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Fractional Strain Tensor and Fractional Elasticity

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In memory of J.L. Ericksen

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Abstract

A new fractional strain tensor $\epsilon^\alpha(u)$ of order α $(0 < \alpha < 1)$ is introduced for a displacement u of a body occupying the entire three-dimensional space. For $\alpha \uparrow 1$, the fractional strain tensor approaches the classical infinitesimal strain tensor of the linear elasticity. It is shown that $\epsilon^\alpha(u)$ satisfies Korn's inequality (in a general L^p version, $1 < p < \infty$) and the fractional analog of Saint-Venant's compatibility condition. The strain $\epsilon^\alpha(u)$ is then used to formulate a three-dimensional fractional linear elasticity theory. The equilibrium of the body in an external force f is determined by the Euler-Lagrange equation of the total energy functional. The solution u is given by Green's function G_α:

$$u(x) = \int_{\mathbf{R}^n} G_\alpha(x-y) f(y)\,dy, \quad x \in \mathbf{R}^3.$$

For an isotropic body the equilibrium equation reads

$$-\mu(-\Delta)^\alpha u + (\lambda + \mu)\nabla^\alpha \operatorname{div}^\alpha u + f = 0$$

where λ, μ are the Lamé moduli of the material and $(-\Delta)^\alpha$, ∇^α and $\operatorname{div}^\alpha$ are the fractional laplacean, gradient and divergence. Green's function can be determined explicitly in this case:

$$G_\alpha(x) = \frac{c_\alpha}{\mu |x|^{3-2\alpha}} \Big(I - \frac{\lambda + \mu}{2\alpha(\lambda + 2\mu)} \big(I + (2\alpha - 3)|x|^{-2} x \otimes x \big) \Big),$$

$x \in \mathbf{R}^3$, $x \neq 0$, where I is the identity tensor (matrix), and c_α a normalization factor (determined below). For $\alpha \uparrow 1$ the function G_α approaches Green's function of the standard linear elasticity. Similar approach applies to the equilibrium solution.

Keywords Fractional strain tensor · Fractional gradient · Fractional calculus · Fractional elasticity · Fractional Green's function

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1 Introduction

It is well-known that long-range interatomic forces, important in many situations,[1] are not accounted for adequately by the classical local elasticity. The nonlocal theories are designed to eliminate this undesirable feature: the behavior at a material point is influenced by the state of all points of the body. The equilibrium displacement is determined by integral equations, rather than by the differential equations of the local theories.

The theory of nonlocal elasticity has attracted the attention of many writers; there are many approaches. The literature is large.[2]

This paper discusses a particular form of nonlocal elasticity, which is based on the fractional vector calculus, i.e., the theory of gradients, divergences, etc. of fractional order. Central to the approach is a novel definition of the fractional strain tensor. For a body that occupies the entire n-dimensional space (which is assumed throughout), the value $\epsilon^\alpha(u)(x)$ of the fractional strain tensor at $x \in \mathbf{R}^n$ is given by

$$\epsilon^\alpha(u)(x) = \frac{1}{2}\mu_\alpha \int_{\mathbf{R}^n} \frac{\big(u(x)-u(y)\big)\otimes(x-y)+(x-y)\otimes\big(u(x)-u(y)\big)}{|x-y|^{n+\alpha+1}}\,dy; \tag{1}$$

here

$$u : \mathbf{R}^n \to \mathbf{R}^n \tag{2}$$

is the displacement of the body,

$$0 < \alpha < 1$$

is a fixed number, called the order of the fractional strain tensor, and

$$\mu_\alpha := 2^\alpha \pi^{-n/2}\Gamma\big((n+\alpha+1)/2\big)/\,\Gamma\big((1-\alpha)/2\big)$$

is a normalization factor; Γ is the gamma function. The field $\epsilon^\alpha(u)$ takes the values in the space of symmetric second-order tensors.

The specific form (1) of $\epsilon^\alpha(u)$ is motivated below.

The reader is referred to [3], [4], [7], [33], [34], [1], [21], [29] and the literature therein for earlier approaches to elasticity based on the fractional calculus.

The theory based on (1) is rotationally invariant, i.e., if u is a displacement as in (2) and if a new displacement $\bar{u} : \mathbf{R}^n \to \mathbf{R}^n$ is given by

$$\bar{u}(x) = qu(q^{\mathrm{T}}x), \quad x \in \mathbf{R}^n,$$

where q is an orthogonal tensor, then

$$\epsilon^\alpha(\bar{u})(x) = q\epsilon^\alpha(u)(q^{\mathrm{T}}x)q^{\mathrm{T}}.$$

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[1] E.g., in thin film mechanics, fracture mechanics, the theory of dislocations etc.

[2] See [26] and [15] for recent reviews.

(This is easily verified by elementary *algebraic* rearrangements of the formulas (1) for $\epsilon^{\alpha}(\bar{u})$ and $\epsilon^{\alpha}(u)$.)

The satisfaction of the fundamental requirement of rotational invariance distinguishes the present work from earlier three-dimensional approaches to fractional elasticity. Indeed, these works use (explicitly or implicitly) the definition of the fractional gradient that is not rotationally invariant. Specifically, let φ be a scalar- or vector-valued function of $x_1, \dots, x_n$. The fractional gradient $\nabla^{\alpha}\varphi$ of φ is defined in a co-ordinate way as the n tuple

$$\nabla^{\alpha}\varphi = \big(\mathrm{D}y^{\alpha}_{x_1}\,\varphi, \dots, \mathrm{D}^{\alpha}_{x_n}\,\varphi\big) \tag{3}$$

of the one-dimensional fractional derivatives $\mathrm{D}^{\alpha}_{x_i}\,\varphi$ with respect to the variable x_i, $i = 1, \dots, n$. Clearly, (3) is laid down by analogy with the classical gradient

$$\nabla\varphi = \big(\mathrm{D}_{x_1}\,\varphi, \dots, \mathrm{D}_{x_n}\,\varphi\big)$$

where $\mathrm{D}_{x_i}\,\varphi$ are the partial derivatives with respect to x_i. The classical gradient is rotationally invariant, i.e., if a new function $\bar{\varphi}$ is given by

$$\bar{\varphi}(x) = \varphi(q^{\mathrm{T}}x), \qquad x \in \mathbf{R}^n,$$

where q is an orthogonal tensor, then

$$\nabla\bar{\varphi}(x) = q\nabla\varphi(q^{\mathrm{T}}x).$$

In contrast, the definition (3) of $\nabla^{\alpha}\varphi$ *is not rotationally invariant;* i.e., it is easy to find a function φ and a point x such that

$$\nabla^{\alpha}\bar{\varphi}(x) \neq q\nabla^{\alpha}\varphi(q^{\mathrm{T}}x),$$

see [30, Sect. 1].[3]

The present paper is based on a rotationally invariant fractional gradient ∇^{α} (see Definition 2.3);[4] similar rotationally invariant definitions are given of the fractional laplacean $(-\Delta)^{\alpha/2}$ and fractional divergence $\operatorname{div}^{\alpha}$. In fact, up to a multiple, the operators $(-\Delta)^{\alpha/2}$, ∇^{α}, and $\operatorname{div}^{\alpha}$ used in this paper are the only operators that are rotationally invariant and have natural scaling properties, as proved in [30].

The operators $(-\Delta)^{\alpha/2}$, ∇^{α}, and $\operatorname{div}^{\alpha}$ are defined in two steps. In Sect. 2 they are defined as elements of appropriate test function spaces of smooth and (slowly) decaying scalar- vector- or tensor-valued functions on $\mathbf{R}^n$. These initial definitions are then extended in Sect. 3 to the duals of test function spaces by dual (weak) definitions. Thus, e.g., ∇^{α} is the formal adjoint of $-\operatorname{div}^{\alpha}$, etc. The space of test functions and its dual are described in detail in Sects. 2 and 3. Here we only mention that the space of test functions contains the Schwartz space of rapidly decaying functions and thus the dual can be interpreted as a subset of the space of tempered distributions.

The integral in (1) converges if the displacement u belongs to the space $\mathscr{T}(\mathbf{C}^n)$ of slowly decaying test functions on $\mathbf{R}^n$ with values in the space $\mathbf{C}^n$ of $n$tuples of complex numbers.[5]

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[3] This negative result is independent of the chosen type of the one-dimensional fractional derivative $\mathrm{D}^{\alpha}_{x_i}\,\varphi$ (Riemann-Liouville, Grünwald-Letnikov, …).

[4] See the references in Sect. 2.

[5] We use complex values in view of the future use of the Fourier transformation (starting from Sect. 4).

Then $\epsilon^\alpha(u)$ belongs to a similar space $\mathscr{T}(\mathrm{Sym})$ of symmetric-tensor-valued test functions. The elements u of the dual space $\mathscr{T}'(\mathbf{C}^n)$ of $\mathscr{T}(\mathbf{C}^n)$ are interpreted as (generalized) displacements. The weak definition (41) then determines $\epsilon^\alpha(u)$ as an element of the dual $\mathscr{T}'(\mathrm{Sym})$ of $\mathscr{T}(\mathrm{Sym})$.[6]

The most interesting cases arise when the weak strain tensor $\epsilon^\alpha(u)$ is restricted in some special way. Two particular cases are immediate: when $\epsilon^\alpha(u)$ is represented by an L^p function, $1 \le p \le \infty$ (see Definition 6.1(ii)), and when $\epsilon^\alpha(u)$ is represented by a measure (see Definition 6.1(iii)). The present paper deals with the first case; the second case will be treated in a future paper.

The particular form (1) of $\epsilon^\alpha(u)$ is motivated by the analogy with the classical strain tensor

$$\epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}}),$$

since in terms of the fractional gradient (9), (1) can be rewritten as

$$\epsilon^\alpha(u) = \frac{1}{2}(\nabla^\alpha u + \nabla^\alpha u^{\mathrm{T}})$$

if $u \in \mathscr{T}(\mathbf{C}^n)$. Moreover, we have the following approach to the classical strain tensor:

$$\epsilon^\alpha(u) \to \epsilon(u) \quad \text{as} \quad \alpha \uparrow 1. \tag{4}$$

Equation (4) is a consequence of the property $\nabla^\alpha u \to \nabla u$ as $\alpha \uparrow 1$ proved (and stated precisely) in [2, Appendix C].

