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The Isotropic Cosserat Shell Model Including Terms up to $O(h^5)$. Part I: Derivation in Matrix Notation

Ionel-Dumitrel Ghiba[1,2] · Mircea Bîrsan[3,1] · Peter Lewintan[3] · Patrizio Neff[4]

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Abstract We present a new geometrically nonlinear Cosserat shell model incorporating effects up to order $O(h^5)$ in the shell thickness h. The method that we follow is an educated 8-parameter ansatz for the three-dimensional elastic shell deformation with attendant analytical thickness integration, which leads us to obtain completely two-dimensional sets of equations in variational form. We give an explicit form of the curvature energy using the orthogonal Cartan-decomposition of the wryness tensor. Moreover, we consider the matrix representation of all tensors in the derivation of the variational formulation, because this is convenient when the problem of existence is considered, and it is also preferential for numerical simulations. The step by step construction allows us to give a transparent approximation of the three-dimensional parental problem. The resulting 6-parameter isotropic shell model combines membrane, bending and curvature effects at the same time. The Cosserat shell model naturally includes a frame of orthogonal directors, the last of which does not necessarily coincide with the normal of the surface. This rotation-field is coupled to the shell-deformation and augments the well-known Reissner-Mindlin kinematics (one independent director) with so-called in-plane drill rotations, the inclusion of which is decisive

✉ I.-D. Ghiba
dumitrel.ghiba@uaic.ro

M. Bîrsan
mircea.birsan@uni-due.de

P. Lewintan
peter.lewintan@uni-due.de

P. Neff
patrizio.neff@uni-due.de

1 Department of Mathematics, Alexandru Ioan Cuza University of Iaşi, Blvd. Carol I, no. 11, 700506 Iaşi, Romania

2 Octav Mayer Institute of Mathematics of the Romanian Academy, Iaşi Branch, 700505 Iaşi, Romania

3 Lehrstuhl für Nichtlineare Analysis und Modellierung, Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann Str. 9, 45127 Essen, Germany

4 Head of Lehrstuhl für Nichtlineare Analysis und Modellierung, Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann Str. 9, 45127 Essen, Germany

for subsequent numerical treatment and existence proofs. As a major novelty, we determine the constitutive coefficients of the Cosserat shell model in dependence on the geometry of the shell which are otherwise difficult to guess.

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1 Introduction

The theory of shells is an important branch of the theory of deformable solids. Its importance resides in the multitude of applications that can be investigated using shell models. In general, the shell and plate theories are intended for the study of *thin bodies*, i.e., bodies in which the thickness in one direction is much smaller than the dimensions in the other two orthogonal directions. As typical examples for shells we mention: roofs of buildings in civil engineering, vehicle bodies in automotive industry, components of wings and propellers in aerospace industry, cell walls and biological membranes.

The large variety of shell-type structures, such as laminated or functionally graded shells made of advanced materials, like polymer foams or cellular materials, as well as the need to fabricate three-dimensional micro- and nanostructures of various shapes leads to the necessity of elaborating new adequate models to describe their mechanical behavior. The process of development of various shell theories is far from being finalized, due to the continuous emergence of new technologies in connection with shell manufacturing. For instance, the emerging need to simulate the mechanical response of highly flexible ultra thin structures (allowing easily for finite rotations) and nano-scale thin structures excludes the use of classical infinitesimal–displacement models, either of Reissner–Mindlin or Kichhoff–Love type. In particular, graphene sheets consisting of monolayer atomic arrangement have non-vanishing "bending stiffness". This is at odds with classical thin shell theory, in which, for $h \to 0$, the bending stiffness should be absent (since the bending terms scale with h^3 against h for the membrane strains). Instead, in an extended continuum model like the *Cosserat-type shell model*, there is a "curvature stiffness" surviving for $h \to 0$ related to a characteristic size $L_c > 0$ (internal length parameter, which is related to the microstructure and in principle independent of the thickness).

1.1 Different Approaches to Shell Modelling

There are several alternative ways to describe the mechanical behavior of shells and to derive the two-dimensional field equations. The method used in our paper is the so-called *derivation approach*. It starts from a given three-dimensional model of the body and reduces it via physically reasonable constitutive assumptions on the kinematics to a two-dimensional model (i.e., *dimensional reduction*). The philosophy behind the derivation approach is expressed clearly by the grandmaster W.T. Koiter as follows [50, p. 93]: "Any two-dimensional theory of thin shells is necessarily of an *approximate character*. An exact two-dimensional theory of shells cannot exist, because the actual body we have to deal with, thin as it may be, is always three-dimensional. [...] Since the theory we have to deal with is approximate in character, we feel that extreme rigour in its development is hardly desirable. [...] *Flexible* *bodies* like thin shells require a *flexible approach*." We mention also that the "rationale" of descend from three to two dimensions should always be complemented by an investigation of the intrinsic mathematical properties of the lower dimensional models.

This procedure is opposed to both the *intrinsic approach* and the *asymptotic methods*. The intrinsic approach is related to the *derivation approach* but it takes the shell a priori to be a two-dimensional surface (appropriate for modelling graphene sheets, with virtually zero thickness) with additional extrinsic directors in the sense of a deformable Cosserat surface [2, 32, 33]. There, two-dimensional equilibrium in appropriate resultant stress and strain variables is postulated ab-initio more or less independently of three-dimensional considerations. For instance, in the shell model elaborated by Zhilin and Altenbach [3, 99] each point of the surface is endowed with a triad of orthonormal vectors (called *directors*), which specify the orientation of the material points and describe the *rotations* of shell filaments. Several interesting applications of this approach (also called *directed surfaces*) have been presented in [6]. In this approach, the constitutive parameters appearing in the a priori chosen two-dimensional energy are obtained by fitting the solutions of some specific problems with the solutions obtained by considering the shell as a three-dimensional body. However, this fitting is usually done only for linear problems, since one reason of being of the nonlinear shell model is that even the classical three-dimensional problem is difficult to be solved for nonlinear problems. Therefore, it is still missing a complete description of the dependence of the constitutive parameters of the reduced nonlinear two-dimensional problem on the constitutive parameters of the parental nonlinear three-dimensional problem and on the mean-curvature H and Gauß-curvature K of the shell's initially stress-free midsurface. It is possible that using the intrinsic approach, certain three-dimensional effects may be missing in the derived shell model. Another approach which is also related to the derivation approach is the *uniform-approximation technique*, mostly motivated by engineering intuition [5, 13, 46, 47, 87, 95, 96]. It uses polynomial expansions in the thickness direction both for the displacements and for the stresses and then it truncates the series expansions.

The procedure of the *asymptotic methods* is to establish two-dimensional equations by formal expansion of the three-dimensional solutions in power series in terms of a small thickness parameter. Using the asymptotic methods, a thorough mathematical analysis of linear, infinitesimal displacement shell theory is presented in [23] and the extensive references therein. Properly invariant, geometrically nonlinear elastic shell theories are derived by formal asymptotic methods in [53, 55], see also [18]. The various shell models based on linearized three-dimensional elasticity proposed in the literature have been rigorously justified in those cases, where some normality assumption is introduced, either a priori or as a result of an asymptotic analysis, see notably the extensive work of Ciarlet and his collaborators [23, 28], see also [90–92]. However, even in the infinitesimal-displacement case it becomes apparent that a model involving membrane and bending simultaneously, cannot be obtained by formal asymptotic methods. In some landmark contributions [41, 51], see also [40, 42] based on the Γ-limit of the three-dimensional model for vanishing thickness, the "membrane-dominated model" and the "flexural-dominated model", respectively, are obtained. The difference between asymptotic methods and Γ-convergence is that the methods based on Γ-convergence [51] lead to another finite-strain membrane term, indicating a non-resistance of the membrane shell in compression. In some examples of careful modelling, the (derivation) Koiter model [49, 89], is simply the sum of the correctly identified membrane and bending contributions, properly scaled with the thickness (the membrane terms scale with h and bending terms with h^3). It is shown in [25] that the Koiter model is asymptotically at least as good as either the membrane model or the bending model in the respective deformation regimes. Regarding the bending term, agreement has been reached that the term which is consistent with the three-dimensional isotropic Saint-Venant–Kirchhoff energy

$$W_{\mathrm{SVK}}(F) = \frac{\mu}{4}\,\|U^2 - \mathbb{1}_3\|^2 + \frac{\lambda}{8}[\mathrm{tr}(U^2 - \mathbb{1}_3)]^2 = \frac{\mu}{4}\,\|C - \mathbb{1}_3\|^2 + \frac{\lambda}{8}[\mathrm{tr}(C - \mathbb{1}_3)]^2, \quad (1.1)$$

where $C = U^2 = F^T F$, is a quadratic expression containing the second fundamental form II_{y_0} of the surface. Nevertheless, the model obtained by energy projection in [42] differs from the results obtained by formal asymptotic analysis in [40]. For the finite-strain membrane model, no rigorous justification of the formal asymptotic approach has been given, because of the lack of a theory which guarantees the well-posedness of the nonlinear three-dimensional problem based on (1.1). Since the membrane terms in a finite-strain properly invariant Kirchhoff-Love shell or finite-strain Reissner-Mindlin model are non-elliptic, the remaining minimization problem is not well-posed even if classical bending is present. By contrast, this is not the case, when a nonlinear three-dimensional problem in the Cosserat theory is considered [70, 93, 94]. By ignoring the Cosserat effects, we will be forced to consider a polyconvex energy [4] in the three-dimensional formulation of the initial problem. These energies do not allow an easy manipulation with respect to the approaches described in this subsection. In this direction, an example is the article [27], see also [16, 29], where the Ciarlet-Geymonat energy [24] is used. In these articles, no reduced completely two-dimensional minimization problem is presented and no through the thickness integration is performed analytically. The obtained problems are "two-dimensional" only in the sense that the final problem is to find three vector fields defined on a bounded open subset of $\mathbb{R}^2$, but all three-dimensional coordinates remain present in the minimization problem. Moreover, the obtained minimization problem is compared with the Koiter model only for small strain-tensors, situation in which the considered energy is actually the isotropic Saint-Venant–Kirchhoff energy.

In applications there are usually regions of a shell where membrane effects dominate, while bending is dominant in others, but both have to be present in the general model. A fully three-dimensional resolution of a thin shell problem remains elusive at present, notwithstanding the increase in computing power. Hence, there is still a need to come up with a sound finite-strain shell model, combining both effects of membrane and bending in one system of equations, as the Koiter model does successfully in the infinitesimal-displacement case.

There are numerous proposals in the engineering literature for such a finite-strain, geometrically nonlinear shell formulation. In many cases, the need has been felt to devote attention to the rotation field $\overline{R} \in \mathrm{SO}(3)$, since rotations are locally the dominant deformation mode of a thin flexible structure. We also mention that considering the new unknown field $\overline{R} \in \mathrm{SO}(3)$, we are able to keep more three-dimensional effects into our two-dimensional variational formulation (local independent rotations in various directions, i.e., each material point of the body has the degrees of freedom of a rigid body), in contrast to the shell models based on classical elasticity. In fact, this is one of the first raison d'être of the Cosserat theory [32, 33]. This has led to shell models which include the so-called *drilling rotations* [98], meaning that in-plane rotations about the shell filament are also taken into account. In [88] it is shown that the inclusion of drilling rotations in the model has a beneficial influence on the numerical implementation. However, a mathematical analysis for such a family of finite-strain curved shell models is, as of yet, still missing.

One of the most general and effective approaches to shells is the so called geometrically nonlinear 6-parameter resultant shell theory, which was originally proposed by Reissner [80] and has subsequently been extended considerably. An account of these developments and main achievements have been presented in the works of Libai and Simmonds [52] and

Pietraszkiewicz et al. [36, 77]. This model involves two independent kinematic fields: the translation vector field and the rotation tensor field (in total six independent scalar kinematic variables). The two-dimensional equilibrium equations and static boundary conditions of the shell are derived exactly by direct through-the-thickness integration of the stresses in the three-dimensional balance laws of linear and angular momentum. The kinematic fields are then constructed directly on the two-dimensional level using the integral identity of the virtual work principle. Following this procedure, the two-dimensional model is expressed in terms of stress resultants and work-averaged deformation fields defined on the shell base surface. The kinematical structure of this 6-parameter model (involving the translation vector and rotation tensor) is identical to the kinematical structure of Cosserat shells (defined as material surfaces endowed with a triad of rigid directors describing the orientation of points). Several developments of this model and applications to complex shell problems, including phase transition and multifold shells, together with the finite element implementation, have been presented by Pietraszkiewicz, Eremeyev and Chróścielewski with their co-workers in a number of papers [17, 22, 36, 37, 77], see also [78].

1.2 The New Shell Model Presented in this Article

In this paper, we extend the Cosserat plate model established by Neff in his habilitation thesis in 2003 [59, 61, 62, 64, 97] to the general case of curved initial shell configurations. The results have been previously announced (using a succinct tensor notation) in [9, 71] and the aim of the current article is to explain the derivation of the model in more details, with added transparency, and using only the matrix representation. Moreover, all our calculations do not use curvilinear coordinates and do not explicitly use an a priori parametrization of the mid-surface, since at the very beginning we are starting by considering the general form of the three-dimensional deformation energy of a fully three-dimensional body, without involving the informations about its mid-surface in the variational formulation. To be more precise, our initial constitutive assumptions (the form of the energy) do not, of course, depend on the shape of the three-dimensional body.

We start with a suitable bulk three-dimensional isotropic Cosserat model written in Cartesian coordinates, we make an appropriate ansatz for the deformation and rotation functions and we perform the integration over the thickness. The form we take for the three-dimensional Cosserat energy is already a strong point of our new approach, since it will help us to give an explicit analytic form of the entire energy of the shell model, and therefore it will be useful in analytical and numerical studies. More precisely, we give an explicit form of the curvature energy using the orthogonal Cartan-decomposition of the wryness tensor (the used curvature tensor of the Cosserat bulk model). In the modelling process we follow the derivation approach as described for planar configurations in [59], but we need to additionally incorporate the curvature effects by using known results from the differential geometry of surfaces in $\mathbb{R}^3$. Thus, we obtain a geometrically nonlinear formulation for Cosserat-type shells with 6 independent kinematical variables: 3 for displacements and 3 for finite rotations (including drilling rotations).

The new model will resolve some shortcomings of classical approaches, which we have mentioned in the previous subsections. In particular, it satisfies the following requirements which we deem necessary for an effective general shell model:

- a geometrically nonlinear formulation that allows for finite rotations.
- the description of transverse shear, drilling rotations, thickness stretch and asymmetric shift of the midsurface.

- a hyperelastic, variational formulation with second-order Euler-Lagrange equations in view of an efficient finite element implementation.
- a dimensionally reduced energy density which is entirely defined in terms of two-dimensional quantities with a clear physical meaning and by a step by step construction.
- well-posedness: existence of solutions, but not unqualified uniqueness in order to be able to describe buckling due to membrane forces (e.g., under lateral compression).
- the consistency with classical shell models for infinitesimal deformations.
- the incorporation of non-classical size effects, such that graphene sheets have bending/curvature resistance.

Since we begin with a Cosserat bulk model which already contains in its formulation the so-called *Cosserat couple modulus* $\mu_c \geq 0$ and an internal length $L_c > 0$ (which is characteristic for the material, e.g., related to the grain size in a polycrystal), the reduced energy density for shells will also include the material parameters μ_c and L_c, in conjunction with specific terms having a clear physical meaning, expressed as functions of two-dimensional quantities. The internal length parameter L_c allows for the incorporation of non-classical size effects in the shell model (in the sense that smaller samples are relatively stiffer than larger samples [31]).

For very irregular and curved initial shell configurations it is not at all clear which terms get small in a thin shell approximation. Moreover, when another dimension (beside the thickness) of the parental three-dimensional body is very small or when the deformations are large compared to the thickness, terms of order $O(h^3)$ may not be sufficient to capture important three-dimensional behaviour. Therefore, we aim to elaborate a complete and consistent model up to the order $O(h^5)$, i.e., we determine all the terms up to the order $O(h^5)$ in both the membrane part and the bending-curvature part of the energy density. Indeed, in a shell model h is very small, but the reason of being of a shell model is to obtain an approximation of the deformation of a three-dimensional body. By considering terms up to the order $O(h^5)$, some additional three-dimensional effects are not omitted in the obtained two-dimensional model. The used method allows this construction in a very transparent way without considering at the very beginning a two-dimensional problem, approach in which terms of order $O(h^5)$ cannot be guessed a priori. Moreover, the coefficients of the terms in the energy density depend on the mean-curvature H and Gauß-curvature K of the shell's midsurface, the calculations showing us that these coefficients have unforeseeable expressions. Thus, we come up with an improved model which should generalize most of the known variants of shell theory (since they consider only terms of order $O(h^3)$, see [89]). In this respect we will deliver more accurate qualitative and numerical results in forthcoming papers.

We regard also other shell models from the literature as particular cases of our formulation. For instance, the 6-parameter resultant shell theory [36] can be viewed as a special case, since it is a theory of order $O(h^3)$, it omits all mixed terms, it is not elaborated starting from a three-dimensional parental problem, and their constitutive coefficients are not expressed in terms of the mean-curvature and Gauß-curvature and not in terms of the constitutive coefficients of the three-dimensional internal energy.

The present paper is completely self-contained and can be read also by researchers not yet accustomed to the specific notation and usages of shell-theory. We arrive at this point at the expense of working, as far as possible, with concepts from 3D-elasticity theory as well as consequently utilizing "reconstructed" 3D-matrix objects. Thus the paper is ideally suited for researchers in need of quickly understanding the basic ingredients of a geometrically nonlinear shell theory. In forthcoming papers we will compare a suitable restriction of our modellling framework with the geometrically nonlinear $O(h^3)$-Koiter model. Preliminary observations suggests that our model (restricted to the same order $O(h^3)$) includes terms not present in the standard Koiter model (isotropic Kirchhoff-Love shell). For developable surfaces (Gauß-curvature $\mathrm{K} = 0$) and after linearisation, both approaches seem to coincide. We will also investigate a corresponding Γ-convergence result, similar to [65, 66, 68].

The considered matrix representation, in the entire derivation of the model, is more convenient when the problem of existence is considered, it is also preferred for numerical simulations in the engineering community and it offers some details about how the elastic shell model obtained in the present article may be extended to an elasto-plastic shell model [58, 60] or to a model which includes residual (initial) stresses [54]. These subjects will be considered in future works based on our model. For instance, in Part II [43] we show that the expression of the energy allows us to have a decent control on each term of the energy density, in order to show the coercivity and the convexity of the energy, and finally to show the existence of minimisers. These types of thin bodies are of great importance nowadays, in view of the new shell manufacturing procedures and we believe that the terms of order $O(h^5)$ included here will play important roles in increasing the accuracy of analytical and numerical predictions in these industrial processes. In forthcoming papers we will prove the existence of the solution of the obtained minimization problem [43] (at order $O(h^3)$ and $O(h^5)$), we will offer some numerical simulations similar to [83] in order to compare our model with some other previous models from literature. Moreover, the pure elastic nonlinear Cosserat shell model will also be extended to viscoelasticity and multiplicative plasticity [61, 82, 85] and it will allow us to discuss residual stress effects in applications to design-control problems of nano-three-dimensional objects [34], situations in which a model up to order $O(h^5)$ is useful since another dimension (beside thickness) may be very small or the deformations are large compared to thickness. Therefore, a model up to order $O(h^5)$ may be very useful in the study of thin bodies with a relative not "so small" thickness compared to the other two dimensions, e.g., in the study of nano-structures.

