulate an energy balance in the form

$$\int_P \rho(\ddot{u}_i\dot{u}_i + \dot{e})dv = \int_P \rho(f_i\dot{u}_i + s\dot{\alpha})dv + \int_{\partial P}(t_i\dot{u}_i + \mu_{ji}\dot{u}_{i,j} + \sigma\dot{\alpha} + H_j\dot{\alpha}_{,j})da, \tag{5}$$

for all regions P of B and every time. Here we have used the notations: u_j is the displacement vector, e is the internal energy per unit mass f_i is the body force per unit mass, t_i is a part of the stress vector associated with the surface $\partial\mathcal{P}$ but measured per unit area of ∂P. Each of terms $\mu_{ji}\dot{u}_{i,j}$ and $H_j\dot{\alpha}_j$ is a rate of work per unit mass. According to Green and Rivlin [13], μ_{ji} is a dipolar surface force and H_j is a monopolar entropy flux, per unit area. In [9], μ_{ji} is called double force per unit area. We assume that the dipolar body force and the spin inertia per unit mass are absent (see [8]). From (5) we obtain

$$\int_P \rho\ddot{u}_i dv = \int_P \rho f_i dv + \int_{\partial P} t_i da, \tag{6}$$

for all regions P of B. Under suitable continuity assumptions, this conservation law yields Cauchy's relations

$$t_i = t_{ji}n_j, \tag{7}$$

and the equations of motion

$$t_{ji,j} + \rho f_i = \rho\ddot{u}_i, \tag{8}$$

where t_{ij} is the stress tensor. In view of (2), (3), (7) and (8), the relation (5) becomes

$$\int_P \rho\dot{e}dv = \int_P [t_{ji}\dot{u}_{i,j} + \Lambda_j\dot{\alpha}_{,j} + \rho(\dot{\eta} - \xi)\dot{\alpha}]dv + \int_{\partial P}(\mu_{ji}\dot{u}_{i,j} + H_j\dot{\alpha}_{,j})da. \tag{9}$$

From (9) we obtain

$$(\mu_{ji} - \mu_{kji}n_k)\dot{u}_{i,j} + (H_j - H_{kj}n_k)\dot{\alpha}_{,j} = 0, \tag{10}$$

where μ_{ijk} is the dipolar stress tensor and H_{kj} is the entropy flux tensor. If we use the relation (10) and the divergence theorem, then from (9) we find the local form of energy balance

$$\rho\dot{e} = \tau_{ji}\dot{u}_{i,j} + \Pi_j\dot{\alpha}_{,j} + H_{ij}\dot{\alpha}_{,ij} + \rho(\dot{\eta} - \xi)\dot{\alpha}, \tag{11}$$

where

$$\tau_{ji} = t_{ji} + \mu_{kji,k},\ \Pi_j = \Lambda_j + H_{kj,k}. \tag{12}$$

Let us consider a motion of the body which differs from the given motion by a superposed uniform rigid body angular velocity, and assume that ρ, $\dot{e}$, τ_{ij}, μ_{ijk}, Π_j, H_{ij} and θ are not affected by such motion. From (11) we get [13]

$$\tau_{ij} = \tau_{ji}. \tag{13}$$

It is usual in the current literature to obtain an equation for balance of energy in terms of the Helmholtz free energy ψ introduced by

$$\psi = e - \theta\eta. \tag{14}$$

We consider the following strain tensors from the linear theory (Mindlin and Eshel [10])

$$2e_{ij} = u_{i,j} + u_{j,i},\ \kappa_{ijk} = u_{k,ij}. \tag{15}$$

The relation (11) can be written in the form

$$\rho(\dot{\psi} + \eta\ddot{\alpha}) + \rho\dot{\alpha}\xi = \tau_{ij}\dot{e}_{ij} + \mu_{kji}\dot{\kappa}_{kji} + \Pi_j\dot{\alpha}_{,j} + H_{ij}\dot{\alpha}_{,ij}. \tag{16}$$

We require constitutive equations for ψ, τ_{ij}, μ_{ijk}, η, Λ_j, H_{kj} and ξ and assume that these are functions of the set of variables $A = (e_{ij}, \kappa_{ijk}, \theta, \alpha_{,j}, \alpha_{,kj})$. For simplicity, we regard the material to be homogeneous and assume that there is no kinematical constraint. If we take into account the relations

$$\psi = \widehat{\psi}(A), \tau_{ij} = \widehat{\tau}_{ij}(A), \ldots, \xi_j = \widehat{\xi}_j(A), \tag{17}$$

then the equation (16) implies that

$$\begin{aligned} &\left(\frac{\partial U}{\partial e_{ij}} - \tau_{ij}\right)\dot{e}_{ij} + \left(\frac{\partial U}{\partial \kappa_{ijk}} - \mu_{ijk}\right)\dot{\kappa}_{ijk} + \left(\frac{\partial U}{\partial \dot{\alpha}} + \rho\eta\right)\ddot{\alpha} \\ &\quad + \left(\frac{\partial U}{\partial \alpha_{,j}} - \Pi_j\right)\dot{\alpha}_{,j} + \left(\frac{\partial U}{\partial \alpha_{,ij}} - H_{ij}\right)\dot{\alpha}_{,ij} + \rho\dot{\alpha}\xi = 0, \end{aligned} \tag{18}$$