The fractional strain tensor satisfies Korn's inequality. Let $1 < p < \infty$ and $0 < \alpha < 1$. For every $u \in \mathscr{T}'(\mathbf{C}^n)$ with $\epsilon^\alpha(u) \in L^p(\mathrm{Sym})$ we have

$$|\epsilon^\alpha(u)|_{L^p} \ge c|\nabla^\alpha u|_{L^p}$$

with some positive constant $c = c(n, p, \alpha)$ independent of u. Here $|\cdot|_{L^p}$ is the usual norm on the space L^p,

$$|\cdot|_{L^p} = \Bigl(\int_{\mathbf{R}^n} |\cdot|^p dx\Bigr)^{1/p}.$$

We then apply the formalism of fractional strain tensors to treat linearly elastic fractional bodies. In Sect. 7 we present the theory for bodies of general symmetry; in this introduction we consider isotropic bodies for simplicity. (See Example 7.5 and its proof for details.) The energy is given by

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^n} \bigl[\lambda\bigl(\operatorname{tr}\epsilon^\alpha(u)\bigr)^2 + 2\mu\epsilon^\alpha(u)\cdot\epsilon^\alpha(u)\bigr]dx - \int_{\mathbf{R}^n} f\cdot u\,dx,$$

where f is the external force and λ, μ are the Lamé moduli of the material, subject to the classical strong ellipticity inequalities (61). The equilibrium displacement is the solution of the corresponding Euler-Lagrange equations

$$-\mu(-\Delta)^\alpha u + (\lambda + \mu)\nabla^\alpha \operatorname{div}^\alpha u + f = 0. \tag{5}$$

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[6] Since the domain of definition of functions from various function spaces is always $\mathbf{R}^n$, we indicate only the ranges: thus, e.g., the symbol $L^p(\mathbf{C}^m)$ denotes the space of $\mathbf{C}^m$-valued functions on $\mathbf{R}^n$ that belong to L^p.

The solution is given in terms of Green's function G_α by

$$u(x) = \int_{\mathbf{R}^n} G_\alpha(x-y) f(y) dy$$

provided that

$$f \in L^q(\mathbf{C}^n) \quad \text{where} \quad 1 < q < n/2\alpha.$$

The solution belongs to the space $R^{2\alpha,q}(\mathbf{C}^n)$ of *Riesz* potentials[7] of order 2α, defined in Sect. 5. The equilibrium equation (5) then has a pointwise meaning almost everywhere.

The existence and properties of Green's function for a body of general symmetry are established in Sect. 7. The explicit form is available for isotropic bodies:

$$G_\alpha(x) = \frac{c_\alpha}{\mu |x|^{n-2\alpha}} \Big(I - \frac{\lambda + \mu}{2\alpha(\lambda + 2\mu)} \big(I + (2\alpha - n)|x|^{-2} x \otimes x \big) \Big) \tag{6}$$

for any $x \in \mathbf{R}^n$, $x \neq 0$, where I is the identity tensor (matrix) and c_α a normalization factor, see (63).

Green's function G_1 of classical linear elasticity is obtained by putting $\alpha = 1$ in (6); thus $G_\alpha \to G_1$ as $\alpha \uparrow 1$ by continuity. Moreover, the displacement $u = u_\alpha$ of fractional elasticity of order α converges for $\alpha \uparrow 1$ to the displacement u_1 given by the standard linear elasticity. This will be shown elsewhere from a broader perspective.[8]

2 Fractional Vector Calculus I (Smooth Case)

In this section we introduce the fractional laplacean $(-\Delta)^{\alpha/2}$, fractional gradient ∇^α and the fractional divergence $\operatorname{div}^\alpha$ for smooth test fields. Here the order α can be any complex number satisfying $\operatorname{Re}\alpha > -n$; see [30]. However, in (8)–(11) below, we give sample formulas only for $0 < \alpha < 1$. At the end of this section, we also recall the Riesz transformation, which will play a crucial role in the proof of fractional Korn's inequality. The operators $(-\Delta)^{\alpha/2}$, ∇^α and $\operatorname{div}^\alpha$ with $\operatorname{Re}\alpha > 0$ will be extended to irregular fields in Sect. 3.

Throughout the paper, Lin denotes the space of all (complex) second-order tensors on an n-dimensional space, interpreted as linear transformations from $\mathbf{C}^n$ into itself; Sym is the subspace of all symmetric tensors. Further, Z denotes a finite-dimensional complex vector space endowed with a *bilinear* form which associates with any $x, y \in Z$ a complex number $x \cdot y$. Below we use the choices $Z = \mathbf{C}$, $\mathbf{C}^n$, Lin, and Sym. The tensor product $Z \otimes \mathbf{C}^n$ is interpreted as the space of all linear transformations from $\mathbf{C}^n$ into Z. If $z \in Z$ and $x \in \mathbf{C}^n$ then $z \otimes x \in Z \otimes \mathbf{C}^n$ is a linear transformation given by $(z \otimes x)y = z(x \cdot y)$ for every $y \in \mathbf{C}^n$, where $x \cdot y = \sum_{i=1}^n x_i y_i$.

We denote by $L^p(Z)$ the set of all Z-valued functions on $\mathbf{R}^n$ integrable with p th power, $1 \le p \le \infty$, by $\mathscr{S}(Z)$ the set of all infinitely differentiable rapidly decaying Z-valued functions on $\mathbf{R}^n$, and by $\mathscr{D}(Z)$ the subset of $\mathscr{S}(Z)$ consisting of functions with compact support.

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[7] The space of Riesz potentials is larger than the more familiar space of Bessel potentials, see (39).

[8] After the completion of the present research, the author has learned of the paper by Evgrafov & Bellido [9] on nonlocal and fractional elasticity. Section 3.5 of their paper has many features in common with the present paper.

Definition 2.1 (Cf. [18], [19]) We denote by $\mathscr{T}(Z)$ the space of all infinitely differentiable maps $f : \mathbf{R}^n \to Z$ whose derivatives $\nabla^i f$ of any order $i \in \mathbf{N}_0 := \{0, 1, \dots\}$ are bounded and integrable on $\mathbf{R}^n$. As a consequence, one obtains that $\nabla^i f(x) \to 0$ as $|x| \to \infty$ for each $i \in \mathbf{N}_0$. We introduce an increasing sequence of norms $\| \cdot \|_k$ on $\mathscr{T}(Z)$, $k \in \mathbf{N}_0$, by

$$\|f\|_k = \max\{|\nabla^i f|_{L^1}, |\nabla^i f|_{L^\infty} : 0 \le i \le k\}.$$

A sequence $\{f_l : l \in \mathbf{N}\}$ of elements of $\mathscr{T}(Z)$ is said to converge to an element $f \in \mathscr{T}(Z)$ if and only if

$$\|f - f_l\|_k \to 0 \ \text{ as } \ l \to \infty \ \text{ for every fixed } \ k \in \mathbf{N}_0.$$

Clearly,

$$\mathscr{S}(Z) \subset \mathscr{T}(Z); \tag{7}$$

it turns out that the imbedding is continuous and dense (even $\mathscr{D}(Z)$ is dense in $\mathscr{T}(Z)$), [19].

Definition 2.2 Let $0 < \alpha < 1$. The fractional laplacean of $f \in \mathscr{T}(Z)$ of order $\alpha/2$ is a function $(-\Delta)^{\alpha/2} f : \mathbf{R}^n \to Z$ defined by

$$(-\Delta)^{\alpha/2} f(x) = \nu_\alpha \int_{\mathbf{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+\alpha}} \, dy \tag{8}$$

for every $x \in \mathbf{R}^n$, where

$$\nu_\alpha := 2^\alpha \pi^{-n/2} \Gamma\big((n + \alpha)/2\big) / \, \Gamma(-\alpha/2).$$

If $\alpha \in \mathbf{C}$ and $\operatorname{Re} \alpha > 0$, the operator $(-\Delta)^{\alpha/2}$ maps the space $\mathscr{T}(Z)$ continuously into itself [19], [30].

The definition (8) is standard; I refer, e.g., to [24, Eq. (25.59)], where the authors use the notation $\mathbf{D}^\alpha$ for $(-\Delta)^{\alpha/2}$.

Definition 2.3 Let $0 < \alpha < 1$. The fractional gradient of order α of $u \in \mathscr{T}(Z)$ is a function $\nabla^\alpha u : \mathbf{R}^n \to Z \otimes \mathbf{C}^n$ defined by

$$\nabla^\alpha u(x) = \mu_\alpha \int_{\mathbf{R}^n} \frac{\big(u(x) - u(y)\big) \otimes (x - y)}{|x - y|^{n+\alpha+1}} \, dy \tag{9}$$

for any $x \in \mathbf{R}^n$. In particular, if $u \in \mathscr{T}(\mathbf{C}^n)$, then (9) defines a function $\nabla^\alpha u : \mathbf{R}^n \to \mathrm{Lin}$ and if $f \in \mathscr{T}(\mathbf{C})$ then (9) simplifies to

$$\nabla^\alpha f(x) = \mu_\alpha \int_{\mathbf{R}^n} \frac{(x - y)\big(f(x) - f(y)\big)}{|x - y|^{n+\alpha+1}} \, dy \tag{10}$$

and $\nabla^\alpha u : \mathbf{R}^n \to \mathbf{C}^n$.[9] Apparently, the symbol ∇^α occurs first in [25, p. 3534]. The term "fractional gradient" appears first in [27]. However, the origins of this notion are found in the works of Horváth [13], [14]. See also [28], [30] and [5], [6].

If $\alpha \in \mathbf{C}$ and $\operatorname{Re} \alpha > 0$, the operator ∇^α maps the space $\mathscr{T}(Z)$ continuously into $\mathscr{T}(Z \otimes \mathbf{C}^n)$.

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[9] Here I use the form (10) given by Comi & Stefani [5, Sect. 2], which is equivalent to the formulas for ∇^α given elsewhere in the literature. A similar remark applies to the fractional divergence $\operatorname{div}^\alpha$, introduced originally in [30, Sect. 2] by an equivalent formula.

