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A Thermodynamic Approach to Rate-Type Models of Elastic-Plastic Materials

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Abstract

Within the framework of continuum thermodynamics, a tensor-valued rate-type model of elastic-plastic materials is established. The evolution of the stress-strain relation is governed by a free-energy function and a hysteretic function proportional to the entropy production density. It is a key point of the present approach that the entropy production is given by a constitutive function consistent with the second-law inequality. The free energy depends on both stress and strain, as well as temperature. In the absence of hysteretic loops the evolution of the stress depends on the values of stress and strain and is affected by the time derivative of the strain. The hysteretic behaviour is modelled in detail in the one-dimensional case. Simple examples are established by a hysteretic function proportional to the absolute value of the strain rate. While the entropy production is formally similar to the widely used dissipation potentials the corresponding approaches are qualitatively different. The entropy production and the free energy potential are functions of the same set of physical variables and no internal variable is involved. The analysis of the second-law inequality leads to the sought constitutive relation, here in the rate-type form, for stress and strain.

Keywords Elastic-plastic materials · Hypoelastic materials · Materials of stress-rate type · Large-strain rate-independent theories · Thermodynamics · Free energies · Hardening variable · Hysteretic function

Mathematics Subject Classification (2010) 74C15 · 74A15 · 74A20 · 74D10 · 80A17

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1 Introduction

The literature on the modelling of plastic behaviour of materials shows different pictures of the material.[1] In many theories of plasticity it is assumed that the displacement gradient $\mathbf{H}$ admits a decomposition $\mathbf{H} = \mathbf{H}^e + \mathbf{H}^p$, where $\mathbf{H}^e$ represents the elastic distortion and $\mathbf{H}^p$ the plastic distortion. Physically $\mathbf{H}^e$ is ascribed to non-dissipative processes and $\mathbf{H}^p$ to dissipative processes. Hence, letting $\mathbf{E}^e$, $\mathbf{E}^p$ be the symmetric parts of $\mathbf{H}^e$, $\mathbf{H}^p$ it is assumed[2] that the second law inequality results in an equality for $\mathbf{E}^e$ and an inequality for $\mathbf{E}^p$. In other approaches [6, 34] the decomposition is placed into the context of the continua with a two-scale representation of the deformation. In large-deformation plasticity a multiplicative decomposition of the deformation gradient $\mathbf{F}$, namely the Kröner-Lee decomposition [22, 23] $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$, is applied; as shown in [6], the Kröner-Lee decomposition is incompatible with the geometry of two-scale continua.

Other models can be viewed within the gradient theory [1, 19]. In [14] the gradient involves the time derivative $\dot{\mathbf{E}}^p$ within a power conjugate to the gradient of $\dot{\mathbf{E}}^p$, via a third-order microscopic hyperstress.

Approaches have been developed also without having recourse to the additive decomposition of strain. This is the case for approaches involving history variables or rheological models especially in the Bouc-Wen form, see, e.g., [20, 50, 51]. In [35] the stretching, not the strain, tensor is considered in the additive elastic-plastic decomposition.

The elastic-plastic decomposition is avoided in some works of Rajagopal and Srinivasa [42, 43]; their approach leads to a rate-type equation which is reviewed in § 11.

The present approach avoids the splitting of the displacement gradient into elastic and plastic parts as well as the multiplicative decomposition of the deformation gradient and the introduction of internal powers due to hyperstresses. Nor will we have recourse to dissipation potentials [26] or to a maximum dissipation criterion [4, 25]. The modelling is strictly based on the second law of thermodynamics which turns out to be a relation involving the free energy and the entropy production. It is a key point of the present approach[3] that the entropy production too is given by a positive-valued constitutive function; the dependence on stress and strain rates of the entropy production is shown to characterize the hysteretic properties of the material. Though this view is at the basis of very general models, here we show that simple, realistic behaviours are obtained by letting hysteretic functions involve the sign of the time derivative of strain (or stress). In addition the simultaneous dependence of the free energy on strain and stress (see § 5) is essential to the modelling of inelastic materials. Throughout we use objective tensors and objective time derivatives. Indeed the selection of objective time derivatives is based on constitutive (and balance) equations in the reference configuration where only the material time derivative occurs. The thermodynamic analysis is developed in general in a three-dimensional setting (§§ 3-5) while detailed hysteretic models are developed in a one-dimensional setting (§§ 6-10).

[1] Chapters 75 to 112 of [15], Chaps. 8 of [18] and 12 of [33], and the book [17] are relevant references to the subject.

[2] The elastic-plastic decomposition $\mathbf{E} = \mathbf{E}^e + \mathbf{E}^p$ is investigated via an approach [11] that does not rely on auxiliary elements such as reference frame and reference configuration.

[3] See § 11.3

2 Balance Laws and Entropy Inequality

We consider a body occupying the time dependent region $\Omega \subset \mathscr{E}^3$. The motion is described by means of the function $\boldsymbol{\chi}(\mathbf{X}, t)$ providing the position vector $\mathbf{x} \in \Omega$ in terms of the position $\mathbf{X}$ in a reference configuration R, and the time t, so that $\Omega = \boldsymbol{\chi}(\mathrm{R}, t)$. The symbols ∇, ∇_R denote the gradient operator with respect to $\mathbf{x} \in \Omega$, $\mathbf{X} \in \mathrm{R}$. The function $\boldsymbol{\chi}$ is assumed to be differentiable; hence we can define the deformation gradient as $\mathbf{F} = \nabla_R \boldsymbol{\chi}$ or, in suffix notation, $F_{iK} = \partial_{X_K} \chi_i$. The invertibility of $\mathbf{X} \mapsto \mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ is guaranteed by letting $J := \det \mathbf{F} > 0$. Let $\mathbf{v}(\mathbf{x}, t)$ be the velocity field, on $\Omega \times \mathbb{R}$. A superposed dot denotes time differentiation following the motion of the body and hence, for any function $f(\mathbf{x}, t)$, we have[4] $\dot{f} = \partial_t f + \mathbf{v} \cdot \nabla f$. We denote by $\mathbf{L}$ the velocity gradient, $L_{ij} = \partial_{x_j} v_i$, and recall that

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}.$$

Moreover $\mathbf{D}$ denotes the stretching tensor, $\mathbf{D} = \mathrm{sym}\mathbf{L}$. In terms of $\mathbf{F}$ the (right) Cauchy-Green and the Green-St. Venant deformation tensors are respectively defined by

$$\mathbf{C} := \mathbf{F}^T\mathbf{F}, \qquad \mathbf{E} := \tfrac{1}{2}(\mathbf{C} - \mathbf{1}).$$

Let ε be the internal energy density (per unit mass), $\mathbf{T}$ the symmetric Cauchy stress, $\mathbf{q}$ the heat flux vector, ρ the mass density, r the (external) heat supply and $\mathbf{b}$ the mechanical body force per unit mass. The balance equations for mass, linear momentum, and energy are taken in the form

$$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0, \qquad \rho \dot{\mathbf{v}} = \nabla \cdot \mathbf{T} + \rho \mathbf{b}, \qquad \rho \dot{\varepsilon} = \mathbf{T} \cdot \mathbf{D} - \nabla \cdot \mathbf{q} + \rho r. \tag{1}$$

Let η be the entropy density and $\theta > 0$ the absolute temperature. As the statement of the second law of thermodynamics we take it that the inequality

$$\sigma := \rho \dot{\eta} + \nabla \cdot (\mathbf{q}/\theta) - \rho r/\theta \geq 0 \tag{2}$$

for the entropy production σ holds for any process compatible with the balance equations. Consequently, admissible constitutive equations are required to satisfy inequality (2).

Multiplying (2) by θ and substituting $\nabla \cdot \mathbf{q} - \rho r$ from the energy equation $(1)_3$ we have

$$\theta \sigma = \rho \theta \dot{\eta} - \rho \dot{\varepsilon} + \mathbf{T} \cdot \mathbf{D} - \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta \geq 0. \tag{3}$$

Letting $\psi = \varepsilon - \theta \eta$ (Helmholtz free energy density) we can rewrite inequality (3) as

$$-\rho(\dot{\psi} + \eta \dot{\theta}) + \mathbf{T} \cdot \mathbf{D} - \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta \geq 0. \tag{4}$$

The modelling of the constitutive properties is made simpler by using referential, Euclidean invariant quantities. Let

$$\mathbf{T}_{RR} := J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}, \qquad \mathbf{q}_R := J\mathbf{F}^{-1}\mathbf{q};$$

[4] Throughout the symbol ∂ with a suffix denotes partial differentiation, e.g., $\partial_t = \partial/\partial t$.

$\mathbf{T}_{RR}$ is then the second Piola-Kirchhoff stress. Both $\mathbf{T}_{RR}$ and $\mathbf{q}_R$ are Euclidean invariants ([49, § 19]; [18, Ch. 5]). Under any change of frame with rotation tensor $\mathbf{Q}$,

$$\mathbf{x} \to \mathbf{x}^*, \qquad \mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x},\ \det \mathbf{Q} = 1,$$

we have $\mathbf{F}^* = \mathbf{QF}$ and hence $\mathbf{T}_{RR}$ and $\mathbf{q}_R$ are invariant,

$$\mathbf{T}^*_{RR} = J(\mathbf{QF})^{-1}\mathbf{QTQ}^T(\mathbf{QF})^{-T} = J\mathbf{F}^{-1}\mathbf{TF}^{-T} = \mathbf{T}_{RR},$$

and likewise $\mathbf{q}^*_R = \mathbf{q}_R$. The tensors $\mathbf{C}$ and $\mathbf{E}$ are invariant too in that

$$\mathbf{C}^* = (\mathbf{QF})^T(\mathbf{QF}) = \mathbf{F}^T\mathbf{F} = \mathbf{C}.$$

Moreover

$$\dot{\mathbf{E}} = \mathbf{F}^T\mathbf{DF}.$$

Accordingly we multiply inequality (4) by J, observe that $J\rho$ is the mass density ρ_R in the reference configuration and find

$$J\theta\sigma = -\rho_R(\dot{\psi} + \eta\dot{\theta}) + \mathbf{T}_{RR} \cdot \dot{\mathbf{E}} - \frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R \theta \geq 0. \tag{5}$$

The entropy inequality can be written in terms of the Gibbs free energy ϕ related to the Helmholtz free energy ψ by[5]

$$\rho_R\phi := \rho_R\psi - \mathbf{T}_{RR} \cdot \mathbf{E}.$$

Hence (4) becomes

$$J\theta\sigma = -\rho_R(\dot{\phi} + \eta\dot{\theta}) - \dot{\mathbf{T}}_{RR} \cdot \mathbf{E} - \frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R \theta \geq 0. \tag{6}$$

3 Constitutive Relations

So as to describe the elastic-plastic effects we let the strain $\mathbf{E}$, the stress $\mathbf{T}_{RR}$, and the derivatives $\dot{\mathbf{E}}$, $\dot{\mathbf{T}}_{RR}$ be among the independent variables.[6] Since also thermal properties are modelled then we let

$$\Xi := (\theta, \mathbf{E}, \mathbf{T}_{RR}, \nabla_R \theta, \dot{\mathbf{E}}, \dot{\mathbf{T}}_{RR})$$

[5] In isothermal elastic processes, ϕ represents the opposite of the complementary strain energy density. In classical thermodynamics $\phi = \psi + p\upsilon$, where p and υ denote the pressure and the specific volume, respectively.

[6] The use of functions dependent on both stress and strain deserves some comments. Rate-type models are developed within macroscopic thermodynamics for viscoelastic fluids where the stress depends on the stretching as in the Maxwell or the Oldroyd model [7, 24, 41, 53]. The common dependence on stress and strain occurs in fluid theories where implicit relations are involved [39, 40]. Here no implicit relation is considered. However in plasticity, and likewise in any hysteretic model, the dependence of $\mathbf{T}_{RR}$ on $\mathbf{E}$ would be given by a multivalued function. To avoid this type of dependence it seems convenient to let the pertinent constitutive equations depend on both stress and strain. Otherwise we might consider an internal variable and correspondingly a generalized stress; see § 11.2.

be the set of independent variables. Accordingly we let ψ, η, $\mathbf{q}_R$ be functions of Ξ and assume η and $\mathbf{q}_R$ are continuous while the free energy ψ is continuously differentiable.

Upon evaluation of $\dot{\psi}$ and substitution in (5) we obtain

$$\rho_R(\partial_\theta\psi + \eta)\dot{\theta} + (\rho_R\partial_{\mathbf{E}}\psi - \mathbf{T}_{RR}) \cdot \dot{\mathbf{E}} + \rho_R\partial_{\mathbf{T}_{RR}}\psi \cdot \dot{\mathbf{T}}_{RR} +$$
$$\rho_R\partial_{\nabla_R\theta}\psi \cdot \nabla_R\dot{\theta} + \partial_{\dot{\mathbf{E}}}\psi \cdot \ddot{\mathbf{E}} + \partial_{\dot{\mathbf{T}}_{RR}}\psi \cdot \ddot{\mathbf{T}}_{RR} + \frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R\theta = -J\theta\sigma \le 0.$$

The linearity and arbitrariness of $\dot{\theta}$, $\nabla_R\dot{\theta}$, $\ddot{\mathbf{E}}$, $\ddot{\mathbf{T}}_{RR}$ imply that ψ is independent of $\nabla_R\theta$, $\dot{\mathbf{E}}$, $\dot{\mathbf{T}}_{RR}$ and hence

$$\psi = \psi(\theta, \mathbf{E}, \mathbf{T}_{RR}), \qquad \eta = -\partial_\theta\psi,$$

and

$$(\rho_R\partial_{\mathbf{E}}\psi - \mathbf{T}_{RR}) \cdot \dot{\mathbf{E}} + \rho_R\partial_{\mathbf{T}_{RR}}\psi \cdot \dot{\mathbf{T}}_{RR} + \frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R\theta \le 0. \tag{7}$$

Further restrictions placed on ψ and $\mathbf{q}_R$ depend on the arbitrariness or possible constraints on $\dot{\mathbf{E}}$, $\dot{\mathbf{T}}_{RR}$, $\nabla_R\theta$. Since ψ is independent of $\nabla_R\theta$, if further $\dot{\mathbf{E}}$ and $\dot{\mathbf{T}}_{RR}$ are independent of each other then it follows

$$\partial_{\mathbf{T}_{RR}}\psi = \mathbf{0}, \qquad \mathbf{T}_{RR} = \rho_R\partial_{\mathbf{E}}\psi, \qquad \mathbf{q}_R \cdot \nabla_R\theta \le 0,$$

as it happens for hyperelastic materials.

Yet, to describe plasticity a connection between $\dot{\mathbf{E}}$ and $\dot{\mathbf{T}}_{RR}$ is in order. Since ψ is independent of $\nabla_R\theta$ then inequality (7) implies that

$$(\rho_R\partial_{\mathbf{E}}\psi - \mathbf{T}_{RR}) \cdot \dot{\mathbf{E}} + \rho_R\partial_{\mathbf{T}_{RR}}\psi \cdot \dot{\mathbf{T}}_{RR} = -J\theta\sigma_T \le 0, \tag{8}$$
$$\mathbf{q}_R \cdot \nabla_R\theta = -J\theta^2\sigma_q \le 0,$$

where σ_T, σ_q are the entropy productions associated with deformation and heat conduction. A Fourier-like relation for $\mathbf{q}_R$,

$$\mathbf{q}_R = -\kappa(\theta, \mathbf{E}, \mathbf{T}_{RR})\nabla_R\theta, \qquad \kappa \ge 0,$$

satisfies $\mathbf{q}_R \cdot \nabla_R\theta \le 0$ with $\sigma_q = \kappa|\nabla\theta|^2/\rho\theta^2$; more involved constitutive equations for the heat conduction might include $\mathbf{q}_R$ among the independent variables and $\dot{\mathbf{q}}_R$ as a constitutive function [29]. A detailed modelling of the mechanical properties—via $\mathbf{T}_{RR}$, $\mathbf{E}$, $\dot{\mathbf{T}}_{RR}$, $\dot{\mathbf{E}}$—is developed on the basis of (8).