where we have introduced the notation $U = \rho\widehat{\psi}$. Using the method in [3], we find that the necesary and sufficient conditions for equation (18) to be satisfied according to the constitutive assumptions above are

$$\begin{aligned} \tau_{ij} &= \frac{\partial U}{\partial e_{ij}}, \mu_{ijk} = \frac{\partial U}{\partial \kappa_{ijk}}, \rho\eta = -\frac{\partial U}{\partial \dot{\alpha}}, \\ \Pi_j &= \frac{\partial U}{\partial \alpha_{,j}}, H_{ij} = \frac{\partial U}{\partial \alpha_{,ij}}, \xi = 0. \end{aligned} \tag{19}$$

For a given deformation, $\dot{u}_{i,j}$ and $\dot{\alpha}_{,j}$ in (10) can be chosen arbitrarily so that, on the basis of the constitutive equations, we get

$$\mu_{ji} = \mu_{kji}n_k, H_j = H_{kj}n_k. \tag{20}$$

We denote

$$\alpha(x, t_0) = \alpha_0, \dot{\alpha}(x, t_0) = T_0, \tag{21}$$

and assume that α_0 and T_0 are given constants. Let us introduce the notations

$$T = \theta - T_0, \varphi = \int_{t_0}^{t} T dt. \tag{22}$$

From (21) and (22) we get

$$\alpha = \varphi + T_0(t - t_0) + \alpha_0, \alpha_{,j} = \varphi_{,j}, \dot{\varphi} = T. \tag{23}$$

In what follows we assume that $u_j = \varepsilon u'_j$, $\varphi = \varepsilon\varphi'$, where ε is a constant small enough for squares and higher power to be neglected, and u'_j and φ' are independent of ε. In the linear theory, we assume that U is a quadratic form of the variables e_{ij}, κ_{ijk}, $\dot{\varphi}$, $\varphi_{,j}$ and $\varphi_{,ij}$. In the case of materials possessing a center of symmetry we have

$$\begin{aligned} 2U = {} & A_{ijrs}e_{ij}e_{rs} + B_{ijkpqr}\kappa_{ijk}\kappa_{pqr} + 2C_{ijrs}e_{ij}\varphi_{,rs} \\ & + D_{ijrs}\varphi_{,ij}\phi_{,rs} + 2E_{ijks}\kappa_{ijk}\varphi_{,s} + K_{ij}\varphi_{,i}\varphi_{,j} \\ & - a\dot{\varphi}^2 - 2b_{ij}e_{ij}\dot{\varphi} - 2c_{ij}\varphi_{,ij}\dot{\varphi}. \end{aligned} \tag{24}$$

The coefficients from (24) have the following properties

$$\begin{aligned} & A_{ijrs} = A_{jirs} = A_{rsij}, B_{ijkpqr} = B_{pqrijk} = B_{jikpqr}, C_{ijrs} = C_{jirs} = C_{ijsr}, \\ & D_{ijrs} = D_{jirs} = D_{rsij}, E_{ijrs} = E_{jirs}, K_{ij} = K_{ji}, b_{ij} = b_{ji}, c_{ij} = c_{ji}. \end{aligned} \tag{25}$$

From (19), (23), and (24) we find that

$$\begin{aligned} & \tau_{ij} = A_{ijrs}e_{rs} + C_{ijrs}\varphi_{,rs} - b_{ij}\dot{\varphi}, \\ & \mu_{ijk} = B_{ijkpqr}\kappa_{pqr} + E_{ijks}\varphi_{,s}, \\ & \rho\eta = b_{ij}e_{ij} + c_{ij}\varphi_{,ij} + a\dot{\varphi}, \\ & \Pi_i = E_{pqri}\kappa_{pqr} + K_{ij}\varphi_{,j}, \\ & H_{ij} = C_{rsij}e_{rs} + D_{ijrs}\varphi_{,rs} - c_{ij}\dot{\varphi}. \end{aligned} \tag{26}$$

In view of (12) and (19), the equations (3) and (8) can be written in the form

$$\begin{aligned} & \tau_{ji,j} - \mu_{kji,kj} + \rho f_i = \rho\ddot{u}_i, \\ & \Pi_{j,j} - H_{kj,kj} + \rho s = \rho\dot{\eta}. \end{aligned} \tag{27}$$

The linear theory is characterized by the following system: the equations of motion (27), the constitutive equations (26) and the geometric equations (15). To derive the form of the boundary conditions we use the method of Toupin [8]. In view of (2), (7), (12) and (23), the surface integral from (5) can be written in the form

$$\begin{aligned} \int_{\partial P} (t_i\dot{u}_i + \mu_{ji}\dot{u}_{i,j} + \sigma\dot{\varphi} + H_j\dot{\varphi}_{,j})da = \int_{\partial P} & [(\tau_{ki} - \mu_{ski,s})\dot{u}_i + \mu_{kji}\dot{u}_{i,j} \\ & + (\Pi_k - H_{sk,s})\dot{\varphi} + H_{kj}\dot{\varphi}_{,j}]n_k da. \end{aligned} \tag{28}$$