Definition 2.4 Let $0 < \alpha < 1$. The fractional divergence of order α of $\sigma \in \mathscr{T}(Z \otimes \mathbf{C}^n)$ is a function $\operatorname{div}^\alpha \sigma : \mathbf{R}^n \to Z$ defined by

$$\operatorname{div}^\alpha \sigma(x) = \mu_\alpha \int_{\mathbf{R}^n} \frac{\big(\sigma(x) - \sigma(y)\big)(x - y)}{|x - y|^{n+\alpha+1}}\, dy, \tag{11}$$

$x \in \mathbf{R}^n$. In particular, if $\sigma \in \mathscr{T}(\mathrm{Lin})$ then (11) defines a function $\operatorname{div}^\alpha \sigma : \mathbf{R}^n \to \mathbf{C}^n$ and if $u \in \mathscr{T}(\mathbf{C}^n)$ then (11) reads

$$\operatorname{div}^\alpha u(x) = \mu_\alpha \int_{\mathbf{R}^n} \frac{(x - y) \cdot \big(u(x) - u(y)\big)}{|x - y|^{n+\alpha+1}} dy$$

and defines a function $\operatorname{div}^\alpha u : \mathbf{R}^n \to \mathbf{C}$.

If $\alpha \in \mathbf{C}$ and $\operatorname{Re} \alpha > 0$, the operator $\operatorname{div}^\alpha$ maps the space $\mathscr{T}(Z \otimes \mathbf{C}^n)$ into $\mathscr{T}(Z)$.

Definition 2.5 The Riesz transform Rf of $f \in \mathscr{T}(\mathbf{C})$ is defined by

$$Rf(x) = \mu_0 \lim_{\epsilon \downarrow 0} \int_{|x-y|>\epsilon} \frac{(x - y) f(y)}{|x - y|^{n+1}}\, dy, \tag{12}$$

$x \in \mathbf{R}^n$. Here μ_0 is the particular case of μ_α.

Theorem 2.6 *Let $1 < p < \infty$. The limit in (12) exists for every $f \in L^p(\mathbf{R})$ and for almost every $x \in \mathbf{R}^n$. This limit defines a continuous linear operator, again denoted by R, from $L^p(\mathbf{R})$ into $L^p(\mathbf{C}^n)$. We write $Rf = (R_1 f, \ldots, R_n f)$, where the components $R_i f$ map $L^p(\mathbf{R})$ into itself.*

3 Fractional Vector Calculus II (Distributions)

We denote by $\mathscr{T}'(Z)$ the topological dual of $\mathscr{T}(Z)$. We write $\langle f, g\rangle$ for the value of $f \in \mathscr{T}'(Z)$ on $g \in \mathscr{T}(Z)$.

In this section we extend the fractional operators $(-\Delta)^{\alpha/2}$, ∇^α and $\operatorname{div}^\alpha$ from the spaces of smooth test functions to their duals by means of weak (distributional) definitions. Weak definitions have been used in the context of fractional calculus in, e.g., [24, Sect. 8.3], [19], [30, Sect. 6] and [5]. I follow [30].

It follows from the continuity of the imbedding (7) that by restricting the domain of definition of a functional $f \in \mathscr{T}'(Z)$ to $\mathscr{S}(Z)$, we obtain a Z-valued tempered distribution $f_0 \in \mathscr{S}'(Z)$; the density of the imbedding shows that conversely f_0 completely determines f. In view of this, we can consider the elements of $\mathscr{T}'(Z)$ as a special class of tempered distributions and write

$$\mathscr{T}'(Z) \subset \mathscr{S}'(Z). \tag{13}$$

Any element $f \in \mathscr{T}(Z)$ determines a linear functional in $\mathscr{T}'(Z)$, again denoted by f, defined by

$$\langle f, g\rangle := \int_{\mathbf{R}^n} f \cdot g\, dx \tag{14}$$

for any $g \in \mathscr{T}(Z)$. Thus we have the imbedding

$$\mathscr{T}(Z) \subset \mathscr{T}'(Z).$$

Further, it is not hard to see that any functional represented by a function in $L^p(Z)$ ($1 \le p \le \infty$) belongs to $\mathscr{T}'(Z)$, and any distribution in $\mathscr{D}'(Z)$ with compact support can be extended to a functional in $\mathscr{T}'(Z)$.

Definition 3.1 The weak fractional laplacean in $\mathscr{T}'(Z)$ is defined as the adjoint of the original fractional laplacean defined for test functions. Hence the weak fractional laplacean $(-\Delta)^{\alpha/2} f$ of $f \in \mathscr{T}'(Z)$ is an element of $\mathscr{T}'(Z)$ satisfying

$$\big\langle (-\Delta)^{\alpha/2} f, g \big\rangle = \big\langle f, (-\Delta)^{\alpha/2} g \big\rangle \tag{15}$$

for any $g \in \mathscr{T}(Z)$, where $(-\Delta)^{\alpha/2} g$ is given by (8). Equation (15) is motivated by the following easily verifiable identity

$$\int_{\mathbf{R}^n} (-\Delta)^{\alpha/2} f \cdot g dx = \int_{\mathbf{R}^n} f \cdot (-\Delta)^{\alpha/2} g dx$$

for any $f, g \in \mathscr{T}(Z)$, where the laplaceans are given by (8). For elements of $\mathscr{T}(Z)$ the weak definition (15) reduces to the original definition (8).

Definition 3.2 The weak fractional gradient on $\mathscr{T}'(Z)$ is defined as minus the adjoint of the fractional divergence on test functions. Thus the weak fractional gradient $\nabla^\alpha u$ of $u \in \mathscr{T}'(Z)$ is an element of $\mathscr{T}'(Z \otimes \mathbf{C}^n)$ satisfying

$$\langle \nabla^\alpha u, \sigma \rangle = - \langle u, \operatorname{div}^\alpha \sigma \rangle \tag{16}$$

for every $\sigma \in \mathscr{T}(Z \otimes \mathbf{C}^n)$, where $\operatorname{div}^\alpha \sigma$ is given by (11). Equation (16) is motivated by the identity

$$\int_{\mathbf{R}^n} \nabla^\alpha u \cdot \sigma dx = - \int_{\mathbf{R}^n} u \cdot \operatorname{div}^\alpha \sigma dx$$

for each $u \in \mathscr{T}(Z)$, $\sigma \in \mathscr{T}(Z \otimes \mathbf{C}^n)$, where $\nabla^\alpha u$ and $\operatorname{div}^\alpha u$ are given by (9) and (11). Again, for elements of $\mathscr{T}(Z)$, the weak fractional gradient reduces to the original definition in Sect. 2.

Definition 3.3 The weak fractional divergence on $\mathscr{T}'(Z \otimes \mathbf{C}^n)$ is defined as minus the adjoint of the fractional gradient on test functions. Thus the weak fractional divergence $\operatorname{div}^\alpha \sigma$ of $\sigma \in \mathscr{T}'(Z \otimes \mathbf{C}^n)$ is an element of $\mathscr{T}'(Z)$ satisfying (16) for every $u \in \mathscr{T}(Z)$, where $\nabla^\alpha u$ is given by (9).

4 Fourier Transformation

This section determines the Fourier transforms of the fractional operators $(-\Delta)^{\alpha/2}$, ∇^α and $\operatorname{div}^\alpha$; see Equations (21)–(25). These Fourier transforms will play basic roles in analyzing equations of the fractional approach to continuous media in Sects. 6 and 7.

4.1 Fourier Transformation in $L^1(Z)$ and $\mathscr{T}(Z)$

The Fourier transform of a function $f \in L^1(Z)$ is the function $Ff \equiv \hat{f}$ defined by

$$Ff(\xi) \equiv \hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) \mathrm{e}^{\mathrm{i}x\cdot\xi}\, dx, \quad \xi \in \mathbf{R}^n; \tag{17}$$

the inverse transform is

$$F^{-1}\hat{f}(x) \equiv f(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \hat{f}(\xi) \mathrm{e}^{-\mathrm{i}x\cdot\xi}\, dx, \quad x \in \mathbf{R}^n, \tag{18}$$

provided $\hat{f} \in L^1(Z)$.

Remark. Definitions (17) and (18) are identical with those in [10, Chapter II, Sect. 1] and [24, p. 484], including the normalization factors (which may vary from author to author).

We denote by $\hat{\mathscr{T}}(Z)$ the set of Fourier transforms of functions from $\mathscr{T}(Z) \subset L^1(Z)$:

$$\hat{\mathscr{T}}(Z) = \{\hat{f} : f \in \mathscr{T}(Z)\}.$$

Proposition 4.2

(i) *Each function* $\varphi \in \hat{\mathscr{T}}(Z)$ *is continuous and rapidly decaying in the sense that for every non-negative integer* i *there exists a constant* $c = c(i, \varphi)$ *such that*

$$|\varphi(\xi)| \le c\bigl(1 + |\xi|^{-i}\bigr)$$

for each $\xi \in \mathbf{R}^n$;

(ii) *We have*

$$\mathscr{S}(Z) \subset \hat{\mathscr{T}}(Z); \tag{19}$$

(iii) *If* $\operatorname{Re}\alpha > 0$ *and* $\varphi \in \hat{\mathscr{T}}(Z)$ *then*

$$|\xi|^\alpha \varphi \in \hat{\mathscr{T}}(Z) \ \textit{and}\ |\xi|^{\alpha-1}\varphi \otimes \xi \in \hat{\mathscr{T}}(Z \otimes \mathbf{C}^n). \tag{20}$$

It follows from (i) and (ii) that the set $\hat{\mathscr{T}}(Z)$ contains non-differentiable functions: if $\varphi \in \mathscr{S}(Z)$ does not vanish at the origin in $\mathbf{R}^n$ and $0 < \operatorname{Re}\alpha < 1$, then $|\xi|^\alpha \varphi \in \hat{\mathscr{T}}(Z)$ is not differentiable at the origin.