4 A Thermodynamic Context for Incremental Models of Inelastic Materials

A class of models, consistent with inequality (8) and describing the visco-elastic-plastic behaviour,[7] is realized by letting $\dot{\mathbf{E}}$ and $\dot{\mathbf{T}}_{RR}$ be related via a nonlinear relation. The basic idea stems from the observation that inequality (8) holds if

$$(\rho_R\partial_{\mathbf{E}}\psi - \mathbf{T}_{RR}) \cdot \dot{\mathbf{E}} + \rho_R\partial_{\mathbf{T}_{RR}}\psi \cdot \dot{\mathbf{T}}_{RR} = -\Gamma(\theta, \mathbf{E}, \mathbf{T}_{RR}, \dot{\mathbf{E}}, \dot{\mathbf{T}}_{RR}), \tag{9}$$

[7] Plastic models are developed in a one-dimensional setting.

where $\Gamma = J\theta\sigma_T \geq 0$. Within the isothermal setting, Γ is usually referred to as *rate of (mechanical) dissipation* [38].

The statement of the second law in terms of an equality is not new in the literature.[8] If we ignore that Γ has to be non-negative we may say that the second law is stated as an equality. However, there are two correct ways of viewing at (9). First, $-\Gamma$ is just a symbol defined by the left-hand side; we then merely require that $\Gamma \geq 0$. Secondly, as we do here, we let Γ be an appropriate constitutive function of the chosen set of independent variables and require that Eq. (9) holds. The selection of the function Γ and the requirement $\Gamma \geq 0$ are the key point of the present approach.

Equation (9) can be viewed as a very general rate-type constitutive relation and the functional dependence of Γ can be used to classify the corresponding incremental (rate-type) models. For instance, $\Gamma = 0$ marks the linear hypoelastic behaviour,[9] whereas a function Γ which is independent of $\dot{\mathbf{E}}$ and $\dot{\mathbf{T}}_{RR}$ is typically related to viscoelastic[10] and rate-dependent viscoplastic[11] models. Indeed, to describe the elastic-plastic behaviour, below we find it operative to let Γ depend on $\dot{\mathbf{E}}$ or $\dot{\mathbf{T}}_{RR}$ via their norms, in the one-dimensional case.

Although (9) was introduced using Lagrangian fields, it can easily be translated into the spatial description. Observe that since $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$ then

$$\mathbf{0} = (\mathbf{F}\mathbf{F}^{-1})^{\cdot} = \dot{\mathbf{F}}\mathbf{F}^{-1} + \mathbf{F}(\mathbf{F}^{-1})^{\cdot} = \mathbf{L} + \mathbf{F}(\mathbf{F}^{-1})^{\cdot}$$

whence

$$(\mathbf{F}^{-1})^{\cdot} = -\mathbf{F}^{-1}\mathbf{L}, \qquad [(\mathbf{F}^{-1})^{\cdot}]^T = -\mathbf{L}^T\mathbf{F}^{-T}.$$

Consequently,

$$\dot{\mathbf{T}}_{RR} = (J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T})^{\cdot} = J\mathbf{F}^{-1}[(\nabla\cdot\mathbf{v})\mathbf{T} - \mathbf{L}\mathbf{T} + \dot{\mathbf{T}} - \mathbf{T}\mathbf{L}^T]\mathbf{F}^{-T} = J\mathbf{F}^{-1}\,\overset{\square}{\mathbf{T}}\,\mathbf{F}^{-T},$$

where $\overset{\square}{\mathbf{T}} = \dot{\mathbf{T}} + (\nabla\cdot\mathbf{v})\mathbf{T} - \mathbf{L}\mathbf{T} - \mathbf{T}\mathbf{L}^T$ is the Truesdell rate of $\mathbf{T}$ [30]. Moreover

$$\partial_{\mathbf{T}}\psi = J\mathbf{F}^{-T}\partial_{\mathbf{T}_{RR}}\psi\,\mathbf{F}^{-1}, \qquad \partial_{\mathbf{E}}\psi = \mathbf{F}^{-1}\partial_{\mathbf{F}}\psi$$

and hence

$$\partial_{\mathbf{T}_{RR}}\psi\cdot\dot{\mathbf{T}}_{RR} = \partial_{\mathbf{T}}\psi\cdot\overset{\square}{\mathbf{T}}.$$

In addition, since

$$\dot{\mathbf{E}} = \mathbf{F}^T\mathbf{D}\mathbf{F}, \qquad \mathbf{T}_{RR}\cdot\dot{\mathbf{E}} = J\mathbf{T}\cdot\mathbf{D},$$

then (9) can be written

$$J[(\rho\mathbf{F}\partial_{\mathbf{E}}\psi\mathbf{F}^T - \mathbf{T})\cdot\mathbf{D} + \rho\partial_{\mathbf{T}}\psi\cdot\overset{\square}{\mathbf{T}}] = -\Gamma. \tag{10}$$

[8] See § 11.3.

[9] See, e.g., [28, 49].

[10] See, e.g., [9, 10].

[11] For the one-dimensional case, see [16, 45].

For formal convenience we now consider the Eulerian Almansi finite strain tensor denoted by $\boldsymbol{\mathcal{E}}$. The definition of $\boldsymbol{\mathcal{E}}$ and its relation with $\mathbf{E}$ ($\boldsymbol{\mathcal{E}}, \mathbf{E} \in \mathrm{Sym}$) are established by looking at a homogeneous deformation so that the position vectors $\mathbf{y} = \boldsymbol{\chi}(\mathbf{Y}, t)$, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, where $\mathbf{Y}, \mathbf{X} \in \mathrm{R}$, satisfy

$$\mathbf{y} - \mathbf{x} = \mathbf{F}(\mathbf{Y} - \mathbf{X}).$$

The difference $|\mathbf{y} - \mathbf{x}|^2 - |\mathbf{Y} - \mathbf{X}|^2$ is given by

$$|\mathbf{y} - \mathbf{x}|^2 - |\mathbf{Y} - \mathbf{X}|^2 = 2(\mathbf{Y} - \mathbf{X}) \cdot \mathbf{E}(\mathbf{Y} - \mathbf{X}) = 2(\mathbf{y} - \mathbf{x}) \cdot \boldsymbol{\mathcal{E}}(\mathbf{y} - \mathbf{x}).$$

Replacing $\mathbf{Y} - \mathbf{X}$ with $\mathbf{F}^{-1}(\mathbf{y} - \mathbf{x})$ and making use of the arbitrariness of $\mathbf{y} - \mathbf{x}$ we conclude that

$$\boldsymbol{\mathcal{E}} = \mathbf{F}^{-T}\mathbf{E}\mathbf{F}^{-1} = \tfrac{1}{2}(\mathbf{1} - \mathbf{B}^{-1}),$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the Eulerian (or left) Cauchy-Green deformation tensor. Consequently

$$\partial_{E_{kl}} \mathcal{E}_{ij} = \partial_{E_{kl}} [F^{-T}_{im} E_{ms} F^{-1}_{sj}] = F^{-T}_{ik} F^{-1}_{lj},$$
$$\partial_{E_{kl}} \psi = \partial_{\mathcal{E}_{ij}} \psi \, \partial_{E_{kl}} \mathcal{E}_{ij} = F^{-1}_{ki} \partial_{\mathcal{E}_{ij}} \psi F^{-T}_{jl},$$

whence $\partial_{\mathbf{E}} \psi = \mathbf{F}^{-1} \partial_{\boldsymbol{\mathcal{E}}} \psi \mathbf{F}^{-T}$. We then obtain that the Cotter-Rivlin rate of $\boldsymbol{\mathcal{E}}$, $\overset{\triangle}{\boldsymbol{\mathcal{E}}} := \dot{\boldsymbol{\mathcal{E}}} + \mathbf{L}^T \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}\mathbf{L}$, yields

$$\overset{\triangle}{\boldsymbol{\mathcal{E}}} = \tfrac{1}{2}(\mathbf{L}^T\mathbf{B}^{-1} + \mathbf{B}^{-1}\mathbf{L}) + \tfrac{1}{2}\mathbf{L}^T(\mathbf{1} - \mathbf{B}^{-1}) + \tfrac{1}{2}(\mathbf{1} - \mathbf{B}^{-1})\mathbf{L} = \mathbf{D}. \tag{11}$$

Accordingly Eq. (10) can be given the form

$$(\rho \partial_{\boldsymbol{\mathcal{E}}} \psi - \mathbf{T}) \cdot \overset{\triangle}{\boldsymbol{\mathcal{E}}} + \rho \partial_{\mathbf{T}} \psi \cdot \overset{\square}{\mathbf{T}} = -\Lambda(\theta, \boldsymbol{\mathcal{E}}, \mathbf{T}, \overset{\triangle}{\boldsymbol{\mathcal{E}}}, \overset{\square}{\mathbf{T}}), \tag{12}$$

where $\Lambda = (1/J)\Gamma(\theta, \mathbf{F}^T\boldsymbol{\mathcal{E}}\mathbf{F}, J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}, \mathbf{F}^T \overset{\triangle}{\boldsymbol{\mathcal{E}}} \mathbf{F}, J\mathbf{F}^{-1} \overset{\square}{\mathbf{T}} \mathbf{F}^{-T})$. Equation (12) is objective and gives an invariant formulation of a wide class of incremental three-dimensional models in the spatial description. Based on the time derivative of referential quantities, $\dot{\mathbf{T}}_{RR}$, $\dot{\mathbf{E}}$, we have found that in the Eulerian description $\overset{\triangle}{\boldsymbol{\mathcal{E}}} = \mathbf{D}$ and $\overset{\square}{\mathbf{T}}$ are privileged derivatives.[12] Unfortunately experimental tests on the appropriate time derivative are immaterial in one-dimensional settings.

By means of the Gibbs free energy

$$\phi(\theta, \mathbf{E}, \mathbf{T}_{RR}) := \psi(\theta, \mathbf{E}, \mathbf{T}_{RR}) - \mathbf{T}_{RR} \cdot \mathbf{E}/\rho_R$$

we can write the entropy inequality

$$-\rho_R(\dot{\psi} + \eta\dot{\theta}) + \mathbf{T}_{RR} \cdot \dot{\mathbf{E}} - \frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R \theta \geq 0$$

in the form

$$-\rho_R(\dot{\phi} + \eta\dot{\theta}) - \dot{\mathbf{T}}_{RR} \cdot \mathbf{E} - \frac{1}{\theta}\mathbf{q}_R \cdot \nabla_R \theta \geq 0.$$

[12] See Remark 2 on $\overset{\circ}{\mathbf{T}}$ as a preferable derivative.

Since $\mathbf{E}$ and $\mathbf{T}_{RR}$ are independent variables then

$$\rho_R \partial_{\mathbf{E}} \phi = \rho_R \partial_{\mathbf{E}} \psi - \mathbf{T}_{RR}, \qquad \rho_R \partial_{\mathbf{T}_{RR}} \phi = \rho_R \partial_{\mathbf{T}_{RR}} \psi - \mathbf{E}. \tag{13}$$

Consequently, (9) can be given the form

$$\rho_R \partial_{\mathbf{E}} \phi \cdot \dot{\mathbf{E}} + (\rho_R \partial_{\mathbf{T}_{RR}} \phi + \mathbf{E}) \cdot \dot{\mathbf{T}}_{RR} = -\Gamma(\theta, \mathbf{E}, \mathbf{T}_{RR}, \dot{\mathbf{E}}, \dot{\mathbf{T}}_{RR}). \tag{14}$$

This equation involves $\mathbf{E}$ and $\mathbf{T}_{RR}$ as Lagrangian independent variables. We now look for the analogue in the (spatial) Eulerian description. Observe

$$\mathbf{T}_{RR} \cdot \mathbf{E} = \tfrac{1}{2} J \mathbf{T} \cdot \mathbf{F}^{-T} (\mathbf{F}^T \mathbf{F} - \mathbf{1}) \mathbf{F}^{-1} = \tfrac{1}{2} J \mathbf{T} \cdot (\mathbf{1} - \mathbf{B}^{-1}) = J \mathbf{T} \cdot \boldsymbol{\mathcal{E}}$$

and hence

$$\rho \phi = \rho \psi - \mathbf{T} \cdot \boldsymbol{\mathcal{E}}.$$

Now, if ψ and ϕ depend on $\boldsymbol{\mathcal{E}}$, $\mathbf{T}$ then

$$\rho \partial_{\boldsymbol{\mathcal{E}}} \phi = \rho \partial_{\boldsymbol{\mathcal{E}}} \psi - \mathbf{T}, \qquad \rho \partial_{\mathbf{T}} \phi = \rho \partial_{\mathbf{T}} \psi - \boldsymbol{\mathcal{E}}$$

Consequently inequality (12) can be written in the form

$$\rho \partial_{\boldsymbol{\mathcal{E}}} \phi \cdot \overset{\triangle}{\boldsymbol{\mathcal{E}}} + (\rho \partial_{\mathbf{T}} \phi + \boldsymbol{\mathcal{E}}) \cdot \overset{\square}{\mathbf{T}} = -\Lambda(\theta, \boldsymbol{\mathcal{E}}, \mathbf{T}, \overset{\triangle}{\boldsymbol{\mathcal{E}}}, \overset{\square}{\mathbf{T}}). \tag{15}$$

On the other hand, since $\partial_{\mathbf{T}} \phi = J \mathbf{F}^{-T} \partial_{\mathbf{T}_{RR}} \phi \mathbf{F}^{-1}$ then

$$\partial_{\mathbf{T}_{RR}} \phi \cdot \dot{\mathbf{T}}_{RR} = \partial_{\mathbf{T}} \phi \cdot \overset{\square}{\mathbf{T}}, \qquad \mathbf{E} \cdot \dot{\mathbf{T}}_{RR} = J \boldsymbol{\mathcal{E}} \cdot \overset{\square}{\mathbf{T}},$$

and (15) follows also from (14) by defining $\Lambda = \Gamma / J$.

If the material is incompressible, $\nabla \cdot \mathbf{v} = 0$, then

$$\overset{\square}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{L}\mathbf{T} - \mathbf{T}\mathbf{L}^T =: \overset{\triangledown}{\mathbf{T}},$$

the Truesdell rate $\overset{\square}{\mathbf{T}}$ equals the Oldroyd rate $\overset{\triangledown}{\mathbf{T}}$. Hence, for incompressible materials,

$$\rho \partial_{\boldsymbol{\mathcal{E}}} \phi \cdot \overset{\triangle}{\boldsymbol{\mathcal{E}}} + (\rho \partial_{\mathbf{T}} \phi + \boldsymbol{\mathcal{E}}) \cdot \overset{\triangledown}{\mathbf{T}} = -\Lambda(\theta, \boldsymbol{\mathcal{E}}, \mathbf{T}, \overset{\triangle}{\boldsymbol{\mathcal{E}}}, \overset{\triangledown}{\mathbf{T}}).$$

Henceforth three simple cases of the incremental constitutive equations (9) and (14) are considered assuming that Γ depends on $\dot{\mathbf{E}}$ or $\dot{\mathbf{T}}_{RR}$ via their norms, or is proportional to $|\mathbf{T}_{RR} \cdot \dot{\mathbf{E}}|$.