2 The Three-Dimensional Formulation

In [59] a physically linear, fully frame-invariant isotropic Cosserat model is introduced. The problem has been posed in a variational setting. We consider a three-dimensional *shell-like thin domain* $\Omega_\xi \subset \mathbb{R}^3$. A generic point of Ω_ξ will be denoted by (ξ_1, ξ_2, ξ_3) in a fixed standard base vector e_1, e_2, e_3 of $\mathbb{R}^3$. The elastic material constituting the shell is assumed to be homogeneous and isotropic and the reference configuration Ω_ξ is assumed to be a natural (stress-free) state. In the rest of the present article we use the notation given in Appendix A.1.

The deformation of the body occupying the domain Ω_ξ is described by a map φ_ξ (*called deformation*) and by a *microrotation* $\overline{R}_\xi$,

$$\varphi_\xi : \Omega_\xi \subset \mathbb{R}^3 \to \mathbb{R}^3, \qquad \overline{R}_\xi : \Omega_\xi \subset \mathbb{R}^3 \to \mathrm{SO}(3)\,. \tag{2.1}$$

We denote the current configuration by $\Omega_c := \varphi_\xi(\Omega_\xi) \subset \mathbb{R}^3$. The deformation and the microrotation is solution of the following *geometrically nonlinear minimization problem* posed on Ω_ξ:

$$\begin{aligned} I(\varphi_\xi, F_\xi, \overline{R}_\xi, \alpha_\xi) = \int_{\Omega_\xi} \left[W_{\mathrm{mp}}(\overline{U}_\xi) + W_{\mathrm{curv}}(\alpha_\xi)\right] dV(\xi) - \Pi(\varphi_\xi, \overline{R}_\xi) \\ \to \ \text{min.} \quad \text{w.r.t.} \quad (\varphi_\xi, \overline{R}_\xi)\,, \end{aligned} \tag{2.2}$$

where

$$
\begin{aligned}
F_\xi &:= \nabla_\xi \varphi_\xi \in \mathbb{R}^{3\times 3} && \text{(the deformation gradient)},\\
\overline{U}_\xi &:= \overline{R}_\xi^T F_\xi \in \mathbb{R}^{3\times 3} && \text{(the non-symmetric Biot-type stretch tensor)},\\
\alpha_\xi &:= \overline{R}_\xi^T \operatorname{Curl}_\xi \overline{R}_\xi \in \mathbb{R}^{3\times 3} && \text{(the second order dislocation density tensor [67])},
\end{aligned}
$$

$$
\begin{aligned}
W_{\mathrm{mp}}(\overline{U}_\xi) :=& \mu \,\|\mathrm{dev}\ \mathrm{sym}(\overline{U}_\xi - \mathbb{1}_3)\|^2 + \mu_c \,\|\mathrm{skew}(\overline{U}_\xi - \mathbb{1}_3)\|^2 \\
&+ \frac{\kappa}{2}\,[\mathrm{tr}(\mathrm{sym}(\overline{U}_\xi - \mathbb{1}_3))]^2 \qquad \text{(physically linear)},\\
W_{\mathrm{curv}}(\alpha_\xi) :=& \mu\, L_c^2 \left(b_1 \,\|\mathrm{dev}\ \mathrm{sym}\, \alpha_\xi\|^2 + b_2\, \|\mathrm{skew}\, \alpha_\xi\|^2 + b_3\, [\mathrm{tr}(\alpha_\xi)]^2\right) \qquad \text{(curvature energy)},
\end{aligned}
\tag{2.3}
$$

and $dV(\xi)$ denotes the volume element in the Ω_ξ-configuration.

The total elastically stored energy $W = W_{\mathrm{mp}} + W_{\mathrm{curv}}$ depends on the deformation gradient F_ξ and microrotations $\overline{R}_\xi$ together with their spatial derivatives. In general, the *Biot-type stretch tensor* $\overline{U}_\xi$ is not symmetric (the first Cosserat deformation tensor [33]). The parameters μ and λ are the *Lamé constants* of classical isotropic elasticity, $\kappa = \frac{2\mu+3\lambda}{3}$ is the *infinitesimal bulk modulus*, b_1, b_2, b_3 are *non-dimensional constitutive curvature coefficients (weights)*, $\mu_c \geq 0$ is called the *Cosserat couple modulus* and $L_c > 0$ introduces an *internal length* which is characteristic for the material, e.g., related to the grain size in a polycrystal. The internal length $L_c > 0$ is responsible for *size effects* in the sense that smaller samples are relatively stiffer than larger samples. If not stated otherwise, we assume that $\mu > 0$, $\kappa > 0$, $\mu_c > 0$, $b_1 > 0$, $b_2 > 0$, $b_3 > 0$.

The form of the curvature energy W_{curv} is not that originally considered in [59]. Indeed, Neff [59] uses a curvature energy expressed in terms of the *third order curvature tensor* $\mathfrak{K} = (\overline{R}_\xi^T \nabla(\overline{R}_\xi.e_1)\,|\,\overline{R}_\xi^T \nabla(\overline{R}_\xi.e_2)\,|\,\overline{R}_\xi^T \nabla(\overline{R}_\xi.e_3))$. As we will remark in Sect. 3, the new form of the energy based on the *second order dislocation density tensor* α_ξ simplifies considerably the representation by admitting to use the orthogonal decomposition

$$
\overline{R}_\xi^T \operatorname{Curl}_\xi \overline{R}_\xi = \alpha_\xi = \mathrm{dev}\ \mathrm{sym}\ \alpha_\xi + \mathrm{skew}\ \alpha_\xi + \frac{1}{3}\ \mathrm{tr}(\alpha_\xi)\mathbb{1}_3. \tag{2.4}
$$

Moreover, it yields an equivalent control of spatial derivatives of rotations [67] and allows us to write the curvature energy in a fictitious Cartesian configuration in terms of the so-called *wryness tensor*. This fact has some further implications, e.g., the coupling between the membrane part, the membrane-bending part, the bending-curvature part and the curvature part of the energy of the shell model is transparent and will coincide with shell-bending curvature tensors elsewhere considered [36].

In (2.2), $\Pi(\varphi_\xi, \overline{R}_\xi)$ is the external loading potential, which admits the following additive decomposition:

$$
\Pi(\varphi_\xi, \overline{R}_\xi) = \Pi_f(\varphi_\xi) + \Pi_t(\varphi_\xi) + \Pi_\Omega(\overline{R}_\xi) + \Pi_{\partial\Omega_t}(\overline{R}_\xi), \tag{2.5}
$$

where

$$
\begin{aligned}
\Pi_f(\varphi_\xi) &:= \int_{\Omega_\xi} \langle f, u\rangle\, dV(\xi) = \text{potential of external applied body forces } f,\\
\Pi_t(\varphi_\xi) &:= \int_{\partial\Omega_t} \langle t, u\rangle\, dS(\xi) = \text{potential of external applied boundary forces } t,
\end{aligned}
\tag{2.6}
$$

$$\Pi_{\Omega}(\overline{R}_\xi) := \text{ potential of external applied body couples}\,,$$

$$\Pi_{\partial\Omega_t}(\overline{R}_\xi) := \text{ potential of external applied boundary couples}\,,$$

and $u = \varphi_\xi - \xi$ is the displacement vector, $\partial\Omega_t$ is a subset of the boundary of Ω_ξ, and $dS(\xi)$ is the area element.

2.1 Relation to the Biot Nonlinear Elasticity Model

The used three-dimensional Cosserat model can be seen as an extension of the geometrically nonlinear isotropic Biot-model. Indeed, letting formally[1] $\mu_c \to +\infty$ and $L_c \to 0$, the independent rotation field $\overline{R} \to \text{polar}(F)$ must coincide with the continuum rotation in the polar decomposition of $F = R\,U = \text{polar}(F)\sqrt{F^T F}$. Since for $L_c \to 0$ curvature is absent, the resulting minimization problem is

$$\int_{\Omega_\xi} \big[W_{\text{Biot}}(F) - \langle f, \varphi\rangle\big]\, dV(\xi) \quad \to \text{ min.} \quad \text{w.r.t.} \quad (\varphi_\xi)\,, \tag{2.7}$$

where

$$\begin{aligned} W_{\text{Biot}}(F) &= \mu\, \|U - \mathbb{1}_3\|^2 + \frac{\lambda}{2}\,[\text{tr}(U - \mathbb{1}_3)]^2 \\ &= \mu\, \|\text{dev sym}(U - \mathbb{1}_3)\|^2 + \frac{\kappa}{2}\,[\text{tr}(\text{sym}(U - \mathbb{1}_3))]^2. \end{aligned} \tag{2.8}$$

Recall that typically, the Koiter shell-model is obtained based on the dimension reduction from the isotropic Saint-Venant–Kirchhoff energy

$$W_{\text{SVK}}(F) = \frac{\mu}{4}\, \|U^2 - \mathbb{1}_3\|^2 + \frac{\lambda}{8}[\text{tr}(U^2 - \mathbb{1}_3)]^2 = \frac{\mu}{4}\, \|C - \mathbb{1}_3\|^2 + \frac{\lambda}{8}[\text{tr}(C - \mathbb{1}_3)]^2, \tag{2.9}$$

where $C = U^2 = F^T F$. Both energies (2.8) and (2.9) are linearisation-equivalent and meant to well-capture the small strain regime expected for the response of a thin shell. However, $W_{\text{SVK}}(F)$ introduces physically unacceptable behaviour under the slightest compression (compression would be softer than tension). Since W_{Biot} does not have this feature, we believe that arriving with our model at W_{Biot} is advantageous.

Preliminary calculations show us that, in some particular cases, the total energy of the Cosserat-shell model constructed by using the Biot energy reduces to quadratic and bilinear forms in terms of the difference of the squares of the first fundamental forms (of the initial configuration and of the current configuration) and/or in terms of the difference of the second fundamental forms. This is consistent with new estimates of the distance between two surfaces obtained in [26, 30] which anticipate the need for this type of energies.

3 Transformed Variational Problem in the Fictitious Configuration Ω_h

In what follows, we assume that the parameter domain $\Omega_h \subset \mathbb{R}^3$ is a right cylinder of the form

$$\Omega_h = \left\{(x_1, x_2, x_3)\,\middle|\,(x_1, x_2) \in \omega,\, -\frac{h}{2} < x_3 < \frac{h}{2}\right\} = \omega \times \left(-\frac{h}{2}, \frac{h}{2}\right),$$

[1] In order to arrive at the limit Biot model for $\lambda = 0$, it is sufficient to consider $L_c \to 0$ and $\mu_c \geq \mu$, see [12, 38, 39, 72].

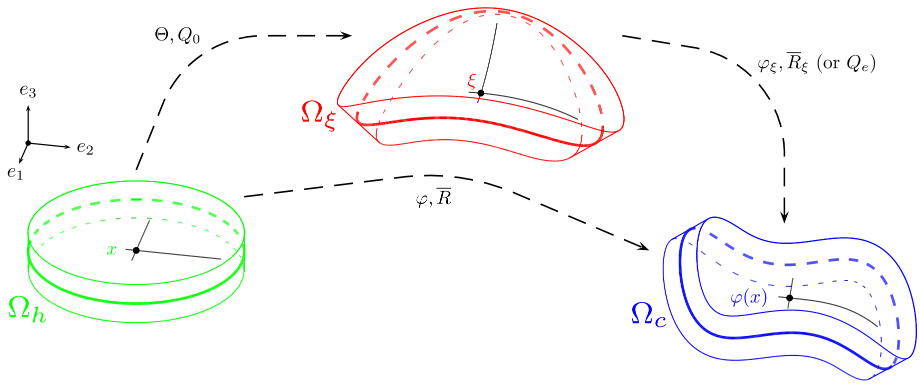

Fig. 1 The shell in its initial configuration Ω_ξ, the shell in the deformed configuration Ω_c, and the fictitious planar cartesian reference configuration Ω_h. Here, $\overline{R}_\xi$ is the elastic rotation field, Q_0 is the initial rotation from the fictitious planar cartesian reference configuration to the initial configuration Ω_ξ, and $\overline{R}$ is the total rotation field from the fictitious planar cartesian reference configuration to the deformed configuration Ω_c

where $\omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\partial\omega$ and the constant length $h > 0$ is the *thickness of the shell*. For shell–like bodies we consider the domain Ω_h to be thin, i.e., the thickness h is small. Thus, the domain Ω_h can be viewed as a *fictitious Cartesian configuration* of the body.

We assume furthermore that there exists a given C^1-diffeomorphism $\Theta : \mathbb{R}^3 \to \mathbb{R}^3$, which maps the fictitious Cartesian parameter space Ω_h with coordinates $(x_1, x_2, x_3) \in \mathbb{R}^3$ onto $\Theta(x_1, x_2, x_3) = (\xi_1, \xi_2, \xi_3)$ such that the initially curved reference configuration of the shell is $\Theta(\Omega_h) = \Omega_\xi$ (see Fig. 1).

Now, let us define the map

$$\varphi : \Omega_h \to \Omega_c, \qquad \varphi(x_1, x_2, x_3) = \varphi_\xi(\Theta(x_1, x_2, x_3)). \tag{3.1}$$

We view φ as a function which maps the fictitious planar reference configuration Ω_h into the deformed (current) configuration Ω_c. Hence, the guiding question is: how can we construct the map φ and a total rotation tensor $\overline{R}$ in order to reduce suitably the three-dimensional problem to a two-dimensional problem? To answer this question (see Fig. 1) we reformulate the minimization principle (2.2) in the fictitious, Cartesian configuration Ω_h. If we construct such mappings, since the diffeomorphism Θ is considered known, then we also know the map which describes the deformation of the initial curved reference configuration Ω_ξ into the current configuration Ω_c of the body.

Assume an underlying three-dimensional deformation of the shell-like body φ_ξ is known and differentiable. Consider a point $\beta = (x_1, x_2, 0) \in \omega \times \{0\}$ and $\Theta(\beta)$. For the moment, we do not assume that $\Theta(\beta)$ is mapped to the midsurface of $\Omega_\xi = \Theta(\Omega_h)$. Consider also the point $\beta_{x_3} = (x_1, x_2, x_3)$, i.e., the line $\beta\beta_{x_3}$ is normal to ω. Then, we have the expansion

$$\varphi_\xi(\Theta(\beta_{x_3})) = \varphi_\xi(\Theta(\beta)) + x_3 \nabla_\xi \varphi_\xi(\Theta(\beta)) \nabla_x \Theta(\beta)\, e_3 + o(x_3) \qquad \text{or} \tag{3.2}$$

$$\varphi_\xi(\Theta(x_1, x_2, x_3)) = \underbrace{\varphi_\xi(\Theta(x_1, x_2, 0))}_{:= m_\xi} + x_3 \nabla_\xi \varphi_\xi(\Theta(x_1, x_2, 0)) \nabla_x \Theta(x_1, x_2, 0) + o(x_3)\,,$$

where $\Theta(x_1, x_2, 0)$ does not belong to the midsurface of Ω_ξ, but to the transformed midsurface $\omega_\xi = \Theta(\omega \times \{0\})$, as long as Θ has not a specific expression.

3.1 Transformation of the Minimization Problem

Consider the *elastic microrotation*

$$\overline{Q}_e : \Omega_h \to \mathrm{SO}(3), \qquad \overline{Q}_e(x_1, x_2, x_3) := \overline{R}_\xi(\Theta(x_1, x_2, x_3)) \tag{3.3}$$

and *the elastic (non-symmetric) Biot-type stretch tensor* (the elastic first Cosserat deformation tensor)

$$\overline{U}_e : \Omega_h \to \mathrm{Sym}(3), \quad \overline{U}_e(x_1, x_2, x_3) := \overline{U}_\xi(\Theta(x_1, x_2, x_3)). \tag{3.4}$$

We use the polar decomposition [69] of $\nabla_x \Theta$ and write

$$\nabla_x \Theta = Q_0 U_0, \quad Q_0 = \mathrm{polar}(\nabla_x \Theta) = \mathrm{polar}([\nabla_x \Theta]^{-T}) \in \mathrm{SO}(3), \tag{3.5}$$
$$U_0 \in \mathrm{Sym}^+(3).$$

Corresponding to the elastic deformation process, we have the total microrotation

$$\overline{R} : \Omega_h \to \mathrm{SO}(3), \qquad \overline{R}(x_1, x_2, x_3) = \overline{Q}_e(x_1, x_2, x_3)\, Q_0(x_1, x_2, x_3). \tag{3.6}$$

Obviously, if we know the total microrotation $\overline{R}$, then we know the microrotation $\overline{R}_\xi$. Using the chain rule

$$\partial_{x_k}\varphi = \sum_{i=1}^{3} \partial_{\xi_i}\varphi_\xi\, \partial_{x_k}\xi_i, \quad \nabla_x\varphi(x_1, x_2, x_3) = \nabla_\xi\varphi_\xi(\Theta(x_1, x_2, x_3))\nabla_x\Theta(x_1, x_2, x_3), \tag{3.7}$$

we deduce (the multiplicative decomposition)

$$F(x_1, x_2, x_3) = F_\xi(\Theta(x_1, x_2, x_3))\, \nabla_x\Theta(x_1, x_2, x_3), \qquad \text{where} \quad F = \nabla_x\varphi, \tag{3.8}$$
$$F_\xi(\Theta(x_1, x_2, x_3)) = F(x_1, x_2, x_3)\, [\nabla_x\Theta(x_1, x_2, x_3)]^{-1}.$$

Therefore, the elastic non-symmetric stretch tensor is given by

$$\overline{U}_e = \overline{Q}_e^T\, F\, [\nabla_x\Theta]^{-1} = Q_0\, \overline{R}^T\, F\, [\nabla_x\Theta]^{-1}. \tag{3.9}$$

As a Lagrangian strain measure for curvature (orientation change) one can also employ the so-called *wryness tensor* (second order tensor) [35, 67]

$$\Gamma_\xi := \Big(\mathrm{axl}(\overline{R}_\xi^T\, \partial_{\xi_1}\overline{R}_\xi)\,|\,\mathrm{axl}(\overline{R}_\xi^T\, \partial_{\xi_2}\overline{R}_\xi)\,|\,\mathrm{axl}(\overline{R}_\xi^T\, \partial_{\xi_3}\overline{R}_\xi)\Big) \in \mathbb{R}^{3\times 3}, \tag{3.10}$$

since (see [67]) the following close relationship between the wryness tensor and the dislocation density tensor holds

$$\alpha_\xi = -\Gamma_\xi^T + \mathrm{tr}(\Gamma_\xi)\, \mathbb{1}_3, \qquad \text{or equivalently,} \qquad \Gamma_\xi = -\alpha_\xi^T + \frac{1}{2}\mathrm{tr}(\alpha_\xi)\, \mathbb{1}_3. \tag{3.11}$$

For infinitesimal strains this formula is well-known under the name Nye's formula, and $-\Gamma$ is also called Nye's curvature tensor. We will use this terminology further on [67].