In the last integral from (28), $u_{i,j}$ is not independent of $\dot{u}_i$, on ∂B; only its normal component $Du_i = u_{i,j}n_j$ is independent. If we introduce the surface gradient $D_i = (\delta_{ij} - n_i n_j)\partial/\partial x_j$, then we get

$$\begin{aligned} & \mu_{kji}\dot{u}_{i,j}n_k = \mu_{kji}n_k n_j D\dot{u}_i - \dot{u}_i D_j(\mu_{kji}n_k) + D_j(\mu_{kji}n_k\dot{u}_i), \\ & H_{ki}\dot{\varphi}_{,i}n_k = H_{kj}n_k n_j D\dot{\varphi} - \dot{\varphi}D_j(H_{kj}n_k) + D_j(H_{kj}n_k\dot{\varphi}). \end{aligned} \tag{29}$$

Following [8] we obtain

$$\int_{\partial P}(t_i\dot{u}_i+\mu_{ji}\dot{u}_{i,j}+\sigma\dot{\varphi}+H_j\dot{\varphi}_{,j})da=\int_{\partial P}(P_i\dot{u}_i+R_iD\dot{u}_i+\Sigma\dot{\varphi}+\Omega D\dot{\varphi})da$$
$$+\int_C(Z_i\dot{u}_i+Y\dot{\varphi})ds, \tag{30}$$

where we have used the notation

$$\begin{aligned}
P_i&=(\tau_{ji}-\mu_{kji,k})n_j-D_j(\mu_{kji}n_k)+(D_jn_j)\mu_{pqi}n_pn_q,\\
R_i&=\mu_{rsi}n_rn_s,\ Z_i=<\mu_{pqi}n_py_q>,\\
\Sigma&=(\Pi_j-H_{kj,k})n_j+(D_jn_j)H_{kp}n_kn_p-D_j(H_{kj}n_k),\\
\Omega&=H_{ij}n_in_j,\ Y=<H_{kj}n_ky_j>,\ y_j=\varepsilon_{jri}n_is_r.
\end{aligned} \tag{31}$$

In this equation, ε_{ijk} is the alternating symbol, s_k are the components of the unit vector tangent to C, and $< f >$ denotes the difference of limits of f from the both sides of C. The first boundary-initial-value problem is characterized by the following boundary conditions

$$u_i=\widetilde{u}_i,\quad Du_i=\widetilde{\omega}_i,\varphi=\widetilde{\varphi},\quad D\varphi=\widetilde{\tau},\ \text{ on }\partial B\times I, \tag{32}$$

where $\widetilde{u}_i$, $\widetilde{\omega}_i$, $\widetilde{\varphi}$ and $\widetilde{\tau}$ are given functions, and $I=(0,\infty)$. In the case of the second boundary-initial-value problem the boundary conditions are

$$P_i=\widetilde{P}_i,R_i=\widetilde{R}_i,\Sigma=\widetilde{\Sigma},\Omega=\widetilde{\Omega}\ \text{ on }\partial B\times I,\quad Z_i=\widetilde{Z}_i,Y=\widetilde{Y}\ \text{ on }C\times I, \tag{33}$$

where $\widetilde{P}_i$, $\widetilde{R}_i$, $\widetilde{\Sigma}$, $\widetilde{\Omega}$, $\widetilde{Z}_i$, $\widetilde{Y}$ are prescribed functions. The initial conditions are

$$u_i(x,0)=u_i^0(x),\dot{u}_i(x,0)=v_i^0(x),\varphi(x,0)=\varphi^0(x),\dot{\varphi}(x,0)=\chi^0(x),x\in B, \tag{34}$$

where the functions u_i^0, v_i^0, φ^0 and χ^0 are prescribed. We assume that: (i) f_i and s are continuous; (ii) ρ is a prescribed positive constant; (iii) the constitutive coefficients satisfy the symmetry relation (25); (iv) $\widetilde{u}_i$ and $\widetilde{\varphi}$ are continuous on $\partial B\times I$; (v) $\widetilde{\omega}_i$, $\widetilde{\tau}$, $\widetilde{P}_i$, $\widetilde{R}_i$, $\widetilde{\Sigma}$, $\widetilde{\Omega}$ are continuous in time and piecewise regular on $\partial B\times I$; (vi) $\widetilde{Z}_i$ and $\widetilde{Y}$ are continuous in time and piecewise regular on $C\times I$; (vii) u_i^0, v_i^0, φ^0 and χ^0 are continuous on B.