5 Duhem-Like Solids

Within models for hysteresis the Duhem model is expressed by a rate equation of the form

$$\dot{x} = f(x, u) g(\dot{u}),$$

where u is the input and x is the response. By analogy, we refer to Duhem-like solids as those modelled by a relation between stress-rate and strain-rate. First we let

$$\Gamma(\theta, \mathbf{E}, \mathbf{T}_{RR}, \dot{\mathbf{E}}, \dot{\mathbf{T}}_{RR}) = \gamma_E |\dot{\mathbf{E}}|, \qquad \gamma_E \geq 0,$$

so that (9) can be written in the form

$$(\rho_R \partial_{\mathbf{E}} \psi - \mathbf{T}_{RR}) \cdot \dot{\mathbf{E}} + \rho_R \partial_{\mathbf{T}_{RR}} \psi \cdot \dot{\mathbf{T}}_{RR} = -\gamma_E |\dot{\mathbf{E}}|. \tag{16}$$

We let γ_E depend on $\mathbf{E}$, $\mathbf{T}_{RR}$ and $\dot{\mathbf{E}}/|\dot{\mathbf{E}}|$ and be parameterized by the temperature θ. Accordingly, this equation is invariant under the time transformation

$$t \to c\, t, \qquad c > 0,$$

and hence it describes a rate-independent behaviour. When the restriction is made to the one-dimensional case, especially in connection with uniaxial tensile tests, Eq. (16) leads to the well-known class of scalar Duhem models.[13]

Since $\dot{\mathbf{E}} \cdot \dot{\mathbf{E}} = \mathrm{tr}\,(\mathbf{BD})^2$ we can translate (16) into the spatial description by way of (12),

$$(\rho \partial_{\boldsymbol{\mathcal{E}}} \psi - \mathbf{T}) \cdot \mathbf{D} + \rho \partial_{\mathbf{T}} \psi \cdot \overset{\square}{\mathbf{T}} = -\frac{\gamma_E}{J} [\mathrm{tr}\,(\mathbf{BD})^2]^{1/2}. \tag{17}$$

Hypo- and hyperelastic models follow from (16) and (17) as Duhem-like models without dissipation. In the spatial description, if $\gamma_E = 0$ and $\partial_{\mathbf{T}} \psi = \mathbf{0}$ it follows

$$(\rho \partial_{\boldsymbol{\mathcal{E}}} \psi - \mathbf{T}) \cdot \mathbf{D} = 0.$$

The arbitrariness of $\mathbf{D}$ implies

$$\mathbf{T} = \rho \partial_{\boldsymbol{\mathcal{E}}} \psi = \rho \partial_{\mathbf{F}} \psi \mathbf{F}^T$$

and $\mathbf{T}_{RR} = \rho_R \partial_{\mathbf{E}} \psi$ in the material description. Accordingly, $\mathbf{T}$ and $\boldsymbol{\mathcal{E}}$ cannot be independent variables. Otherwise, if ψ is a C^2 function we find a contradiction, namely

$$\partial^2_{\mathbf{T}\boldsymbol{\mathcal{E}}} \psi = \frac{1}{\rho} \mathbf{1} \otimes \mathbf{1}, \qquad \partial^2_{\boldsymbol{\mathcal{E}}\mathbf{T}} \psi = \mathbf{0}.$$

Accordingly, the dependence of $\mathbf{T}$ on $\boldsymbol{\mathcal{E}}$ must be expressed by a constitutive relation describing hyperelastic solids. The analogous conclusion follows in the material description.

If, instead, $\gamma_E = 0$ but $\partial_{\mathbf{T}} \psi \neq \mathbf{0}$ ($\partial_{\mathbf{T}_{RR}} \psi \neq \mathbf{0}$), then (9) and (12) can be written in the form

$$(\rho_R \partial_{\mathbf{E}} \psi - \mathbf{T}_{RR}) \cdot \dot{\mathbf{E}} + \rho_R \partial_{\mathbf{T}_{RR}} \psi \cdot \dot{\mathbf{T}}_{RR} = 0, \quad (\rho \partial_{\boldsymbol{\mathcal{E}}} \psi - \mathbf{T}) \cdot \mathbf{D} + \rho \partial_{\mathbf{T}} \psi \cdot \overset{\square}{\mathbf{T}} = 0. \tag{18}$$

We emphasize that these expressions are consistent, in that

$$\mathbf{T}_{RR} \cdot \dot{\mathbf{E}} = J \mathbf{T} \cdot \mathbf{D}, \qquad \partial_{\mathbf{E}} \psi \cdot \dot{\mathbf{E}} = \partial_{\boldsymbol{\mathcal{E}}} \psi \cdot \mathbf{D}, \qquad \partial_{\mathbf{T}_{RR}} \psi \cdot \dot{\mathbf{T}}_{RR} = \partial_{\mathbf{T}} \psi \cdot \overset{\square}{\mathbf{T}}.$$

As a consequence, $\dot{\mathbf{T}}_{RR}$ and $\overset{\square}{\mathbf{T}}$ can be given a linear representation in $\dot{\mathbf{E}}$ and $\mathbf{D}$, respectively. In this connection, we take advantage of the following

[13] See, e.g., [52], pp. 130-150.

Proposition 1 *Given a second-order tensor* $\mathbf{A}$ *let* $\mathbf{N} = \mathbf{A}/|\mathbf{A}|$. *If* $\mathbf{Z}$ *is a second-order tensor such that only* $\mathbf{Z} \cdot \mathbf{N}$ *is known then*[14]

$$\mathbf{Z} = (\mathbf{Z} \cdot \mathbf{N})\mathbf{N} + (\mathsf{I} - \mathbf{N} \otimes \mathbf{N})\mathbf{G} \tag{19}$$

for any second-order tensor $\mathbf{G}$.

Proof To see this we first observe that, for any second-order tensor $\mathbf{Y}$,

$$[(\mathsf{I} - \mathbf{N} \otimes \mathbf{N})\mathbf{Y}] \cdot \mathbf{N} = 0;$$

in words, the projection onto $\mathbf{N}$ of the orthogonal part of $\mathbf{Y}$ with respect to $\mathbf{N}$ is identically zero. Hence,

$$\mathbf{Y} = \mathbf{Y}_\parallel + \mathbf{Y}_\perp, \qquad \mathbf{Y}_\parallel := (\mathbf{Y} \cdot \mathbf{N})\mathbf{N}, \quad \mathbf{Y}_\perp := (\mathsf{I} - \mathbf{N} \otimes \mathbf{N})\mathbf{Y},$$

and $\mathbf{Y}_\perp \cdot \mathbf{N} = 0$.

If $\mathbf{Z}$ is a tensor such that $\mathbf{Z}_\parallel$ is given while $\mathbf{Z}_\perp$ is undetermined then the representation of $\mathbf{Z}$ allows for any tensor $\mathbf{Z}_\perp$ subject to $\mathbf{Z}_\perp \cdot \mathbf{N} = 0$. Now, since $[(\mathsf{I} - \mathbf{N} \otimes \mathbf{N})\mathbf{G}] \cdot \mathbf{N} = 0$ for any second-order tensor $\mathbf{G}$, then $[(\mathsf{I} - \mathbf{N} \otimes \mathbf{N})\mathbf{G}]$ is any possible value of $\mathbf{Z}_\perp$. Hence the relation $\mathbf{Z} = \mathbf{Z}_\parallel + \mathbf{Z}_\perp$ results in the representation (19). □

Upon the identifications

$$\mathbf{A} = \partial_{\mathbf{T}_{RR}} \psi, \qquad \mathbf{Z} = \dot{\mathbf{T}}_{RR}, \qquad \mathbf{G} = \mathsf{G}_{RR} \dot{\mathbf{E}},$$

we now apply the representation (19) to $\dot{\mathbf{T}}_{RR}$ subject to $(18)_1$. It follows

$$\dot{\mathbf{T}}_{RR} = \frac{1}{\rho_R} [(\mathbf{T}_{RR} - \rho_R \partial_{\mathbf{E}} \psi) \cdot \dot{\mathbf{E}}]\mathbf{N} + (\mathsf{I} - \mathbf{N} \otimes \mathbf{N})\mathsf{G}_{RR} \dot{\mathbf{E}},$$

where $\mathbf{N} = \partial_{\mathbf{T}_{RR}} \psi / |\partial_{\mathbf{T}_{RR}} \psi|$ and $\mathsf{G}_{RR}(\theta, \mathbf{E}, \mathbf{T}_{RR})$ is any arbitrary fourth-order tensor-valued function. Hence, letting

$$\mathsf{C}_{RR}(\theta, \mathbf{E}, \mathbf{T}_{RR}) = \mathsf{G}_{RR} + \frac{1}{\rho_R |\partial_{\mathbf{T}_{RR}} \psi|^2} \partial_{\mathbf{T}_{RR}} \psi \otimes (\mathbf{T}_{RR} - \rho_R \partial_{\mathbf{E}} \psi - \rho_R \mathsf{G}_{RR}^T \partial_{\mathbf{T}_{RR}} \psi)$$

we can write

$$\dot{\mathbf{T}}_{RR} = \mathsf{C}_{RR} \dot{\mathbf{E}}. \tag{20}$$

In general, the fourth-order tensor C_{RR} enjoys the minor symmetries, not the major one.

By the arbitrariness of G_{RR} it follows that there are infinitely many tensors C_{RR} compatible with a given free energy ψ. It is apparent that this representation relies crucially on the assumption $\partial_{\mathbf{T}_{RR}} \psi \neq \mathbf{0}$ and then cannot hold for hyperelastic materials.

If, instead, a constitutive relation

$$\dot{\mathbf{T}}_{RR} = \hat{\mathsf{C}}_{RR} \dot{\mathbf{E}}, \tag{21}$$

[14] The symbol I denotes the fourth-order identity tensor.

is assigned in advance then from $(18)_1$ we obtain

$$(\rho_R \partial_{\mathbf{E}} \psi - \mathbf{T}_{RR}) \cdot \dot{\mathbf{E}} + \rho_R \partial_{\mathbf{T}_{RR}} \psi \cdot \hat{\mathsf{C}}_{RR} \dot{\mathbf{E}} = 0.$$

The arbitrariness of $\dot{\mathbf{E}}$ then implies

$$\mathbf{T}_{RR} - \rho_R \partial_{\mathbf{E}} \psi = \rho_R \hat{\mathsf{C}}_{RR}^T \partial_{\mathbf{T}_{RR}} \psi. \tag{22}$$

In [16] the existence of a thermodynamic potential ψ satisfying $(18)_1$ is investigated by exploiting the overdetermined system (22) with a given tensor $\hat{\mathsf{C}}_{RR}$. In the particular case when $\partial_{\mathbf{T}_{RR}} \psi = \mathbf{0}$, Eq. (22) becomes $\mathbf{T}_{RR} = \rho_R \partial_{\mathbf{E}} \psi$ and this relation is compatible with (21) provided that ψ depends only on $\mathbf{E}$ and $\hat{\mathsf{C}}_{RR}^T = \rho_R \partial^2_{\mathbf{E}} \psi$.

In the general case, when $\gamma_E \neq 0$ and $\partial_{\mathbf{T}_{RR}} \psi \neq \mathbf{0}$, by paralleling previous arguments we can rewrite (16) as

$$\dot{\mathbf{T}}_{RR} = \mathsf{C}_{RR} \dot{\mathbf{E}} + \boldsymbol{\Gamma}_E |\dot{\mathbf{E}}|, \qquad \boldsymbol{\Gamma}_E = -\gamma_E \frac{\partial_{\mathbf{T}_{RR}} \psi}{\rho_R |\partial_{\mathbf{T}_{RR}} \psi|^2}. \tag{23}$$

Observe that sometimes an additive decomposition of the stress is introduced a priori [26]. Here the decomposition (23) is obtained as a result for the rate $\dot{\mathbf{T}}_{RR}$.

Applying to $(18)_2$ the representation (19) with $\mathbf{A} = \partial_{\mathbf{T}} \psi$, $\mathbf{Z} = \overset{\square}{\mathbf{T}}$ and $\mathbf{G} = \mathbf{GD}$, in the spatial description we find

$$\overset{\square}{\mathbf{T}} = \mathsf{C}\, \mathbf{D}, \tag{24}$$

where

$$\mathsf{C}(\theta, \boldsymbol{\mathcal{E}}, \mathbf{T}) = \mathsf{G} + \frac{1}{\rho |\partial_{\mathbf{T}} \psi|^2}\, \partial_{\mathbf{T}} \psi \otimes (\mathbf{T} - \rho \partial_{\boldsymbol{\mathcal{E}}} \psi - \rho\, \mathsf{G}^T \partial_{\mathbf{T}} \psi).$$

As well as C_{RR}, the tensor C enjoys the minor symmetries, not the major one.

As an example, let G be any non-singular, symmetric fourth-order tensor (possibly parameterized by θ) and let

$$\rho \psi = \rho \psi_0(\theta) + \tfrac{1}{2} \mathsf{G}^{-1} \mathbf{T} \cdot \mathbf{T}.$$

Then, $\partial_{\mathbf{T}} \psi = \mathsf{G}^{-1} \mathbf{T}$ and applying (24) it follows $\overset{\square}{\mathbf{T}} = \mathsf{G}\mathbf{D}$. After replacing this relation into $(18)_2$ and exploiting the expression of $\rho\psi$ we obtain the identity

$$-\mathbf{T} \cdot \mathbf{D} + \mathsf{G}^{-1} \mathbf{T} \cdot \mathsf{G}\mathbf{D} = 0.$$

In the general case, when $\gamma_E \neq 0$ and $\partial_{\mathbf{T}} \psi \neq \mathbf{0}$, (17) yields

$$\overset{\square}{\mathbf{T}} = \mathsf{C}\, \mathbf{D} + \boldsymbol{\Lambda}_E [\mathrm{tr}\, (\mathbf{BD})^2]^{1/2}, \qquad \boldsymbol{\Lambda}_E = -\gamma_E \frac{\partial_{\mathbf{T}} \psi}{\rho_R |\partial_{\mathbf{T}} \psi|^2}. \tag{25}$$

5.1 Hypoelastic Solids

Following Truesdell [47, 48], materials characterized by relations of the form (24) are said to be hypoelastic if the tensor C is possibly dependent on the temperature θ and the stress

$\mathbf{T}$ but is independent of the strain. Further the solid is said to be hypoelastic of grade zero if $\mathbf{C}$ is independent of $\mathbf{T}$. Instead Noll [31, 32], Thomas [46] and Ericksen [8] define and investigate hypoelastic solids in the form

$$\overset{\circ}{\mathbf{T}} = \mathbf{B}\mathbf{D}, \tag{26}$$

$\overset{\circ}{\mathbf{T}}$ being the corotational derivative of $\mathbf{T}$ and $\mathbf{B}$ is a fourth-order tensor function of $\mathbf{T}$. So different rates of the Cauchy stress are considered, the Truesdell rate and the corotational rate. With reference to the present approach we can then say that the constitutive equation (21) (and likewise, in the spatial description, Eq. (24)) describes hypoelastic solids if $\mathbf{C}_{RR}$ is independent of the strain measure $\mathbf{E}$, which is the case if $\partial_{\mathbf{E}}\psi = \mathbf{0}$.