Proof (i): Since any derivative $\nabla^i f$ of an $f \in \mathscr{T}(Z)$ is integrable, its Fourier transform $\xi^i \hat{f}(\xi)$ is bounded and continuous.

(ii): Equation (19) follows from (7) and $F\mathscr{S}(Z) = \mathscr{S}(Z)$.

(iii): Equation $(20)_{1,2}$ follow from (21) and (23), below. □

4.3 Fourier Transforms of Fractional Differential Operators

In the following table, the left column displays the value Oa of a fractional differential operator O on a function a, while the right column displays the Fourier transform of Oa on the Fourier transform $\hat{a}$ of a. Let $\operatorname{Re}\alpha > 0$.

$$(-\Delta)^{\alpha/2} f \qquad\qquad |\xi|^\alpha \hat{f}, \tag{21}$$

$$\nabla^\alpha f \qquad\qquad -\mathrm{i}\xi|\xi|^{\alpha-1}\hat{f}, \tag{22}$$

$$\nabla^\alpha u \qquad -\mathrm{i}|\xi|^{\alpha-1}\hat{u}\otimes\xi, \tag{23}$$

$$\operatorname{div}^\alpha u \qquad -\mathrm{i}|\xi|^{\alpha-1}\xi\cdot\hat{u}, \tag{24}$$

$$\operatorname{div}^\alpha \sigma \qquad -\mathrm{i}|\xi|^{\alpha-1}\hat{\sigma}\xi, \tag{25}$$

$$Rf \qquad \mathrm{i}\xi|\xi|^{-1}\hat{f}, \tag{26}$$

here

(21) *holds for* f *from* $\mathscr{T}(\mathrm{Z})$,

(22) *and* (26) *hold for* f *from* $\mathscr{T}(\mathbf{C})$,

(23) *and* (24) *hold for* u *from* $\mathscr{T}(\mathbf{C}^n)$ *or* from $\mathscr{T}(\mathrm{Z})$,

and (25) *holds for* σ *from* $\mathscr{T}(\mathrm{Lin})$ *or from* $\mathscr{T}(\mathrm{Z}\otimes\mathbf{C}^n)$.

Equation (21) is well-known; see e.g., [24, Eq. (25.62)]. Equations (22)–(24) are direct consequences of (21). For example, to obtain Equation (22), we first use (8) and (10) to prove that

$$\nabla^\alpha f = \nabla(-\Delta)^{(\alpha-1)/2} f; \tag{27}$$

see [30]. Then replace α by $\alpha - 1$ in (21) to obtain the right column entry in the form $|\xi|^{\alpha-1}\hat{f}(\xi)$. Finally, we use the well-known fact that the Fourier transformation changes the differentiation into the multiplication by $-\mathrm{i}\xi$ to convert $|\xi|^{\alpha-1}\hat{f}(\xi)$ into the right-entry of (22). The reader is referred to [27] for the first occurrence of (22) and a different proof. Formula (25) is proved similarly; only (27) is replaced by

$$\operatorname{div}^\alpha u = \operatorname{div}(-\Delta)^{(\alpha-1)/2} u.$$

Finally, (26) is proved in [8, Eq. (4.8), p. 76] and in [32, Eq. (8), p. 58].

4.4 Fourier Transformation in $\mathscr{T}'(\mathbf{Z})$

We define the Fourier transform of $f \in \mathscr{T}'(\mathrm{Z})$ as the functional $Ff \equiv \hat{f}$ on $\hat{\mathscr{T}}(\mathrm{Z})$ given by

$$\left\langle \hat{f}, \varphi \right\rangle = \langle f, \hat{\varphi} \rangle$$

for every $\varphi \in \hat{\mathscr{T}}(\mathrm{Z})$, where $\hat{\varphi} \in \mathscr{T}(\mathrm{Z})$ is the (direct) Fourier transform of φ. We denote by $\hat{\mathscr{T}}'(\mathrm{Z})$ the image of $\mathscr{T}'(\mathrm{Z})$ under the Fourier transform:

$$\hat{\mathscr{T}}'(\mathrm{Z}) = \{\hat{f} : f \in \mathscr{T}'(\mathrm{Z})\}.$$

Alternatively, in view of (19) we can restrict the domain of any functional $\Lambda \in \hat{\mathscr{T}}'(\mathrm{Z})$ to obtain a tempered distribution $\Lambda_0 \in \mathscr{S}'(Z)$. This allows to interpret $\hat{\mathscr{T}}'(Z)$ as a special class of Z-valued tempered distributions, i.e.,

$$\hat{\mathscr{T}}'(\mathrm{Z}) \subset \mathscr{S}'(\mathrm{Z}). \tag{28}$$

This is also consistent with the inclusion (13), since the Fourier transformation maps the space $\mathscr{S}'(Z)$ onto itself. The interpretation (28) is sometimes convenient.

Let Λ be a linear functional on $\hat{\mathscr{T}}(Z)$ and Ξ a linear functional on $\hat{\mathscr{T}}(Z \otimes \mathbf{C}^n)$. Proposition 4.2(iii) allows to define the products $|\xi|^\alpha \Lambda$, $|\xi|^{\alpha-1} \Lambda \otimes \xi$ and $|\xi|^{\alpha-1} \Xi \xi$ as linear functionals on $\mathscr{S}(Z)$, $\mathscr{S}(Z \otimes \mathbf{C}^n)$ and $\mathscr{S}(Z)$, respectively, by

$$\langle |\xi|^\alpha \Lambda, \varphi \rangle = \langle \Lambda, |\xi|^\alpha \varphi \rangle, \tag{29}$$

$$\left\langle |\xi|^{\alpha-1} \Lambda \otimes \xi, \psi \right\rangle = \left\langle \Lambda, |\xi|^{\alpha-1} \psi \xi \right\rangle \tag{30}$$

and

$$\left\langle |\xi|^{\alpha-1} \Xi \xi, \varphi \right\rangle = \left\langle \Xi, |\xi|^{\alpha-1} \varphi \otimes \xi \right\rangle \tag{31}$$

for every $\varphi \in \hat{\mathscr{T}}(Z)$ and $\psi \in \hat{\mathscr{T}}(Z \otimes \mathbf{C}^n)$. The definitions (29)–(31) in particular apply when Λ or Ξ are identified with the Fourier transforms of the elements of $\mathscr{T}'(Z)$ or $\mathscr{T}'(Z \otimes \mathbf{C}^n)$. With the definitions (29)–(31), Equations (21)–(25) can be extended to the elements f, u and σ of the duals $\mathscr{T}'(Z)$.

5 The Space of Riesz Potentials

We now introduce the basic space of solutions of the equilibrium problem of the fractional elasticity (Sect. 7).

Definition 5.1 (Samko [23], Samko, Kilbas & Marichev [24, § 26]) Let $0 < \alpha < n$ and $1 < p < n/\alpha$. We define the space of Z-valued Riesz potentials $R^{\alpha,p}(Z)$ as the set of all convolutions

$$f = I_\alpha * \varphi \quad \text{where} \quad \varphi \in L^p(\mathbf{R}^n, Z) \tag{32}$$

and I_α is the Riesz kernel,

$$I_\alpha(x) = \nu_{-\alpha} |x|^{\alpha-n}, \qquad 0 \neq x \in \mathbf{R}^n.$$

We define the norm $|\cdot|_{\alpha,p}$ on $R^{\alpha,p}(Z)$ by

$$|f|_{\alpha,p} = |\varphi|_{L^p} \tag{33}$$

for any f as in (32). This norm renders $R^{\alpha,p}(Z)$ a Banach space.

We refer to Herz [11], Maz'ya & Havin [20], Johnson [16], Peetre [22] for additional references on spaces of Riesz potentials.

Recall that in Sect. 3 we defined the fractional laplacean and fractional gradient of any $f \in \mathscr{T}'(Z)$ as elements of $\mathscr{T}'(Z)$ and $\mathscr{T}'(Z \otimes \mathbf{C}^n)$, respectively. We now consider particular cases when the fractional laplacean and fractional gradient are represented by integrable functions.

Definitions 5.2 Let $\alpha > 0$ and $1 \le p \le \infty$.

(i) We say that an element f of $\mathscr{T}'(Z)$ has a weak fractional laplacean of order $\alpha/2$ in L^p if there exists an element $(-\Delta)^{\alpha/2} f \in L^p(Z)$ such that

$$\left\langle f, (-\Delta)^{\alpha/2} g \right\rangle = \int_{\mathbf{R}^n} (-\Delta)^{\alpha/2} f \cdot g dx$$

for every $g \in \mathscr{T}(Z)$. If this is the case, we write $(-\Delta)^{\alpha/2} f \in L^p(Z)$.

(ii) We say that an element f of $\mathscr{T}'(Z)$ has a weak fractional gradient of order α in L^p if there exists an element $\nabla^\alpha f \in L^p(Z \otimes \mathbf{C}^n)$ such that

$$\langle f, \operatorname{div}^\alpha g \rangle = -\int_{\mathbf{R}^n} \nabla^\alpha f \cdot g dx$$

for every $g \in \mathscr{T}(Z \otimes \mathbf{C}^n)$. If this is the case, we write $\nabla^\alpha f \in L^p(Z \otimes \mathbf{C}^n)$.

Definitions of the type 5.2(i) and (ii) occur in [5, Sects. 2 and 3].

Proposition 5.3 *If* $1 < p < \infty, \alpha > 0$ *and* $f \in \mathscr{T}'(Z)$ *then*

$$(-\Delta)^{\alpha/2} f \in L^p(Z) \quad \Leftrightarrow \quad \nabla^\alpha f \in L^p(Z \otimes \mathbf{C}^n).$$

Cf. [27], [28] and [5].