Note that $\partial_{x_k}\overline{Q}_e=\sum_{i=1}^3\partial_{\xi_i}\overline{R}_\xi\,\partial_{x_k}\xi_i$, $\partial_{\xi_k}\overline{R}_\xi=\sum_{i=1}^3\partial_{x_i}\overline{Q}_e\,\partial_{\xi_k}x_i$ and

$$\overline{R}_\xi^T\,\partial_{\xi_k}\overline{R}_\xi=\sum_{i=1}^3(\overline{Q}_e^T\,\partial_{x_i}\overline{Q}_e)\,\partial_{\xi_k}x_i=\sum_{i=1}^3\,(\overline{Q}_e^T\,\partial_{x_i}\overline{Q}_e)([\nabla_x\Theta]^{-1})_{ik} \tag{3.12}$$

$$\operatorname{axl}\big(\overline{R}_\xi^T\,\partial_{\xi_k}\overline{R}_\xi\big)=\sum_{i=1}^3\operatorname{axl}\big(\overline{Q}_e^T\,\partial_{x_i}\overline{Q}_e\big)([\nabla_x\Theta]^{-1})_{ik}.$$

Thus, we have the chain rule

$$\begin{aligned}\Gamma_\xi&=\Big(\sum_{i=1}^3\operatorname{axl}\Big(\overline{Q}_e^T\,\partial_{x_i}\overline{Q}_e\Big)([\nabla_x\Theta]^{-1})_{i1}\,\Big|\,\sum_{i=1}^3\operatorname{axl}\big(\overline{Q}_e^T\,\partial_{x_i}\overline{Q}_e\big)([\nabla_x\Theta]^{-1})_{i2}\,\Big|\\&\quad\times\sum_{i=1}^3\operatorname{axl}\big(\overline{Q}_e^T\,\partial_{x_i}\overline{Q}_e\big)([\nabla_x\Theta]^{-1})_{i3}\Big)\\&=\Big(\operatorname{axl}(\overline{Q}_e^T\,\partial_{x_1}\overline{Q}_e)\,|\,\operatorname{axl}(\overline{Q}_e^T\,\partial_{x_2}\overline{Q}_e)\,|\,\operatorname{axl}(\overline{Q}_e^T\,\partial_{x_3}\overline{Q}_e)\Big)[\nabla_x\Theta]^{-1}.\end{aligned} \tag{3.13}$$

Define $\Gamma_e:=\Big(\operatorname{axl}(\overline{Q}_e^T\,\partial_{x_1}\overline{Q}_e)\,|\,\operatorname{axl}(\overline{Q}_e^T\,\partial_{x_2}\overline{Q}_e)\,|\,\operatorname{axl}(\overline{Q}_e^T\,\partial_{x_3}\overline{Q}_e)\Big)$, $\alpha_e:=\overline{Q}_e^T\operatorname{Curl}_x\overline{Q}_e$. Using Nye's formula for α_e and Γ_e, we have the correspondence

$$\alpha_e=-\Gamma_e^T+\operatorname{tr}(\Gamma_e)\,\mathbb{1}_3,\qquad\text{or equivalently,}\qquad\Gamma_e=-\alpha_e^T+\frac{1}{2}\operatorname{tr}(\alpha_e)\,\mathbb{1}_3. \tag{3.14}$$

In view of (3.11) and (3.13), we can write

$$\begin{aligned}\alpha_\xi&=-\Gamma_\xi^T+\operatorname{tr}(\Gamma_\xi)\,\mathbb{1}_3=-(\Gamma_e\,[\nabla_x\Theta]^{-1})^T+\operatorname{tr}(\Gamma_e\,[\nabla_x\Theta]^{-1})\,\mathbb{1}_3\\&=-[\nabla_x\Theta]^{-T}\,\Gamma_e^T+\operatorname{tr}(\Gamma_e)\,[\nabla_x\Theta]^{-T}-\operatorname{tr}(\Gamma_e)\,[\nabla_x\Theta]^{-T}+\operatorname{tr}(\Gamma_e\,[\nabla_x\Theta]^{-1})\,\mathbb{1}_3\\&=[\nabla_x\Theta]^{-T}\alpha_e-\operatorname{tr}(\Gamma_e)\,[\nabla_x\Theta]^{-T}+\operatorname{tr}(\Gamma_e\,[\nabla_x\Theta]^{-1})\,\mathbb{1}_3.\end{aligned} \tag{3.15}$$

Moreover

$$\begin{aligned}\operatorname{tr}(\Gamma_e)&=-\operatorname{tr}(\alpha_e)+\frac{3}{2}\operatorname{tr}(\alpha_e)=\frac{1}{2}\operatorname{tr}(\alpha_e),\\\Gamma_e\,[\nabla_x\Theta]^{-1}&=-\alpha_e^T\,[\nabla_x\Theta]^{-1}+\frac{1}{2}\operatorname{tr}(\alpha_e)\,[\nabla_x\Theta]^{-1},\\\operatorname{tr}(\Gamma_e\,[\nabla_x\Theta]^{-1})&=-\operatorname{tr}(\alpha_e^T\,[\nabla_x\Theta]^{-1})+\frac{1}{2}\operatorname{tr}(\alpha_e)\,\operatorname{tr}([\nabla_x\Theta]^{-1}).\end{aligned} \tag{3.16}$$

We deduce

$$\begin{aligned}\alpha_\xi&=[\nabla_x\Theta]^{-T}\alpha_e-\frac{1}{2}\operatorname{tr}(\alpha_e)\,[\nabla_x\Theta]^{-T}-\operatorname{tr}(\,[\nabla_x\Theta]^{-T}\alpha_e)\,\mathbb{1}_3+\frac{1}{2}\operatorname{tr}(\alpha_e)\,\operatorname{tr}([\nabla_x\Theta]^{-1})\,\mathbb{1}_3\\&=[\nabla_x\Theta]^{-T}\alpha_e-\operatorname{tr}(\alpha_e^T\,[\nabla_x\Theta]^{-1})\,\mathbb{1}_3-\frac{1}{2}\operatorname{tr}(\alpha_e)\,\Big\{[\nabla_x\Theta]^{-T}-\operatorname{tr}([\nabla_x\Theta]^{-1})\,\mathbb{1}_3\Big\}.\end{aligned} \tag{3.17}$$

However, we will not use this formula to rewrite the curvature energy in the fictitious Cartesian configuration Ω_h, since it is easier to use (from (3.11))

$$\operatorname{sym}\alpha_\xi=-\operatorname{sym}\Gamma_\xi+\operatorname{tr}(\Gamma_\xi)\,\mathbb{1}_3=-\operatorname{sym}(\Gamma_e\,[\nabla_x\Theta]^{-1})+\operatorname{tr}(\Gamma_e\,[\nabla_x\Theta]^{-1})\,\mathbb{1}_3,$$

$$\begin{aligned}\text{dev sym}\,\alpha_\xi &= -\,\text{dev sym}\,\Gamma_\xi = -\text{dev sym}(\Gamma_e\,[\nabla_x\Theta]^{-1}),\\ \text{skew}\,\alpha_\xi &= -\,\text{skew}\,\Gamma_\xi = -\,\text{skew}(\Gamma_e\,[\nabla_x\Theta]^{-1}),\\ \text{tr}(\alpha_\xi) &= -\,\text{tr}(\Gamma_\xi)+3\,\text{tr}(\Gamma_\xi) = 2\,\text{tr}(\Gamma_\xi) = 2\,\text{tr}(\Gamma_e\,[\nabla_x\Theta]^{-1}),\end{aligned} \tag{3.18}$$

and to express the curvature energy in terms of $\Gamma_e\,[\nabla_x\Theta]^{-1}$ as

$$\begin{aligned}W_{\text{curv}}(\alpha_\xi) =\,&\mu\,L_{\text{c}}^2\big(b_1\,\|\text{dev sym}(\Gamma_e\,[\nabla_x\Theta]^{-1})\|^2+b_2\,\|\text{skew}(\Gamma_e\,[\nabla_x\Theta]^{-1})\|^2\\ &+4\,b_3\,[\text{tr}(\Gamma_e\,[\nabla_x\Theta]^{-1})]^2\big).\end{aligned} \tag{3.19}$$

Note that using

$$\overline{Q}_e^T\,\partial_{x_i}\overline{Q}_e = Q_0\,\overline{R}^T\,\partial_{x_i}(\overline{R}\,Q_0^T) = Q_0(\overline{R}^T\,\partial_{x_i}\overline{R})\,Q_0^T - Q_0(Q_0^T\partial_{x_i}Q_0)\,Q_0^T, \quad i=1,2,3 \tag{3.20}$$

and the identity

$$\text{axl}(Q\,A\,Q^T) = Q\,\text{axl}(A) \qquad \forall\,Q\in\text{SO}(3) \quad \text{and} \quad \forall\,A\in\mathfrak{so}(3), \tag{3.21}$$

we obtain the following form of the wryness tensor

$$\begin{aligned}\Gamma(x_1,x_2,x_3) &:= \Gamma_\xi(\Theta(x_1,x_2,x_3)) = \Gamma_e\,[\nabla_x\Theta]^{-1}\\ &= Q_0\Big[\Big(\text{axl}(\overline{R}^T\,\partial_{x_1}\overline{R})\,|\,\text{axl}(\overline{R}^T\,\partial_{x_2}\overline{R})\,|\,\text{axl}(\overline{R}^T\,\partial_{x_3}\overline{R})\Big)\\ &\quad -\Big(\text{axl}(Q_0^T\,\partial_{x_1}Q_0)\,|\,\text{axl}(Q_0^T\,\partial_{x_2}Q_0)\,|\,\text{axl}(Q_0^T\,\partial_{x_3}Q_0)\Big)\Big]\,[\nabla_x\Theta]^{-1}.\end{aligned} \tag{3.22}$$

Applying the change of variables formula we obtain now a new form of the energy functional I which suggests to seek the unknown functions φ and $\overline{R}$ as solutions of the following minimization problem

$$I = \int_{\Omega_h}\Big[\widetilde{W}_{\text{mp}}(F,\overline{R})+\widetilde{W}_{\text{curv}}(\Gamma)\Big]\det(\nabla_x\Theta)\,dV - \widetilde{\Pi}(\varphi,\overline{R}) \quad \to \quad \text{min. w.r.t.} \quad (\varphi,\overline{R}), \tag{3.23}$$

where dV denotes the volume element $dx_1dx_2dx_3$ and

$$\begin{aligned}\widetilde{W}_{\text{mp}}(F,\overline{R}) &= W_{\text{mp}}(\overline{U}_e)\\ &= \mu\,\|\text{sym}(\overline{U}_e-\mathbb{1}_3)\|^2+\mu_{\text{c}}\,\|\text{skew}(\overline{U}_e-\mathbb{1}_3)\|^2+\frac{\lambda}{2}\,[\text{tr}(\text{sym}(\overline{U}_e-\mathbb{1}_3))]^2\\ &= \mu\,\|\text{dev sym}(\overline{U}_e-\mathbb{1}_3)\|^2+\mu_{\text{c}}\,\|\text{skew}(\overline{U}_e-\mathbb{1}_3)\|^2+\frac{\kappa}{2}\,[\text{tr}(\text{sym}(\overline{U}_e-\mathbb{1}_3))]^2,\\ \widetilde{W}_{\text{curv}}(\Gamma) &= \mu\,L_{\text{c}}^2\,\big(b_1\,\|\text{dev sym}\,\Gamma\|^2+b_2\,\|\text{skew}\,\Gamma\|^2+4\,b_3\,[\text{tr}(\Gamma)]^2\big)\,.\end{aligned}$$

The external loading potential can be written as

$$\widetilde{\Pi}(\varphi,\overline{R}) = \widetilde{\Pi}_f(\varphi)+\widetilde{\Pi}_t(\varphi)+\widetilde{\Pi}_{\Omega_h}(\overline{R})+\widetilde{\Pi}_{\Gamma_t}(\overline{R})\,, \tag{3.24}$$

with

$$\widetilde{\Pi}_f(\varphi) := \Pi_f(\varphi_\xi) = \int_{\Omega_\xi}\big\langle f,u\big\rangle dV(\xi) = \int_{\Omega_h}\big\langle \tilde{f},\tilde{v}\big\rangle dV\,,$$

$$\widetilde{\Pi}_t(\varphi) := \Pi_t(\varphi_\xi) = \int_{\partial\Omega_t} \langle t, u\rangle \, dS(\xi) = \int_{\Gamma_t} \langle \tilde{t}, \tilde{v}\rangle \, dS\,, \tag{3.25}$$
$$\widetilde{\Pi}_{\Omega_h}(\overline{R}) := \Pi_\Omega(\overline{R}_\xi)\,, \qquad \widetilde{\Pi}_{\Gamma_t}(\overline{R}) := \Pi_{\partial\Omega_t}(\overline{R}_\xi)\,,$$

where $\tilde{v}(x_i) = \varphi(x_i) - \Theta(x_i)$ is the displacement vector and the vector fields $\tilde{f}$ and $\tilde{t}$ can be determined in terms of f and t, respectively, for instance $(\tilde{f}(x))_i = (f(\Theta(x)))_i \det(\nabla_x\Theta)$. Here, Γ_t and Γ_d are nonempty subsets of the boundary of Ω_h such that $\Gamma_t \cup \Gamma_d = \partial\Omega_h$ and $\Gamma_t \cap \Gamma_d = \emptyset$. On Γ_t we consider traction boundary conditions, while on Γ_d we have Dirichlet-type boundary conditions (i.e., φ and $\overline{R}$ are prescribed on Γ_d). We assume that Γ_d has the form $\Gamma_d = \gamma_d \times (-\frac{h}{2}, \frac{h}{2})$, where the curve γ_d is a subset of $\partial\omega$ with length $(\gamma_d) > 0$. Accordingly, the boundary subset Γ_t has the form $\Gamma_t = \Big(\gamma_t \times (-\frac{h}{2}, \frac{h}{2})\Big) \cup \Big(\omega \times \Big\{\frac{h}{2}\Big\}\Big) \cup \Big(\omega \times \Big\{-\frac{h}{2}\Big\}\Big)$ and $\Theta(\Gamma_t) = \partial\Omega_t$.

3.2 Useful Tensors Defined Through the Diffeomorphism Θ

For our purpose, the diffeomorphism $\Theta : \mathbb{R}^3 \to \mathbb{R}^3$ describing the reference configuration (i.e., the curved surface of the shell), will be chosen in the specific form

$$\Theta(x_1, x_2, x_3) = y_0(x_1, x_2) + x_3\, n_0(x_1, x_2), \qquad n_0 = \frac{\partial_{x_1} y_0 \times \partial_{x_2} y_0}{\|\partial_{x_1} y_0 \times \partial_{x_2} y_0\|}\,, \tag{3.26}$$

where $y_0 : \omega \to \mathbb{R}^3$ is a function of class $C^2(\omega)$. This specific form of the diffeomorphism Θ maps the midsurface ω of the fictitious Cartesian configuration parameter space Ω_h onto the midsurface $\omega_\xi = y_0(\omega)$ of Ω_ξ and n_0 is the unit normal vector to ω_ξ. For simplicity and where no confusions may arise, further on we will omit to write explicitly the arguments (x_1, x_2, x_3) of the diffeomorphism Θ or we will specify only its dependence on x_3. Remark that

$$\begin{aligned} \nabla_x\Theta(x_3) &= (\nabla y_0 | n_0) + x_3(\nabla n_0 | 0) \ \ \forall x_3 \in \left(-\frac{h}{2}, \frac{h}{2}\right), \\ \nabla_x\Theta(0) &= (\nabla y_0 | \, n_0), \ \ [\nabla_x\Theta(0)]^{-T} e_3 = n_0, \end{aligned} \tag{3.27}$$

and $\det \nabla_x\Theta(0) = \det(\nabla y_0 | n_0) = \sqrt{\det[(\nabla y_0)^T \nabla y_0]}$ represents the surface element.

In the following we identify the *Weingarten map (or shape operator)* on $y_0(\omega)$ with its associated matrix $\mathrm{L}_{y_0} \in \mathbb{R}^{2\times 2}$ defined in Appendix A.2 by $\mathrm{L}_{y_0} = \mathrm{I}_{y_0}^{-1}\mathrm{II}_{y_0}$, where I_{y_0} and II_{y_0} are the matrix representations of the *first fundamental form (metric)* and the *second fundamental form* on $y_0(\omega)$, respectively. Then, the *Gauß curvature* K of the surface $y_0(\omega)$ is determined by $\mathrm{K} = \det(\mathrm{L}_{y_0})$ and the *mean curvature* H through $2\,\mathrm{H} := \mathrm{tr}(\mathrm{L}_{y_0})$. We denote the principal curvatures of the surface by κ_1 and κ_2.

For our purpose we will write the expressions of $\nabla_x\Theta$, $\det(\nabla_x\Theta)$, $[\nabla_x\Theta]^{-1}$ corresponding to the special form of the map Θ given by (3.26), as well as some other of its properties, see Appendix A.3. We have $[\nabla_x\Theta(x_3)]\, e_3 = n_0$. Let us recall that $X \in \mathrm{GL}^+(3)$ satisfies the ***G****eneralized* ***K****irchhoff* ***C****onstraint* (GKC) [60] if $X \in \mathrm{GKC} := \{X \in \mathrm{GL}^+(3) \mid X^T X\, e_3 = \varrho^2 e_3,\ \ \varrho \in \mathbb{R}^+\}$. For all $X \in \mathrm{GKC}$ with the polar decomposition $X = R\, U_0$, if follows that $U_0 \in \mathrm{GKC}$. In view of this property and $\nabla\Theta(x_3) = Q_0(x_3) U_0(x_3)$, it follows[2]

[2]In the rest of the paper $*$ denotes quantities having expressions which are not relevant for our calculations.