Remark 1 Under the crucial assumption $\partial_{\mathbf{T}}\psi \neq \mathbf{0}$, the absence of dissipation, locally in space and time, holds true even for the general scheme in the spatial description. Indeed, replacing the constitutive relation (24) into the thermodynamic relation (12) we obtain the rate of dissipation

$$\Lambda = (\rho\partial_{\boldsymbol{\mathcal{E}}}\psi - \mathbf{T}) \cdot \mathbf{D} + \rho\partial_{\mathbf{T}}\psi \cdot \mathbf{C}\mathbf{D}.$$

Applying the definition of $\mathbf{C}$ it follows

$$\begin{aligned}\Lambda &= (\rho\partial_{\boldsymbol{\mathcal{E}}}\psi - \mathbf{T}) \cdot \mathbf{D} + \rho\partial_{\mathbf{T}}\psi \cdot \left[\mathbf{G}\mathbf{D} + \frac{(\mathbf{T} - \rho\partial_{\boldsymbol{\mathcal{E}}}\psi) \cdot \mathbf{D}}{\rho|\partial_{\mathbf{T}}\psi|^2}\partial_{\mathbf{T}}\psi - \frac{\partial_{\mathbf{T}}\psi \cdot (\mathbf{G}\mathbf{D})}{|\partial_{\mathbf{T}}\psi|^2}\partial_{\mathbf{T}}\psi\right] \\ &= (\rho\partial_{\boldsymbol{\mathcal{E}}}\psi - \mathbf{T}) \cdot \mathbf{D} + \rho\partial_{\mathbf{T}}\psi \cdot (\mathbf{G}\mathbf{D}) + (\mathbf{T} - \rho\partial_{\boldsymbol{\mathcal{E}}}\psi) \cdot \mathbf{D} - \rho\partial_{\mathbf{T}}\psi \cdot (\mathbf{G}\mathbf{D}) = 0.\end{aligned}$$

The result (24) resembles the Lagrangian Jaumann (LJ) formulation of inelastic strain. Now, in the present notation the LJ formulation, e.g., Eq. (1) of [21], reads as (26), $\mathbf{B}$ being a constant fourth-order tensor. As shown, e.g., by [21], in the LJ formulation a residual stress remains at the end of an elastic closed path, for finite deformations, so that the work done may not be zero.

This is not a contradiction. Within the present approach both stress and strain are independent variables and $\partial_{\mathbf{T}}\psi \neq \mathbf{0}$. Therefore the work done must be calculated in a closed path in the strain-stress space; the presence of a residual stress prevents the path to be closed. As a comment, we observe that the stretching $\mathbf{D}$ is not the corotational derivative of a deformation measure; instead $\mathbf{D} = \overset{\triangle}{\boldsymbol{\mathcal{E}}}$ according to (11). The use of $\overset{\circ}{\mathbf{T}}$ in the left-hand side as an objective stress rate is not motivated by the right-hand side. Instead, the thermodynamic analysis leads to (24) and hence the Truesdell derivative appears as the appropriate stress rate.

A different approach consists in assuming that a hypoelastic constitutive relation (of grade zero)

$$\overset{\square}{\mathbf{T}} = \hat{\mathbf{C}}\,\mathbf{D}, \tag{27}$$

is assigned in advance with a constant elastic tensor $\hat{\mathbf{C}}$. Then from $(18)_2$ we obtain

$$(\rho\partial_{\boldsymbol{\mathcal{E}}}\psi - \mathbf{T}) \cdot \mathbf{D} + \rho\partial_{\mathbf{T}}\psi \cdot \hat{\mathbf{C}}\,\mathbf{D} = 0.$$

The arbitrariness of $\mathbf{D}$ then implies

$$\mathbf{T} - \rho\partial_{\boldsymbol{\mathcal{E}}}\psi = \rho\hat{\mathbf{C}}^T\partial_{\mathbf{T}}\psi. \tag{28}$$

In the particular case when $\partial_{\mathbf{T}}\psi = \mathbf{0}$, Eq. (28) becomes

$$\mathbf{T} = \rho \partial_{\boldsymbol{\varepsilon}} \psi. \tag{29}$$

Unfortunately, contrary to what occurs in the material description, this relation is incompatible with (27). Xiao et al. [3, 27, 55–57] presented explicit, integrable-exactly hypoelastic relations like (26) based on the logarithmic stress rate and proved that this is the only choice among all infinitely many objective corotational stress rates compatible with (29).

The derivative $\dot{\mathbf{K}} - \boldsymbol{\Sigma}\mathbf{K} - \mathbf{K}\boldsymbol{\Sigma}^T$ is objective subject to the transformation law $\boldsymbol{\Sigma}^* = \mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}^T - \boldsymbol{\Omega}$, $\boldsymbol{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^T$, $\mathbf{Q}$ being the rotation tensor [30]. This happens for the corotational derivative $\overset{\circ}{\mathbf{K}}$, where $\boldsymbol{\Sigma} = \mathbf{W}$, or any derivative obtained by adding to $\boldsymbol{\Sigma}$ an objective tensor. This is the case for the Truesdell and Oldroyd rates and, e.g., the Hill rate $\overset{\circ}{\mathbf{K}} - m(\mathbf{KD} + \mathbf{DK})$, m being any real number, or the logarithmic rate [2, 54] where

$$\boldsymbol{\Sigma} = \boldsymbol{\Omega}^{\log} = \mathbf{W} + \sum_{\sigma \neq \tau = 1}^{m} \left(\frac{1 + (b_\sigma / b_\tau)}{1 - b_\sigma / b_\tau} + \frac{2}{\ln(b_\sigma / b_\tau)} \right) \mathbf{B}_\sigma \mathbf{D} \mathbf{B}_\tau,$$

$b_1, \ldots, b_m$ being the m distinct eigenvalues and $\mathbf{B}_1, \ldots, \mathbf{B}_m$ the corresponding eigenprojections of $\mathbf{B} = \mathbf{F}\mathbf{F}^T$. The logarithmic derivative is motivated also by integrability conditions for every process of elastic deformation.

Remark 2 In ref. [36] Prager examines the condition that the stress field has to be independent of time, when the continuum performs a rigid body motion, relative to an observer at rest with the continuum. This requirement in turn implies that the yield function is stationary when the stress rate vanishes. Based on this requirement Prager infers that, in connection with the modelling of plastic behaviour, the corotational derivative is preferable to the other definitions of stress rate (Oldroyd and Truesdell) examined in [36]. We observe that in rigid motions $\mathbf{D} = \mathbf{0}$ (see, e.g., [15, § 10.1]) and then the Cotter-Rivlin, Oldroyd, Truesdell and logarithmic derivatives reduce to the corotational derivative. It seems then worth accounting also for thermodynamic requirements to select appropriate objective time derivatives.

Finally, we also remark that owing to the relation

$$\overset{\square}{\mathbf{T}} = \overset{\circ}{\mathbf{T}} - \mathbf{DT} - \mathbf{TD} + (\mathrm{tr}\,\mathbf{D})\mathbf{T}$$

the rate equation (24) can be written in the form (26), namely

$$\overset{\circ}{\mathbf{T}} = \mathsf{C}^* \mathbf{D}, \qquad \mathsf{C}^* = \mathsf{C} + \mathsf{A},$$

where

$$\mathsf{A}_{ikpq} = T_{ip}\delta_{kq} + T_{qk}\delta_{ip} - T_{ik}\delta_{pq}.$$

Hence, there is no contradiction between Prager's analysis and rate equations in the form (24) when C^* is allowed to depend on $\mathbf{T}$.

5.2 An Additive Decomposition of Rates

An analogous setting is now established by means of the Gibbs free energy ϕ. As a result a rate-type equation for $\mathbf{E}$ is obtained which eventually does not require the dependence of ψ on the stress.

Consider (14) and let

$$\Gamma(\theta, \mathbf{E}, \mathbf{T}_{RR}, \dot{\mathbf{E}}, \dot{\mathbf{T}}_{RR}) = \gamma_T |\dot{\mathbf{T}}_{RR}|, \qquad \gamma_T \geq 0,$$

where γ_T is allowed to depend on $\mathbf{E}$, $\mathbf{T}_{RR}$ and $\dot{\mathbf{T}}_{RR}/|\dot{\mathbf{T}}_{RR}|$ and is parameterized by the temperature θ. Accordingly,

$$\rho_R \partial_{\mathbf{E}}\phi \cdot \dot{\mathbf{E}} + (\rho_R \partial_{\mathbf{T}_{RR}}\phi + \mathbf{E}) \cdot \dot{\mathbf{T}}_{RR} = -\gamma_T |\dot{\mathbf{T}}_{RR}|. \tag{30}$$

As with (16), this equation is rate-independent. Since

$$\dot{\mathbf{T}}_{RR} \cdot \dot{\mathbf{T}}_{RR} = J^2 (\mathbf{F}^{-1} \overset{\square}{\mathbf{T}} \mathbf{F}^{-T}) \cdot (\mathbf{F}^{-1} \overset{\square}{\mathbf{T}} \mathbf{F}^{-T}) = J^2 \mathrm{tr} \left(\mathbf{B}^{-1} \overset{\square}{\mathbf{T}}\right)^2$$

then (30) in the spatial description becomes

$$\partial_{\boldsymbol{\mathcal{E}}}\phi \cdot \mathbf{D} + (\rho \partial_{\mathbf{T}}\phi + \boldsymbol{\mathcal{E}}) \cdot \overset{\square}{\mathbf{T}} = -\gamma_T \left[\mathrm{tr}\left(\mathbf{B}^{-1} \overset{\square}{\mathbf{T}}\right)^2\right]^{1/2}.$$

If $\gamma_T = 0$ and $\partial_{\mathbf{E}}\phi = \mathbf{0}$ then (30) reduces to

$$(\rho_R \partial_{\mathbf{T}_{RR}}\phi + \mathbf{E}) \cdot \dot{\mathbf{T}}_{RR} = 0.$$

The arbitrariness of $\dot{\mathbf{T}}_{RR}$ implies $\mathbf{E} = \rho_R \partial_{\mathbf{T}_{RR}}\phi$ ($\boldsymbol{\mathcal{E}} = -\rho \partial_{\mathbf{T}}\phi$ in the spatial description) and hence $\mathbf{T}_{RR}$ and $\mathbf{E}$ cannot be independent.

If, instead, $\gamma_T = 0$ but $\partial_{\mathbf{E}}\phi \neq \mathbf{0}$ ($\partial_{\boldsymbol{\mathcal{E}}}\phi \neq \mathbf{0}$), then (14) and (15) reduce to

$$\rho_R \partial_{\mathbf{E}}\phi \cdot \dot{\mathbf{E}} + (\rho_R \partial_{\mathbf{T}_{RR}}\phi + \mathbf{E}) \cdot \dot{\mathbf{T}}_{RR} = 0, \qquad \rho \partial_{\boldsymbol{\mathcal{E}}}\phi \cdot \mathbf{D} + (\rho \partial_{\mathbf{T}}\phi + \boldsymbol{\mathcal{E}}) \cdot \overset{\square}{\mathbf{T}} = 0. \tag{31}$$

Then, $\dot{\mathbf{E}}$ and $\mathbf{D}$ can be given a linear representation in $\dot{\mathbf{T}}_{RR}$ and $\overset{\square}{\mathbf{T}}$, respectively. By paralleling the procedure of the previous section, we can write

$$\begin{aligned} &\dot{\mathbf{E}} = \mathsf{K}_{RR} \dot{\mathbf{T}}_{RR}, \\ &\mathsf{K}_{RR} = \mathsf{H}_{RR} - \frac{1}{\rho_R |\partial_{\mathbf{E}}\phi|^2} \partial_{\mathbf{E}}\phi \otimes (\mathbf{E} + \rho_R \partial_{\mathbf{T}_{RR}}\phi - \rho_R \mathsf{H}_{RR}^T \partial_{\mathbf{E}}\phi) \end{aligned} \tag{32}$$

and

$$\mathbf{D} = \mathsf{K} \overset{\square}{\mathbf{T}}, \qquad \mathsf{K} = \mathsf{H} - \frac{1}{\rho |\partial_{\boldsymbol{\mathcal{E}}}\phi|^2} \partial_{\boldsymbol{\mathcal{E}}}\phi \otimes (\boldsymbol{\mathcal{E}} + \rho \partial_{\mathbf{T}}\phi - \rho \mathsf{H}^T \partial_{\boldsymbol{\mathcal{E}}}\phi),$$

H_{RR} and H being arbitrary fourth-order tensor-valued functions independent of the rates. The additive decomposition of K_{RR} and K induces the corresponding decomposition of $\dot{\mathbf{E}}$ and $\mathbf{D}$.

The following example shows a possible form of K_{RR}. Let

$$\rho_R \phi(\theta, \mathbf{E}, \mathbf{T}_{RR}) = \rho_R \phi_0(\theta) + \tfrac{1}{2} \int_0^{|\mathbf{T}_{RR}|^2} [\nu(\theta) + u\beta(\theta, u)] du - \mathbf{T}_{RR} \cdot \mathbf{E},$$

ν and β being real-valued functions. Consequently

$$\rho_R \partial_{\mathbf{T}_{RR}} \phi = [\nu(\theta) + |\mathbf{T}_{RR}|^2 \beta(\theta, |\mathbf{T}_{RR}|^2)]\mathbf{T}_{RR} - \mathbf{E}, \qquad \rho_R \partial_{\mathbf{E}} \phi = -\mathbf{T}_{RR},$$

whence

$$\mathbf{E} + \rho_R \partial_{\mathbf{T}_{RR}} \phi = [\nu(\theta) + |\mathbf{T}_{RR}|^2 \beta(\theta, |\mathbf{T}_{RR}|^2)]\rho_R \partial_{\mathbf{E}} \phi.$$

Let $\mathsf{H}_{RR} = \mu \mathsf{I}$. It follows from (32) that

$$\mathsf{K}_{RR}(\theta, \mathbf{T}_{RR}) = \mu \mathsf{I} + \beta(\theta, |\mathbf{T}_{RR}|^2)\mathbf{T}_{RR} \otimes \mathbf{T}_{RR} + (\nu - \mu)\frac{\mathbf{T}_{RR}}{|\mathbf{T}_{RR}|} \otimes \frac{\mathbf{T}_{RR}}{|\mathbf{T}_{RR}|}$$

and then K_{RR} enjoys the major symmetry. Choosing $\mu = \nu$ reduces K_{RR} to

$$\mathsf{K}_{RR} = \nu(\theta)\mathsf{I}_{RR} + \beta(\theta, |\mathbf{T}_{RR}|^2)\mathbf{T}_{RR} \otimes \mathbf{T}_{RR}.$$

The resulting constitutive equation (32) may be viewed as generated by Prandtl-Reuss plasticity theory provided that we let $\operatorname{tr}\mathbf{E} = \operatorname{tr}\mathbf{T}_{RR} = 0$.[15]

In the general case, when $\gamma_T \neq 0$ and $\partial_{\mathbf{E}}\phi = \partial_{\mathbf{E}}\psi - \mathbf{T}_{RR} \neq \mathbf{0}$, by paralleling previous arguments we can rewrite (30) as

$$\dot{\mathbf{E}} = \mathsf{K}_{RR}\dot{\mathbf{T}}_{RR} + \boldsymbol{\Gamma}_T |\dot{\mathbf{T}}_{RR}|, \qquad \boldsymbol{\Gamma}_T = -\gamma_T \frac{\partial_{\mathbf{E}}\phi}{\rho_R |\partial_{\mathbf{E}}\phi|^2}. \tag{33}$$

Taking into account (13), both K_{RR} and $\boldsymbol{\Gamma}_T$ can be represented using ψ rather than ϕ. A similar result holds true in the spatial description, namely

$$\mathbf{D} = \mathsf{K}\overset{\square}{\mathbf{T}} + \boldsymbol{\Lambda}_E \left[\operatorname{tr}\left(\mathbf{B}^{-1}\overset{\square}{\mathbf{T}}\right)^2\right]^{1/2}, \qquad \boldsymbol{\Lambda}_E = -\gamma_T \frac{\partial_{\boldsymbol{\varepsilon}}\phi}{\rho |\partial_{\boldsymbol{\varepsilon}}\phi|^2}.$$

Remark 3 By Eq. (33), the strain rate $\dot{\mathbf{E}}$ is the sum of two terms

$$\mathsf{K}_{RR}\dot{\mathbf{T}}_{RR}, \qquad \boldsymbol{\Gamma}_T |\dot{\mathbf{T}}_{RR}|.$$

A similar decomposition also holds for the stretching tensor $\mathbf{D}$. In a qualitative sense, we might view this splitting as the analogue of the standard incremental relation $\dot{\mathbf{E}} = \dot{\mathbf{E}}^e + \dot{\mathbf{E}}^p$ of classical formulations of plasticity where

$$\dot{\mathbf{E}}^e := \mathsf{K}_{RR}\dot{\mathbf{T}}_{RR}, \qquad \dot{\mathbf{E}}^p := \boldsymbol{\Gamma}_T |\dot{\mathbf{T}}_{RR}|.$$

It should be noted that this additive decomposition does not require the assumption of small deformations, but is valid in the more general context of large deformations. However this view is merely formal in that (33) is non-integrable and we cannot recover an additive decomposition of $\mathbf{E}$ from it.