Proof We deduce from (21), (22) and (26) that

$$\nabla^\alpha f = R(-\Delta)^{\alpha/2} f \tag{34}$$

for any $f \in \mathscr{T}(Z)$. By the weak definitions 5.2(i), (ii), Equation (34) extends to $f \in \mathscr{T}'(Z)$ as in the statement of the proposition. □

Theorem 5.4 (Characterization of Riesz potentials) *We have*

$$\begin{aligned} R^{\alpha,p}(Z) &= \{f \in L^{p^*}(Z) : (-\Delta)^{\alpha/2} f \in L^p(Z)\} \\ &= \{f \in L^{p^*}(Z) : \nabla^\alpha f \in L^p(Z \otimes \mathbf{C}^n)\} \end{aligned} \tag{35}$$

for any $0 < \alpha < n$ *and* $1 < p < n/\alpha$, *where*

$$p^* = np/(n - \alpha p).$$

Moreover, the operator $(-\Delta)^{\alpha/2}$ *is the left inverse of the operator* I_α *on* $L^p(Z)$, *i.e.,*

$$(-\Delta)^{\alpha/2}(I_\alpha * \varphi) = \varphi \ \textit{ for every } \ \varphi \in L^p(Z). \tag{36}$$

In view of (36), Equation (32) can be rewritten formally as

$$f = (-\Delta)^{-\alpha/2} \varphi.$$

Proof The reader is referred to [24, Theorem 26.8] for the proof of $(35)_1$; Equation $(35)_2$ is then a consequence of Proposition 5.3. For the proof of (36), see [24, Theorem 26.3]. □

Remark 5.5 (Imbedding of $R^{\alpha,p}$ with respect to α) If $0 < \alpha < n$, $0 < p < n/\alpha$, and $0 \le \beta \le \alpha$ then

$$R^{\alpha,p}(Z) \subset R^{\beta,q}(Z) \quad \text{where} \quad q = \frac{np}{n - (\alpha - \beta)p}.$$

See [23, Theorem 6].

The space of Riesz potentials can be compared with the more familiar space of Bessel potentials.

Definition 5.6 Let $\alpha > 0$ and $1 < p < \infty$. We define the space of Z-valued Bessel potentials $L^{\alpha,p}(Z)$ as the set of all convolutions

$$f = G_\alpha * \varphi \quad \text{where} \quad \varphi \in L^p(\mathbf{R}^n, Z) \tag{37}$$

and G_α is the Bessel kernel, i.e., the inverse Fourier transform (in the sense of distributions) of the function $\hat{G}_\alpha(\xi) := (1 + |\xi|^2)^{-\alpha/2}$. We define the norm $|\cdot|'_{\alpha,p}$ on $L^{\alpha,p}(Z)$ by

$$|f|'_{\alpha,p} = |\varphi|_{L^p}$$

for any f as in (37). This norm renders $L^{\alpha,p}(Z)$ a Banach space. We refer, e.g., to [24, Sect. 27.1] for the basic information about spaces of Bessel potentials.

Theorem 5.7 (Characterization of Bessel potentials)) *We have*

$$\begin{aligned} L^{\alpha,p}(Z) &= \{f \in L^p(Z) : (-\Delta)^{\alpha/2} f \in L^p(Z)\} \\ &= \{f \in L^p(Z) : \nabla^\alpha f \in L^p(Z \otimes \mathbf{C}^n)\} \end{aligned} \tag{38}$$

for any $\alpha > 0$ *and* $1 < p < \infty$. *Consequently, if* $0 < \alpha < n$ *and* $1 < p < n/\alpha$, *then*

$$L^{\alpha,p}(Z) = R^{\alpha,p}(Z) \cap L^p(Z). \tag{39}$$

Proof For the proof of $(38)_1$, see [31] for $0 < \alpha < 2$ and [24, Theorem 27.3] for the general case. Equation $(38)_2$ is then a consequence of Proposition 5.3. ☐

6 Weak Fractional Strain Tensor. Fractional Korn's Inequality

The fractional gradient (see (9)) allows to rewrite the definition (1) of the fractional strain tensor as

$$\epsilon^\alpha(u) = \frac{1}{2}(\nabla^\alpha u + \nabla^\alpha u^{\mathrm{T}}) \tag{40}$$

for every $u \in \mathscr{T}(\mathbf{C}^n)$, in analogy to the relation

$$\epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$$

for the classical strain tensor.

We now give the weak definition of the fractional strain tensor and two particular cases.

Definitions 6.1 Let $0 < \alpha < 1$.

(i) If $u \in \mathscr{T}'(\mathbf{C}^n)$, we define the weak fractional strain tensor $\epsilon^\alpha(u)$ of u as an element of $\mathscr{T}'(\mathrm{Sym})$ given by

$$\langle \epsilon^\alpha(u), \sigma \rangle = -\langle u, \operatorname{div}^\alpha \sigma \rangle \tag{41}$$

for any $\sigma \in \mathscr{T}(\mathrm{Sym})$.

(ii) Let $1 \le p \le \infty$. A deformation $u \in \mathscr{T}'(\mathbf{C}^n)$ is said to have the weak strain tensor of order α in L^p if there exists $\epsilon^\alpha(u) \in L^p(\mathrm{Sym})$ satisfying (41) for any $\sigma \in \mathscr{T}(\mathrm{Sym})$. If this is the case, we write $\epsilon^\alpha(u) \in L^p(\mathrm{Sym})$.

(iii) We say that $u \in L^1(\mathbf{R}^n, \mathbf{R}^n)$ is a displacement of bounded fractional deformation if the weak fractional strain tensor $\epsilon^\alpha(u)$ is represented by a Sym-valued measure on $\mathbf{R}^n$, again denoted by $\epsilon^\alpha(u)$. Equation (41) then reads

$$\int_{\mathbf{R}^n} \sigma \cdot d\,\epsilon^\alpha(u) = -\int_{\mathbf{R}^n} u \cdot \operatorname{div}^\alpha \sigma\, d\mathscr{L}^n$$

for every $\sigma \in \mathscr{T}(\mathrm{Sym})$. If this is the case, we write $\epsilon^\alpha(u) \in \mathscr{M}(\mathrm{Sym})$.

We often omit the modifier "weak" and speak of "fractional strain tensor." With Definition 6.1(i), Equation (40) continues to hold provided that ∇^α is interpreted as the weak fractional gradient.

Theorem 6.2 (Fractional Korn's inequality) *If* $0 < \alpha < 1$ *and* $1 < p < \infty$, *there exists a positive constant* $c = c(n, p, \alpha)$ *such that for every* $u \in \mathscr{T}'(\mathbf{C}^n)$ *with* $\epsilon^\alpha(u) \in L^p(\mathrm{Sym})$ *we have* $\nabla^\alpha u \in L^p(\mathrm{Lin})$ *and*

$$|\epsilon^\alpha(u)|_{L^p} \geq c|\nabla^\alpha u|_{L^p}. \tag{42}$$

Proof Prove preliminarily that any $u \in \mathscr{T}'(\mathbf{C}^n)$ with $\epsilon^\alpha(u) \in L^p(\mathrm{Sym})$ satisfies $(-\Delta)^{\alpha/2} u \in L^p(\mathbf{C}^n)$ and

$$(-\Delta)^{\alpha/2} u_i = R_k\big(2\epsilon^\alpha_{ik} + R_i R_l \epsilon^\alpha_{kl}\big) \tag{43}$$

$1 \leq i \leq n$, where we use the summation convention and denote by R_i the components of the Riesz transformation (Theorem 2.6). To prove (43), we denote by $\hat{u}$ and $\hat{\epsilon}^\alpha$ the Fourier transforms of u and $\epsilon^\alpha(u)$. By (23), the Fourier transform of $\nabla^\alpha u$ is the function $\xi \mapsto -\mathrm{i}|\xi|^{\alpha-1}\hat{u} \otimes \xi$ and hence (40) gives

$$\hat{\epsilon}^\alpha = -\frac{\mathrm{i}}{2}\big(|\xi|^{\alpha-1}\hat{u} \otimes \xi + |\xi|^{\alpha-1}\xi \otimes \hat{u}\big) \tag{44}$$

or in components

$$\hat{\epsilon}^\alpha_{ik} = -\frac{\mathrm{i}}{2}\big(|\xi|^{\alpha-1}\hat{u}_i \xi_k + |\xi|^{\alpha-1}\xi_i \hat{u}_k\big). \tag{45}$$

By (26), the Fourier transform of $R_k f$ is the function $\xi \mapsto \mathrm{i}\xi_k|\xi|^{-1}\hat{f}$ and thus by (45),

$$\hat{R}_i \hat{R}_k \hat{R}_l \hat{\epsilon}^\alpha_{kl} = -|\xi|^{\alpha-2}\xi_i(\hat{u} \cdot \xi)$$

and

$$\hat{R}_k \hat{\epsilon}^\alpha_{ik} = \frac{1}{2}\big(|\xi|^\alpha \hat{u}_i + |\xi|^{\alpha-2}\xi_i(\hat{u} \cdot \xi)\big).$$

Hence

$$|\xi|^\alpha \hat{u}_i = 2\hat{R}_k \hat{\epsilon}^\alpha_{ik} + \hat{R}_i \hat{R}_k \hat{R}_l \hat{\epsilon}^\alpha_{kl}$$

and (43) follows. As a consequence,

$$\nabla^\alpha u_{ij} = -2R_k R_j \epsilon^\alpha_{ik}(u) - R_i R_j R_k R_l \epsilon^\alpha_{kl}(u), \tag{46}$$

$1 \leq i, j \leq n$.

By Theorem 2.6, every component R_i of the Riesz transformation maps $L^p(\mathbf{R})$ continuously into itself if $p > 1$. One can thus estimate each term in (46) to obtain (42) for each $u \in \mathscr{T}(\mathbf{C}^n)$ with some constant $c = c(n, p, \alpha)$ that is determined by the norm of the Riesz transformation. $\square$

Proposition 6.3 (Compatibility conditions) *If* $0 < \alpha < 1$ *and* $1 < p < \infty$ *then for any* $u \in \mathscr{T}'(\mathbf{C}^n)$ *with* $\epsilon^\alpha(u) \in L^p(\mathrm{Sym})$ *and for any collection of integers* $i, j, k, l \in \{1, \ldots, n\}$ *we have*

$$\nabla_l^\alpha \nabla_j^\alpha \epsilon_{ik}^\alpha + \nabla_k^\alpha \nabla_i^\alpha \epsilon_{jl}^\alpha - \nabla_l^\alpha \nabla_i^\alpha \epsilon_{jk}^\alpha - \nabla_k^\alpha \nabla_j^\alpha \epsilon_{il}^\alpha = 0 \tag{47}$$

in the weak sense, i.e.,

$$\left\langle \epsilon_{ik}^\alpha, \nabla_j^\alpha \nabla_l^\alpha \varphi \right\rangle + \left\langle \epsilon_{jl}^\alpha, \nabla_i^\alpha \nabla_k^\alpha \varphi \right\rangle - \left\langle \epsilon_{jk}^\alpha, \nabla_i^\alpha \nabla_l^\alpha \varphi \right\rangle - \left\langle \epsilon_{il}^\alpha, \nabla_j^\alpha \nabla_k^\alpha \varphi \right\rangle = 0 \tag{48}$$

for every $\varphi \in \mathscr{T}(\mathbf{C})$, *where we use the notation* (14).