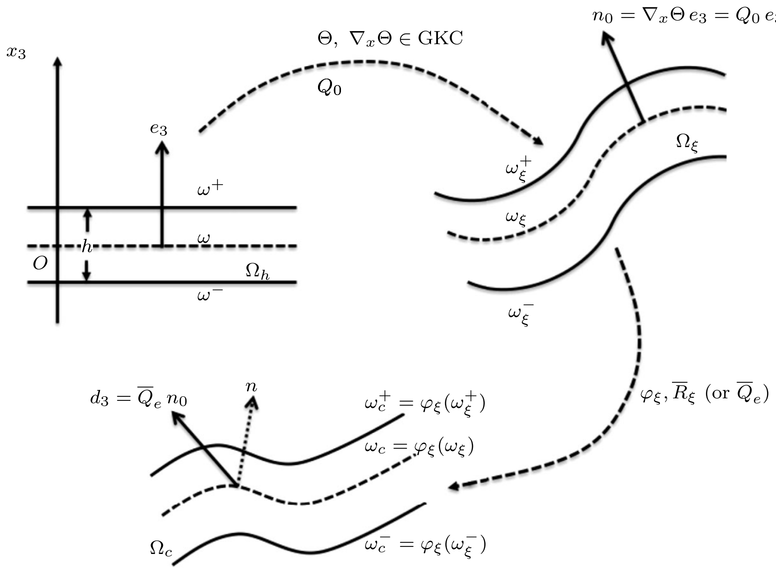

Fig. 2 Transverse section in the shell. The shell is stress free at the upper and lower surface in the current configuration Ω_c. With regard to the first Piola-Kirchhoff tensor S_1 this is equivalent to condition (3.34). Transverse shear is automatically included in the model allowing the unit vector $d_3 = \overline{Q}_e.n_0$ not to coincide with unit vector n, the normal to the deformed midsurface

$U_0(x_3) = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which implies that

$$d_3^0(x_3) := Q_0(x_3).e_3 = Q_0(x_3)\,U_0(x_3).e_3 = \nabla_x\Theta(x_3).e_3 = n_0. \tag{3.28}$$

This means that the initial director d_3^0 is chosen along the normal to the reference midsurface (the "*material filament*" of the shell, see Fig. 2), while $d_\alpha^0 := Q_0.e_\alpha$, for $\alpha = 1, 2$, is an orthonormal basis in the tangent plane of ω_ξ. In the current configuration Ω_c the director $d_3 := \overline{Q}_e.d_3^0$ is no longer orthogonal to the deformed surface $\omega_c := \varphi_\xi(\omega_\xi)$ and the directors $d_\alpha := \overline{Q}_e.d_\alpha^0$, for $\alpha = 1, 2$, are not tangent to this surface. The deviation of the director d_3 from the normal vector to ω_c describes the *transverse shear deformation* of shells. Moreover, the rotations of d_1, d_2 about the director d_3 describe the so-called *drilling rotations* in shells (see [8, 98]).

Let us introduce the tensors[3] defined by:

$$\mathrm{A}_{y_0} := (\nabla y_0|0)\,[\nabla_x\Theta(0)]^{-1} \in \mathbb{R}^{3\times 3}, \qquad \mathrm{B}_{y_0} := -(\nabla n_0|0)\,[\nabla_x\Theta(0)]^{-1} \in \mathbb{R}^{3\times 3}, \tag{3.29}$$

and the so-called *alternator tensor* C_{y_0} of the surface [100]

$$\mathrm{C}_{y_0} := \det(\nabla_x\Theta(0))\,[\nabla_x\Theta(0)]^{-T} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} [\nabla_x\Theta(0)]^{-1}. \tag{3.30}$$

[3]These tensors are usually called the first fundamental form and the second fundamental form, respectively. However, we will not use this terminology since it may lead to some confusions. The relation between these tensors are explained in Proposition A.3.

The introduced tensors have the properties given by Proposition A.3 from Appendix A.4, which are essential in the derivation of the model entirely in matrix representation.

3.3 Plane Stress Conditions in the Curved (Reference) Configuration

The *first Piola-Kirchhoff stress tensor* in the reference (curved) configuration Ω_ξ is given by $S_1(F_\xi, \overline{R}_\xi) = D_{F_\xi} \widetilde{W}_{\rm mp}(F_\xi, \overline{R}_\xi)$. We also consider the *Biot-type stress tensor* from classical elasticity theory $T_{\rm Biot}\big(\overline{U}_\xi\big) := D_{\overline{U}_\xi} W_{\rm mp}(\overline{U}_\xi)$. Since $D_{F_\xi} \overline{U}_\xi.X = \overline{R}_\xi^T X$ and $\big\langle D_{F_\xi} \widetilde{W}_{\rm mp}(F_\xi, \overline{R}_\xi), X\big\rangle = \big\langle D_{\overline{U}_\xi} W_{\rm mp}(\overline{U}_\xi), D_{F_\xi} \overline{U}_\xi.X\big\rangle$, for all $X \in \mathbb{R}^{3\times 3}$, we note that $D_{F_\xi} \widetilde{W}_{\rm mp}(F_\xi, \overline{R}_\xi) = \overline{R}_\xi D_{\overline{U}_\xi} W_{\rm mp}(\overline{U}_\xi)$. Hence, we have

$$S_1(F_\xi, \overline{R}_\xi) = \overline{R}_\xi\, T_{\rm Biot}(\overline{U}_\xi), \qquad\qquad T_{\rm Biot}(\overline{U}_\xi) = \overline{R}_\xi^T\, S_1(F_\xi, \overline{R}_\xi)\,. \tag{3.31}$$

The Biot-type stress tensor $T_{\rm Biot}$ is given by

$$T_{\rm Biot}(\overline{U}_\xi) = 2\,\mu\, {\rm sym}(\overline{U}_\xi - \mathbb{1}_3) + 2\,\mu_{\rm c}\, {\rm skew}(\overline{U}_\xi - \mathbb{1}_3) + \lambda\, {\rm tr}({\rm sym}(\overline{U}_\xi - \mathbb{1}_3))\mathbb{1}_3\,, \tag{3.32}$$

while, using (3.31), we obtain that the first Piola-Kirchhoff tensor S_1 has the following form

$$\begin{aligned} S_1(F_\xi, \overline{R}_\xi) = \overline{R}_\xi \Big[& 2\,\mu\, {\rm sym}(\overline{R}_\xi^T F_\xi - \mathbb{1}_3) + 2\,\mu_{\rm c}\, {\rm skew}(\overline{R}_\xi^T F_\xi - \mathbb{1}_3) \\ & + \lambda\, {\rm tr}({\rm sym}(\overline{R}_\xi^T F_\xi - \mathbb{1}_3))\mathbb{1}_3 \Big]. \end{aligned} \tag{3.33}$$

As usual in the development of shell theories, we assume that the normal stress (Piola-Kirchhoff stress tensor in the normal direction n_0) on the transverse boundaries (*upper and lower faces* ω_ξ^+ and ω_ξ^-, respectively, of the curved reference configuration Ω_ξ) are vanishing, i.e.,

$$S_1(F_\xi, \overline{R}_\xi)\Big|_{\omega_\xi^\pm} .\,(\pm n_0) = 0 \qquad \text{“zero normal stresses” on upper and lower faces.} \tag{3.34}$$

In the limit case $h \to 0$, these conditions imply $S_1(F_\xi, \overline{R}_\xi)\Big|_{\omega_\xi} .\,(\pm n_0) = 0$, i.e.,“zero normal stresses” on the midsurface $\omega_\xi = \Theta(\omega \times \{0\})$, but the reverse of this implication is not valid.

In fact, (3.34) is equivalent to the assumption that the Biot-stress tensor in the normal direction n_0 is vanishing, since $T_{\rm Biot}(\overline{U}_\xi)\Big|_{\omega_\xi^\pm} .\,(\pm n_0) = \overline{R}_\xi^T\Big|_{\omega_\xi^\pm} S_1(F_\xi, \overline{R}_\xi))\Big|_{\omega_\xi^\pm} .(\pm n_0) = 0$, and this implies, after scalar multiplication with n_0

$$\Big\langle T_{\rm Biot}(\overline{U}_\xi)\Big|_{\omega_\xi^\pm} .\, n_0, n_0\Big\rangle = 0 \qquad \text{“zero normal tractions” on upper and lower faces.} \tag{3.35}$$

3.4 Neumann Boundary Conditions in the Fictitious Cartesian Configuration

Using the coordinates of the fictitious Cartesian configuration, the plane stress conditions (3.35) are written in the form

$$\Big\langle T_{\rm Biot}\Big(\overline{U}_e\big(x_1, x_2, \pm\frac{h}{2}\big)\Big)\, n_0, n_0\Big\rangle = 0\,. \tag{3.36}$$

A simplified approximated form of (3.36) can be written in the limit case $h \to 0$ as in the following. Let us define the function

$$f(x_3) := \big\langle T_{\text{Biot}}(\overline{U}_e(x_1, x_2, x_3))\, n_0, n_0 \big\rangle, \qquad \forall\, x_3 \in \left[-\frac{h}{2}, \frac{h}{2}\right]. \tag{3.37}$$

The Taylor expansion of $f(x_3)$ in $x_3 = 0$ leads to

$$\begin{aligned} f\left(\frac{h}{2}\right) + f\left(-\frac{h}{2}\right) &= 2\, f(0) + O(h^2), \\ f\left(\frac{h}{2}\right) - f\left(-\frac{h}{2}\right) &= h\, f'(0) + O(h^3), \end{aligned} \tag{3.38}$$

where

$$f'(0) = \big\langle \partial_{x_3} T_{\text{Biot}}(\overline{U}_e(x_1, x_2, x_3))\Big|_{x_3=0} n_0, n_0 \big\rangle. \tag{3.39}$$

In view of the boundary conditions (3.36) we have $f\left(\frac{h}{2}\right) = 0 = f\left(-\frac{h}{2}\right)$ and the relations (3.38) yield $f(0) = O(h^2)$ and $f'(0) = O(h^2)$. In the limit case $h \to 0$ one obtains the following approximated form of the conditions (3.36)

$$\begin{aligned} &\big\langle T_{\text{Biot}}(\overline{U}_e(x_1, x_2, 0))\, n_0, n_0 \big\rangle = 0 \\ &\qquad \text{"zero normal tractions" on the midsurface } \omega_\xi = \Theta(\omega), \\ &\big\langle \partial_{x_3} T_{\text{Biot}}(\overline{U}_e(x_1, x_2, x_3))\Big|_{x_3=0} n_0, n_0 \big\rangle = 0 \\ &\qquad \text{"zero variations of normal tractions" on the midsurface } \omega_\xi = \Theta(\omega). \end{aligned} \tag{3.40}$$

These relations represent a first approximation in the dimensional reduction procedure and they will be used further on. In addition, in Appendix A.5 we prove that, for our method, this first approximation leads to the same results (but in a simpler way) as is the case when the complete Neumann condition (3.34) are used, and then the limit $h \to 0$ is considered.

4 The Two-Dimensional Approximation

4.1 The 8-Parameter Ansatz for the Two-Dimensional Approximation

In the following, we want to find a *reasonable approximation* of $(\varphi, \overline{R})$ involving only two-dimensional quantities. Following the formal dimensional reduction procedure for the Cosserat elastic plates given in [59], we consider that the rotation $\overline{R} : \Omega_h \to \text{SO}(3)$ in the thin shell does not depend on the thickness variable x_3

$$\overline{R}(x_1, x_2, x_3) = \overline{R}_s(x_1, x_2)\,, \tag{4.1}$$

in line with the assumed thinness and material homogeneity of the structure. Moreover, an approximation of the elastic rotation $\overline{Q}_e : \Omega_h \to \text{SO}(3)$ will be given by $\overline{Q}_{e,s}$

$$\overline{Q}_{e,s}(x_1, x_2) = \overline{R}_s(x_1, x_2)\, Q_0^T(x_1, x_2, 0)\,. \tag{4.2}$$

Taking into account[4] that $\nabla_x\Theta \in \mathrm{GKC}$, with $\varrho = 1$, we have $[\nabla_x\Theta]^{-1}\,[\nabla_x\Theta]^{-T}\,e_3 = e_3$. In view of the properties of GKC, it follows

$$\begin{aligned}\overline{R}_s(x_1,x_2)\,e_3 = \overline{R}_s(x_1,x_2)\,U_0(x_1,x_2,0)\,e_3 &= \overline{Q}_{e,s}(x_1,x_2)\,Q_0(x_1,x_2,0)\,U_0(x_1,x_2,0)\,e_3\,.\\ &= \overline{Q}_{e,s}(x_1,x_2)\,\nabla_x\Theta(x_1,x_2,0)\,e_3 = \overline{Q}_{e,s}(x_1,x_2)\,(\nabla y_0|n_0)\,e_3\\ &= \overline{Q}_{e,s}(x_1,x_2)\,n_0. \end{aligned} \tag{4.3}$$

Since $\overline{Q}_{e,s}\,Q_0 \in \mathrm{SO}(3)$ and with (3.28) we have

$$Q_0(x_1,x_2,x_3)\,e_3 = n_0 = Q_0(x_1,x_2,0)\,e_3\,. \tag{4.4}$$

In the engineering shell community it is well known [19, 75, 86] that the ansatz for the deformation over the thickness should be at least quadratic in order to avoid the so called *Poisson thickness locking* and to fully capture the three-dimensional kinematics without artificial modification of the material laws,[5] see the detailed discussion of this point in [11] and compare with [10, 14, 15, 81, 84].

We consider therefore the following *8-parameter quadratic ansatz* in the thickness direction for the reconstructed total deformation $\varphi_s : \Omega_h \subset \mathbb{R}^3 \to \mathbb{R}^3$ of the shell-like structure

$$\varphi_s(x_1,x_2,x_3) = m(x_1,x_2) + \left(x_3\varrho_m(x_1,x_2) + \frac{x_3^2}{2}\varrho_b(x_1,x_2)\right)\overline{Q}_{e,s}(x_1,x_2)\nabla_x\Theta(x_1,x_2,0)\,e_3\,, \tag{4.5}$$

where $m : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ takes on the role of the deformation of the midsurface of the shell viewed as a parametrized surface, the yet indeterminate functions $\varrho_m, \varrho_b : \omega \subset \mathbb{R}^2 \to \mathbb{R}$ allow in principal for symmetric thickness stretch ($\varrho_m \neq 1$) and asymmetric thickness stretch ($\varrho_b \neq 0$) about the midsurface.

To construct the new geometrically nonlinear Cosserat shell model we start with an 8-parameter ansatz (8 'dof': 3 components of the membrane deformation m, 3 degrees of freedom for $\overline{R} \in \mathrm{SO}(3)$, 2 degrees of freedom over the thickness ϱ_m and ϱ_b) but in the end we will arrive at a *6-parameter model*. This will be possible because in the isotropic case the two scalar parameters ϱ_m and ϱ_b (the degrees of freedom over the thickness) can analytically be condensed out.

In view of (4.4), the above 8-parameter quadratic ansatz in the thickness direction can be written as

$$\varphi_s(x_1,x_2,x_3) = m(x_1,x_2) + \left(x_3\varrho_m(x_1,x_2) + \frac{x_3^2}{2}\varrho_b(x_1,x_2)\right)\overline{Q}_{e,s}(x_1,x_2)\,n_0\,. \tag{4.6}$$

With regard to the total deformation, this is then a kind of plate formulation since the midsurface of the fictitious Cartesian reference configuration $\omega \subset \mathbb{R}^2$ is assumed to lie in the plane. This implies for the total (reconstructed) deformation gradient of the shell

[4] The definition of the set GKC and its properties are presented in Appendix A.4

[5] Let us quote from [86]: "Due to bending this change of length is generally asymmetric about (the midsurface) and leads to a shift of the original midsurfaces.... This asymmetry requires at least a quadratic representation of the (deformation in thickness direction)."

$$
\begin{aligned}
F_s &= \nabla_x \varphi_s(x_1, x_2, x_3) \\
&= (\nabla m|\, \varrho_m\, \overline{Q}_{e,s}(x_1, x_2)\nabla_x \Theta(x_1, x_2, 0)\, e_3) \qquad (4.7)\\
&\quad + x_3\, (\nabla \left[\varrho_m\, \overline{Q}_{e,s}(x_1, x_2)\nabla_x \Theta(x_1, x_2, 0)\, e_3\right] |\varrho_b\, \overline{Q}_{e,s}(x_1, x_2)\nabla_x \Theta(x_1, x_2, 0)\, e_3) \\
&\quad + \frac{x_3^2}{2} (\nabla \left[\varrho_b\, \overline{Q}_{e,s}(x_1, x_2)\nabla_x \Theta(x_1, x_2, 0)\, e_3\right] |0)
\end{aligned}
$$

and especially

$$
\begin{aligned}
F_s\, e_3 &= \varrho_m\, \overline{Q}_{e,s}(x_1, x_2)\nabla_x \Theta(x_1, x_2, 0)\, e_3 + x_3\, \varrho_b\, \overline{Q}_{e,s}(x_1, x_2)\nabla_x \Theta(x_1, x_2, 0)\, e_3 \\
&\overset{(4.3)}{=} (\varrho_m + x_3\, \varrho_b)\, \overline{R}_s(x_1, x_2)\, e_3. \qquad (4.8)
\end{aligned}
$$

If $\overline{Q}_e = \mathbb{1}_3$, $\varrho_m = 1$, $\varrho_b = 0$, $m = y_0$ (as in the reference configuration Ω_ξ), then $F_s = \nabla_x \Theta$.

In the rest of the paper we do not write explicitly the dependence of these functions on x_1 and x_2.