The analogies suggested by Remark 3 lead us to introduce the notion of *hardening variable* S directly from the model, without adding new internal variables. To this end we let

$$\dot{\mathsf{S}} = \boldsymbol{\Gamma}_T |\dot{\mathbf{T}}_{RR}| \cdot \frac{\mathbf{T}_{RR} - \rho_R \partial_{\mathbf{E}}\psi}{|\mathbf{T}_{RR} - \rho_R \partial_{\mathbf{E}}\psi|}$$

[15] We mention [2], where the Prandtl-Reuss theory is applied to finite deformations, and [28, § C], where the complementary strain energy, $-\phi$, has the same form exhibited here.

Since $\rho_R \partial_{\mathbf{E}} \phi = \rho_R \partial_{\mathbf{E}} \psi - \mathbf{T}_{RR}$ then $\dot{\mathsf{S}}$ is a non-negative quantity strictly related to the rate of dissipation Γ. In fact, from (33) it follows

$$\dot{\mathsf{S}} = \gamma_T \frac{|\dot{\mathbf{T}}_{RR}|}{|\mathbf{T}_{RR} - \rho_R \partial_{\mathbf{E}} \psi|} = \frac{\Gamma}{|\mathbf{T}_{RR} - \rho_R \partial_{\mathbf{E}} \psi|} \geq 0.$$

Owing to this relation, the quantity $\mathsf{S}(t) = \int_0^t \dot{\mathsf{S}}(\tau) d\tau$ may be viewed as the *accumulated dissipation* up to t.

5.3 A Three-Dimensional Model of Inelastic Solid

Another class of models is now determined by letting Γ depend on $\dot{\mathbf{E}}$ in the form

$$\Gamma = \mathcal{H} \mathbf{T}_{RR} \cdot \dot{\mathbf{E}}, \qquad \mathcal{H} = H(\mathbf{T}_{RR} \cdot \dot{\mathbf{E}}) H\big(|\mathbf{T}_{RR}^{\mathrm{dev}}|^2 - \tfrac{2}{3} T_y^2\big),$$

where $\mathbf{T}_{RR}^{\mathrm{dev}}$ represents the deviatoric part of $\mathbf{T}_{RR}$, T_y denotes the tensile yield strength of the material[16] and H is the Heaviside step function. Accordingly Eq. (9) can be written in the form

$$(\rho_R \partial_{\mathbf{E}} \psi - [1 - \mathcal{H}] \mathbf{T}_{RR}) \cdot \dot{\mathbf{E}} + \rho_R \partial_{\mathbf{T}_{RR}} \psi \cdot \dot{\mathbf{T}}_{RR} = 0.$$

By arguing as in §5.1 it follows that

$$\dot{\mathbf{T}}_{RR} = \mathsf{C}_{RR} \dot{\mathbf{E}}$$

where

$$\mathsf{C}_{RR} = \mathsf{G}_{RR} + \frac{1}{\rho_R |\partial_{\mathbf{T}_{RR}} \psi|^2} \partial_{\mathbf{T}_{RR}} \psi \otimes \big([1 - \mathcal{H}] \mathbf{T}_{RR} - \rho_R \partial_{\mathbf{E}} \psi - \rho_R \mathsf{G}_{RR}^T \partial_{\mathbf{T}} \psi\big).$$

Likewise, owing to the work-conjugacy relation, in the spatial description we obtain

$$\Lambda := \Gamma / J = \mathcal{H} \mathbf{T} \cdot \mathbf{D}, \qquad \mathcal{H} = H(\mathbf{T} \cdot \mathbf{D}) H\big(|\mathbf{T}_*^{\mathrm{dev}}|^2 - \tfrac{2}{3} T_y^2\big),$$

where $\mathbf{T}_* := J \mathbf{T} \mathbf{B}^{-1}$, $\mathbf{B}$ being the left Cauchy-Green deformation tensor.[17] From equation (12) we then obtain the representation (24), $\overset{\square}{\mathbf{T}} = \mathsf{C}\, \mathbf{D}$, where

$$\mathsf{C}(\theta, \mathcal{E}, \mathbf{T}) = \mathsf{G} + \frac{1}{\rho |\partial_{\mathbf{T}} \psi|^2} \partial_{\mathbf{T}} \psi \otimes ([1 - \mathcal{H}] \mathbf{T} - \rho \partial_{\mathcal{E}} \psi - \rho\, \mathsf{G}^T \partial_{\mathbf{T}} \psi).$$

As an example, let X be a symmetric fourth-order tensor and

$$\rho \partial_{\mathbf{T}} \psi = \mathsf{X} \mathbf{T}, \quad \partial_{\mathcal{E}} \psi = \mathbf{0}, \quad \mathsf{G} = [1 - \mathcal{H}] \mathsf{X}^{-1}.$$

Then we have $\mathsf{C} = \mathsf{G} = [1 - \mathcal{H}] \mathsf{X}^{-1}$ and hence

$$\mathsf{X}\, \overset{\square}{\mathbf{T}} = [1 - H(\mathbf{T} \cdot \mathbf{D}) H\big(|\mathbf{T}_*^{\mathrm{dev}}|^2 - \tfrac{2}{3} T_y^2\big)] \mathbf{D}$$

[16] This choice is consistent with von Mises' standard criterion.

[17] Here we applied the identities $\mathrm{tr}\, \mathbf{T}_{RR} = J \mathrm{tr}\, (\mathbf{T}\mathbf{B}^{-1})$ and $|\mathbf{T}_{RR}|^2 = J^2 \mathrm{tr}\, [(\mathbf{T}\mathbf{B}^{-1})^2]$.

If, for simplicity, we let $\mathbf{X}$ be the isotropic tensor[18]

$$\mathbf{X} = \frac{1}{E}[(1+\nu)\mathsf{I} - \nu \mathbf{1} \otimes \mathbf{1}]$$

then we find

$$\frac{1}{E}[(1+\nu)\overset{\square}{\mathbf{T}} - \nu(\mathrm{tr}\,\overset{\square}{\mathbf{T}})\mathbf{1}] = [1 - H(\mathbf{T}\cdot\mathbf{D})H\big(|\mathbf{T}_*^{\mathrm{dev}}|^2 - \tfrac{2}{3}T_y^2\big)]\mathbf{D} \tag{34}$$

and in this special case the free energy density is given by

$$\rho\psi = \rho\psi_0(\theta) + \frac{1}{2E}[(1+\nu)|\mathbf{T}|^2 - \nu(\mathrm{tr}\,\mathbf{T})^2].$$

As a comment, if $\overset{\square}{\mathbf{T}}$ vanishes then $\dot{\mathbf{T}}_{RR}$ does the same,[19] so that $|\mathbf{T}_*|^2 = |\mathbf{T}_{RR}|^2$ holds constant and the yield surface is well defined. Moreover, Eq. (34) reduces to Eq. (5.4) of [5] when the linear approximation is considered.

6 One-Dimensional Models

Restrict attention to one-dimensional models associated with strain and stress in the direction $\mathbf{e}$ so that

$$\mathbf{E} = E\,\mathbf{e}\otimes\mathbf{e}, \qquad \mathbf{T}_{RR} = S\,\mathbf{e}\otimes\mathbf{e}.$$

The notation S for the component of $\mathbf{T}_{RR}$ is consistent with the *engineering-stress* considered in the literature as the ratio of the axial force over the reference area ([15, § 74]).

Hence we regard ψ as a function of θ, E, S and write (9) in the form

$$\partial_S\psi\,\dot{S} + (\partial_E\psi - S)\dot{E} = -\Gamma(\theta, E, S, \dot{E}, \dot{S}). \tag{35}$$

At constant temperature $\dot{\psi} = \partial_S\psi\,\dot{S} + \partial_E\psi\,\dot{E}$ an hence integration of (35), as $t \in [t_1, t_2]$, along a closed curve in the $E - S$ plane results in

$$0 \le \int_{t_1}^{t_2} \Gamma\,dt = \int_{t_1}^{t_2} [-\partial_S\psi\,\dot{S} + (S - \partial_E\psi)\dot{E}]dt = \int_{t_1}^{t_2} S\,\dot{E}\,dt = \oint S\,dE, \tag{36}$$

$\oint$ denoting the integral along the closed curve. By (36) we have $\oint S\,dE \ge 0$ and this implies that the closed curve is run in the clockwise sense.

Consistent with (16) we let

$$\Gamma(\theta, E, S, \dot{E}, \dot{S}) = \gamma_E|\dot{E}|,$$

$\gamma_E \ge 0$ being possibly dependent on E, S, $\mathrm{sgn}\,\dot{E}$ and parameterized by θ. Equation (35) is invariant under the time transformation $t \mapsto ct$, $c > 0$, and hence the associated model is *rate-independent* because so is γ_E. It describes plastic behaviour when γ_E is smooth, but

[18] E and ν denote the Young modulus and the Poisson ratio, respectively.

[19] A straightforward calculations leads to $J^2|\overset{\square}{\mathbf{T}}|^2 = \mathrm{tr}\,[(\dot{\mathbf{T}}_{RR}\mathbf{C})^2]$.

when γ_E is piecewise smooth and vanishes in a suitable open region (called *elastic region*) it describes an elastic-plastic behaviour.

Look at time intervals where $\dot{E} \neq 0$. Since $\partial_S \psi \neq 0$, divide Eq. (35) by $\partial_S \psi \dot{E}$ to obtain

$$\frac{\dot{S}}{\dot{E}} = \frac{S - \partial_E \psi}{\partial_S \psi} - \frac{\gamma_E}{\partial_S \psi} \operatorname{sgn} \dot{E}. \tag{37}$$

As $\dot{E} \neq 0$, E changes in time and S changes in time correspondingly. We can then view S as a function of E and

$$\dot{S} = \frac{dS}{dE} \dot{E}, \qquad \frac{\dot{S}}{\dot{E}} = \frac{dS}{dE}.$$

Now let

$$\chi_1 = \frac{S - \partial_E \psi}{\partial_S \psi}, \qquad \chi_2 = -\frac{\gamma_E}{\partial_S \psi}. \tag{38}$$

Accordingly, χ_1 is a function of E and S parameterized by θ, whereas χ_2 can also depend on $\operatorname{sgn} \dot{E}$. The uniaxial stress-strain slope is then given by

$$\frac{dS}{dE} = \chi_1 + \chi_2 \operatorname{sgn} \dot{E}. \tag{39}$$

If $\gamma_E = 0$, and hence $\chi_2 = 0$, dS/dE gives the slope of the hypoelastic curve and

$$\frac{dS}{dE} = \chi_1(\theta, E, S) > 0$$

means that the slope at E depends also on S; hence χ_1 is the positive slope of the (engineering) stress-strain curve. Accordingly the elastic model is recovered by letting

$$\gamma_E = 0, \qquad \partial_S \psi = 0, \qquad S = \partial_E \psi.$$

Since the slope of the stress-strain curve is supposed to be non-negative, in addition to the positivity of χ_1 we assume that

$$\chi_1 + \chi_2 \operatorname{sgn} \dot{E} \geq 0. \tag{40}$$

It is worth remarking that the monotonicity condition (40) follows from physical arguments about the stress-strain curve but are not required by thermodynamics.

We now proceed to establish some models of elastic-plastic rate-independent materials by specifying the functions χ_1 and χ_2. In this regard we observe that χ_1 is fully determined by the free energy ψ while χ_2 depends also on γ_E. Consequently different models can be determined by using the same free energy ψ but different functions γ_E. Owing to its effect on the hysteretic properties, γ_E is referred to as *hysteretic function.*

To determine the functions χ_1 and χ_2, we start with the definitions of *elastic point*, *elastic curve* and *elastic region*.[20] At a given temperature θ, any pair (E, S) such that $\gamma_E = 0$ is

[20] These notions are quite similar to those of equilibrium point, equilibrium curve and equilibrium region introduced in connection with viscoelastic and viscoplastic materials [16].

named elastic point and the set $\mathbb{E}$ of all elastic points is referred to as elastic region. If γ_E is smooth, we assume that there exists a non-decreasing smooth function,[21]

$$S = \mathscr{S}(E),$$

whose graph $\mathfrak{S} = \{(E, \mathscr{S}(E)) : E \in \mathbb{R}\}$ is referred to as *elastic curve* and coincides with the elastic region $\mathbb{E}$. On the other hand, if γ_E is piecewise smooth with respect to E and S, we assume that there exist two non-decreasing smooth functions, $S = \mathscr{S}^+(E)$ and $S = \mathscr{S}^-(E)$, which bound the elastic region $\mathbb{E}$.

In the absence of hysteretic effects, and hence when $\chi_2 = \gamma_E = 0$, by the thermodynamic condition (35) it follows

$$\partial_S \psi \, \dot{S} + (\partial_E \psi - S)\dot{E} = 0. \tag{41}$$

Denoting by the suffix $\mathbb{E}$ the restriction to the elastic region, we observe that

$$\frac{dS}{dE}\Big|_{\mathbb{E}} = \frac{S - \partial_E \psi}{\partial_S \psi}\Big|_{\mathbb{E}} = \chi_1\big|_{\mathbb{E}}$$

describes the non-hysteretic behaviour and accordingly χ_1 can be viewed as the *elastic differential stiffness* (tangent stiffness). Equation (41) in itself gives a constitutive equation of the form $\dot{S} = \hat{S}(S, E, \dot{E})$, thus within the class of hypoelastic materials [49].

Experiments show that the hysteretic effect, due to the clockwise run, results in a decrease of the slope when S is greater than $\mathscr{S}^+(E)$ and E increases, whereas the slope increases when E decreases. Likewise, when $S < \mathscr{S}^-(E)$ the slope decreases if E decreases and increases if E increases. Around both corners, the slope has a positive jump. Consequently any model should satisfy the conditions

$$\chi_2(E, S) \begin{cases} < 0 & \text{if} \quad S > \mathscr{S}^+(E), \\ = 0 & \text{if} \quad \mathscr{S}^-(E) \le S \le \mathscr{S}^+(E), \\ > 0 & \text{if} \quad S < \mathscr{S}^-(E). \end{cases} \tag{42}$$

The main conceptual blocks in the construction of hysteretic models are given by the following scheme.

- elastic regime: $\gamma_E = 0$, $\partial_S \psi = 0$, $S = \partial_E \psi$,

- hypoelastic regime: $\gamma_E = 0$, $\partial_S \psi \neq 0$, $\frac{dS}{dE} = \frac{S - \partial_E \psi}{\partial_S \psi}$,

- hysteretic regime: $\gamma_E \neq 0$, $\partial_S \psi \neq 0$, $\frac{dS}{dE} = \frac{S - \partial_E \psi}{\partial_S \psi} - \frac{\gamma_E}{\partial_S \psi} \operatorname{sgn} \dot{E}$.

The next investigation deals with non-elastic bodies and hence is developed under the assumption that $\partial_S \psi \neq 0$.