In the case of homogeneous and isotropic solids the constitutive equations become [10]

$$\begin{aligned}
\tau_{ij}&=\lambda e_{rr}d_{ij}+2\mu e_{ij}-b\dot{\varphi}\delta_{ij}+\beta_1\delta_{ij}\Delta\varphi+2\beta_2\varphi_{,ij},\\
\mu_{ijk}&=\frac{1}{2}\alpha_1(\kappa_{rri}\delta_{jk}+2\kappa_{krr}\delta_{ij}+\kappa_{rrj}\delta_{ik})+\alpha_2(\kappa_{irr}\delta_{jk}+\kappa_{jrr}\delta_{ik})+2\alpha_3\kappa_{rrk}\delta_{ij}\\
&\quad+2\alpha_4\kappa_{ijk}+\alpha_5(\kappa_{kji}+\kappa_{kij})+\gamma_1\delta_{ij}\varphi_{,k}+\gamma_2(\delta_{jk}\varphi_{,i}+\delta_{ik}\varphi_{,j}),\\
\rho\eta&=be_{rr}+a\dot{\varphi}+c\Delta\varphi,\\
\Pi_i&=\gamma_1\kappa_{ssi}+2\gamma_2\kappa_{iss}+k\varphi_{,i},\\
H_{ij}&=\beta_1\delta_{ij}e_{rr}+2\beta_2e_{ij}+d_1\delta_{ij}\Delta\varphi+d_2\varphi_{,ij}-c\dot{\varphi}\delta_{ij},
\end{aligned} \tag{35}$$

where Δ is the Laplacian, δ_{ij} is the Kronecker's delta and λ, μ, a, b, c, k, α_m, $(m=1,2,\ldots,5)$, β_ρ, γ_ρ, d_ρ are constitutive coefficients. If we use (16) and (29), then we can express the equations (27) in terms of unknown functions u_j and φ. The resulting equations are

$$
\begin{aligned}
&(\mu - v_1\Delta)\Delta u_i + (\lambda + \mu - v_2\Delta)u_{j,ji} + (\beta - \gamma)\Delta\phi_{,j} - b\dot{\varphi}_{,i} + \rho f_i = \rho\ddot{u}_i,\\
&d\Delta\Delta\varphi - k\Delta\varphi + (\beta - \gamma)\Delta u_{j,j} + a\ddot{\varphi} + b\dot{u}_{j,j} = \rho s,
\end{aligned}
\tag{36}
$$

where we have used the notation

$$
\begin{aligned}
&v_1 = 2(\alpha_3 + \alpha_4), v_2 = 2(\alpha_1 + \alpha_2 + \alpha_5),\\
&\beta = \beta_1 + 2\beta_2, \gamma = \gamma_1 + 2\gamma_2, d = d_1 + d_2.
\end{aligned}
\tag{37}
$$

The second equation in (36) implies the following coupled heat equation

$$
d\Delta\Delta T - k\Delta T + (\beta - \gamma)\Delta\dot{u}_{j,j} + a\ddot{T} + b\ddot{u}_{j,j} = \rho\dot{s}. \tag{38}
$$

In the case of rigid heat conductors we obtain a fourth order equation for temperature.

3 Uniqueness

In this section we present a uniqueness result in the context of the dynamic theory. By an admissible process $p = \{u_i, \varphi, e_{ij}, \kappa_{ijk}, \tau_{ij}, \mu_{ijk}, \eta, \Pi_j, H_{ij}\}$ we mean an ordered array of functions u_i, φ, e_{ij}, κ_{ijk}, τ_{ij}, μ_{ijk}, η, Π_j and H_{ij} defined on $B \times [0, \infty)$ with the following properties: (i) $u_i \in C^{4,2}$; $\varphi \in C^{4,2}$; $e_{ij}, \kappa_{ijk} \in C^{2,0}$; $\tau_{ij}, \Pi_j, H_{ji} \in C^{1,0}$; $\mu_{ijk} \in C^{2,0}$; $\eta \in C^{0,1}$ on $B \times I$; (ii) u_i, $\dot{u}_i$, $\ddot{u}_i$, φ, $\dot{\varphi}$, $\ddot{\varphi}$, $u_{i,j}$, $u_{i,jk}$, $\varphi_{,ij}$, τ_{ij}, $\tau_{ij,i}$, μ_{ijk}, $\mu_{ijk,ij}$, Π_j, $\Pi_{j,j}$, H_{ij}, $H_{ij,i}$, η and $\dot{\eta}$ are continuous on $B \times [0, \infty)$. By a solution of the first boundary-value problem we mean an admissible process which satisfies the equations (15), (26) and (27) on $B \times I$, the boundary conditions (32) and the initial conditions (34). Similarly, we can define the solution of the second boundary-initial-value problem.

We define the functions W and E by

$$
\begin{aligned}
2W &= A_{ijrs}e_{ij}e_{rs} + B_{ijkpqr}\kappa_{ijk}\kappa_{pqr} + 2C_{ijrs}e_{ij}\varphi_{,rs}\\
&\quad + 2E_{ijkr}\kappa_{ijk}\varphi_{,r} + K_{ij}\varphi_{,i}\varphi_{,j} + D_{ijrs}\varphi_{,ij}\varphi_{,rs},\\
E &= \frac{1}{2}\int_B (\rho\dot{u}_i\dot{u}_i + a\dot{\varphi}^2 + 2W)dv.
\end{aligned}
\tag{39}
$$

Theorem 1 *Assume that*

(i) *ρ and a are strictly positive*;
(ii) *the relations (25) hold*;
(iii) *W is a positive semidefinite quadratic form.*

Then, the boundary-initial-value problems have at most one solution.