4.2 From an 8-Parameter Ansatz to a 6-Parameter Model via the Fictitious Boundary Conditions

We intend to find ϱ_m and ϱ_b such that the boundary conditions (3.40) in the fictitious configuration

$$
\left\langle T_{\mathrm{Biot}}(\overline{U}_{e,s})\, n_0, n_0 \right\rangle = 0, \qquad \left\langle \partial_{x_3} T_{\mathrm{Biot}}(\overline{U}_{e,s})\, n_0, n_0 \right\rangle = 0 \quad \text{for} \quad x_3 = 0
$$

are satisfied. These boundary conditions are equivalent to

$$
\left\langle T_{\mathrm{Biot}}(\overline{U}_{e,s})\, Q_0\, e_3,\, Q_0\, e_3 \right\rangle = 0, \qquad \left\langle \partial_{x_3} T_{\mathrm{Biot}}(\overline{U}_{e,s})\, Q_0\, e_3,\, Q_0\, e_3 \right\rangle = 0 \quad \text{for} \quad x_3 = 0, \quad (4.9)
$$

and further to

$$
\left\langle Q_0^T\, T_{\mathrm{Biot}}(\overline{U}_{e,s})\, Q_0\, e_3, e_3 \right\rangle = 0, \qquad \left\langle Q_0^T\, \partial_{x_3} T_{\mathrm{Biot}}(\overline{U}_{e,s})\, Q_0\, e_3, e_3 \right\rangle = 0 \quad \text{for} \quad x_3 = 0, \quad (4.10)
$$

where

$$
\begin{aligned}
\overline{U}_{e,s} &= \overline{Q}_{e,s}^T\, F_s\, [\nabla_x \Theta]^{-1} = Q_0\, \overline{R}^T\, F_s\, [\nabla_x \Theta]^{-1}, \qquad (4.11)\\
T_{\mathrm{Biot}}(\overline{U}_{e,s}) &= 2\,\mu\, \mathrm{sym}(\overline{U}_{e,s} - \mathbb{1}_3) + 2\,\mu_{\mathrm{c}}\, \mathrm{skew}(\overline{U}_{e,s} - \mathbb{1}_3) + \lambda\, \mathrm{tr}(\mathrm{sym}(\overline{U}_{e,s} - \mathbb{1}_3))\mathbb{1}_3\,.
\end{aligned}
$$

To this aim, we calculate

$$
\begin{aligned}
2\, Q_0^T\, \mathrm{sym}(\overline{U}_{e,s} - \mathbb{1}_3)\, Q_0 &= Q_0^T \Big(\overline{Q}_{e,s}^T F_s [\nabla_x \Theta]^{-1} + [\nabla_x \Theta]^{-T} F_s^T \overline{Q}_{e,s} - 2\mathbb{1}_3\Big) Q_0 \qquad (4.12)\\
&= Q_0^T \overline{Q}_{e,s}^T F_s U_0^{-1} Q_0^T\, Q_0 + Q_0^T\, Q_0 U_0^{-T}\, F_s^T \overline{Q}_{e,s}\, Q_0 - 2\mathbb{1}_3 \\
&= Q_0^T \overline{Q}_{e,s}^T F_s U_0^{-1} + U_0^{-T}\, F_s^T \overline{Q}_{e,s}\, Q_0 - 2\mathbb{1}_3\,, \\
2\, Q_0^T\, \mathrm{skew}(\overline{U}_{e,s} - \mathbb{1}_3)\, Q_0 &= Q_0^T \Big(\overline{Q}_{e,s}^T F_s [\nabla_x \Theta]^{-1} - [\nabla_x \Theta]^{-T} F_s^T \overline{Q}_{e,s}\Big) Q_0 \qquad (4.13)\\
&= Q_0^T \overline{Q}_{e,s}^T F_s U_0^{-1} Q_0^T\, Q_0 - Q_0^T\, Q_0 U_0^{-T}\, F_s^T \overline{Q}_{e,s}\, Q_0 \\
&= Q_0^T \overline{Q}_{e,s}^T F_s U_0^{-1} - U_0^{-T}\, F_s^T \overline{Q}_{e,s}\, Q_0\,,
\end{aligned}
$$

and

$$\begin{aligned}\big\langle Q_0^T\,\mathrm{sym}(\overline{U}_{e,s}-\mathbb{1}_3)\,Q_0\,e_3,e_3\big\rangle &= \big\langle (Q_0^T\overline{Q}_{e,s}^T F_s U_0^{-1}+U_0^{-T}F_s^T\overline{Q}_{e,s}Q_0-2\mathbb{1}_3)\,e_3,e_3\big\rangle\\ &= 2\big\langle (U_0^{-1}F_s^T\overline{Q}_{e,s}Q_0-\mathbb{1}_3)\,e_3,e_3\big\rangle = 2\big\langle U_0^{-1}F_s^T\overline{Q}_{e,s}Q_0\,e_3,e_3\big\rangle-2,\\ &= 2\big\langle F_s^T\overline{Q}_{e,s}Q_0\,e_3,U_0^{-T}\,e_3\big\rangle-2 = 2\big\langle F_s^T\overline{Q}_{e,s}Q_0\,e_3,e_3\big\rangle-2\\ &\overset{(4.3)}{=} 2\big\langle F_s^T\overline{R}_s\,e_3,e_3\big\rangle-2 = 2\big\langle \overline{R}_s\,e_3,F_s\,e_3\big\rangle-2\\ &\overset{(4.8)}{=} 2\big\langle \overline{R}_s\,e_3,(\varrho_m+x_3\,\varrho_b)\,\overline{R}_s\,e_3\big\rangle-2 = 2\,(\varrho_m+x_3\,\varrho_b-1)\end{aligned}\tag{4.14}$$

where we have used the special structure of $U_0=\begin{pmatrix}* & * & 0\\ * & * & 0\\ 0 & 0 & 1\end{pmatrix}$ and $\overline{R}_s\in\mathrm{SO}(3)$. Furthermore, we have

$$\big\langle Q_0^T\,\mathrm{skew}(\overline{U}_{e,s}-\mathbb{1}_3)\,Q_0\,e_3,e_3\big\rangle = \big\langle (Q_0^T\overline{Q}_{e,s}^T F_s U_0^{-1}-U_0^{-T}F_s^T\overline{Q}_{e,s}Q_0)\,e_3,e_3\big\rangle = 0,\tag{4.15}$$

and

$$\begin{aligned}&\big\langle Q_0^T\,\mathrm{tr}(\mathrm{sym}(\overline{U}_{e,s}-\mathbb{1}_3))\,\mathbb{1}_3\,Q_0\,e_3,e_3\big\rangle\\ &\quad= \mathrm{tr}(\mathrm{sym}(\overline{U}_{e,s}-\mathbb{1}_3))\big\langle Q_0^T\,Q_0\,e_3,e_3\big\rangle = \mathrm{tr}(\mathrm{sym}(\overline{U}_{e,s}-\mathbb{1}_3))\\ &\quad= \big\langle \mathrm{sym}(\overline{U}_{e,s}-\mathbb{1}_3),\mathbb{1}_3\big\rangle = \big\langle Q_0^T\overline{Q}_{e,s}^T F_s U_0^{-1}+U_0^{-T}F_s^T\overline{Q}_{e,s}Q_0-2\mathbb{1}_3,\mathbb{1}_3\big\rangle\\ &\quad= 2\big\langle U_0^{-1}F_s^T\overline{Q}_{e,s}Q_0-\mathbb{1}_3,\mathbb{1}_3\big\rangle = 2\big\langle F_s^T\overline{Q}_{e,s}Q_0,U_0^{-1}\big\rangle-6\\ &\quad= 2[\big\langle(\nabla m|0)^T\overline{Q}_{e,s}Q_0(x_3),U_0^{-1}\big\rangle+\varrho_m+x_3\varrho_m\big\langle(\nabla(\overline{Q}_{e,s}Q_0(x_3)\,e_3)|0)^T\overline{Q}_{e,s}Q_0(x_3),U_0^{-1}\big\rangle\\ &\qquad+x_3\varrho_b+\frac{x_3^2}{2}\varrho_b\big\langle(\nabla(\overline{Q}_{e,s}Q_0(x_3)\,e_3)|0)^T\overline{Q}_{e,s}Q_0(x_3),U_0^{-1}\big\rangle-3]\\ &\quad= 2[\big\langle(\nabla m|0)^T\overline{Q}_{e,s},[\nabla_x\Theta(x_3)]^{-1}\big\rangle+\varrho_m\\ &\qquad+x_3\varrho_m\big\langle(\nabla(\overline{Q}_{e,s}Q_0(x_3)\,e_3)|0)^T\overline{Q}_{e,s},[\nabla_x\Theta(x_3)]^{-1}\big\rangle\\ &\qquad+x_3\varrho_b+\frac{x_3^2}{2}\varrho_b\big\langle(\nabla(\overline{Q}_{e,s}Q_0(x_3)\,e_3)|0)^T\overline{Q}_{e,s},[\nabla_x\Theta(x_3)]^{-1}\big\rangle-3].\end{aligned}\tag{4.16}$$

We deduce

$$\begin{aligned}\big\langle Q_0^T\,T_{\mathrm{Biot}}(\overline{U}_{e,s})\,Q_0\,e_3,e_3\big\rangle &= \mu[2(\varrho_m-1)+2x_3\varrho_b]+\lambda\Big[\big\langle(\nabla m|0)^T\overline{Q}_{e,s},[\nabla_x\Theta(x_3)]^{-1}\big\rangle+\varrho_m\\ &\quad+x_3\varrho_m\big\langle(\nabla(\overline{Q}_{e,s}Q_0(x_3)\,e_3)|0)^T\overline{Q}_{e,s},[\nabla_x\Theta(x_3)]^{-1}\big\rangle\\ &\quad+x_3\varrho_b+\frac{x_3^2}{2}\varrho_b\big\langle\overline{Q}_{e,s}^T(\nabla(\overline{Q}_{e,s}Q_0(x_3)\,e_3)|0),[\nabla_x\Theta(x_3)]^{-T}\big\rangle-3\Big].\end{aligned}\tag{4.17}$$

The requirement $(4.10)_1$ leads to the "plane stress" requirement for $x_3=0$ (zero normal tractions on the upper and lower surface)

$$2\,\mu\,(\varrho_m-1)+\lambda\Big[\big\langle(\nabla m|0)^T\overline{Q}_{e,s},[\nabla_x\Theta(0)]^{-1}\big\rangle+\varrho_m-3\Big]=0,\tag{4.18}$$

which, considering $\nabla_x\Theta = (\nabla y_0|n_0) + x_3(\nabla n_0|0)$, is equivalent to

$$2\,\mu\,(\varrho_m - 1) + \lambda\,[\langle\overline{Q}_{e,s}^T(\nabla m|0), (\nabla y_0|n_0)^{-T}\rangle + \varrho_m - 3] = 0 \tag{4.19}$$

and we obtain

$$\varrho_m = 1 - \frac{\lambda}{\lambda+2\mu}[\langle\overline{Q}_{e,s}^T(\nabla m|0)[\nabla_x\Theta(0)]^{-1}, \mathbb{1}_3\rangle - 2]. \tag{4.20}$$

Now, let us consider the boundary conditions (4.10)$_2$ and observe

$$\begin{aligned}\langle Q_0^T\,\partial_{x_3} T_{\mathrm{Biot}}(\overline{U}_{e,s})\,Q_0\,e_3, e_3\rangle &= \langle\partial_{x_3} Y_{\mathrm{Biot}}(\overline{U}_{e,s})\,Q_0\,e_3,\,Q_0\,e_3\rangle\\ &= \partial_{x_3}\langle Q_0^T\,T_{\mathrm{Biot}}(\overline{U}_{e,s})\,Q_0\,e_3, e_3\rangle,\end{aligned} \tag{4.21}$$

since $Q_0\,e_3 = n_0$ and therefore it is independent of x_3. We deduce

$$\begin{aligned}&\partial_{x_3}\langle Q_0^T\,T_{\mathrm{Biot}}(\overline{U}_{e,s})\,Q_0\,e_3, e_3\rangle\\ &= 2\,\mu\,\varrho_b + \lambda[\langle\overline{Q}_{e,s}^T(\nabla m|0), \partial_{x_3}[\nabla_x\Theta(x_3)]^{-T}\rangle\\ &\quad + \varrho_m\langle\overline{Q}_{e,s}^T(\nabla(\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)|0), [\nabla_x\Theta(x_3)]^{-T}\rangle\\ &\quad + x_3\varrho_m\langle\overline{Q}_{e,s}^T(\nabla(\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)|0), \partial_{x_3}[\nabla_x\Theta(x_3)]^{-T}\rangle\\ &\quad + \varrho_b + x_3\varrho_b\langle\overline{Q}_{e,s}^T(\nabla(\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)|0), [\nabla_x\Theta(x_3)]^{-T}\rangle - 3]\\ &\quad + \frac{x_3^2}{2}\varrho_b\langle\overline{Q}_{e,s}^T(\nabla(\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)|0), \partial_{x_3}[\nabla_x\Theta(x_3)]^{-T}\rangle]\end{aligned} \tag{4.22}$$

and the boundary condition (4.21)$_2$ becomes

$$\begin{aligned}&2\,\mu\,\varrho_b + \lambda[\langle\overline{Q}_{e,s}^T(\nabla m|0), \partial_{x_3}[\nabla_x\Theta(x_3)]^{-T}\rangle\Big|_{x_3=0}\\ &\quad + \varrho_m\langle\overline{Q}_{e,s}^T(\nabla(\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)|0), [\nabla_x\Theta(0)]^{-T}\rangle + \varrho_b] = 0.\end{aligned} \tag{4.23}$$

Here we have used that $\partial_{x_3}[\nabla_x\Theta]^{-1}\Big|_{x_3=0}$ is finite, since $\det(\nabla_x\Theta(x_3))$ has a third order polynomial expression, see Proposition A.2, and $\det(\nabla y_0|n_0) \neq 0$. We also remark that

$$\begin{aligned}\partial_{x_3}[\nabla_x\Theta(x_3)]^{-1}\Big|_{x_3=0} &= -[\nabla_x\Theta(0)]^{-1}\partial_{x_3}[\nabla_x\Theta(x_3)]\Big|_{x_3=0}[\nabla_x\Theta(0)]^{-1}\\ &= -(\nabla y_0|n_0)^{-1}\,(\nabla n_0|0)\,(\nabla y_0|n_0)^{-1}.\end{aligned}$$

Therefore, equation (4.21)$_2$ is equivalent to

$$\begin{aligned}&2\,\mu\,\varrho_b + \lambda[-\langle\overline{Q}_{e,s}^T(\nabla m|0)\,[\nabla_x\Theta(0)]^{-1}\,(\nabla n_0|0)\,[\nabla_x\Theta(0)]^{-1}, \mathbb{1}_3\rangle\\ &\qquad + \varrho_m\langle\overline{Q}_{e,s}^T(\nabla(\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)|0), [\nabla_x\Theta(0)]^{-T}\rangle + \varrho_b] = 0,\end{aligned} \tag{4.24}$$

which yields

$$\begin{aligned}
\varrho_b =& \frac{\lambda}{\lambda+2\mu}\big\langle \overline{Q}_{e,s}^T(\nabla m|0)\,[\nabla_x\Theta(0)]^{-1}\,(\nabla n_0|0)\,[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\big\rangle\\
&-\varrho_m\,\frac{\lambda}{\lambda+2\mu}\big\langle \overline{Q}_{e,s}^T(\nabla(\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)|0)\,[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\big\rangle\\
=& \frac{\lambda}{\lambda+2\mu}\big\langle \overline{Q}_{e,s}^T(\nabla m|0)\,[\nabla_x\Theta(0)]^{-1}\,(\nabla n_0|0)\,[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\big\rangle\\
&-\frac{\lambda}{\lambda+2\mu}\big\langle \overline{Q}_{e,s}^T(\nabla(\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)|0)\,[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\big\rangle\\
&+\frac{\lambda^2}{(\lambda+2\mu)^2}\Big[\big\langle \overline{Q}_{e,s}^T(\nabla(\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)|0)\,[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\big\rangle\Big]\\
&\times\Big[\big\langle \overline{Q}_{e,s}^T(\nabla m|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\big\rangle-2\Big].
\end{aligned}\tag{4.25}$$

Remark 4.1 The term $\frac{\lambda^2}{(\lambda+2\mu)^2}\langle \overline{Q}_{e,s}^T(\nabla(\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle[\langle \overline{Q}_{e,s}^T(\nabla m|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle-2]$ represents a nonlinear coupling between midsurface in-plane (membrane) strain and normal curvature, a result of the derivation not present in the underlying three-dimensional theory where only products of deformation gradient and rotations occur.[6] Since we have in mind a small strain situation, this product is one order smaller than $\frac{\lambda}{\lambda+2\mu}\langle(\nabla(\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)|0)^T\ \overline{Q}_{e,s}[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle$. Therefore, we neglect this term. The presence of the term $-\frac{\lambda}{\lambda+2\mu}\langle \overline{Q}_{e,s}^T(\nabla m|0)(\nabla y_0|n_0)^{-1}(\nabla n_0|0)(\nabla y_0|n_0)^{-1},\mathbb{1}_3\rangle$ is not in contradiction with the Cosserat-plate model [59] because in the plate case it is automatically zero, since $\nabla n_0=\nabla e_3\equiv 0$.

Thus, our considered form for ϱ_m and ϱ_b will be

$$\begin{aligned}
\varrho_m^e =& 1-\frac{\lambda}{\lambda+2\mu}[\langle \overline{Q}_{e,s}^T(\nabla m|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle-2]\,,\\
\varrho_b^e =& -\frac{\lambda}{\lambda+2\mu}\langle \overline{Q}_{e,s}^T(\nabla(\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle\\
&+\frac{\lambda}{\lambda+2\mu}\langle \overline{Q}_{e,s}^T(\nabla m|0)[\nabla_x\Theta(0)]^{-1}(\nabla n_0|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle.
\end{aligned}\tag{4.26}$$

Remark that the condition (3.35) is not satisfied exactly. However, the formula $(4.26)_1$ has a clear physical significance: in-plane stretch leads to thickness reduction.

Now, our final aim in the determination of ϱ_m^e and ϱ_b^e is to compute them for $\overline{Q}_{e,s}=\mathbb{1}_3$ and $m=y_0$, which means the elastic deformation is absent, i.e., we compute

$$\varrho_m^0 = 1-\frac{\lambda}{\lambda+2\mu}[\langle(\nabla y_0|0)(\nabla y_0|n_0)^{-1},\mathbb{1}_3\rangle-2],\tag{4.27}$$

[6] In addition, this term is not invariant under reflection across the midsurface, i.e. $\overline{Q}_e=(\overline{Q}_{e,1},\overline{Q}_{e,2},\overline{Q}_{e,3})\mapsto(\overline{Q}_{e,1},\overline{Q}_{e,2},-\overline{Q}_{e,3})$ [56].

$$\begin{aligned}\varrho_b^0 &= -\frac{\lambda}{\lambda+2\mu}\big\langle (\nabla(\nabla_x\Theta(0)\,e_3)|0)[\nabla_x\Theta(0)]^{-1}, \mathbb{1}_3\big\rangle\\&\quad+\frac{\lambda}{\lambda+2\mu}\big\langle (\nabla y_0|0)[\nabla_x\Theta(0)]^{-1}(\nabla n_0|0)[\nabla_x\Theta(0)]^{-1}, \mathbb{1}_3\big\rangle\\&= -\frac{\lambda}{\lambda+2\mu}\big\langle (\nabla n_0|0)[\nabla_x\Theta(0)]^{-1}, \mathbb{1}_3\big\rangle\\&\quad+\frac{\lambda}{\lambda+2\mu}\big\langle (\nabla y_0|0)[\nabla_x\Theta(0)]^{-1}(\nabla n_0|0)[\nabla_x\Theta(0)]^{-1}, \mathbb{1}_3\big\rangle.\end{aligned}\tag{4.28}$$

The identity $\mathrm{tr}(\mathrm{A}_{y_0}) = \big\langle (\nabla y_0|0)(\nabla y_0|n_0)^{-1}, \mathbb{1}_3\big\rangle = 2$, (see Proposition A.3) implies that $\varrho_m^0 = 1$. Next, we compute ϱ_b^0. With the help of the curvature tensors $\mathrm{A}_{y_0}, \mathrm{B}_{y_0}$ (see Proposition A.3) we have

$$\mathrm{tr}[(\nabla y_0|0)(\nabla y_0|n_0)^{-1}(\nabla n_0|0)(\nabla y_0|n_0)^{-1}] = -2\,\mathrm{H}.\tag{4.29}$$

Hence, we deduce

$$\varrho_b^0 = \frac{\lambda}{(\lambda+2\mu)}\mathrm{tr}[\mathrm{B}_{y_0}] - \frac{\lambda}{(\lambda+2\mu)}\mathrm{tr}[\mathrm{L}_{y_0}] = 2\frac{\lambda}{(\lambda+2\mu)}\,\mathrm{H} - 2\frac{\lambda}{(\lambda+2\mu)}\,\mathrm{H} = 0.\tag{4.30}$$

Thus, the reference values ϱ_m^0 and ϱ_b^0 of the parameters ϱ_m^e and ϱ_b^e are given by $\varrho_m^0 = 1$, $\varrho_b^0 = 0$, which means that in the absence of elastic deformation the ansatz (4.6) $\varphi_s^0(x_1,x_2,x_3) = y_0(x_1,x_2) + x_3\,n_0(x_1,x_2) = \Theta(x_1,x_2,x_3)$ is exact.