Alternatively, a family of one-dimensional models consistent with an additive decomposition of the strain rate follows by letting $\phi = \psi - ES$ and assuming $\partial_E \phi = \partial_E \psi - S \neq 0$. In fact, after dividing (39) by $\partial_E \psi - S$ and assuming $\Gamma = \gamma_T |\dot{S}|$ we obtain

$$\dot{E} = \xi_1 \dot{S} + \xi_2 |\dot{S}|, \qquad \xi_1 = \frac{1}{\chi_1} = \frac{\partial_S \psi}{S - \partial_E \psi}, \qquad \xi_2 = \frac{\gamma_T}{S - \partial_E \psi}, \tag{43}$$

[21] The dependence on the constant temperature θ is understood and not written.

$\gamma_T \geq 0$ being possibly dependent on E, S, $\mathrm{sgn}\,\dot{S}$ and parameterized by θ. This splitting mimics the standard relation $\dot{E} = \dot{E}^e + \dot{E}^p$ of the one-dimensional incremental theory of plasticity and suggests that we introduce a hardening variable S by means of the differential relation

$$\dot{\mathsf{S}} = \mathrm{sgn}\,(S - \partial_E \psi)\dot{E}^p = \mathrm{sgn}\,(S - \partial_E \psi)\xi_2|\dot{S}|.$$

This quantity turns out to be always non-negative and strictly related to the rate of dissipation. Indeed,

$$\dot{\mathsf{S}} = \frac{\gamma_T}{|S - \partial_E \psi|}|\dot{S}| = \frac{\Gamma}{|S - \partial_E \psi|}.$$

Looking at time intervals where $\dot{S} \neq 0$, from (43) the uniaxial strain-stress slope follows

$$\frac{dE}{dS} = \xi_1 + \xi_2\,\mathrm{sgn}\,\dot{S} \tag{44}$$

and the monotonicity requirement becomes

$$\xi_1 + \xi_2\,\mathrm{sgn}\,\dot{S} \geq 0. \tag{45}$$

The notions of elastic point, elastic curve and elastic region correspond to the vanishing of γ_T.

Possible models of the constitutive relations (39) and (44) are established below by determining the free energy and the hysteretic functions γ_E and γ_T.

Remark 4 Equation (35) is the one-dimensional version of the thermodynamic conditions (8), (9). It is worth observing that (35) has a similarity with a model by Puzrin and Houlsby [37] which traces back to an approach by Ziegler [58]. The variables are the stress σ and an internal variable α or the strain ε and α. In terms of the potential $f(\varepsilon, \alpha)$ they derive $\sigma = \partial_\varepsilon f$ and the generalized stress $\bar{\chi} = -\partial_\alpha f$. Moreover the model involves a dissipation function $d(\dot{\alpha})$ which generates the dissipative generalized stress $\chi = \partial_{\dot{\alpha}} d$. The physical meaning of α, χ, $\bar{\chi}$ then need to be determined.

7 The Helmholtz Free Energy

To determine the function ψ appearing into the model (38)-(39) we start with the generic assumption

$$\psi(S, E) = \mathcal{L}(S - \mathcal{G}(E)) + \mathcal{F}(S) + \mathcal{H}(E), \tag{46}$$

$\mathcal{L}$, $\mathcal{G}$, $\mathcal{F}$, and $\mathcal{H}$ being undetermined differentiable functions, possibly dependent on θ; the dependence on the parameter (temperature) θ is understood and not written. Substitution of $\partial_S \psi$ and $\partial_E \psi$ in (38) yields

$$\chi_1 = \frac{S + \mathcal{L}'(S - \mathcal{G}(E))\mathcal{G}'(E) - \mathcal{H}'(E)}{\mathcal{L}'(S - \mathcal{G}(E)) + \mathcal{F}'(S)}, \qquad \chi_2 = -\frac{\gamma_E}{\mathcal{L}'(S - \mathcal{G}(E)) + \mathcal{F}'(S)}.$$

The function χ_1 is assumed to be non-negative. Furthermore for simplicity we let χ_1 depend only on E,

$$\chi_1 = g(E) \geq 0.$$

Consequently we obtain the requirement

$$S - \mathcal{F}'(S)g(E) - \mathcal{L}'(S - \mathcal{G}(E))[g(E) - \mathcal{G}'(E)] = \mathcal{H}'(E)$$

which is satisfied by letting $\mathcal{F}'(S) = 0$ and

$$g(E) - \mathcal{G}'(E) = \alpha, \qquad \mathcal{H}'(E) = \mathcal{G}(E), \qquad \mathcal{L}'(S - \mathcal{G}(E)) = \frac{1}{\alpha}(S - \mathcal{G}(E)),$$

for some $\alpha \neq 0$. Hence χ_2 becomes

$$\chi_2 = -\gamma_E \alpha \frac{1}{S - \mathcal{G}(E)}.$$

To fulfill the condition (42) outside the elastic region where $\gamma_E > 0$, some restrictions on α and $\mathcal{G}$ are required. For simplicity we let

$$\alpha > 0, \qquad \mathcal{G} = \mathscr{S}.$$

If γ_E is smooth, then $\mathscr{S}$ represents the elastic function. Otherwise, if γ_E is piecewise smooth and there exist infinitely many elastic curves lying within the elastic region $\mathscr{E}$, we are allowed to choose an arbitrary increasing smooth function such that $\mathscr{S}^- \leq \mathscr{S} \leq \mathscr{S}^+$.

To summarize the properties of the model we state that

1) the free energy ψ, defined by (46) to within an additive constant (possibly parameterized by the temperature θ), is characterized by the elastic function $\mathscr{S}$ and parameterized by $\alpha > 0$ and the temperature θ in the form

$$\psi(S, E) = \frac{1}{2\alpha}[S - \mathscr{S}(E)]^2 + \mathcal{H}(E), \tag{47}$$

where $\mathcal{H}'(E) = \mathscr{S}(E)$;
2) the characteristic functions χ_1 and χ_2 are given by

$$\chi_1 = \mathscr{S}'(E) + \alpha \geq 0, \qquad \chi_2 = -\frac{\alpha \gamma_E}{S - \mathscr{S}(E)}. \tag{48}$$

The model is then fully characterized by the elastic function $\mathscr{S}(E)$, the positive parameter α, and the hysteretic function γ_E. The positive parameter α gives the difference between the elastic differential stiffness χ_1 and the slope of the elastic function $\mathscr{S}(E)$. The occurrence of both $\mathscr{S}$ and α allows a greater flexibility in the modelling of hysteretic materials as will be shown in §§ 8-10.

Properties 1) can also be obtained starting from (44). A characterization of ξ_1 and ξ_2 similar to property 2) occurs provided that γ_E is replaced by γ_T.

8 Examples of Smooth Hysteretic Function γ_E

To determine a definite hysteretic behaviour via an explicit representation of χ_2, we now investigate some expressions of γ_E. We assume that γ_E is a smooth function. Accordingly, the elastic region reduces to a smooth curve in the (E, S)-plane and the resulting hysteretic behaviour provides some models of plastic flow. In the modelling process the emphasis is on the thermodynamic consistency rather than on the fit of experimental data. However, for definiteness, having in mind metals we take the unit for the variable E equal to 0.01 and that for S equal to 100 MPa.

8.1 Plastic Flow with Asymptotic Strength

Let $\mathscr{S}(E) \equiv 0$ so that $\psi = S^2/2\alpha$ and $\chi_1 = \alpha$. Then take the hysteretic factor γ_E as a smooth function in the form

$$\gamma_E(S) = \frac{1}{S_u} S^2, \qquad S_u > 0.$$

Hence $S = \mathscr{S}(E) \equiv 0$ gives exactly the elastic function. Consequently (37) becomes

$$\frac{dS}{dE} = \frac{\alpha}{S_u}(S_u - S \,\mathrm{sgn}\, \dot{E}).$$

Since $dS/dE \to 0$ as $S \to \pm S_u$, hysteresis loops are confined to the open strip $|S| < S_u$. The elastic region shrinks to a line; hardening occurs as soon as the system leaves the elastic line.

The monotonicity requirement holds in that $|S| \le S_u$. The model is characterized by the positive parameters α, S_u; α denotes the slope of the stress-strain path across the elastic curve $\mathfrak{S}$ while S_u denotes the asymptotic strength, as $|E| \to \infty$, or the ultimate tensile stress: the maximum stress that a material can withstand while being stretched or pulled before breaking.

Hysteresis loops in Fig. 1 are determined by solving the system of equations

$$\begin{cases} \dot{E} = \omega \mathfrak{E} \cos \omega t, \\ \dot{S} = \alpha (S_u \dot{E} - S|\dot{E}|)/S_u, \end{cases}$$

with initial values E_0, S_0. The rate independence of the model allows the loops to be independent of the angular frequency ω.

8.2 Plastic Flow with Linear Elastic Function

The elastic function is taken to be linear and increasing, $\mathscr{S}(E) = \kappa E$, $\kappa > 0$ being possibly dependent on the temperature θ. Hence Eqs. (47) and (48) imply

$$\psi = \frac{1}{2\alpha}(S - \kappa E)^2 + \tfrac{1}{2}\kappa E^2$$

and $\chi_1 = \kappa + \alpha$. The hysteretic function γ_E is taken in the form

$$\gamma_E(E, S) = \frac{1}{\sigma_0}(S - \kappa E)^2, \qquad \sigma_0 > 0.$$

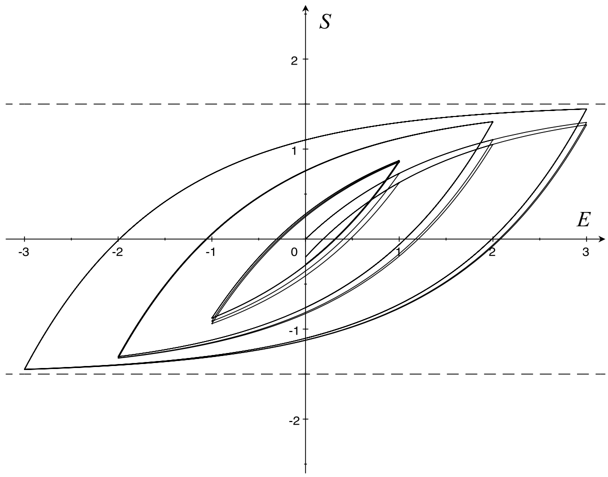

Fig. 1 Plastic model: hysteresis loops (solid) and asymptotic bounds $|S| = S_u$ (dashed) with $\alpha = 1$, $S_u = 1.5$, $\mathfrak{E} = 1, 2, 3$ and starting from (E_0, S_0) with $E_0 = 0$, $S_0 = 0, -0.1$

Hence, the slope of the hysteretic path is given by

$$\frac{dS}{dE} = \kappa + \frac{\alpha}{\sigma_0}[\sigma_0 - (S - \kappa E)\operatorname{sgn}\dot{E}]. \tag{49}$$

All loops are within the strip $|S - \kappa E| \le \sigma_0$ and the positive value of κ warrants the monotonicity condition. So the model is characterized by the positive parameters α, κ, and σ_0. As in previous example the elastic region reduces to a single line, here $S = \kappa E$. Figure 2 shows an example of the loops obtained by solving the system of equations

$$\begin{cases} \dot{E} = \omega \mathfrak{E} \cos \omega t, \\ \dot{S} = (\kappa + \alpha)\dot{E} - \dfrac{\alpha}{\sigma_0}(S - \kappa E)|\dot{E}|, \end{cases}$$

with initial values E_0, S_0. Again, the rate independence of the model allows the loops to be independent of the angular frequency ω.

8.3 Plastic Flow with Nonlinear Elastic Function

The elastic function $\mathscr{S}$ and the hysteretic coefficient γ_E are taken in the form

$$\mathscr{S}(E) = \kappa \tanh(\zeta E), \qquad \kappa, \zeta > 0,$$

$$\gamma_E(E, S) = \frac{1}{\sigma_0}[S - \kappa \tanh(\zeta E)]^2, \qquad \sigma_0 > 0.$$

From (47) it follows that

$$\psi = \frac{1}{2\alpha}[S - \kappa \tanh(\zeta E)]^2 + \frac{\kappa}{\zeta} \ln[\cosh(\zeta E)],$$

while

$$\chi_1 = \zeta \kappa [1 - \tanh^2(\zeta E)] + \alpha.$$

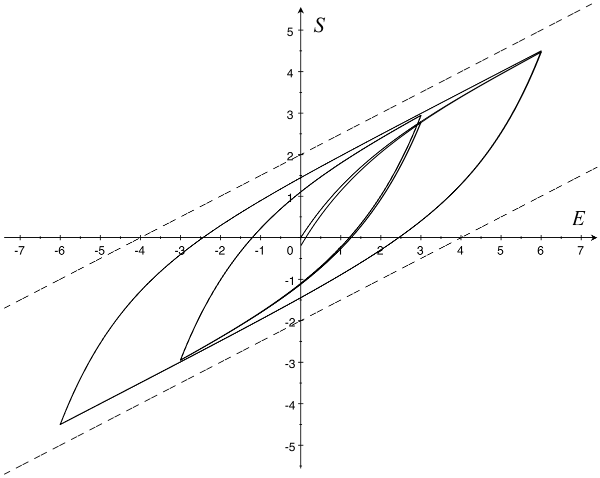

Fig. 2 Plastic model with linear elastic function; hysteresis loops (solid) and asymptotic-strength bounds (dashed) are determined by letting $\alpha = 1$, $\kappa = 0.5$, $\sigma_0 = 2$; the amplitudes of the oscillations are $\mathfrak{E} = 3$ and $\mathfrak{E} = 6$ with $E_0 = 0$, $S_0 = 0, -0.1$

Accordingly, the slope of any stress-strain path is given by

$$\frac{dS}{dE} = \zeta\kappa[1 - \tanh^2(\zeta E)] + \frac{\alpha}{\sigma_0}[\sigma_0 - (S - \kappa\tanh(\zeta E))\operatorname{sgn}\dot{E}]. \tag{50}$$

The elastic region reduces to the curve $S = \kappa\tanh(\zeta E)$. Here, the choice of a function $\mathscr{S}$ with horizontal asymptotes, $\mathscr{S}(E) = \pm\kappa$, forces the loops to rely within a horizontal strip, indeed $|S| \le \kappa + \sigma_0$. Therefore, the value $\kappa + \sigma_0$ takes the meaning of asymptotic strength (or ultimate tensile stress) of the model. The monotonicity condition (40) gives

$$|S - \kappa\tanh(\zeta E)| \le \frac{\zeta\kappa\sigma_0}{\alpha}[1 - \tanh^2(\zeta E)] + \sigma_0$$

and is satisfied provided that

$$|S - \kappa\tanh(\zeta E)| \le \sigma_0.$$

This inequality identifies a region which is included within the strip $|S| \le \kappa + \sigma_0$. Figure 3 shows some loops obtained from (50).

9 Models with Piecewise Smooth Hysteretic Function γ_E

Some models are now developed where a nontrivial elastic region exists. Accordingly, they describe an elastic-plastic behaviour. Henceforth S_y denotes the yield limit.