Proof Let us denote

$$
F = \tau_{ij}\dot{e}_{ij} + \mu_{ijk}\dot{\kappa}_{ijk} + \rho\dot{\eta}\dot{\varphi} + \Pi_j\dot{\varphi}_{,j} + H_{ji}\dot{\varphi}_{,ij}. \tag{40}
$$

By using (25), (26), and (39) we obtain

$$F=\frac{1}{2}\frac{\partial}{\partial t}(a\dot{\varphi}^2+2W). \tag{41}$$

On the other hand, by (15) and (27) we get

$$F=[(\tau_{jk}-\mu_{jik,i})\dot{u}_k+\mu_{ijk}\dot{u}_{k,i}+\Pi_j\dot{\varphi}+H_{ij}\dot{\varphi}_{,i}-H_{ji,i}\dot{\varphi}]_{,j}+\rho(f_j\dot{u}_j+s\dot{\varphi})-\rho\ddot{u}_i\dot{u}_i. \tag{42}$$

From (12), (41) and (42) we find that

$$\frac{1}{2}\frac{\partial}{\partial t}(a\dot{\varphi}^2+2W)=[t_{jk}\dot{u}_k+\mu_{jki}\dot{u}_{i,k}+\Lambda_j\dot{\varphi}+H_{kj}\dot{\varphi}_{,k}]_{,j}+\rho(f_j\dot{u}_j+s\dot{\varphi})-\rho\ddot{u}_i\dot{u}_i. \tag{43}$$

If we integrate the relation (43) over B and use the divergence theorem and the relations (2), (7), (20) and (39) then we obtain

$$\dot{E}=\int_{\partial B}(t_i\dot{u}_i+\mu_{ji}\dot{u}_{i,j}+\sigma\dot{\varphi}+H_j\dot{\varphi}_{,j})da+\int_B\rho(f_j\dot{u}_j+s\dot{\varphi})dv. \tag{44}$$

With the help of (30) we arrive at

$$\dot{E}=\int_{\partial B}(P_i\dot{u}_i+R_iD\dot{u}_i+\Sigma\dot{\varphi}+\Omega D\dot{\varphi})da+\int_C(Z_i\dot{u}_i+Y\dot{\varphi})ds+\int_B\rho(f_j\dot{u}_j+s\dot{\varphi})dv. \tag{45}$$

Suppose that there are two solutions. Then, their difference $p^*=\{u_i^*,\varphi^*,e_{ij}^*,\kappa_{ijk}^*,\tau_{ij}^*,\mu_{ijk}^*,$ $\eta^*,\Pi_j^*,H_{ij}^*\}$ corresponds to null data, so that

$$P_i^*\dot{u}_i^*+R_i^*D\dot{u}_i^*+\Sigma^*\dot{\varphi}^*+\Omega^*D\dot{\varphi}^*=0 \text{ on } \partial B\times I,\ \ Z_i^*\dot{u}_i^*+Y^*\dot{\varphi}^*=0, \text{ on } C\times I, \tag{46}$$

and

$$u_i^*(x,0)=0,\quad \dot{u}_i^*(x,0)=0,\quad \varphi^*(x,0)=0,\quad \dot{\varphi}^*(x,0)=0 \text{ on } x\in\overline{B}. \tag{47}$$

The functions W and E associated with process p^* will be denoted by W^* and E^*, respectively. The conditions (47) imply the following relations

$$e_{ij}^*(x,0)=0,\quad \kappa_{ijk}^*(x,0)=0,\quad \varphi_{,i}^*(x,0)=0,\quad \varphi_{,ij}^*(x,0)=0. \tag{48}$$

It follows from (39) and (48) that

$$W^*(x,0)=0. \tag{49}$$

With the help of (39), (47) and (49) we get

$$E^*(0)=0. \tag{50}$$

From (45), (46) and (50) we obtain

$$E^*(t)=0,\ t\in I. \tag{51}$$

By using the hypotheses of the theorem we find that $\dot{u}_i^*=0$ and $\dot{\varphi}^*=0$ on $B\times I$. Since u_i^* and φ^* vanish initially we conclude that $u_i^*=0$ and $\varphi^*=0$ on $B\times I$. □

4 Existence Theorem

In this section we provide an existence theorem of solutions for the problem determined by equations (15), (26) and (27) with the initial conditions (34) and the boundary conditions

$$u_i = 0, \quad Du_i = 0, \quad \varphi = 0, \quad D\varphi = 0 \text{ on } \partial B \times I. \tag{52}$$

In this section we assume that conditions (i) and (ii) of Theorem 1 are maintained and instead of condition (iii) we assume that:

(iii') The function W defined at (39) is strictly positive definite.

We will transform our problem into an abstract Cauchy problem on the Hilbert space $\mathcal{H}$ defined by:

$$\mathcal{H} = \mathbf{W}_0^{2,2}(B) \times \mathbf{L}^2(B) \times W_0^{2,2}(B) \times L^2(B), \tag{53}$$

where $\mathbf{W}_0^{2,2}(B) = [W_0^{2,2}(B)]^3$ and $\mathbf{L}^2(B) = [L^2(B)]^3$. Here $W_0^{2,2}(B)$ and $L^2(B)$ are the usual Sobolev spaces. Then, we will show the existence of a semigroup of linear operators defining the solutions of the problem (see [14]). This kind of arguments are usual in the study of well posed thermoelastic problems.