4.3 The Ansatz for the Deformation Gradient

Having obtained a suitable form of the relevant coefficients ϱ_m^e, ϱ_b^e, it is expedient to base the expansion of the three-dimensional elastic Cosserat energy on a further simplified expression, please compare with (4.7), namely

$$\begin{aligned}F_s &= \nabla_x\varphi_s(x_1,x_2,x_3)\\&= (\nabla m|\,\varrho_m\,\overline{Q}_{e,s}(x_1,x_2)\nabla_x\Theta(x_1,x_2,0)\,e_3)\\&\quad+x_3\,(\nabla[\underbrace{\cancel{\varrho_m}}_{\cong\varrho_m^0=1}\ \overline{Q}_{e,s}(x_1,x_2)\nabla_x\Theta(x_1,x_2,0)\,e_3]|\varrho_b\,\overline{Q}_{e,s}(x_1,x_2)\nabla_x\Theta(x_1,x_2,0)\,e_3)\\&\quad+\frac{x_3^2}{2}(\nabla[\underbrace{\cancel{\varrho_b}}_{\cong\varrho_b^0=0}\ \overline{Q}_{e,s}(x_1,x_2)\nabla_x\Theta(x_1,x_2,0)\,e_3]|0)\,.\\&\cong(\nabla m|\,\varrho_m^e\,\overline{Q}_{e,s}(x_1,x_2)\nabla_x\Theta(x_1,x_2,0)\,e_3)\\&\quad+x_3(\nabla\big[\,\overline{Q}_{e,s}(x_1,x_2)\nabla_x\Theta(x_1,x_2,0)\,e_3\big]\,|\varrho_b^e\,\overline{Q}_{e,s}(x_1,x_2)\nabla_x\Theta(x_1,x_2,0)\,e_3).\end{aligned}\tag{4.31}$$

Remark 4.2 (Raison d'être) 1) The reduced model should at no place contain space derivatives of the thickness stretch ϱ_m^e, since in the underlying three-dimensional Cosserat model curvature is only present through the dislocation density tensor α_ξ (or through the wryness tensor Γ_ξ) related only to rotations $\overline{Q}_e$.

2) If we blithely use the fully reconstructed deformation gradient F_s and integrate analytically through the thickness, we would obtain second order derivatives in the energy (through derivatives on ϱ_m^e and ϱ_b^e) both for the midsurface m and microrotation $\overline{Q}_e$, leading to a coupled fourth order problem, a situation which has to be avoided for simplicity and efficiency in a subsequent numerical implementation, taking also in consideration the second order Cosserat bulk problem.
3) Keeping the quadratic ansatz (4.7) but neglecting only the derivatives of ϱ_m^e and ϱ_b^e, i.e., basing the integration through the thickness instead on (4.7), the reduced ansatz (4.31) would already lead to a second order equilibrium problem and entitle us to skip the quadratic term altogether, since either h^5-bending terms appear or h^3-product of membrane and bending appear, which can be dominated through Youngs-inequality by a sum of h^2-membrane and h^4-bending terms, which themselves are subordinate (for small h) to the already appearing h-membrane and h^3-bending terms.
4) The error induced by the modified ansatz (4.31) in the energy density will be of higher order under the assumption of small elastic midsurface strain.
5) Finally, it should be observed that by using (4.31) we are consistent with John's general result [44, 45] that the stress distribution through the thickness is approximately linear for a thin shell.

Motivated by the above remarks on the ansatz for the (reconstructed) deformation gradient (4.31), the chain rule leads to the approximation

$$\begin{aligned} F_{s,\xi} &= \nabla_x \varphi_s(x_1,x_2,x_3)[\nabla_x \Theta(x_1,x_2,x_3)]^{-1} \\ &\cong \widetilde{F}_{e,s} := (\nabla m|\,\varrho_m^e\, \overline{Q}_{e,s}(x_1,x_2)\nabla_x\Theta(x_1,x_2,0)\,e_3)[\nabla_x\Theta(x_1,x_2,x_3)]^{-1} \\ &\quad + x_3(\nabla\big[\,\overline{Q}_{e,s}(x_1,x_2)\nabla_x\Theta(x_1,x_2,0)\,e_3\big]\,|\varrho_b^e\,\overline{Q}_{e,s}(x_1,x_2)\nabla_x\Theta(x_1,x_2,0)\,e_3)[\nabla_x\Theta(x_1,x_2,x_3)]^{-1}. \end{aligned} \tag{4.32}$$

Our model will be constructed under the following assumption upon the thickness

$$h\,|\kappa_1| < \frac{1}{2}, \qquad h\,|\kappa_2| < \frac{1}{2}, \tag{4.33}$$

where κ_1, κ_2 are the principal curvatures.

In consequence, using (iii) from Proposition A.2, we find that the (reconstructed) deformation gradient is given by

$$\begin{aligned} \widetilde{F}_{e,s} &= \frac{1}{b(x_3)}\big[(\nabla m|\,\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3) + x_3(\nabla\big[\,\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3\big]\,|0) \\ &\qquad + (\varrho_m^e - 1 + x_3\varrho_b^e)(0|0|\,\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)\big] \\ &\qquad \times \left[\mathbb{1}_3 + x_3(\mathrm{L}_{y_0}^\flat - 2\,\mathrm{H}\,\mathbb{1}_3) + x_3^2\,\mathrm{K}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right][\nabla_x\Theta(0)]^{-1}, \end{aligned} \tag{4.34}$$

where we have set $b(x_3) := 1 - 2\,\mathrm{H}\,x_3 + \mathrm{K}\,x_3^2$. Next, we need to express the tensors

$$\begin{aligned} \widetilde{\mathcal{E}}_s &:= \overline{U}_{e,s} - \mathbb{1}_3 = \overline{Q}_{e,s}^T\,\widetilde{F}_{e,s} - \mathbb{1}_3, \\ \Gamma_s &:= (\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_1}\overline{Q}_e)\,|\,\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_2}\overline{Q}_e)\,|0)[\nabla_x\Theta(x_3)]^{-1} \end{aligned} \tag{4.35}$$

with the help of the usual strain measures in the nonlinear 6-parameter shell theory [36], see Sect. 6. Therefore, we introduce the following tensor fields on the surface ω_ξ [7, 8, 20, 36, 52]

$$
\begin{aligned}
\mathcal{E}_{m,s} &:= \overline{Q}_{e,s}^T(\nabla m|\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)[\nabla_x\Theta(0)]^{-1} - \mathbb{1}_3 \\
&\quad \text{(the elastic shell strain tensor)}, \\
\mathcal{K}_{e,s} &:= (\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_1}\overline{Q}_e)\,|\,\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_2}\overline{Q}_e)\,|0)[\nabla_x\Theta(0)]^{-1} \\
&\quad \text{(elastic shell bending–curvature tensor)}.
\end{aligned}
\tag{4.36}
$$

Lemma 4.3 *The following identities are satisfied*

i) $\mathcal{E}_{m,s} = (\overline{Q}_{e,s}^T\,\nabla m - \nabla y_0|0)[\nabla_x\Theta(0)]^{-1} = Q_0(\overline{R}_s^T\,\nabla m - Q_0^T\nabla y_0|0)[\nabla_x\Theta(0)]^{-1}$;
ii) $\mathcal{E}_{m,s}\mathrm{A}_{y_0} = \mathcal{E}_{m,s}$;
iii) $\mathcal{K}_{e,s}\mathrm{A}_{y_0} = \mathcal{K}_{e,s}$; *iv)* $\overline{Q}_{e,s}^T\,(\nabla[\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3]\,|\,0)\,[\nabla_x\Theta(0)]^{-1} = \mathrm{C}_{y_0}\mathcal{K}_{e,s} - \mathrm{B}_{y_0}$.

Proof Starting with i), we observe

$$
\begin{aligned}
\mathcal{E}_{m,s} &= (\overline{Q}_{e,s}^T\nabla m|n_0)[\nabla_x\Theta(0)]^{-1} - \mathbb{1}_3 \\
&= (\overline{Q}_{e,s}^T\nabla m|n_0)[\nabla_x\Theta(0)]^{-1} - (\nabla y_0|n_0)\,[\nabla_x\Theta(0)]^{-1}.
\end{aligned}
\tag{4.37}
$$

Hence, we obtain ii) and iii) with Proposition A.3 (v).

The last item follows from the same procedure we used to establish (3.22). We have

$$
\begin{aligned}
\mathcal{K}_{e,s} \;=\; Q_0\Big[&\Big(\mathrm{axl}(\overline{R}_s^T\,\partial_{x_1}\overline{R}_s)\,|\,\mathrm{axl}(\overline{R}_s^T\,\partial_{x_2}\overline{R}_s)\,|\,0\Big) \\
&- \Big(\mathrm{axl}(Q_0^T(0)\,\partial_{x_1}Q_0(0))\,|\,\mathrm{axl}(Q_0^T(0)\,\partial_{x_2}Q_0(0))\,|\,0\Big)\Big]\,[\nabla_x\Theta(0)]^{-1}.
\end{aligned}
\tag{4.38}
$$

We compute

$$
\begin{aligned}
\overline{R}_s^T\,\partial_{x_\alpha}\overline{R}_s &= (d_1\,|\,d_2\,|\,d_3)^T\,(\partial_{x_\alpha}d_1\,|\,\partial_{x_\alpha}d_2\,|\,\partial_{x_\alpha}d_3) \\
&= \begin{pmatrix} 0 & \langle d_1,\partial_{x_\alpha}d_2\rangle & \langle d_1,\partial_{x_\alpha}d_3\rangle \\ \langle d_2,\partial_{x_\alpha}d_1\rangle & 0 & \langle d_2,\partial_{x_\alpha}d_3\rangle \\ \langle d_3,\partial_{x_\alpha}d_1\rangle & \langle d_3,\partial_{x_\alpha}d_2\rangle & 0 \end{pmatrix}, \\
\mathrm{axl}(\overline{R}_s^T\,\partial_{x_\alpha}\overline{R}_s) &= \Big(-\langle d_2,\partial_{x_\alpha}d_3\rangle\,|\,\langle d_1,\partial_{x_\alpha}d_3\rangle\,|\,-\langle d_1,\partial_{x_\alpha}d_2\rangle\Big)^T, \qquad \alpha = 1,2.
\end{aligned}
\tag{4.39}
$$

Thus, we deduce

$$
\begin{aligned}
&\mathrm{C}_{y_0}\,Q_0(0)\,\Big(\mathrm{axl}(\overline{R}_s^T(\,\partial_{x_1}\overline{R}_s)\,|\,\mathrm{axl}(\overline{R}_s^T\,\partial_{x_2}\overline{R}_s)\,|\,0\Big)\,[\nabla_x\Theta(0)]^{-1} \\
&\quad = Q_0(0)\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -\langle d_2,\partial_{x_1}d_3\rangle & -\langle d_2,\partial_{x_2}d_3\rangle & 0 \\ \langle d_1,\partial_{x_1}d_3\rangle & \langle d_1,\partial_{x_2}d_3\rangle & 0 \\ -\langle d_1,\partial_{x_1}d_2\rangle & -\langle d_1,\partial_{x_2}d_2\rangle & 0 \end{pmatrix}[\nabla_x\Theta(0)]^{-1} \\
&\quad = Q_0(0)\,\overline{R}_s^T\,(\partial_{x_1}d_3\,|\,\partial_{x_2}d_3\,|\,0)\,[\nabla_x\Theta(0)]^{-1} \\
&\quad = \overline{Q}_{e,s}^T\,(\nabla[\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3]\,|\,0)\,[\nabla_x\Theta(0)]^{-1}.
\end{aligned}
\tag{4.40}
$$

Using Proposition A.3 and (4.38) we conclude

$$\begin{aligned}\mathrm{C}_{y_0}\mathcal{K}_{e,s} &= \overline{Q}_{e,s}^{T}\,(\nabla[\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3]\,|0)\,[\nabla_x\Theta(0)]^{-1}+\mathrm{B}_{y_0}\\ &= [\overline{Q}_{e,s}^{T}\,(\nabla[\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3]\,|0) - \nabla_x\Theta(0)\mathrm{L}_{y_0}^{\flat}]\,[\nabla_x\Theta(0)]^{-1}. \qquad \square\end{aligned}$$

Accordingly, for the strain tensor corresponding to $\widetilde{\mathcal{E}}_s$ and using Lemma 4.3 we find the following expression of the tensor $\widetilde{\mathcal{E}}_s$ defined by (4.35)

$$\begin{aligned}\widetilde{\mathcal{E}}_s := \frac{1}{b(x_3)}\Big\{&\mathcal{E}_{m,s} + (\varrho_m^e - 1)\overline{Q}_{e,s}^{T}(0|0|\,\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)[\nabla_x\Theta(0)]^{-1}\\ &+ x_3\Big[2\,\mathrm{H}\,\mathbb{1}_3 + \overline{Q}_{e,s}^{T}(\nabla m|\,\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)(\mathrm{L}_{y_0}^{\flat} - 2\,\mathrm{H}\,\mathbb{1}_3)[\nabla_x\Theta(0)]^{-1}\\ &\quad + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s} - \mathrm{B}_{y_0} + (\varrho_m^e - 1)\overline{Q}_{e,s}^{T}(0|0|\,\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)\\ &\quad \times\,(\mathrm{L}_{y_0}^{\flat} - 2\,\mathrm{H}\,\mathbb{1}_3)[\nabla_x\Theta(0)]^{-1}\\ &\quad + \varrho_b^e\,\overline{Q}_{e,s}^{T}(0|0|\,\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)[\nabla_x\Theta(0)]^{-1}\Big] \qquad (4.41)\\ &+ x_3^2\Big[\overline{Q}_{e,s}^{T}(\nabla m|\,\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)\mathrm{K}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}[\nabla_x\Theta(0)]^{-1}\\ &\quad - \mathrm{K}\,\mathbb{1}_3 + \overline{Q}_{e,s}^{T}(\nabla\left[\,\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3\right]|0)(\mathrm{L}_{y_0}^{\flat} - 2\,\mathrm{H}\,\mathbb{1}_3)[\nabla_x\Theta(0)]^{-1}\\ &\quad + (\varrho_m^e - 1)\overline{Q}_{e,s}^{T}(0|0|\,\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)\,\mathrm{K}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}[\nabla_x\Theta(0)]^{-1}\\ &\quad + \varrho_b^e\,\overline{Q}_{e,s}^{T}(0|0|\,\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)\,(\mathrm{L}_{y_0}^{\flat} - 2\,\mathrm{H}\,\mathbb{1}_3)[\nabla_x\Theta(0)]^{-1}\Big]\\ &+ x_3^3\Big[\,\overline{Q}_{e,s}^{T}(\nabla\left[\,\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3\right]|0)\mathrm{K}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}[\nabla_x\Theta(0)]^{-1}\\ &\quad + \varrho_b^e\,\overline{Q}_{e,s}^{T}(0|0|\,\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)\,\mathrm{K}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}[\nabla_x\Theta(0)]^{-1}\Big]\Big\}.\end{aligned}$$

From Proposition A.3 we have

$$\begin{aligned}\overline{Q}_{e,s}^{T}(\nabla m|\,\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3)\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}[\nabla_x\Theta(0)]^{-1} &= (0|0|\,\nabla_x\Theta(0)\,e_3)\,[\nabla_x\Theta(0)]^{-1}\\ &= \mathbb{1}_3 - \mathrm{A}_{y_0}, \qquad (4.42)\end{aligned}$$

which, together with Lemma 4.3 and $\mathrm{B}_{y_0} = [\nabla_x\Theta(0)]\,\mathrm{L}_{y_0}^{\flat}\,[\nabla_x\Theta(0)]^{-1}$ leads to

$$\begin{aligned}\widetilde{\mathcal{E}}_s = \frac{1}{b(x_3)}\Big\{&\mathcal{E}_{m,s} + (\varrho_m^e - 1)(0|0|\,\nabla_x\Theta(0)\,e_3)[\nabla_x\Theta(0)]^{-1} \qquad (4.43)\\ &+ x_3\Big[2\,\mathrm{H}\,\mathbb{1}_3 + (\mathcal{E}_{m,s} + \mathbb{1}_3)\,(\mathrm{B}_{y_0} - 2\,\mathrm{H}\,\mathbb{1}_3)\\ &\quad + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s} - \mathrm{B}_{y_0} + [2\,\mathrm{H}\,(1 - \varrho_m^e) + \varrho_b^e](0|0|\,\nabla_x\Theta(0)\,e_3)[\nabla_x\Theta(0)]^{-1}\Big]\end{aligned}$$

$$
\begin{aligned}
&+ x_3^2\Big[\mathrm{K}\,\mathbb{1}_3 - \mathrm{K}\,\mathrm{A}_{y_0} - \mathrm{K}\,\mathbb{1}_3 + [\mathrm{C}_{y_0}\,\mathcal{K}_{e,s} - \mathrm{B}_{y_0}](\mathrm{B}_{y_0} - 2\,\mathrm{H}\,\mathbb{1}_3)\\
&\qquad + (\varrho_m^e - 1)(0|0|\,\nabla_x\Theta(0)\,e_3)\,\mathrm{K}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}[\nabla_x\Theta(0)]^{-1}\\
&\qquad + \varrho_b^e\,(0|0|\,\nabla_x\Theta(0)\,e_3)\,(\mathrm{L}^\flat_{y_0} - 2\,\mathrm{H}\,\mathbb{1}_3)[\nabla_x\Theta(0)]^{-1}\Big]\\
&+ x_3^3\Big[0_3 + \varrho_b^e\,(0|0|\,\nabla_x\Theta(0)\,e_3)\,\mathrm{K}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}[\nabla_x\Theta(0)]^{-1}\Big]\Big\}.
\end{aligned}
$$

Further, using the Cayley-Hamilton type equation and item i) from Proposition A.3, item ii) of Lemma 4.3 and $\mathcal{E}_{m,s}\,(0|0|\,\overline{Q}_{e,s}\nabla_x\Theta(0)\,e_3) = 0_3$ we deduce

$$
\begin{aligned}
\widetilde{\mathcal{E}}_s = \frac{1}{b(x_3)}\Big\{&\mathcal{E}_{m,s} + (\varrho_m^e - 1)(0|0|\,\nabla_x\Theta(0)\,e_3)[\nabla_x\Theta(0)]^{-1}\\
&+ x_3\Big[\mathcal{E}_{m,s}(\mathrm{B}_{y_0} - 2\,\mathrm{H}\,\mathrm{A}_{y_0}) + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s} + A_1\,(0|0|\,\nabla_x\Theta(0)\,e_3)[\nabla_x\Theta(0)]^{-1}\Big]\\
&+ x_3^2\Big[\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}(\mathrm{B}_{y_0} - 2\,\mathrm{H}\,\mathrm{A}_{y_0}) + A_2\,(0|0|\,\nabla_x\Theta(0)\,e_3)[\nabla_x\Theta(0)]^{-1}\Big] \qquad (4.44)\\
&+ x_3^3\,\mathrm{K}\,\varrho_b^e(0|0|\,\nabla_x\Theta(0)\,e_3)[\nabla_x\Theta(0)]^{-1}\Big\},
\end{aligned}
$$

where

$$
A_1 := 2\,\mathrm{H}\,(1 - \varrho_m^e) + \varrho_b^e, \qquad A_2 := \mathrm{K}\,(\varrho_m^e - 1) - 2\,\mathrm{H}\,\varrho_b^e. \tag{4.45}
$$