9.1 Elastic-Perfectly Plastic Model

Letting $\mathscr{S}(E) \equiv 0$ it follows that $\psi = S^2/2\alpha$ and $\chi_1 = \alpha$. In addition we let

$$\gamma_E(S, \operatorname{sgn}\dot{E}) = \begin{cases} |S| & \text{if } |S| = S_y \text{ and } \operatorname{sgn}(S\dot{E}) > 0, \\ 0 & \text{if } |S| < S_y \text{ or } |S| = S_y \text{ and } \operatorname{sgn}(S\dot{E}) \le 0. \end{cases}$$

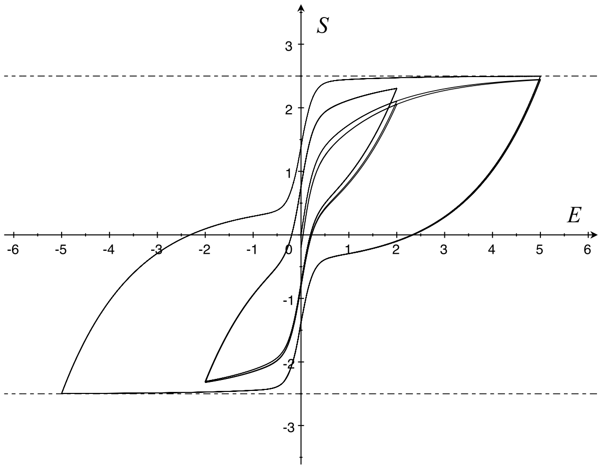

Fig. 3 Plastic model with nonlinear elastic function. Hysteresis loops (solid) and asymptotic-strength bounds $|S| = \kappa + \sigma_0$ (dashed) are determined by letting $\alpha = 1$, $\zeta = 4$, $\kappa = 1$, $\sigma_0 = 1.5$; the amplitudes of the oscillations are $\mathfrak{E} = 2, 5$ with $E_0 = 0$, $S_0 = 0, -0.1$

Correspondingly the open strip $\mathbb{E}_y = \{(E, S) : |S| < S_y\}$ gives the elastic region and $\mathscr{S}(E) \equiv 0$ represents its midline. The stress-strain evolution is given by

$$\frac{dS}{dE} = \begin{cases} 0 & \text{if } |S| = S_y \text{ and } \operatorname{sgn}(S\dot{E}) > 0, \\ \alpha & \text{if } |S| < S_y \text{ or } |S| = S_y \text{ and } \operatorname{sgn}(S\dot{E}) \le 0. \end{cases} \tag{51}$$

Hence the model satisfies the condition (40). Since, by assumption, $\mathscr{S}(E) \equiv 0$, it follows that all hysteretic patterns rely within the closed strip $|S| \le S_y$ which is referred to as the set of all admissible states.

Consider oscillations of E with amplitude $\mathfrak{E}$ and angular frequency ω. Hysteresis loops in the (E, S)-plane are obtained by solving the system of equations

$$\begin{cases} \dot{E} = \omega \mathfrak{E} \cos \omega t, \\ \dot{S} = \alpha(\dot{E} - \gamma_E |\dot{E}|/S), \end{cases}$$

with various initial states (E_0, S_0). Due to the rate-independence property, cycles are invariant with respect to the angular frequency ω (see Fig. 4).

This model is characterized by the positive parameters α and S_y; α represents the stiffness of the linear elastic response. The material behaves elastically ($\chi_2 = 0$) within the elastic region.

9.2 Bilinear Model with Strain Hardening

As in example 8.2, letting $\mathscr{S}(E) = \kappa E$, $\kappa > 0$, we have

$$\psi = \frac{1}{2\alpha}(S - \kappa E)^2 + \tfrac{1}{2}\kappa E^2, \qquad \chi_1 = \kappa + \alpha.$$

In addition, we let $\sigma_0 > 0$ and

$$\gamma_E(E, S, \operatorname{sgn} \dot{E}) = \begin{cases} |S - \kappa E)| & \text{if } |S - \kappa E| \ge \sigma_0 \text{ and } \operatorname{sgn}([S - \kappa E]\dot{E}) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

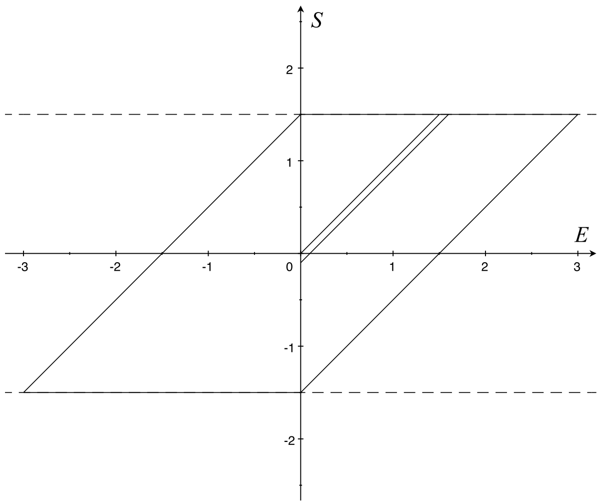

Fig. 4 Elastic-perfectly plastic model: hysteresis loops (solid) and yield-strength bounds $|S| = S_y$ (dashed) with $\alpha = 1$, $S_y = 1.5$, $\mathfrak{E} = 3$ and $E_0 = 0$, $S_0 = 0, -0.1$

It follows that

$$\frac{dS}{dE} = \begin{cases} \kappa & \text{if } |S - \kappa E| \geq \sigma_0 \text{ and } \operatorname{sgn}([S - \kappa E]\dot{E}) > 0, \\ \kappa + \alpha & \text{otherwise.} \end{cases} \tag{52}$$

The loops are comprised within the closed strip $|S - \kappa E| \leq \sigma_0$. The positive value of κ warrants the monotonicity condition. The (linear) elastic behaviour is confined within the open strip $|S - \kappa E| < \sigma_0$ which therefore represents the elastic region. In particular, $\mathscr{S}^+(E) = \kappa E + \sigma_0$ and $\mathscr{S}^-(E) = \kappa E - \sigma_0$ are the upper and lower elastic bounds at every strain E.

This model is characterized by the positive parameters α, κ, and σ_0. Figure 5 shows an example of cycles determined by solving the following system of equations with given values of α, κ, σ_0 and initial values (E_0, S_0),

$$\begin{cases} \dot{E} = \omega \mathfrak{E} \cos \omega t, \\ \dot{S} = \alpha[\dot{E} - \gamma_E |\dot{E}|/(S - \kappa E)]. \end{cases}$$

Using the Heaviside step function H we can write the system in the form

$$\begin{cases} \dot{E} = \omega \mathfrak{E} \cos \omega t, \\ \dot{S} = \big\{\kappa + \alpha[1 - H([S - \kappa E]\dot{E})\, H(|S - \kappa E| - \sigma_0)]\big\}\dot{E}. \end{cases}$$

9.3 Elastic-Plastic Model

As in example 9.1 we let $\mathscr{S}(E) \equiv 0$ and $\psi = S^2/2\alpha$, $\chi_1 = \alpha$. Now let S_y be the yield limit $S_u > S_y > 0$ and choose the hysteretic function in the form

$$\gamma_E(S, \operatorname{sgn} \dot{E}) = \begin{cases} \dfrac{|S| - S_y}{S_u - S_y} |S| & \text{if } |S| \geq S_y \text{ and } \operatorname{sgn}(S\dot{E}) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

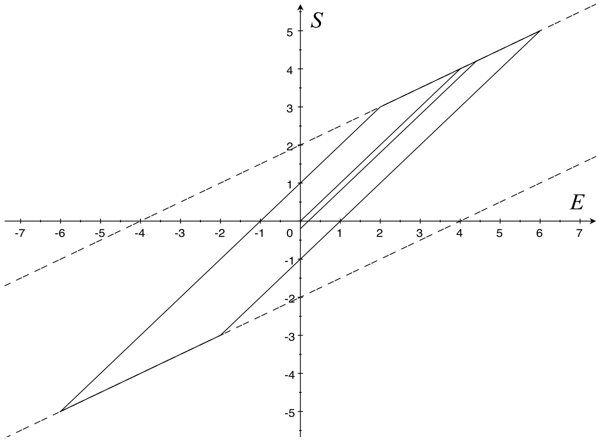

Fig. 5 Elastic-perfectly plastic model with linear strain hardening: hysteresis loops (solid) and yield-strength bounds $|S-\kappa E|=\sigma_0$ (dashed) are determined by $\alpha=1$, $\kappa=0.5$, $\sigma_0=2$; the amplitude of the oscillations is $\mathfrak{E}=6$ with $E_0=0$, $S_0=0,-0.1$

Since $S_u - S_y - (|S| - S_y)\,\mathrm{sgn}\,(S\dot{E}) = S_u - |S|$ when $\mathrm{sgn}\,(S\dot{E}) > 0$, then equation (39) becomes

$$\frac{dS}{dE} = \begin{cases} \alpha\,\dfrac{S_u - |S|}{S_u - S_y} & \text{if } |S| \geq S_y \text{ and } \mathrm{sgn}\,(S\dot{E}) > 0, \\ \alpha & \text{otherwise.} \end{cases}$$

This scheme is quite realistic, although it is unable to capture the Bauschinger effect. It combines the properties of the previous models, a plastic flow with asymptotic strength along with elastic-plastic behaviour, and is characterized by the positive parameters α, S_y, S_u. The loops are placed within the strip $|S| < S_u$. Inside the open strip $|S| < S_y$, $S_y < S_u$, the body behaves elastically, therefore this region is exactly the same elastic region as in example 9.1. Within the strips $|S| \in (S_y, S_u)$ the linear behaviour occurs only during unloading; instead, during loading the material behaves nonlinearly.

Figure 6 shows an example of the cyclic process obtained by solving the system of equations

$$\begin{cases} \dot{E} = \omega\mathfrak{E}\cos\omega t, \\ \dot{S} = \alpha\big(\dot{E} - \gamma_E|\dot{E}|/S\big), \end{cases}$$

with initial value $(E_0, S_0) = (0, 0)$. Asymptotically, as $t \to +\infty$, the hysteretic path tends to a closed loop which is symmetric with respect to the origin.

Making use of the Heaviside step function H we can write the system in the form

$$\begin{cases} \dot{E} = \omega\mathfrak{E}\cos\omega t \\ \dot{S} = \left[1 - \dfrac{|S| - S_y}{S_u - S_y} H(S\dot{E})\, H(|S| - S_y)\right]\alpha\dot{E}. \end{cases}$$

10 Models with Additive Decomposition of the Strain Rate

Some models are now developed starting from the additive decomposition of the strain rate represented by (44) where ξ_1 is induced by the free energy ψ and ξ_2 by the dissipation rate

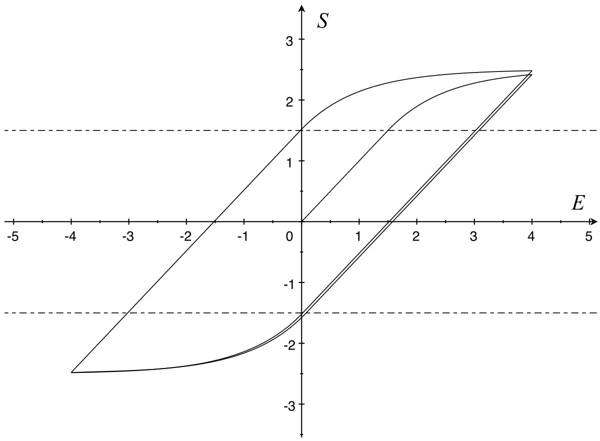

Fig. 6 Elastic-plastic model: hysteresis loops (solid) and yield-strength bounds $|S| = S_y$ (dashed) with $\alpha = 1$, $S_y = 1.5$, $S_u = 2.5$, $\mathfrak{E} = 4$

$\gamma_T|\dot{S}|$. Though they are within the present thermodynamic context, the following models show connections with the one-dimensional incremental theory of plasticity (see, e.g., [15, 33]).

10.1 Elastic-Plastic Model with Linear Strain Hardening

By analogy with example 9.2, we now let $\mathscr{S}(E) \equiv 0$ so that $\psi = S^2/2\alpha$. Hence, $\xi_1 = 1/\alpha$, $\xi_2 = \gamma_T/S$. In addition we let

$$\gamma_T(S, \operatorname{sgn} \dot{S}) = \begin{cases} |S|/h & \text{if } |S| \geq S_y \text{ and } \operatorname{sgn}(S\dot{S}) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $h > 0$ represent the hardening parameter. Correspondingly the stress-strain evolution is given by

$$\frac{dE}{dS} = \begin{cases} 1/\alpha + 1/h & \text{if } |S| \geq S_y \text{ and } \operatorname{sgn}(S\dot{S}) > 0, \\ 1/\alpha & \text{otherwise.} \end{cases} \tag{53}$$

Hence the model is characterized by the positive parameters α, h and S_y. The monotonicity requirement (45) is trivially satisfied. The elastic region is identified with the horizontal strip $\mathbb{E} = \{(E, S) : |S| \leq S_y\}$. The loops are bilinear as in example 9.2, but are not confined in a strip of the (E, S)-plane.

It is of interest to examine the cyclic stress-driven processes

$$\begin{cases} \dot{S} = \omega \mathfrak{S} \cos \omega t, \\ \dot{E} = \left\{ \dfrac{1}{\alpha} + \dfrac{1}{h} H(S\dot{S}) H(|S| - S_y) \right\} \dot{S}. \end{cases}$$

Yet to get a close connection of the resulting loops with those obtained in § 9.2 we show the stress-strain path corresponding to harmonic oscillations of E by means of Fig. 7. Owing to the monotonicity condition ($\operatorname{sgn} \dot{S} = \operatorname{sgn} \dot{E}$) we can write the system in the form

$$\begin{cases} \dot{E} = \omega \mathfrak{E} \cos \omega t, \\ \dot{S} = \left\{ h\alpha/[h + \alpha\, H(S\dot{E}) H(|S| - S_y)] \right\} \dot{E}. \end{cases}$$

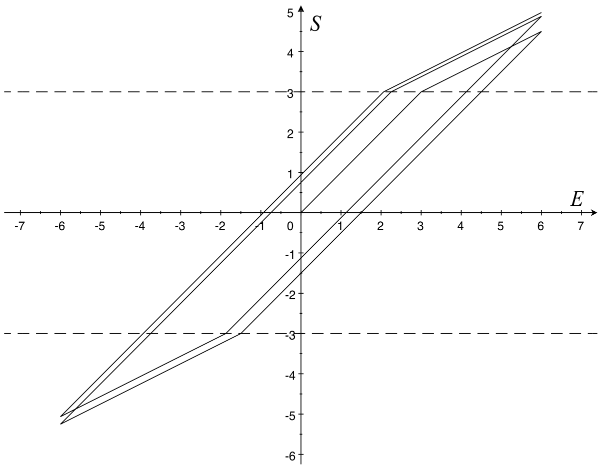

Fig. 7 Elastic-plastic model with linear strain hardening: hysteresis loops (solid) and yield-strength bounds $|S| = S_y$ (dashed) are determined by $\alpha = 1$, $h = 1$, $S_y = 3$; the amplitude of the oscillations is $\mathfrak{E} = 6$

The slopes of the stress-strain relation are α, the elastic modulus, within the elastic region and $\alpha h/(\alpha + h) < \alpha$, the elastic-plastic tangent modulus, outside this region.

In light of (53) we might view $\dot{E}$ as an additive decomposition via

$$\dot{E}^e = \frac{1}{\alpha}\dot{S}, \qquad \dot{E}^p = \begin{cases} \dot{S}/h & \text{if } |S| \geq S_y \text{ and sgn}\,(S\dot{S}) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

To describe the plastic flow a hardening variable S, viewed as a scalar internal variable, is sometimes considered such that its evolution is governed by a first-order differential equation for S parameterized by the derivative of $\mathbf{E}^p$ [15]. Here we find an analogous differential equation involving the engineering stress. Moreover, we now establish a connection between the hardening variable rate and the entropy production. In relation to the one-dimensional incremental theory of plasticity, we can define the rate of the hardening variable[22] S as follows

$$\dot{\mathsf{S}} = (\text{sgn}\, S)\dot{E}^p = \begin{cases} \dfrac{1}{h}|\dot{S}| & \text{if } |S| \geq S_y \text{ and sgn}\,(S\dot{S}) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{54}$$

By (54) $\dot{\mathsf{S}}$ turns out to be always non-negative and strictly related to the rate of dissipation. Indeed, thanks to the definition of $\gamma_{\scriptscriptstyle T}$,

$$\dot{\mathsf{S}}|S| = \gamma_{\scriptscriptstyle T}|\dot{S}| = \Gamma.$$

The definition of S, the cumulative dissipation function, is completed by letting $\mathsf{S}(t_0) = 0$ at the initial time t_0. Consequently, for all $t \geq t_0$

$$\mathsf{S}(t) = \int_{t_0}^{t} \dot{\mathsf{S}}(\tau)\, d\tau = \int_{t_0}^{t} \frac{\Gamma(\tau)}{|S(\tau)|}\, d\tau = \int_{t_0}^{t} \gamma_{\scriptscriptstyle T}(\tau)\frac{|\dot{S}(\tau)|}{|S(\tau)|}\, d\tau.$$

[22] This quantity is often introduced as an additional internal variable. See, e.g., [33, § 8.6.1], where the hardening variable is denoted by α.