An element in this Hilbert space has the form $(\boldsymbol{u}, \boldsymbol{v}, \varphi, \theta)$. In this space we consider the inner product associated with the norm

$$||(\mathbf{u}, \mathbf{v}, \varphi, \theta)||^2 = \frac{1}{2} \int_B \left(\rho v_i v_i + a\theta^2 + 2W\right) dv. \tag{54}$$

It is clear that (54) defines a norm which is equivalent to the usual one in the Hilbert space. We define the operator

$$\mathcal{A} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \varphi \\ \theta \end{pmatrix} = \begin{pmatrix} v_i \\ M_i \\ \theta \\ N \end{pmatrix} \tag{55}$$

where M_i and N are given by

$$M_i = \rho^{-1}[(A_{ijrs} u_{r,s} + C_{ijrs} \varphi_{,rs} - b_{ij}\theta)_{,j} - (B_{kjipqr} u_{r,pq} + E_{kjis}\varphi_{,s})_{,kj}],$$

and

$$N = a^{-1}[(E_{pqri} u_{r,pq} + K_{ij}\varphi_{,j} - b_{ij} v_{,j})_{,i} - (C_{rskj} u_{r,s} + D_{kjrs}\varphi_{,rs})_{,kj}].$$

It is worth noting that $\mathbf{v} \in \mathbf{W}_0^{2,2}(B)$, $\theta \in W_0^{2,2}(B)$, $(M_i) \in \mathbf{L}^2(B)$, $N \in L^2(B)$. It is clear that the domain of the operator is a dense subspace of the Hilbert space.

We can write the basic equations and initial conditions as

$$\frac{dU}{dt} = \mathcal{A}U + \mathcal{F}(t), \quad U(0) = (\mathbf{u}^0, \mathbf{v}^0, \varphi, \theta^0), \tag{56}$$

where $\mathcal{F}$ is given by

$$\mathcal{F} = \begin{pmatrix} 0 \\ f_i \\ 0 \\ s \end{pmatrix}.$$

Now, we will prove that the operator defines a contractive semigroup of linear operators and the existence, uniqueness and continuous dependence of solutions will be concluded.

Lemma 1 *For every $U = (\mathbf{u}, \mathbf{v}, \varphi, \theta)$ at the domain of the operator $(\mathcal{A})$, the following equality*

$$< \mathcal{A}U, U >= 0,$$

holds.

Proof If we apply the definition of the operator and the boundary conditions, after the use of the divergence theorem we obtain the desired equality. □

Lemma 2 *Zero belongs to the resolvent of the operator $\mathcal{A}$.*

Proof Let us consider $(\mathbf{g}_1, \mathbf{g}_2, g_3, g_4)$ an element in our Hilbert space. We have to solve the system

$$\mathbf{v} = \mathbf{g}_1, \ \ \mathbf{M} = \mathbf{g}_2, \ \ \theta = g_3, \ \ N = g_4.$$

We can obtain $\mathbf{v}$ and θ directly. Then, we obtain the system

$$(A_{ijrs}u_{r,s} + C_{ijrs}\varphi_{,rs})_{,j} - (B_{kjipqr}u_{r,pq} + E_{kjis}\varphi_{,s})_{,kj} = \rho g_{2i} + b_{ij}g_{3,j},$$
$$(E_{pqri}u_{r,pq} + K_{ij}\varphi_{,j} - b_{ij}v_j)_{,i} - (C_{rskj}u_{r,s} + D_{kjrs}\varphi_{,rs})_{,kj} = ag_4 + b_{ij}g_{1i,j}$$

To solve this system we can apply the Lax-Milgram lemma (see [15]). To this end, we define the bilinear form

$$\mathcal{B}[(\mathbf{u}, \mathbf{v}, \varphi, \theta), (\mathbf{u}^*, \mathbf{v}^*, \varphi^*, \theta^*)] = \int_B I dv.$$

where I is given by

$$I = A_{ijrs}u_{i,j}u^*_{r,s} + B_{ijkpqr}u_{k,ij}u^*_{r,pq} + C_{ijrs}(u_{i,j}\varphi^*_{,rs} + u^*_{i,j}\varphi_{,rs})$$
$$+ E_{ijkr}(u_{k,ij}\varphi^*_{,r} + u^*_{k,ij}\varphi_{,r}) + K_{ij}\varphi_{,i}\varphi^*_{,j} + D_{ijrs}\varphi_{,ij}\varphi^*_{,rs}.$$

It is clear that $\mathcal{B}$ is bounded on $\mathbf{W}_0^{2,2} \times W_0^{2,2}$ and, in view of the assumption (iii'), it is coercive in this space. On the other side, it is clear that

$$(\rho g_{2i} + b_{ij}g_{3,j}, \ ag_4 + b_{ij}g_{1i,j})$$

belongs to $\mathbf{W}^{-2,2} \times W^{-2,2}$. The Lax-Milgram lemma allow us to conclude the existence of solutions. Indeed we can obtain that the solutions $(\mathbf{u}, \mathbf{v}, \varphi, \theta)$ satisfies the estimate

$$||(\mathbf{u}, \mathbf{v}, \varphi, \theta)|| \leq K||(\mathbf{g}_1, \mathbf{g}_2, g_3, g_4)||,$$

where K is independent of the solution. Therefore, we have proved the lemma. □

Thus, we have

Theorem 2 *The operator $\mathcal{A}$ generates a contractive semigroup on the Hilbert space.*