Using (4.26), we are able to express ϱ_m^e and ϱ_b^e also in terms of $\mathcal{E}_{m,s}, \mathrm{B}_{y_0}, \mathrm{A}_{y_0}, \mathrm{C}_{y_0}$ and $\mathcal{K}_{e,s}$

$$
\begin{aligned}
\varrho_m^e &= 1 - \frac{\lambda}{\lambda + 2\mu}[\langle \mathcal{E}_{m,s} + \mathbb{1}_3 - (0|0|\nabla_x\Theta(0)\,e_3)[\nabla_x\Theta(0)]^{-1}, \mathbb{1}_3\rangle - 2]\\
&= 1 - \frac{\lambda}{\lambda + 2\mu}\,\mathrm{tr}(\mathcal{E}_{m,s}),\\
\varrho_b^e &= -\frac{\lambda}{\lambda + 2\mu}\langle \mathrm{C}_{y_0}\mathcal{K}_{e,s} - \mathrm{B}_{y_0}, \mathbb{1}_3\rangle\\
&\quad - \frac{\lambda}{\lambda + 2\mu}\langle (\mathcal{E}_{m,s} + \mathbb{1}_3 - (0|0|\nabla_x\Theta(0)\,e_3)[\nabla_x\Theta(0)]^{-1})\,\mathrm{B}_{y_0}, \mathbb{1}_3\rangle\\
&= -\frac{\lambda}{\lambda + 2\mu}\langle \mathrm{C}_{y_0}\mathcal{K}_{e,s} + \mathcal{E}_{m,s}\mathrm{B}_{y_0}, \mathbb{1}_3\rangle + \frac{\lambda}{\lambda + 2\mu}\langle (0|0|\nabla_x\Theta(0)\,e_3)[\nabla_x\Theta(0)]^{-1}\,\mathrm{B}_{y_0}, \mathbb{1}_3\rangle\\
&= -\frac{\lambda}{\lambda + 2\mu}\mathrm{tr}[\mathrm{C}_{y_0}\mathcal{K}_{e,s} + \mathcal{E}_{m,s}\mathrm{B}_{y_0}]\,.
\end{aligned} \tag{4.46}
$$

Therefore, in view of (4.46) and item ii) of Lemma 4.3, the coefficients A_1 and A_2 are given by

$$
A_1 = -\frac{\lambda}{\lambda + 2\mu}\,\mathrm{tr}\Big(\mathcal{E}_{m,s}\big(\mathrm{B}_{y_0} - 2\,\mathrm{H}\,\mathrm{A}_{y_0}\big) + \mathrm{C}_{y_0}\mathcal{K}_{e,s}\Big), \tag{4.47}
$$

$$A_2 = \frac{\lambda}{\lambda+2\mu}\Big[2\,\mathrm{H}\,\mathrm{tr}\big(\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\mathcal{K}_{e,s}\big)-\mathrm{K}\,\mathrm{tr}\,\mathcal{E}_{m,s}\Big].$$

We also note that

$$(0|0|\,\nabla_x\Theta(0)\,e_3)[\nabla_x\Theta(0)]^{-1} = (0|0|n_0)(0|0|n_0)^T = n_0\otimes n_0. \tag{4.48}$$

In conclusion, the tensor $\widetilde{\mathcal{E}}_s$ defined by (4.35) is completely expressed in terms of $\mathcal{E}_{m,s}, \mathrm{B}_{y_0}, \mathrm{A}_{y_0}, \mathrm{C}_{y_0}, \mathcal{K}_{e,s}$ and $n_0\otimes n_0$. Similarly, we write Γ_s defined by (4.35) in terms of B_{y_0} and $\mathcal{K}_{e,s}$

$$\begin{aligned}
\Gamma_s &= \frac{1}{b(x_3)}\,(\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_1}\overline{Q}_e)\,|\,\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_2}\overline{Q}_{e,s})\,|0)\\
&\qquad\times\left[\mathbb{1}_3+x_3(\mathrm{L}^\flat_{y_0}-2\,\mathrm{H}\,\mathbb{1}_3)+x_3^2\,\mathrm{K}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right][\nabla_x\Theta(0)]^{-1}\\
&= \frac{1}{b(x_3)}\,\mathcal{K}_{e,s}\,[\nabla_x\Theta(0)]\Big[\mathbb{1}_3+x_3(\mathrm{L}^\flat_{y_0}-2\,\mathrm{H}\,\mathbb{1}_3)\Big][\nabla_x\Theta(0)]^{-1} \qquad (4.49)\\
&\qquad+x_3^2\,\mathrm{K}\,\frac{1}{b(x_3)}\,(\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_1}\overline{Q}_{e,s})\,|\,\mathrm{axl}(\overline{Q}_{e,s}^T\,\partial_{x_2}\overline{Q}_{e,s})\,|0)\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}[\nabla_x\Theta(0)]^{-1}\\
&= \frac{1}{b(x_3)}\Big[\mathcal{K}_{e,s}+x_3\,(\mathcal{K}_{e,s}\mathrm{B}_{y_0}-2\,\mathrm{H}\,\mathcal{K}_{e,s})\Big].
\end{aligned}$$

4.4 Dimensionally Reduced Energy: Analytical Integration Through the Thickness

In what follows, we find the expression of the strain energy density $W = W_{\mathrm{mp}}(\widetilde{\mathcal{E}}_s) + W_{\mathrm{curv}}(\Gamma_s)$ and integrate it over the thickness. To this aim, we introduce the following bilinear forms

$$\begin{aligned}
\mathcal{W}_{\mathrm{mp}}(X,Y) &= \mu\,\big\langle\mathrm{sym}\,X,\mathrm{sym}\,Y\big\rangle+\mu_{\mathrm{c}}\big\langle\mathrm{skew}\,X,\mathrm{skew}\,Y\big\rangle+\frac{\lambda}{2}\,\mathrm{tr}(X)\,\mathrm{tr}(Y)\\
&= \mu\,\big\langle\mathrm{dev\,sym}\,X,\mathrm{dev\,sym}\,Y\big\rangle+\mu_{\mathrm{c}}\big\langle\mathrm{skew}\,X,\mathrm{skew}\,Y\big\rangle+\frac{\kappa}{2}\,\mathrm{tr}(X)\,\mathrm{tr}(Y),\\
\mathcal{W}_{\mathrm{curv}}(X,Y) &= \mu\,L_{\mathrm{c}}^2\Big(b_1\,\big\langle\mathrm{dev\ sym}\,X,\mathrm{dev\ sym}\,Y\big\rangle+b_2\,\big\langle\mathrm{skew}\,X,\mathrm{skew}\,Y\big\rangle\\
&\qquad+\,4\,b_3\,\mathrm{tr}(X)\,\mathrm{tr}\,(Y)\Big)
\end{aligned} \tag{4.50}$$

for any $X, Y\in\mathbb{R}^{3\times3}$. We remark the identities $W_{\mathrm{mp}}(X) = \mathcal{W}_{\mathrm{mp}}(X,X)$, $W_{\mathrm{curv}}(X) = \mathcal{W}_{\mathrm{curv}}(X,X)$. Thus, using (4.44), Lemma 4.3 and the notations (4.47), we obtain

$$\begin{aligned}
W_{\mathrm{mp}}(\widetilde{\mathcal{E}}_s) = \frac{1}{b^2(x_3)}\,W_{\mathrm{mp}}\Big(&\big[\mathcal{E}_{m,s}+(\varrho_m^e-1)n_0\otimes n_0\big]+x_3\big[\big(\mathcal{E}_{m,s}\,\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\mathcal{K}_{e,s}\big)\\
&-2\,\mathrm{H}\,\mathcal{E}_{m,s}+A_1 n_0\otimes n_0\big]\\
&+x_3^2\big[\mathrm{C}_{y_0}\mathcal{K}_{e,s}\mathrm{B}_{y_0}-2\,\mathrm{H}\,\mathrm{C}_{y_0}\mathcal{K}_{e,s}+A_2 n_0\otimes n_0\big]+x_3^3\,\mathrm{K}\,\varrho_b^e\,n_0\otimes n_0\Big).
\end{aligned} \tag{4.51}$$

In order to perform the analytical integration over the thickness, we write $W_{\rm mp}(\widetilde{\mathcal{E}}_s)$ as a polynomial in x_3 with the coefficients C_k, i.e.,

$$W_{\rm mp}(\widetilde{\mathcal{E}}_s) = \frac{1}{b^2(x_3)}\Big(\sum_{k=0}^{6} C_k(x_1,x_2)\,x_3^k\Big), \tag{4.52}$$

where

$$\begin{aligned}
C_0(x_1,x_2) &= W_{\rm mp}\Big(\mathcal{E}_{m,s} + (\varrho_m^e - 1)n_0\otimes n_0\Big),\\
C_1(x_1,x_2) &= 2\,\mathcal{W}_{\rm mp}\Big(\mathcal{E}_{m,s} + (\varrho_m^e - 1)n_0\otimes n_0,\ \big(\mathcal{E}_{m,s}\,\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\mathcal{K}_{e,s}\big)\\
&\qquad - 2\,\mathrm{H}\,\mathcal{E}_{m,s} + A_1 n_0\otimes n_0\Big),\\
C_2(x_1,x_2) &= W_{\rm mp}\Big(\big(\mathcal{E}_{m,s}\,\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\mathcal{K}_{e,s}\big) - 2\,\mathrm{H}\,\mathcal{E}_{m,s} + A_1 n_0\otimes n_0\Big)\\
&\quad + 2\,\mathcal{W}_{\rm mp}\Big(\mathcal{E}_{m,s} + (\varrho_m^e - 1)n_0\otimes n_0,\ \mathrm{C}_{y_0}\mathcal{K}_{e,s}\mathrm{B}_{y_0} - 2\,\mathrm{H}\,\mathrm{C}_{y_0}\mathcal{K}_{e,s} + A_2 n_0\otimes n_0\Big),\\
C_3(x_1,x_2) &= 2\,\mathcal{W}_{\rm mp}\Big(\mathcal{E}_{m,s} + (\varrho_m^e - 1)n_0\otimes n_0,\ \mathrm{K}\,\varrho_b^e\, n_0\otimes n_0\Big) \qquad (4.53)\\
&\quad + 2\,\mathcal{W}_{\rm mp}\Big(\big(\mathcal{E}_{m,s}\,\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\mathcal{K}_{e,s}\big) - 2\,\mathrm{H}\,\mathcal{E}_{m,s} + A_1 n_0\otimes n_0,\ \mathrm{C}_{y_0}\mathcal{K}_{e,s}\mathrm{B}_{y_0}\\
&\qquad - 2\,\mathrm{H}\,\mathrm{C}_{y_0}\mathcal{K}_{e,s} + A_2 n_0\otimes n_0\Big),\\
C_4(x_1,x_2) &= W_{\rm mp}\Big(\mathrm{C}_{y_0}\mathcal{K}_{e,s}\mathrm{B}_{y_0} - 2\,\mathrm{H}\,\mathrm{C}_{y_0}\mathcal{K}_{e,s} + A_2 n_0\otimes n_0\Big)\\
&\quad + 2\,\mathcal{W}_{\rm mp}\Big(\big(\mathcal{E}_{m,s}\,\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\mathcal{K}_{e,s}\big) - 2\,\mathrm{H}\,\mathcal{E}_{m,s} + A_1 n_0\otimes n_0,\ \mathrm{K}\,\varrho_b^e\, n_0\otimes n_0\Big),\\
C_5(x_1,x_2) &= 2\,\mathcal{W}_{\rm mp}\Big(\mathrm{C}_{y_0}\mathcal{K}_{e,s}\mathrm{B}_{y_0} - 2\,\mathrm{H}\,\mathrm{C}_{y_0}\mathcal{K}_{e,s} + A_2 n_0\otimes n_0,\ \mathrm{K}\,\varrho_b^e\, n_0\otimes n_0\Big),\\
C_6(x_1,x_2) &= W_{\rm mp}\Big(\mathrm{K}\,\varrho_b^e\, n_0\otimes n_0\Big).
\end{aligned}$$

Making use of the expansion (since $x_3 \in \big(-\frac{h}{2}, \frac{h}{2}\big)$ and h is small)

$$\begin{aligned}
\frac{1}{b(x_3)} &= \frac{1}{1 - 2\,\mathrm{H}\,x_3 + \mathrm{K}\,x_3^2}\\
&= 1 + 2\,\mathrm{H}\,x_3 + (4\,\mathrm{H}^2 - \mathrm{K})\,x_3^2 + (8\,\mathrm{H}^3 - 4\,\mathrm{H}\,\mathrm{K})\,x_3^3 + (\mathrm{K}^2 - 12\,\mathrm{H}^2\,\mathrm{K} + 16\,\mathrm{H}^4)\,x_3^4\\
&\quad + O(x_3^5),
\end{aligned} \tag{4.54}$$

and of the relations (3.23) and Proposition A.2 i), the integration can be pursued as follows

$$\begin{aligned}
&\int_{\Omega_h} W_{\rm mp}(\widetilde{\mathcal{E}}_s)\,\det\big[\nabla_x\Theta(x)\big]\,\mathrm{d}V\\
&\quad = \int_{\Omega_h}\Big(\sum_{k=0}^{6} C_k(x_1,x_2)\,x_3^k\Big)\Big[1 + 2\,\mathrm{H}\,x_3 + (4\,\mathrm{H}^2 - \mathrm{K})\,x_3^2 + (8\,\mathrm{H}^3 - 4\,\mathrm{H}\,\mathrm{K})x_3^3
\end{aligned}$$

$$
\begin{aligned}
&+ (\mathrm{K}^2 - 12\,\mathrm{H}^2\,\mathrm{K} + 16\,\mathrm{H}^4)\, x_3^4 + O(x_3^5)\Big]\det(\nabla y_0|n_0)\,\mathrm{d}V\\
&= \int_\omega \Big\{ h\, C_0 + \frac{h^3}{12}\Big[(4\,\mathrm{H}^2 - \mathrm{K})C_0 + 2\,\mathrm{H}\, C_1 + C_2\Big]\\
&\quad + \frac{h^5}{80}\Big[(\mathrm{K}^2 - 12\,\mathrm{H}^2\,\mathrm{K} + 16\,\mathrm{H}^4)\, C_0 + (8\,\mathrm{H}^3 - 4\,\mathrm{H}\,\mathrm{K})\, C_1\\
&\quad + (4\,\mathrm{H}^2 - \mathrm{K})\, C_2 + 2\,\mathrm{H}\, C_3 + C_4\Big]\Big\}\det(\nabla y_0|n_0)\,\mathrm{d}a + O(h^7),
\end{aligned}
\tag{4.55}
$$

where $\mathrm{d}a = \mathrm{d}x_1 \mathrm{d}x_2$.

In view of (4.55), we need to find appropriate expressions for the coefficients C_0, C_1, C_2, C_3, C_4 defined by (4.53). In this line, we designate by $\mathcal{W}_{\rm shell}(X, Y)$ the bilinear form

$$
\begin{aligned}
\mathcal{W}_{\rm shell}(X, Y) &= \mu\,\langle \mathrm{sym}\, X, \mathrm{sym}\, Y\rangle + \mu_c\langle \mathrm{skew}\, X, \mathrm{skew}\, Y\rangle + \frac{\lambda\,\mu}{\lambda + 2\mu}\,\mathrm{tr}(X)\,\mathrm{tr}(Y)\\
&= \mu\,\langle \mathrm{dev\,sym}\, X, \mathrm{dev\,sym}\, Y\rangle + \mu_c\langle \mathrm{skew}\, X, \mathrm{skew}\, Y\rangle\\
&\quad + \frac{2\,\mu\,(2\,\lambda + \mu)}{3(\lambda + 2\,\mu)}\,\mathrm{tr}(X)\,\mathrm{tr}(Y),\\
W_{\rm shell}(X) &= \mathcal{W}_{\rm shell}(X, X)
\end{aligned}
\tag{4.56}
$$

and we observe that

$$
\mathcal{W}_{\rm shell}(X, Y) + \frac{\lambda^2}{2\,(\lambda + 2\mu)}\,\mathrm{tr}(X)\,\mathrm{tr}(Y) = \mathcal{W}_{\rm mp}(X, Y), \tag{4.57}
$$

since $\frac{\kappa}{2} - \frac{\lambda^2}{2\,(\lambda+2\mu)} = \frac{2\,\mu\,(2\lambda+\mu)}{3\,(\lambda+2\mu)}$. Using the notations (4.50), (4.56), we obtain

Lemma 4.4 *The following identities*

$$
\begin{aligned}
&\mathcal{W}_{\rm mp}\big(X + \alpha\, n_0\otimes n_0,\ Y + \beta\, n_0 \otimes n_0\big)\\
&\quad = \mathcal{W}_{\rm mp}(X, Y) + \frac{\lambda}{2}\,\big(\alpha\,\mathrm{tr}(Y) + \beta\,\mathrm{tr}(X)\big) + \frac{\lambda + 2\mu}{2}\,\alpha\,\beta,\\
&\mathcal{W}_{\rm mp}\Big(X - \frac{\lambda}{\lambda + 2\mu}\,\big(\mathrm{tr}(X)\big) n_0\otimes n_0,\ Y + \beta\, n_0\otimes n_0\Big) = \mathcal{W}_{\rm shell}(X, Y),
\end{aligned}
\tag{4.58}
$$

hold true for all tensors $X,\ Y \in \mathbb{R}^{3\times 3}$ *of the form* $(*|*|0)\cdot[\nabla_x\Theta(0)]^{-1}$ *and all* $\alpha, \beta \in \mathbb{R}$.