Proof We first prove (47) for any $u \in \mathscr{T}(\mathbf{C}^n)$. Passing to the Fourier transforms, we observe that (45) gives

$$\xi_l \xi_j \hat{\epsilon}_{ik}^\alpha = -\frac{\mathrm{i}}{2} \left(|\xi|^{\alpha-1} \hat{u}_i \xi_k \xi_l \xi_j + |\xi|^{\alpha-1} \xi_i \hat{u}_k \xi_l \xi_j \right).$$

Making the permutations indicated in (47), and summing as suggested there, one obtains

$$-\xi_l \xi_j |\xi|^{2\alpha-2} \hat{\epsilon}_{ik}^\alpha - \xi_k \xi_i |\xi|^{2\alpha-2} \hat{\epsilon}_{jl}^\alpha + \xi_l \xi_i |\xi|^{2\alpha-2} \hat{\epsilon}_{jk}^\alpha + \xi_k \xi_j |\xi|^{2\alpha-2} \hat{\epsilon}_{il}^\alpha = 0$$

The left-hand side of this equation is the Fourier transform of the left-hand side of (47). A multiplication by $\varphi \in \mathscr{T}(\mathbf{C})$ and an integration by parts establishes (48) for every $u \in \mathscr{T}(\mathbf{C}^n)$. This is extended to all u from $R^{\alpha,p}(\mathbf{C}^n)$ by density ([23, Theorem 9]). $\square$

7 Fractional Linear Elasticity

We consider a fractional elastic body of order α which occupies the whole n-dimensional space $\mathbf{R}^n$. The body is described by the strain tensor $\epsilon^\alpha(u)$. Throughout the section we assume that $n \geq 2$ and $0 < \alpha < 1$.

7.1 Energy

The behavior of the body is determined by the fourth-order tensor of elastic constants C, which we interpret as a linear transformation from Sym into Sym. We enclose the argument of C in square brackets, i.e., $\mathsf{C}[a]$ is the value of C on $a \in \mathrm{Sym}$. We assume that C has the major symmetry

$$\mathsf{C}[a] \cdot b = \mathsf{C}[b] \cdot a \quad \text{for every} \quad a, b \in \mathrm{Sym},$$

where $\cdot$ denotes the scalar product on Sym, given by $a \cdot b = \mathrm{tr}(ab)$ for every $a, b \in \mathrm{Sym}$. Throughout the section we assume that C is strongly elliptic, i.e.,

$$\frac{1}{2} \mathsf{C}[v \otimes w + w \otimes v] w \cdot v > 0 \tag{49}$$

for every v, $w \in \mathbf{R}^n$, $v \neq 0 \neq w$. Since $\mathsf{C}[v \otimes w + w \otimes v]$ is symmetric, (49) can be restated equivalently as

$$\mathsf{C}[\operatorname{sym}(v \otimes w)] \cdot \operatorname{sym}(v \otimes w) > 0 \tag{50}$$

where $\operatorname{sym}(a) = (a + a^{\mathrm{T}})/2$ for any $a \in \mathrm{Lin}$.

If the body is subjected to the body force $f : \mathbf{R}^n \to \mathbf{R}^n$, the total energy is given by

$$E(u) = \frac{1}{2} \int_{\mathbf{R}^n} \mathsf{C}[\epsilon^\alpha(u)] \cdot \epsilon^\alpha(u)\, dx - \int_{\mathbf{R}^n} f \cdot u\, dx. \tag{51}$$

The corresponding Euler-Lagrange equations (56) and (60) are used to define equilibrium displacements.

7.2 Fractional Green's Function

The acoustic tensor of the material is the function $A : \mathbf{R}^n \to \mathrm{Sym}$ satisfying

$$b \cdot A(\xi)a = \frac{1}{2}\mathsf{C}[a \otimes \xi + \xi \otimes a]\xi \cdot b$$

for every $\xi \in \mathbf{R}^n$ and every $a, b \in \mathbf{R}^n$. The strong ellipticity of C implies the existence of a positive constant c_1 such that

$$a \cdot A(\xi)a \geq c_1 |\xi|^2 |a|^2$$

for all ξ and a in $\mathbf{R}^n$. Consequently, $A(\xi)$ is invertible for every $\xi \neq 0$. The inverse $B(\xi)$ satisfies

$$|B(\xi)| \leq c_2 |\xi|^{-2} \tag{52}$$

for some c_2 and all $\xi \neq 0$; moreover, since $A(\xi)$ is quadratic in ξ, the function B is infinitely differentiable on $\mathbf{R}^n \setminus \{0\}$ and positively homogeneous of degree -2.

Let $\hat{G}_\alpha : \mathbf{R}^n \setminus \{0\} \to \mathrm{Sym}$ be given by

$$\hat{G}_\alpha(\xi) = |\xi|^{2-2\alpha} B(\xi), \tag{53}$$

$0 \neq \xi \in \mathbf{R}^n$. By (52) we have $|\hat{G}_\alpha(\xi)| \leq c' |\xi|^{-2\alpha}$, which shows $\hat{G}$ determines a tempered distribution. Fractional Green's function G_α of the material is the inverse Fourier transform of the distribution $\hat{G}_\alpha$:

$$G_\alpha = F^{-1} \hat{G}_\alpha. \tag{54}$$

Since B is positively homogeneous of degree -2, Equation (53) shows that $\hat{G}_\alpha$ is positively homogeneous degree -2α. Elementary scaling properties of the Fourier transformation show that then G_α is positively homogeneous degree $2\alpha - n$ and hence of the form

$$G_\alpha(x) = |x|^{2\alpha - n} F_\alpha(x), \quad 0 \neq x \in \mathbf{R}^n, \quad 0 < \alpha \leq 1,$$

where F_α is a positively homogeneous degree 0 function on $\mathbf{R}^n \setminus \{0\}$. It follows from the theorem of Lemoine (see Theorem A.3) that F_α is bounded and infinitely differentiable. Indeed, the infinite differentiability of B on $\mathbf{R}^n \setminus \{0\}$ and Equation (53) show that $\hat{G}_\alpha$ is a homogeneous distribution of degree 2α that is locally in the space of Bessel potentials $L^{s,2}(\mathrm{Sym})$ for all $s > 0$ (see Definition A.2). By Theorem A.3 then Fourier transform G_α is locally in the space $L^{s-n/2+2\alpha,2}(\mathrm{Sym})$. It follows that G_α is locally in $L^{t,2}(\mathrm{Sym})$ for all $t > 0$. Hence G_α is infinitely differentiable on $\mathbf{R}^n \setminus \{0\}$ and consequently F_α is bounded and infinitely differentiable. As a consequence, we have

$$|G_\alpha(x)| \le c_3 |x|^{2\alpha-n} \tag{55}$$

for some constant c_3 and all $x \neq 0$.

Theorem 7.3 *Let $1 < q < n/2\alpha$. For each $f \in L^q(\mathbf{C}^n)$ there exists a unique $u \in R^{2\alpha,q}(\mathbf{C}^n)$ which satisfies the equilibrium equation*

$$\operatorname{div}^\alpha \mathsf{C}[\epsilon^\alpha(u)] + f = 0 \tag{56}$$

at almost every point of $\mathbf{R}^n$. The function u is given by

$$u(x) = \int_{\mathbf{R}^n} G_\alpha(x-y) f(y)\, dy, \tag{57}$$

where G_α is fractional Green's function of the material.

Remarks 7.4

(i) Inequality (55) shows that G_α is majorized by Riesz kernel of order 2α; thus the assumption $f \in L^q(\mathbf{C}^n)$ and Sobolev's inequality imply that the integral in (57) converges at almost every $x \in \mathbf{R}^n$ and defines a function that belongs to $L^{q^*}(\mathbf{C}^n)$, where

$$q^* = qn/(n - 2\alpha q).$$

(ii) Remark 5.5 shows that

$$R^{2\alpha,q}(\mathbf{C}^n) \subset R^{\alpha,p}(\mathbf{C}^n) \subset L^{q^*}(\mathbf{C}^n) \tag{58}$$

where

$$p = nq/(n - \alpha q). \tag{59}$$

(iii) Under the assumptions $f \in L^q(\mathbf{C}^n)$ and $u \in R^{2\alpha,q}(\mathbf{C}^n)$, the pointwise form (56) of the equilibrium equation is equivalent to the weak form, i.e.,

$$\int_{\mathbf{R}^n} \mathsf{C}[\epsilon^\alpha(u)] \cdot \nabla^\alpha v\, dx - \int_{\mathbf{R}^n} f \cdot v\, dx = 0 \tag{60}$$

for every infinitely differentiable function $v : \mathbf{R}^n \to \mathbf{R}^n$ with compact support.

Example 7.5 (Fractional Green's function of an isotropic body) For an isotropic body the tensor C takes the form

$$\mathsf{C}[a] = \lambda(\operatorname{tr} a) I + 2\mu a, \quad a \in \mathrm{Sym},$$

where λ, μ are Lamé's moduli of the material. The tensor C is strongly elliptic if and only if

$$\lambda + 2\mu > 0, \quad \mu > 0. \tag{61}$$

The fractional Green's function is given by

$$G_\alpha(x) = \frac{c_\alpha}{\mu|x|^{n-2\alpha}}\Big(I - \frac{\lambda+\mu}{2\alpha(\lambda+2\mu)}\big(I + (2\alpha - n)|x|^{-2}x \otimes x\big)\Big) \tag{62}$$

for any $x \in \mathbf{R}^n$, $x \neq 0$, where

$$c_\alpha := 2^{-2\alpha}\pi^{-n/2}\Gamma(n/2-\alpha)/\,\Gamma(\alpha). \tag{63}$$

Proof (Theorem 7.3) We shall first show that any solution of (56) must be given by (57). Then we shall reverse the arguments and show, with some care, that any displacement u given by (57) satisfies (56).