Theorem 3 *Assume that* $(\mathbf{f}, s)$ *are smooth on* $L^2(B)$ *and continuous in* $W^{2,2}(B)$ *and that* U_0 *belongs to the domain of the operator. Then, there exists a solution* $U(t)$ *to the Cauchy problem which is smooth in the Hilbert space and takes values in the domain of the operator.*

Since the solutions are defined by means of a semigroup of contractions we can conclude the estimate

$$||U(t)|| \leq ||U(0)|| + \int_0^t ||(\mathbf{f}(\tau), s(\tau))||_{L^2} d\tau.$$

This inequality gives the continuous dependence on the solutions with respect to initial data and supply terms. Therefore, under our assumptions the problem of the second strain gradient thermoelasticity without energy dissipation is well posed in the sense of Hadamard.

The results presented in this section extend those established in [4] for the strain gradient theory in the case that we do not consider high order effects in the thermal displacement.

5 General Solution of the Field Equations

In this section we establish a solution of the field equations that is analogous to Cauchy-Kowalewski-Somigliana solution in the dynamic theory of classical elasticity [16]. In the case of isothermal elasticity, the solutions of Galerkin type in the context of strain gradient elasticity have been presented in [9, 17]. The field equations for isotropic and homogeneous materials can be expressed in terms of the functions u_j and φ in the form

$$\begin{aligned} &L_1 \Delta \boldsymbol{u} + L_2 \text{grad}\,\text{div}\,\boldsymbol{u} + L_3\,\text{grad}\,\varphi + \rho \boldsymbol{f} = \rho \ddot{\boldsymbol{u}}, \\ &L_4 \text{div}\,\boldsymbol{u} + L_5 \varphi = \rho s, \end{aligned} \tag{57}$$

where we have used the notations

$$\begin{aligned} &L_1 = \mu - \nu_1 \Delta,\, L_2 = \lambda + \mu - \nu_2 \Delta,\, L_3 = (\beta - \gamma)\Delta - b\frac{\partial}{\partial t}, \\ &L_4 = (\beta - \gamma)\Delta + b\frac{\partial}{\partial t},\, L_5 = d\Delta\Delta - k\Delta + a\frac{\partial^2}{\partial t^2}. \end{aligned} \tag{58}$$

Let us introduce the notation

$$A_1 = L_1 \Delta - \rho \frac{\partial^2}{\partial t^2},\, A_2 = L_3 L_4 \Delta - L_5 A_3,\, A_3 = (L_1 + L_2)\Delta - \rho \frac{\partial^2}{\partial t^2}. \tag{59}$$

Theorem 4 *Let*

$$\begin{aligned} &\boldsymbol{u} = -A_2 \boldsymbol{F} + (L_3 L_4 - L_2 L_5)\,\text{grad}\,\mathit{div}\,\boldsymbol{F} + L_3\,\text{grad}\,G, \\ &\varphi = -A_1 L_4 \mathit{div}\,\boldsymbol{F} - A_3 G, \end{aligned} \tag{60}$$

where the fields F_j *of class* C^{12} *and* G *of class* C^8 *on* $B \times I$ *satisfy the equations*

$$A_1 A_2 \boldsymbol{F} = \rho \boldsymbol{f}, \;\; A_2 G = \rho s. \tag{61}$$

Then $\boldsymbol{u}$ *and* φ *satisfy the equations* (57).

Proof A straightforward calculation yields

$$L_2A_2 + L_3L_4A_1 = A_3(L_3L_4 - L_2L_5), (L_1 + L_2)\Delta - A_3 = \rho\frac{\partial^2}{\partial t^2},$$
$$A_2 + L_5A_1 = (L_3L_4 - L_2L_5)\Delta. \tag{62}$$

If we substitute $\boldsymbol{u}$ and φ given by (60) into the equations (57) and use (58) and (62), we obtain

$$\begin{aligned}
&L_1\Delta\boldsymbol{u} + L_2\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} + L_3\,\mathrm{grad}\,\varphi - \rho\ddot{\boldsymbol{u}} = -L_1A_2\Delta\boldsymbol{F} + A_2\rho\ddot{\boldsymbol{F}}\\
&\quad + \{(L_3L_4 - L_2L_5)A_3 - L_2A_2 - L_3L_4A_1\}\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{F}\\
&\quad + L_3[L_1\Delta + L_2\Delta - A_3 - \rho\frac{\partial^2}{\partial t^2}]\,\mathrm{grad}\,G \qquad (63)\\
&\quad = -A_1A_2\boldsymbol{F},\\
&L_4\mathrm{div}\,\boldsymbol{u} + L_5\varphi = L_4[(L_3L_4 - L_2L_5)\Delta - A_2 - L_5A_1]\mathrm{div}\,\boldsymbol{F} + (L_3L_4\Delta - L_5A_1)G = A_2G.
\end{aligned}$$

If we use (61) and (63), then we obtain the desired result. □

The solutions of Galerkin type are used to study the deformations produced by concentrated loads [9, 16].