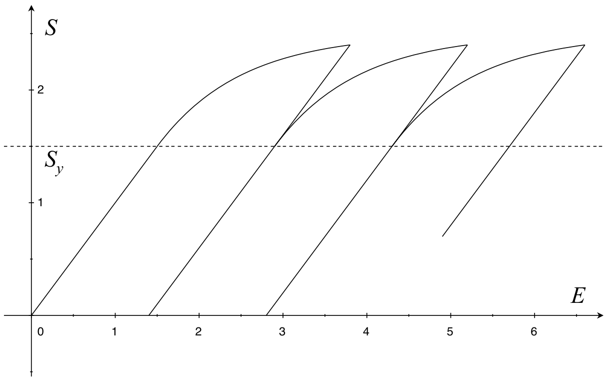

Fig. 8 Elastic-plastic model with nonlinear strain hardening: hysteresis cycles (solid) and yield limit $S = S_y$ are determined by $\alpha = 1$, $S_y = 1.5$, $S_u = 2.5$; the amplitude of the stress oscillation is $\mathfrak{S} = 2.4$

Hence, by exploiting (54) we obtain

$$\mathsf{S}(t) = \frac{1}{h}\int_{\mathcal{I}_{(t_0,t)}} |\dot{S}(\tau)|\, d\tau$$

where $\mathcal{I}_{(t_0,t)} = \{\tau \in (t_0, t) : |S(\tau)| \geq S_y \text{ and } \operatorname{sgn}(S(\tau)\dot{S}(\tau)) > 0\}$.

10.2 Elastic-Plastic Model with Nonlinear Strain Hardening

As in the previous case we let $\mathscr{S}(E) \equiv 0$ and $\psi = S^2/2\alpha$. Hence, $\xi_1 = 1/\alpha$, $\xi_2 = \gamma_T/S$. In addition we let $S_u > S_y > 0$ and choose the hysteretic function in the form

$$\gamma_T(S, \operatorname{sgn}\dot{S}) = \begin{cases} \dfrac{|S| - S_y}{\alpha(S_u - |S|)}|S| & \text{if } |S| \geq S_y \text{ and } \operatorname{sgn}(S\dot{S}) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We point out that $\gamma_T \geq 0$ only in the open strip $|S| < S_u$. With this constraint, equation (44) becomes

$$\frac{dE}{dS} = \begin{cases} \dfrac{S_u - S_y}{\alpha(S_u - |S|)} & \text{if } |S| \geq S_y \text{ and } \operatorname{sgn}(S\dot{S}) > 0, \\ 1/\alpha & \text{otherwise.} \end{cases}$$

The resulting deformation-driven loops are just shown in Fig. 6. Rather stress-driven cycles between $S = 0$ and $S = \mathfrak{S}$ are generated by the system

$$\begin{cases} \dot{S} = \frac{1}{2}\omega\mathfrak{S}\sin\omega t, \\ \dot{E} = \left\{\dfrac{1}{\alpha} + \dfrac{|S| - S_y}{\alpha(S_u - |S|)} H(S\dot{S})H(|S| - S_y)\right\}\dot{S} \end{cases}$$

and are shown in Fig. 8, with $E_0, S_0 = 0$.

Formally we can indicate an additive decomposition of the strain rates by means of the identifications

$$\dot{E}^e = \frac{1}{\alpha}\dot{S}, \qquad \dot{E}^p = \begin{cases} \dfrac{|S| - S_y}{\alpha(S_u - |S|)}\dot{S} & \text{if } |S| \geq S_y \text{ and } \operatorname{sgn}(S\dot{S}) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Accordingly, the rate of the hardening variable S is given by

$$\dot{\mathsf{S}} = (\operatorname{sgn} S)\dot{E}^p = \begin{cases} \dfrac{|S| - S_y}{\alpha(S_u - |S|)}|\dot{S}| & \text{if } |S| \geq S_y \text{ and } \operatorname{sgn}(S\dot{S}) > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{55}$$

and satisfies the relation $\dot{\mathsf{S}}|S| = \gamma_T|\dot{S}| = \Gamma$. Upon an integration of (55) over (t_0, t) we have

$$\mathsf{S}(t) = \int_{\mathcal{I}_{(t_0,t)}} \frac{|S(\tau)| - S_y}{\alpha(S_u - |S(\tau)|)}|\dot{S}(\tau)|\, d\tau,$$

where $\mathcal{I}_{(t_0,t)} = \{\tau \in (t_0, t) : |S(\tau)| \geq S_y \text{ and } \operatorname{sgn}(S(\tau)\dot{S}(\tau)) > 0\}$.

11 Relation to Other Approaches

11.1 Solids Described by Implicit Constitutive Equations

Among the models appeared in the literature we find it interesting to consider equations given in [42]; in the present notation, equations (12) and (15) of [42] read[23]

$$\begin{aligned} \dot{S} - \lambda\dot{E} &= -\lambda H(S\dot{E})H(|S| - S_y)\dot{E}, \\ -a(S + bS^3)\dot{S} + \dot{E} &= f(S)H(S\dot{E})S\dot{E}, \end{aligned} \tag{56}$$

where λ, a, b are constants and $f(S) = -f(-S) \geq 0$ as $S \geq 0$. Consequently

$$\frac{dS}{dE} = \lambda - \lambda H(S\dot{E})H(|S| - S_y), \qquad \frac{dS}{dE} = \frac{1}{a(S + bS^3)} - \frac{f(S)\,H(S\dot{E})}{a(1 + bS^2)}.$$

To ascertain the thermodynamic consistency we check whether Eqs. (56) have the form (37). Relative to $(56)_1$ we then look for the possible function ψ by letting

$$\frac{S - \partial_E\psi}{\partial_S\psi} = \lambda.$$

As a simple guess, let $\partial_E\psi = 0$. Accordingly $\partial_S\psi = S/\lambda$ and we have

$$\frac{dS}{dE} = \lambda - \frac{\lambda\gamma_E}{S}\operatorname{sgn}\dot{E}.$$

Equation $(56)_1$ is recovered by letting

$$\gamma_E = |S|H(|S| - S_y)H(S\dot{E})$$

in that

$$-\frac{\gamma_E}{\partial_S\psi}\operatorname{sgn}\dot{E} = -\lambda \operatorname{sgn} S \operatorname{sgn}\dot{E}\; H(|S| - S_y)H(S\dot{E}).$$

Hence $(56)_1$ is consistent with thermodynamics ($\gamma_E \geq 0$).

[23] Owing to a misprint, the factor λ in front of $H(S\dot{E})H(|S| - S_y)$ is missing in [42].

Concerning (56)$_2$ we require that

$$\frac{S - \partial_E \psi}{\partial_S \psi} = \frac{1}{a(S + bS^3)}.$$

Letting $\partial_E \psi = 0$ we have

$$\partial_S \psi = aS^2(1 + bS^2).$$

Hence we check whether

$$\frac{\gamma_E \operatorname{sgn} \dot{E}}{\partial_S \psi} = \frac{f(S)H(S\dot{E})S\dot{E}}{aS(1 + bS^2)}.$$

This relation holds if

$$\gamma_E = |S|\, f(S)S\, H(S\dot{E})$$

and then $\gamma_E \geq 0$ in that $S\, f(S) \geq 0$. Hence Eq. (56)$_2$ is consistent with thermodynamics.

The approach of [42, 43] is claimed to be consistent with a three-dimensional thermodynamic setting [5]. The setting in [5] starts from a Helmholtz free energy $\psi(\theta, \mathbf{B}_e)$, where $\mathbf{B}_e$ is an appropriate tensor. Next the evolution equation for the entropy is considered in the form

$$\rho\theta\dot{\eta} = \mathbf{T} \cdot \mathbf{D} - \rho\partial_{\mathbf{B}_e}\psi \cdot \dot{\mathbf{B}}_e - \nabla \cdot \mathbf{q}.$$

Upon some developments the entropy production is defined to be (Eq. (26) of [5])

$$\mathbf{T} \cdot \mathbf{D} - \rho\partial_{\mathbf{B}_e}\psi \cdot \dot{\mathbf{B}}_e = H(\mathbf{T} \cdot \mathbf{D})H(|\mathbf{T}| - T_y)\mathbf{T} \cdot \mathbf{D}$$

which is evidently non-negative. Consequently the evolution equation of η, Eq. (27) of [5], becomes

$$\rho\theta\dot{\eta} + \nabla \cdot \mathbf{q} = H(\mathbf{T} \cdot \mathbf{D})H(|\mathbf{T}| - T_y)\mathbf{T} \cdot \mathbf{D}. \tag{57}$$

In light of the form of the right-hand sides of (56) and (57) the corresponding models are said to be consistent with thermodynamics.

Though we have proved the thermodynamic consistency of (56) we observe that the entropy production σ has to satisfy (2).

11.2 Entropy Production and Dissipation Potential

There are similarities and differences among the various approaches based on dissipation potentials and the present approach. For definiteness we examine the scheme of ref. [26]. Let $\mathbf{S}$ and $\mathbf{E}$ be generalized stress and strain measures. To quantify inelastic processes a set of internal variables $\mathbf{I}$ is considered. Then the additive decomposition $\mathbf{S} = \mathbf{S}^{\text{en}} + \mathbf{S}^{\text{dis}}$ is assumed, $\mathbf{S}^{\text{dis}} := \partial_{\mathbf{E}}\psi$, and the reduced dissipation inequality

$$d = \mathbf{S}^{\text{dis}} \cdot \dot{\mathbf{E}} - \partial_{\mathbf{I}}\psi \cdot \dot{\mathbf{I}} \geq 0 \tag{58}$$

follows. The dissipation potential $\pi(\dot{\mathbf{E}}, \dot{\mathbf{I}})$ is then introduced so that $\mathbf{S}^{\text{dis}}$ and $\mathbf{X}^{\text{dis}} := -\partial_{\mathbf{I}}\psi$ are subdifferentials of π with respect of $\dot{\mathbf{E}}$ and $\dot{\mathbf{I}}$.

The similarity with the present approach is given by the reduced dissipation inequality; the dissipation (rate) d in (58) is the analogue of (θ times) the entropy production. Yet there are conceptual differences. In (9) we have a thermodynamic restriction for the functions ψ and $\Gamma = J\theta\sigma_T \geq 0$ which depend on the same set of physical variables, here θ, $\mathbf{E}$, $\mathbf{T}_{RR}$, $\dot{\mathbf{E}}$, $\dot{\mathbf{T}}_{RR}$. Once ψ and Γ are established we have a model involving only physical variables; here, in one dimension, we obtain $\dot{S}$ in terms of S, E, $\dot{E}$. Moreover, in the present approach the second law inequality is a single relation for ψ and Γ. Instead, the approaches involving dissipation potentials consider the separate additive contributions to stress of the free energy and of the dissipation potential.

11.3 Entropy Production as a Constitutive Function

The idea that the entropy production σ be given by a constitutive equation traces back to Green and Naghdi [13]; they do not require that $\sigma \geq 0$ for any process. Later on Rajagopal and Srinivasa [38], in connection with shape memory wires, considered the rate of mechanical dissipation ξ; by definition $\xi = J\theta\sigma$ ($\equiv \Gamma$ in the present notation). Then they let α be a mass fraction and assume that ξ is a function of θ, α, $\dot{\alpha}$, while the stress and the free energy depend on F, α, θ. As a consequence they find that ξ governs the evolution of α via $-\rho_0 \partial_\alpha \psi\, \dot{\alpha} = \xi$, and then any appropriate choice of ξ completes the model.

Conceptually our approach extends this idea assuming from the outset that all constitutive functions depend on the common set of (physical) variables, say Ξ. Next the second law inequality is investigated so that both requirements in (3) are satisfied with σ, η, $\mathbf{q}$, ε, $\mathbf{T}$ functions of Ξ. Accordingly it may happen that the resulting material properties originate from the joint structure of the potential, e.g., $\psi = \varepsilon - \theta\eta$, and the entropy production σ. Equations (9) and (35) are clear examples of joint contributions of ψ and σ. We mention that a similar approach was developed in [12] for ferroelectric materials.

12 Conclusions

In essence the approach developed in this paper is based on the following points. First, the constitutive equations involve both the deformation gradient and the stress among the independent variables via the set

$$\Xi = (\theta, \mathbf{E}, \mathbf{T}_{RR}, \nabla_R \theta, \dot{\mathbf{E}}, \dot{\mathbf{T}}_{RR}).$$

The invariance of the Green-St. Venant tensor $\mathbf{E}$ and the second Piola stress $\mathbf{T}_{RR}$ makes Ξ invariant and hence objective. Second, the requirement of the second law inequality implies that the free energy ψ depends only on θ, $\mathbf{E}$, $\mathbf{T}_{RR}$ and satisfies inequality (8). The rate of dissipation Γ (see (9)) is allowed to depend on Ξ and can be specified as a constitutive function. In particular, Γ is taken in the form $\gamma_E|\dot{\mathbf{E}}|$, $\gamma_T|\dot{\mathbf{T}}_{RR}|$ or $\mathcal{H}_R \mathbf{T}_{RR} \cdot \dot{\mathbf{E}}$, where γ_E, γ_T and $\mathcal{H}_R$ are nonnegative rate-independent functions. This general approach is consistent with large deformations and can be easily translated into the spatial formulation. Third, attention is then restricted to one-dimensional settings and the resulting scheme is given by (47) and (48). Models are then characterized by the elastic function $\mathscr{S}(E)$, the parameter α, and the hysteretic function γ_E (or γ_T). The engineering stress S is given by the differential equation (37) quite analogous to the evolution equation for the hardening variable (Eq. (76.7) of [15]). In some one-dimensional example, we establish a connection between the dissipation rate Γ and the derivative of the hardening variable defined as proportional to the plastic deformation rate $\dot{\mathbf{E}}^p$ (see Eq. (54)). Hysteretic loops in the (E, S) plane are shown for some Duhem-like models by appropriately selecting $\mathscr{S}$, α, and γ_E. The existence of an elastic domain, where the material behaves as a linear hypoelastic solid, is related to the existence of an open region where $\gamma_E = 0$ (or $\gamma_T = 0$).

Following are some features of the present approach. Any model obeying (47) and (48) is consistent with thermodynamics; a fit of the behaviour is obtained by selecting appropriate functions $\mathscr{S}$ and γ_E (or γ_T). A nontrivial elastic region occurs when the hysteretic function is piecewise smooth rather than smooth. Unlike the schemes adopted in most theories of plasticity, no decomposition is made *a priori* between the elastic and plastic strain $\mathbf{E}^e$ and $\mathbf{E}^p$ or the elastic and plastic stress $\mathbf{T}^e$ and $\mathbf{T}^p$. Further, while the use of dissipation potentials leads to additional generalized stresses, the present analysis based on the entropy production allows more detailed constitutive equations for physical stress in terms of the macroscopic strain.

We observe that all the models devised in this paper are rate-independent. As a natural improvement we think to the development of rate-dependent models so that the present approach can be applied to obtain viscoplastic models. Furthermore we remark that detailed models are given in the one-dimensional setting while only a structure of three-dimensional models is established in § 5.3. Detailed three-dimensional models might be determined and the papers [43, 44] are useful references in this connection.

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