We employ the notation of Sect. 7.2.

Thus assume that $u \in \mathscr{T}'(\mathbf{C}^n)$ satisfies (56). Passing to the Fourier transforms of u and f and using (23) and (25) to calculate $\operatorname{div}^\alpha \mathsf{C}[\epsilon^\alpha(u)]$, one finds that (56) reads

$$|\xi|^{2\alpha-2}A(\xi)\hat{u}(\xi) = \hat{f}(\xi), \tag{64}$$

for every $\xi \in \mathbf{R}^n$. The invertibility of the acoustic tensor then yields

$$\hat{u}(\xi) = \hat{G}_\alpha(\xi)\hat{f}(\xi) \tag{65}$$

where $\hat{G}_\alpha$ is given by (53). The inverse Fourier transform and the definition (54) then give (57).

Let us show that if u is given by (57) with $f \in L^q(\mathbf{C}^n)$, then u belongs to $R^{2\alpha,q}(\mathbf{C}^n)$ and satisfies the equilibrium equation (56). It has already been shown in Remark 7.4(i) that u belongs to $L^{q^*}(\mathbf{C}^n)$. We note first that generally the Fourier transform $\hat{f}$ of a general $f \in L^q(\mathbf{C}^n)$ is not represented by a function if $q > 2$; i.e., $\hat{f}$ is generally only a tempered distribution, as explained in [12, Sect. 7.9].

Therefore, to avoid this complication, assume fist that

$$f \in \mathscr{U}(\mathbf{C}^n), \tag{66}$$

where the space $\mathscr{U}(\mathbf{C}^n)$ is defined by (73) with $Z = \mathbf{C}^n$. Since $\mathscr{U}(\mathbf{C}^n)$ is a subset of $\mathscr{S}(\mathbf{C}^n)$, we have $\hat{f} \in \mathscr{S}(\mathbf{C}^n)$. Then the Fourier transform of the right-hand side of (57) is the right-hand side of (65). Since $\hat{f}$ vanishes in some neighborhood of the origin in $\mathbf{R}^n$, by (65), also $\hat{u}$ vanishes in the same neighborhood. Since $\hat{G}_\alpha$ is infinitely differentiable on $\mathbf{R}^n \setminus \{0\}$, we see that $\hat{u}$ is infinitely differentiable on $\mathbf{R}^n$; hence $u \in \mathscr{U}(\mathbf{C}^n)$. As $\mathscr{U}(\mathbf{C}^n) \subset \mathscr{T}(\mathbf{C}^n)$, we deduce that $\epsilon^\alpha(u) \in \mathscr{T}(\mathrm{Sym})$ and $\mathsf{C}[\epsilon^\alpha(u)] \in \mathscr{T}(\mathrm{Sym})$ and hence $\operatorname{div}^\alpha \mathsf{C}[\epsilon^\alpha(u)] \in \mathscr{T}(\mathbf{C}^n)$. Thus we have (56).

Let us show that there exists a constant c such that

$$|u|_{2\alpha,q} \le c|f|_{L^q} \tag{67}$$

provided that f satisfies (66). Indeed, by (21), the Fourier transform of $(-\Delta)^{2\alpha}u$ is

$$F\big((-\Delta)^{2\alpha}u\big) = |\xi|^{2\alpha}\hat{u}(\xi) = |\xi|^{2\alpha}\hat{G}_\alpha(\xi)\hat{f}(\xi) = |\xi|^2 B(\xi)\hat{f}(\xi) \tag{68}$$

by (53). By Sect. 7.2, the function $\xi \mapsto m(\xi) = |\xi|^2 B(\xi)$ is bounded and infinitely differentiable positively homogeneous degree 0. Thus it satisfies the hypothesis of Mikhlin's multiplier theorem (see, e.g., [8, Example 8.12(2)]). By Mikhlin's multiplier theorem then the map $f \mapsto F^{-1}mFf$ maps $L^q(\mathbf{C}^n)$ continuously into itself for any $q \in (1,\infty)$; hence (68) gives

$$|u|_{2\alpha,q} = |(-\Delta)^{2\alpha}u|_{L^q} \le c|f|_{L^q}.$$

Thus (33) and (36) give (67).

This completes the proof of Theorem 7.3 under the assumption (66). The general case of $f \in L^q(\mathbf{C}^n)$ follows by the density of $\mathscr{U}(\mathbf{C}^n)$ in $L^q(\mathbf{C}^n)$, as asserted by Proposition B.1. □

Proof (Example 7.5) One finds that for an isotropic material the acoustic tensor is given by

$$A(\xi) = \mu|\xi|^2 I + (\lambda+\mu)\xi \otimes \xi$$

for any $\xi \in \mathbf{R}^n$. The Fourier transform of the equilibrium equation (64) then takes the form

$$|\xi|^{2\alpha}\hat{u}(\xi) + |\xi|^{2\alpha-2}\xi\big(\xi \cdot \hat{u}(\xi)\big) = \hat{f}(\xi).$$

Using (21), (24), and (22) to return to the variable x, one obtains the equilibrium equation (5).

The inverse of the acoustic tensor is given by

$$B(\xi) = \mu^{-1}|\xi|^{-4}\big(|\xi|^2 I - c\xi \otimes \xi\big)$$

for any $0 \ne \xi \in \mathbf{R}^n$, where $c = (\lambda+\mu)/(\lambda+2\mu)$. Hence

$$\hat{G}_\alpha(\xi) = \mu^{-1}|\xi|^{-2-2\alpha}\big(|\xi|^2 I - c\xi \otimes \xi\big)$$

by (53). Using Formulas (21), (24), and (22) again, one finds that the inverse Fourier transform of $\hat{G}_\alpha$ satisfies

$$G_\alpha(x) = \mu^{-1}\nu_{-2\alpha-2}\big(-\Delta + c\nabla^2\big)|x|^{2+2\alpha-n}. \tag{69}$$

To obtain the explicit form, we use the formulas

$$\Delta|x|^{2+2\alpha-n} = 2\alpha(2+2\alpha-n)|x|^{-n+2\alpha}, \tag{70}$$

$$\nabla^2|x|^{2+2\alpha-n} = (2+2\alpha-n)|x|^{-n+2\alpha}\big(I + (2\alpha-n)x \otimes x/|x|^2\big), \tag{71}$$

and simplify the expression

$$\nu_{-2\alpha-2} := 2^{-2\alpha-2}\pi^{-n/2}\Gamma(n/2-\alpha-1)/\Gamma(\alpha+1)$$

to

$$\nu_{-2\alpha-2} = c_\alpha/\big(2\alpha(n-2\alpha-2)\big) \tag{72}$$

by using

$$\alpha\Gamma(\alpha) = \Gamma(\alpha+1), \quad (n/2-\alpha-1)\Gamma(n/2-\alpha-1) = \Gamma(n/2-\alpha).$$

Formulas (69)–(72) provide (62). □

Proposition 7.6 *The value* $q = 2n/(n + 2\alpha)$ *satisfies the hypothesis of Theorem 7.3. The unique solution* u *of (56) corresponding to* $f \in L^q(\mathbf{C}^n)$ *belongs to the space* $R^{\alpha,2}(\mathbf{C}^n)$ *and minimizes the energy on* $R^{\alpha,2}(\mathbf{C}^n)$*, i.e.,*

$$E(v) \ge E(u) \quad \textit{for every} \quad v \in R^{\alpha,2}(\mathbf{C}^n).$$

Proof One finds that for $q = 2n/(n+2\alpha)$, the value of p from (59) is equal to 2 and thus the inclusion $u \in R^{\alpha,2}(\mathbf{C}^n)$ follows from $(58)_1$. We express the functional $E(u)$ from (51) as the sum $E_0(u) + E_1(u)$ of the quadratic and linear terms, respectively. Clearly, E_0 and E_1 are continuous functionals on $R^{\alpha,2}(\mathbf{C}^n)$. The continuity of E_0 follows from $|\epsilon^\alpha(u)| \le |\nabla^\alpha u|$, while the continuity of E_1 follows from the imbedding $(58)_2$ and the duality between the spaces $L^{q^*}(\mathbf{C}^n)$ and $L^q(\mathbf{C}^n)$.

Parseval's equality gives

$$E_0(u) = \frac{(2\pi)^n}{2} \int_{\mathbf{R}^n} \mathsf{C}[\hat\epsilon^\alpha(\xi)] \cdot \bar{\hat\epsilon}^\alpha(\xi) d\xi$$

for every $u \in \mathscr{T}(\mathbf{C}^n)$, where $\hat\epsilon^\alpha$ is the Fourier transform of $\epsilon^\alpha(u)$ and $\bar{\hat\epsilon}^\alpha$ is the matrix of complex conjugates of the components of $\hat\epsilon^\alpha$. Since by (44),

$$\mathsf{C}[\hat\epsilon^\alpha] \cdot \bar{\hat\epsilon}^\alpha = |\xi|^{2\alpha-2} \mathsf{C}\big[\mathrm{sym}(\hat u \otimes \xi)\big] \cdot \mathrm{sym}(\bar{\hat u} \otimes \xi)$$

where $\hat u$ is the Fourier transform of u, the strong ellipticity (50) implies

$$\mathsf{C}[\hat\epsilon^\alpha] \cdot \bar{\hat\epsilon}^\alpha \ge 0.$$

Thus E_0 and hence E is a positive semidefinite quadratic functional on $\mathscr{T}(\mathbf{C}^n)$ and hence on $R^{\alpha,2}(\mathbf{C}^n)$ by density ([23, Theorem 9]). Equation (60) is then a necessary and sufficient condition for the minimum of E. ☐

Remark 7.7 The strict version of the strong ellipticity condition (50) actually gives that E_0 is coercive on $R^{\alpha,2}(\mathbf{C}^n)$ in the sense that

$$E_0(u) \ge c|u|^2_{\alpha,2}$$

for some $c > 0$ and every $u \in R^{\alpha,2}(\mathbf{C}^n)$. Thus the existence and uniqueness of a minimizer of E follows directly from the Lax-Milgram theorem without recourse to Theorem 7.3.