6 Effects of a Concentrated Heat Supply

In this section we use the solution (60) to investigate the effects of a concentrated external heat source in an infinite space. We consider an isotropic and homogeneous solid subjected to the following loads

$$\boldsymbol{f} = \boldsymbol{0}, \quad \rho s = Q(r, t), \tag{64}$$

where $r = [(x_j - y_j)(x_j - y_j)]^{1/2}$, y is a fixed point, and Q is a given function. The conditions at infinity are given by

$$u_i \to 0, u_{i,j} \to 0, u_{i,jk} \to 0, \varphi \to 0, \varphi_{,j} \to 0, \varphi_{,jk} \to 0 \ \text{ for } r \to \infty.$$

In view of (61) and (64) we can take $\boldsymbol{F} = \boldsymbol{0}$ and $G = \chi(r, t)$, where χ satisfies the equation

$$A_2\chi = Q. \tag{65}$$

In what follows we consider the case of steady vibrations. We assume that

$$Q = \Re e[Q^*(r)\exp(-i\omega t)], \tag{66}$$

where ω is the frequency of vibration, i is the imaginary unit, $\Re e[f]$ is the real part of f, and Q^* is a prescribed function. Let us introduce the notations

$$A_2^* = L_2^*L_4^*\Delta - L_5^*A_3^*, L_3^* = (\beta - \gamma)\Delta + ib\omega, L_4^* = (\beta - \gamma)\Delta - ib\omega,$$
$$L_5^* = d\Delta\Delta - k\Delta - a\omega^2, A_3^* = (\xi - \zeta\Delta)\Delta + \rho\omega^2, \xi = \lambda + 2\mu, \zeta = \nu_1 + \nu_2. \tag{67}$$

If we assume that

$$\chi = \Re e[\chi^*(r,\omega)\exp(-i\omega t)], \tag{68}$$

then from (58), (59) we find the following equation for amplitude χ^*,

$$A_2^* \chi^* = Q^*. \tag{69}$$

We denote by κ_j^2, $(j = 1, 2, 3, 4)$, the roots of the equation

$$d\zeta x^4 + p_1 x^3 + p_2 x^2 - p_3 x + \rho a \omega^4 = 0, \tag{70}$$

where we have introduced the notation

$$p_1 = \xi d + \zeta k - (\beta - \gamma)^2,\ p_2 = k\xi - \rho\omega^2 d - a\zeta\omega^2,\ p_3 = \omega^2(b^2 + \rho k + a\xi). \tag{71}$$

Then, the equation (69) can be expressed in the form

$$(\Delta + \kappa_1^2)(\Delta + \kappa_2^2)(\Delta + \kappa_3^2)(\Delta + \kappa_4^2)\chi^* = eQ^*, \tag{72}$$

where we have used the notation $e = (\xi d)^{-1}$. In what follows we denote by κ_s, $(s = 1, 2, 3, 4)$, the roots with positive real parts and assume that these roots are distinct. If the functions χ_k, $(k = 1, 2, 3, 4)$, satisfy the equations

$$(\Delta + \kappa_j^2)\chi_j = eQ^*,\ \ (\text{no sum};\ j = 1, 2, 3, 4), \tag{73}$$

then the function χ^* can be expressed in the form

$$\chi^* = \sum_{s=1}^{4} h_s \chi_s, \tag{74}$$

where

$$h_s^{-1} = \prod_{j=1 (j\neq s)}^{4} (\kappa_s^2 - \kappa_j^2),\ \ (s = 1, 2, 3, 4). \tag{75}$$

Let ua assume now that $Q^* = \delta(\boldsymbol{x} - \boldsymbol{y})$, where δ is the Dirac measure and $\boldsymbol{y}$ is a fixed point. Then, from (73) we obtain

$$\chi_s = -\frac{e}{4\pi r}\exp(i\kappa_s r),\ \ (s = 1, 2, 3, 4). \tag{76}$$

It follows from (74) and (76) that

$$\chi^* = -\frac{e}{4\pi r}\sum_{s=1}^{4} h_s \exp(i\kappa_s r). \tag{77}$$

If we seek the solution of the form

$$\boldsymbol{u} = \Re e[\boldsymbol{u}^*(r,\omega)\exp(-i\omega t)],\ \varphi = \Re e[\varphi^*(r,\omega)\exp(-i\omega t)],$$

then from (60) and (68) we find that

$$\boldsymbol{u}^* = L_3^* \operatorname{grad} \chi^*, \varphi^* = -A_3^* \chi^*.$$

It is easy to see that the conditions at infinity are satisfied. In classical thermoelasticity, the problem of concentrated loads in the case of steady vibrations has been studied in various works (see, e.g., [18, 19] and references therein).

7 Summary

The results presented in this paper can be summarized as follows:

(a) We use the Green-Naghdi theory of thermomechanics to establish a second gradient theory of thermoelasticity that leads to a fourth-order equations for the temperature.
(b) We establish boundary conditions for thermal displacement and formulate the boundary-initial-value problems.
(c) We present the field equations for homogeneous and isotropic solids.
(d) We establish a uniqueness result for the basic boundary-initial-value problems.
(e) We establish an existence result for the first boundary-initial-value problem.
(f) We investigate the problem of a concentrated heat source.

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