8 Conclusions

The paper introduces a nonlocal fractional strain tensor $\epsilon^\alpha(u)$ of order α, $0 < \alpha < 1$. At variance with the previous proposals, the present fractional strain tensor transforms correctly under rotation.[10] A nonlocal linear constitutive equation is formulated in terms of $\epsilon^\alpha(u)$. The corresponding equilibrium equation with a given external force is formulated using the fractional divergence. The fractional Green function is introduced and used to solve the equilibrium equation. In the particular case of an isotropic material the equilibrium equation and the Green function take explicit forms. The methods of fractional calculus and the Fourier transform are used in the proofs.

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[10] The restriction $0 < \alpha < 1$ is imposed for notation simplification only; the definition can be generalized to arbitrary orders $\alpha > 0$ either by iterating the definition (1) or by admitting fractional gradients of any order in (40).

Appendix A: Fourier Transformation of Homogeneous Distributions

Definition A.1 A distribution f from $\mathscr{S}'(Z)$ is said to be homogeneous of degree $\lambda \in \mathbf{C}$ if

$$\langle f, g \circ \eta_t \rangle = t^{-\lambda - n} \langle f, g \rangle$$

for every $t > 0$ and every $g \in \mathscr{S}(Z)$, where $\eta_t : \mathbf{R}^n \to \mathbf{R}^n$ is given by

$$\eta_t(x) = tx, \quad x \in \mathbf{R}^n.$$

Definition A.2 ([17, Definition 3.2.5]) A distribution $f \in \mathscr{S}(Z)$ is said to be locally in the space of Bessel potentials $L^{s,2}(Z)$ if gf belongs to $L^{s,2}(Z)$ for every infinitely differentiable function $g : \mathbf{R}^n \to \mathbf{C}$ with compact support that is contained in $\mathbf{R}^n \setminus \{0\}$.

Theorem A.3 ([17, Corollary 3.2.6]) *The Fourier transform of a homogeneous distribution f of degree $\lambda \in \mathbf{C}$ that is locally in $L^{s,2}(Z)$ is a homogeneous distribution $\hat{f}$ of degree $-\lambda - n$ that is locally in $L^{s - \mathrm{Re}\,\lambda - n/2, 2}(Z)$.*

Appendix B: The Space $\mathscr{U}(Z)$

This appendix introduces the space of test functions whose Fourier transforms can be safely divided by $|\xi|$ and its positive powers, as needed in the proof of Theorem 7.3.

Proposition B.1 *The set*

$$\mathscr{U}(Z) := \{f \in \mathscr{S}(Z) : \hat{f} = 0 \textit{ in some neighborhood of } 0\} \tag{73}$$

is dense in $L^p(Z)$ for every $p \in (1, \infty)$.

Proof It suffices to prove that any $f \in \mathscr{S}(Z)$ can be approximated by a sequence $f_k \in \mathscr{U}(Z)$ in the L^p norm. Let $\psi : \mathbf{R} \to \mathbf{C}$ be a function whose Fourier transform $\hat{\psi}$ is infinitely differentiable and satisfies $\hat{\psi} = 1$ on $B(0, 1)$ and $\hat{\psi} = 0$ on $\mathbf{R}^n \setminus B(0, 2)$. Let $\psi_t(x) = t^n \psi(tx)$, $x \in \mathbf{R}^n$, $t > 0$. One has

$$|\psi_t|_{L^p}^p = t^{np} \int_{\mathbf{R}^n} |\psi(tx)|^p dx = t^{n(p-1)} \int_{\mathbf{R}^n} |\psi(y)|^p dy$$

and thus if $1 < p < \infty$, $|\psi_t|_{L^p} \to 0$ for $t \to 0$. Since every $f \in \mathscr{S}$ is integrable, Young's convolution inequality implies that $|f * \psi_t|_{L^p} \to 0$ for $t \to 0$, i.e., the function $f_t := f - f * \psi_t$ satisfies

$$|f - f_t|_{L^p} \to 0 \quad \text{for} \quad t \to 0.$$

To show that $f_t \in \mathscr{U}(Z)$, we note that the well-known rules for the Fourier transformation under scaling and convolution give

$$(f_t)\hat{} = \hat{f} - \hat{f}\hat{\psi}_t$$

where $\hat{\psi}_t(\xi) = \hat{\psi}(\xi/t)$ for every $\xi \in \mathbf{R}^n$ and $t > 0$. One finds that $\hat{\psi}_t = 1$ on $B(0, t)$ and hence $(f_t)\hat{} = 0$ on $B(0, t)$. □

Declarations

Competing interests The authors declare no competing interests.

References

1. Alotta, G., Di Paola, M., Pinnola, F.P.: An unified formulation of strong non-local elasticity with fractional order calculus. Meccanica **57**, 793–805 (2022)
2. Brue, E., Calzi, M., Comi, G.E., Stefani, G.: A distributional approach to fractional Sobolev spaces and fractional variation: asymptotics II (2020). (Preprint)
3. Carpinteri, A., Cornetti, P., Sapora, A.: A fractional calculus approach to nonlocal elasticity. Eur. Phys. J. Spec. Top. **193**, 193–204 (2011)
4. Carpinteri, A., Cornetti, P., Sapora, A.: Nonlocal elasticity: an approach based on fractional calculus. Meccanica **49**, 2551–2569 (2014)
5. Comi, G.E., Stefani, G.: A distributional approach to fractional Sobolev spaces and fractional variation: existence of blow-up. J. Funct. Anal. **277**, 3373–3435 (2019)
6. Comi, G.E., Stefani, G.: A distributional approach to fractional Sobolev spaces and fractional variation: asymptotics I (2019). (Preprint)
7. Drapaca, C.S., Sivaloganathan, S.: A fractional model of continuum mechanics. J. Elast. **107**, 105–123 (2012)
8. Duoandikoetxea, J.: Fourier Analysis. Am. Math. Soc., Providence (2001)
9. Evgrafov, A., Bellido, J.C.: From non-local Eringen's model to fractional elasticity. Math. Mech. Solids **24**, 1935–1953 (2019)
10. Gel'fand, I.M., Shilov, G.E.: Generalized Functions I. Properties and Operations. Academic Press, New York (1964)
11. Herz, C.S.: Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms. J. Math. Mech. **18**, 283–324 (1968)
12. Hörmander, L.: The Analysis of Partial Differential Operators I. Distribution Theory and Fourier Analysis, 2nd edn. Springer, Berlin (1990)
13. Horváth, J.: On some composition formulas. Proc. Am. Math. Soc. **10**, 433–437 (1959)
14. Horváth, J.: Composition of hypersingular integral operators. Appl. Anal. **7**, 171–190 (1978)
15. Javili, A., Morasata, R., Oterkus, E., Oterkus, S.: Peridyn. Rev. Math. Mech. Solids **24**, 3714–3739 (2019)
16. Johnson, R.: Temperatures, Riesz potentials, and the Lipschitz spaces of Herz. Proc. Lond. Math. Soc. **27**, 290–316 (1973)
17. Lemoine, C.: Fourier transforms of homogeneous distribution. Ann. Sc. Norm. Super. Pisa, Cl. Sci. 3e Sér. **26**, 117–149 (1972)
18. Martínez, C., Sanz, M.: The Theory of Fractional Powers of Operators Elsevier. Elsevier, Amsterdam (2001)
19. Martínez, C., Sanz, M., Periago, F.: Distributional fractional powers of the Laplacean. Riesz potentials. Stud. Math. **135**, 253–271 (1999)
20. Maz'ya, V.G., Havin, V.P.: Non-linear potential theory. Usp. Mat. Nauk **27**, 67–138 (1972) (Russian)
21. Patnaik, S., Jokar, M., Semperlotti, F.: Variable-order approach to nonlocal elasticity: theoretical formulation, order identification via deep learning, and applications. Comput. Mech. **69**, 267–298 (2021)
22. Peetre, J.: New Thoughts on Besov Spaces. Duke University, Durham (1976)
23. Samko, S.G.: On spaces of Riesz potentials. Izv. Akad. Nauk SSSR, Ser. Mat. **40**, 1143–1172 (1976)
24. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives. Theory and Applications. Gordon & Breach, Amsterdam (1993)
25. Schikorra, A.: ϵ-regularity for systems involving non-local, antisymmetric operators. Calc. Var. **54**, 3531–3570 (2015)
26. Shaat, M., Ghavanloo, E., Fazelzadeh, S.A.: Review on nonlocal continuum mechanics: physics, material applicability, and mathematics. Mech. Mater. **150**, 103587 (2020)
27. Shieh, T.-T., Spector, D.E.: On a new class of fractional partial differential equations. Adv. Calc. Var. **8**, 321–336 (2015)
28. Shieh, T.-T., Spector, D.E.: On a new class of fractional partial differential equations II. Adv. Calc. Var. **11**, 289–307 (2018)
29. Sidhardh, S., Patnaik, S., Semperlotti, F.: Thermodynamics of fractional-order nonlocal continua and its application to the thermoelastic response of beams. Eur. J. Mech. A, Solids **88**, 104238 (2021)

30. Šilhavý, M.: Fractional vector analysis based on invariance requirements (critique of coordinate approaches. Contin. Mech. Thermodyn. **32**, 207–228 (2020)
31. Stein, E.M.: The characterization of functions arising as potentials. Bull. Am. Math. Soc. **67**, 102–104 (1961)
32. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970)
33. Sumelka, W.: Fractional calculus for continuum mechanics-anisotropic non-locality. Bull. Pol. Acad. Sci., Tech. Sci. **64**, 361–372 (2016)
34. Sumelka, W., Blaszczyk, T.: Fractional continua for linear elasticity. Arch. Mech. **66**, 147–172 (2014)