Proof In view of the definition (4.50) we see that

$$
\begin{aligned}
&\mathcal{W}_{\rm mp}\big(X + \alpha\, n_0\otimes n_0,\ Y + \beta\, n_0\otimes n_0\big)\\
&\quad = \mu\,\langle \mathrm{sym}\, X + \alpha\, n_0\otimes n_0, \mathrm{sym}\, Y + \beta\, n_0 \otimes n_0\rangle\\
&\qquad + \mu_c\langle \mathrm{skew}\, X, \mathrm{skew}\, Y\rangle + \frac{\lambda}{2}\,\big(\mathrm{tr}(X) + \alpha\big)\,\big(\mathrm{tr}(Y) + \beta\big).
\end{aligned}
\tag{4.59}
$$

According to (4.48), $n_0 \otimes n_0 = (0|0|n_0)\cdot[\nabla_x\Theta(0)]^{-1}$. Since for $X = (*|*|0)\cdot[\nabla_x\Theta(0)]^{-1}$ we have

$$\begin{aligned}\big\langle \operatorname{sym} X, n_0\otimes n_0\big\rangle &= \big\langle (*|*|0)\,[\nabla_x\Theta(0)]^{-1}, (0|0|n_0)\,[\nabla_x\Theta(0)]^{-1}\big\rangle \\ &= \big\langle (0|0|n_0)^T\,(*|*|0), [[\nabla_x\Theta(0)]^T\,\nabla_x\Theta(0)]^{-1}\big\rangle = \Big\langle \begin{pmatrix}0&0&0\\0&0&0\\ *&*&0\end{pmatrix}, \widehat{\mathrm{I}}_{y_0}^{-1}\Big\rangle = 0,\end{aligned} \tag{4.60}$$

we get

$$\begin{aligned}&\mathcal{W}_{\mathrm{mp}}\big(X+\alpha\, n_0\otimes n_0,\ Y+\beta\, n_0\otimes n_0\big)\\ &\quad= \mu\,\big\langle \operatorname{sym} X, \operatorname{sym} Y\big\rangle + \mu\,\alpha\,\beta\,\big\langle n_0\otimes n_0, n_0\otimes n_0\big\rangle + \mu_{\mathrm{c}}\big\langle \operatorname{skew} X, \operatorname{skew} Y\big\rangle\\ &\qquad+\frac{\lambda}{2}\operatorname{tr}(X)\operatorname{tr}(Y)+\frac{\lambda}{2}\big(\alpha\operatorname{tr}(Y)+\beta\operatorname{tr}(X)\big)+\frac{\lambda}{2}\,\alpha\,\beta\\ &\quad= \mathcal{W}_{\mathrm{mp}}(X,Y)+\frac{\lambda}{2}\big(\alpha\operatorname{tr}(Y)+\beta\operatorname{tr}(X)\big)+\frac{\lambda+2\mu}{2}\,\alpha\,\beta.\end{aligned} \tag{4.61}$$

This means that the relation $(4.58)_1$ holds true, for any $\alpha,\beta\in\mathbb{R}$.

If we write $(4.58)_1$ with $\alpha = \frac{-\lambda}{\lambda+2\mu}\big(\operatorname{tr}(X)\big)$, then we obtain

$$\begin{aligned}&\mathcal{W}_{\mathrm{mp}}\Big(X-\tfrac{\lambda}{\lambda+2\mu}\big(\operatorname{tr}(X)\big)n_0\otimes n_0,\ Y+\beta\, n_0\otimes n_0\Big) =\\ &\quad= \mathcal{W}_{\mathrm{mp}}(X,Y)+\tfrac{\lambda}{2}\Big(\tfrac{-\lambda}{\lambda+2\mu}\operatorname{tr}(X)\operatorname{tr}(Y))+\beta\operatorname{tr}(X)\Big)+\tfrac{\lambda+2\mu}{2}\cdot\tfrac{-\lambda}{\lambda+2\mu}\operatorname{tr}(X)\,\beta\\ &\quad= \mathcal{W}_{\mathrm{mp}}(X,Y)-\frac{\lambda^2}{2\,(\lambda+2\,\mu)}\operatorname{tr}(X)\operatorname{tr}(Y) = \mathcal{W}_{\mathrm{shell}}(X,Y),\end{aligned}$$

where we have used the formula (4.57). Thus, the relation $(4.58)_2$ is also proved. □

By virtue of $(4.58)_2$, $(4.46)_1$, (4.47), (4.36) and Lemma 4.3 we get

$$\begin{aligned}&\mathcal{W}_{\mathrm{mp}}\Big(\mathcal{E}_{m,s}+\big(\varrho_m^e-1\big)n_0\otimes n_0,\ Y+\beta\, n_0\otimes n_0\Big) = \mathcal{W}_{\mathrm{shell}}(\mathcal{E}_{m,s},Y),\\ &\mathcal{W}_{\mathrm{mp}}\Big(\big(\mathcal{E}_{m,s}\,\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\mathcal{K}_{e,s}\big)-2\,\mathrm{H}\,\mathcal{E}_{m,s}+A_1 n_0\otimes n_0, Y+\beta\, n_0\otimes n_0\Big)\\ &\quad= \mathcal{W}_{\mathrm{shell}}\big(\mathcal{E}_{m,s}\,\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\mathcal{K}_{e,s}-2\,\mathrm{H}\,\mathcal{E}_{m,s}, Y\big),\\ &W_{\mathrm{mp}}\Big(\mathrm{C}_{y_0}\mathcal{K}_{e,s}\mathrm{B}_{y_0}-2\,\mathrm{H}\,\mathrm{C}_{y_0}\mathcal{K}_{e,s}+A_2 n_0\otimes n_0\Big)\\ &\quad= W_{\mathrm{shell}}\big(\mathrm{C}_{y_0}\mathcal{K}_{e,s}\mathrm{B}_{y_0}-2\,\mathrm{H}\,\mathrm{C}_{y_0}\mathcal{K}_{e,s}\big)+\frac{\lambda^2}{2(\lambda+2\mu)}\Big[\operatorname{tr}\big((\mathcal{E}_{m,s}\,\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\mathcal{K}_{e,s})\mathrm{B}_{y_0}\big)\Big]^2,\end{aligned} \tag{4.62}$$

where for the last identity we used $(4.58)_1$ and the relation

$$\mathcal{E}_{m,s}\,\mathrm{B}_{y_0}^2 = 2\,\mathrm{H}\,\mathcal{E}_{m,s}\,\mathrm{B}_{y_0}-\mathrm{K}\,\mathcal{E}_{m,s}\,\mathrm{A}_{y_0} = 2\,\mathrm{H}\,\mathcal{E}_{m,s}\,\mathrm{B}_{y_0}-\mathrm{K}\,\mathcal{E}_{m,s}.$$

From (4.53) and (4.62) we get

$$\begin{aligned}C_0 &= W_{\mathrm{shell}}\big(\mathcal{E}_{m,s}\big),\\ C_1 &= 2\,\mathcal{W}_{\mathrm{shell}}\big(\mathcal{E}_{m,s},\ \mathcal{E}_{m,s}\,\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\mathcal{K}_{e,s}-2\,\mathrm{H}\,\mathcal{E}_{m,s}\big)\\ &= -4\,\mathrm{H}\,W_{\mathrm{shell}}\big(\mathcal{E}_{m,s}\big)+2\,\mathcal{W}_{\mathrm{shell}}\big(\mathcal{E}_{m,s},\ \mathcal{E}_{m,s}\,\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\mathcal{K}_{e,s}\big),\\ C_2 &= W_{\mathrm{shell}}\big(\mathcal{E}_{m,s}\,\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\mathcal{K}_{e,s}-2\,\mathrm{H}\,\mathcal{E}_{m,s}\big)+2\,\mathcal{W}_{\mathrm{shell}}\big(\mathcal{E}_{m,s},\ \mathrm{C}_{y_0}\mathcal{K}_{e,s}\mathrm{B}_{y_0}-2\,\mathrm{H}\,\mathrm{C}_{y_0}\mathcal{K}_{e,s}\big),\end{aligned} \tag{4.63}$$

$$C_3 = 2\mathcal{W}_{\text{shell}}\big(\mathcal{E}_{m,s}\,\text{B}_{y_0} + \text{C}_{y_0}\mathcal{K}_{e,s} - 2\,\text{H}\,\mathcal{E}_{m,s},\ \text{C}_{y_0}\mathcal{K}_{e,s}\text{B}_{y_0} - 2\,\text{H}\,\text{C}_{y_0}\mathcal{K}_{e,s}\big),$$

$$C_4 = W_{\text{shell}}\big(\text{C}_{y_0}\mathcal{K}_{e,s}\text{B}_{y_0} - 2\,\text{H}\,\text{C}_{y_0}\mathcal{K}_{e,s}\big) + \frac{\lambda^2}{2(\lambda+2\mu)}\Big[\text{tr}\big((\mathcal{E}_{m,s}\,\text{B}_{y_0} + \text{C}_{y_0}\mathcal{K}_{e,s})\text{B}_{y_0}\big)\Big]^2.$$

With the relations (4.63) we can replace the coefficients C_0, C_1, C_2, C_3, C_4 appearing in (4.55) and we obtain

$$\begin{aligned}
(4\,\text{H}^2 - \text{K})C_0 + 2\,\text{H}\,C_1 + C_2 &= -\text{K}\,W_{\text{shell}}\big(\mathcal{E}_{m,s}\big) + W_{\text{shell}}\big(\mathcal{E}_{m,s}\,\text{B}_{y_0} + \text{C}_{y_0}\mathcal{K}_{e,s}\big)\\
&\quad + 2\,\mathcal{W}_{\text{shell}}\big(\mathcal{E}_{m,s},\ \text{C}_{y_0}\mathcal{K}_{e,s}\text{B}_{y_0} - 2\,\text{H}\,\text{C}_{y_0}\mathcal{K}_{e,s}\big),\\
(\text{K}^2 - 12\,\text{H}^2\,\text{K} + 16\,\text{H}^4)C_0 &+ (8\,\text{H}^3 - 4\,\text{H}\,\text{K})\,C_1 + (4\,\text{H}^2 - \text{K})\,C_2 + 2\,\text{H}\,C_3 + C_4\\
&= -\text{K}\,W_{\text{shell}}\big(\mathcal{E}_{m,s}\,\text{B}_{y_0} + \text{C}_{y_0}\mathcal{K}_{e,s}\big) + W_{\text{shell}}\big((\mathcal{E}_{m,s}\,\text{B}_{y_0} + \text{C}_{y_0}\mathcal{K}_{e,s})\text{B}_{y_0}\big)\\
&\quad + \frac{\lambda^2}{2(\lambda+2\mu)}\Big[\text{tr}\big((\mathcal{E}_{m,s}\,\text{B}_{y_0} + \text{C}_{y_0}\mathcal{K}_{e,s})\text{B}_{y_0}\big)\Big]^2\\
&= -\text{K}\,W_{\text{shell}}\big(\mathcal{E}_{m,s}\,\text{B}_{y_0} + \text{C}_{y_0}\mathcal{K}_{e,s}\big) + W_{\text{mp}}\big((\mathcal{E}_{m,s}\,\text{B}_{y_0} + \text{C}_{y_0}\mathcal{K}_{e,s})\text{B}_{y_0}\big).
\end{aligned} \tag{4.64}$$

Inserting (4.64) into (4.55) (and neglecting the terms of order $O(h^7)$) we obtain the following result of the integration

$$\begin{aligned}
\int_{\Omega_h} W_{\text{mp}}(\widetilde{\mathcal{E}}_s)\,\text{det}\big[\nabla_x\Theta(x)\big]\,\text{d}V &= \int_\omega \Big[\Big(h - \text{K}\,\frac{h^3}{12}\Big)\,W_{\text{shell}}\big(\mathcal{E}_{m,s}\big)\\
&\quad + \Big(\frac{h^3}{12} - \text{K}\,\frac{h^5}{80}\Big)\,W_{\text{shell}}\big(\mathcal{E}_{m,s}\,\text{B}_{y_0} + \text{C}_{y_0}\mathcal{K}_{e,s}\big)\\
&\quad + \frac{h^3}{12}\,2\,\mathcal{W}_{\text{shell}}\big(\mathcal{E}_{m,s},\ \text{C}_{y_0}\mathcal{K}_{e,s}\text{B}_{y_0} - 2\,\text{H}\,\text{C}_{y_0}\mathcal{K}_{e,s}\big)\\
&\quad + \frac{h^5}{80}\,W_{\text{mp}}\big((\mathcal{E}_{m,s}\,\text{B}_{y_0} + \text{C}_{y_0}\mathcal{K}_{e,s})\text{B}_{y_0}\big)\Big]\,\text{det}(\nabla y_0|n_0)\,\text{d}a.
\end{aligned} \tag{4.65}$$

Remark that using Lemma 4.3 and Proposition A.3 we deduce

$$\begin{aligned}
\text{C}_{y_0}\mathcal{K}_{e,s}\text{B}_{y_0} - 2\,\text{H}\,\text{C}_{y_0}\,\mathcal{K}_{e,s} &= \text{C}_{y_0}\,\mathcal{K}_{e,s}(\text{B}_{y_0} - 2\,\text{H}\,\text{A}_{y_0})\\
&= (\mathcal{E}_{m,s}\text{B}_{y_0} + \text{C}_{y_0}\,\mathcal{K}_{e,s})(\text{B}_{y_0} - 2\,\text{H}\,\text{A}_{y_0}) - \mathcal{E}_{m,s}\text{B}_{y_0}(\text{B}_{y_0} - 2\,\text{H}\,\text{A}_{y_0})\\
&= (\mathcal{E}_{m,s}\text{B}_{y_0} + \text{C}_{y_0}\,\mathcal{K}_{e,s})(\text{B}_{y_0} - 2\,\text{H}\,\text{A}_{y_0}) + \text{K}\,\mathcal{E}_{m,s}.
\end{aligned} \tag{4.66}$$

Therefore, using again Lemma 4.3 and Proposition A.3, the energy density is rewritten in the following form

$$\begin{aligned}
&\Big(h - \text{K}\,\frac{h^3}{12}\Big)\,W_{\text{shell}}\big(\mathcal{E}_{m,s}\big) + \Big(\frac{h^3}{12} - \text{K}\,\frac{h^5}{80}\Big)\,W_{\text{shell}}\big(\mathcal{E}_{m,s}\,\text{B}_{y_0} + \text{C}_{y_0}\mathcal{K}_{e,s}\big)\\
&\qquad + \frac{h^3}{12}\,2\,\mathcal{W}_{\text{shell}}\big(\mathcal{E}_{m,s},\ \text{C}_{y_0}\mathcal{K}_{e,s}\text{B}_{y_0} - 2\,\text{H}\,\text{C}_{y_0}\mathcal{K}_{e,s}\big) + \frac{h^5}{80}\,W_{\text{mp}}\big((\mathcal{E}_{m,s}\,\text{B}_{y_0} + \text{C}_{y_0}\mathcal{K}_{e,s})\text{B}_{y_0}\big)\\
&= \Big(h + \text{K}\,\frac{h^3}{12}\Big)\,W_{\text{shell}}\big(\mathcal{E}_{m,s}\big) + \Big(\frac{h^3}{12} - \text{K}\,\frac{h^5}{80}\Big)\,W_{\text{shell}}\big(\mathcal{E}_{m,s}\,\text{B}_{y_0} + \text{C}_{y_0}\mathcal{K}_{e,s}\big)\\
&\quad + \frac{h^3}{12}\,2\,\mathcal{W}_{\text{shell}}\big(\mathcal{E}_{m,s},\ (\mathcal{E}_{m,s}\text{B}_{y_0} + \text{C}_{y_0}\,\mathcal{K}_{e,s})(\text{B}_{y_0} - 2\,\text{H}\,\text{A}_{y_0})\big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{h^5}{80}\,W_{\rm mp}\big((\mathcal{E}_{m,s}\,{\rm B}_{y_0}+{\rm C}_{y_0}\mathcal{K}_{e,s}){\rm B}_{y_0}\big)\\
&=\Big(h+{\rm K}\,\frac{h^3}{12}\Big)\,W_{\rm shell}\big(\mathcal{E}_{m,s}\big)+\Big(\frac{h^3}{12}-{\rm K}\,\frac{h^5}{80}\Big)\,W_{\rm shell}\big(\mathcal{E}_{m,s}\,{\rm B}_{y_0}+{\rm C}_{y_0}\mathcal{K}_{e,s}\big) \qquad (4.67)\\
&-\frac{h^3}{3}{\rm H}\,\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s},\,\mathcal{E}_{m,s}{\rm B}_{y_0}+{\rm C}_{y_0}\,\mathcal{K}_{e,s}\big)+\frac{h^3}{6}\,\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s},\,(\mathcal{E}_{m,s}{\rm B}_{y_0}+{\rm C}_{y_0}\,\mathcal{K}_{e,s}){\rm B}_{y_0}\big)\\
&+\frac{h^5}{80}\,W_{\rm mp}\big((\mathcal{E}_{m,s}\,{\rm B}_{y_0}+{\rm C}_{y_0}\mathcal{K}_{e,s}){\rm B}_{y_0}\big).
\end{aligned}
$$

Analogously, using (4.49), we integrate the curvature part of the strain energy density

$$
\begin{aligned}
&\int_{\Omega_h} W_{\rm curv}(\Gamma_s)\,\det\big[\nabla_x\Theta(x)\big]\,{\rm d}V\\
&=\int_{\Omega_h} W_{\rm curv}\big(\mathcal{K}_{e,s}+x_3\,(\mathcal{K}_{e,s}{\rm B}_{y_0}-2\,{\rm H}\,\mathcal{K}_{e,s})\big)\,\frac{\det(\nabla y_0|n_0)}{b(x_3)}\,{\rm d}V \qquad (4.68)\\
&=\int_{\Omega_h}\Big(D_0+D_1\,x_3+D_2\,x_3^2\Big)\Big[1+2\,{\rm H}\,x_3+(4\,{\rm H}^2-{\rm K})\,x_3^2+(8\,{\rm H}^3-4\,{\rm H}\,{\rm K})x_3^3\\
&\qquad\qquad +({\rm K}^2-12\,{\rm H}^2\,{\rm K}+16\,{\rm H}^4)\,x_3^4+O(x_3^5)\Big]\det(\nabla y_0|n_0)\,{\rm d}V\\
&=\int_{\omega}\Big\{h\,D_0+\frac{h^3}{12}\Big[(4\,{\rm H}^2-{\rm K})D_0+2\,{\rm H}\,D_1+D_2\Big]+\frac{h^5}{80}\Big[({\rm K}^2-12\,{\rm H}^2\,{\rm K}+16\,{\rm H}^4)\,D_0\\
&\qquad\qquad +(8\,{\rm H}^3-4\,{\rm H}\,{\rm K})\,D_1+(4\,{\rm H}^2-{\rm K})\,D_2\Big]\Big\}\det(\nabla y_0|n_0)\,{\rm d}a+O(h^7),
\end{aligned}
$$

where we have denoted by D_k the coefficients of x_3^k $(k = 0, 1, 2)$ in the expression

$$
\begin{aligned}
&W_{\rm curv}\big(\mathcal{K}_{e,s}+x_3\,(\mathcal{K}_{e,s}{\rm B}_{y_0}-2\,{\rm H}\,\mathcal{K}_{e,s})\big)=D_0(x_1,x_2)+D_1(x_1,x_2)\,x_3+D_2(x_1,x_2)\,x_3^2,\quad\text{with}\\
&D_0=W_{\rm curv}\big(\mathcal{K}_{e,s}\big),\qquad D_1=2\,\mathcal{W}_{\rm curv}\big(\mathcal{K}_{e,s},\,\mathcal{K}_{e,s}{\rm B}_{y_0}-2\,{\rm H}\,\mathcal{K}_{e,s}\big), \qquad (4.69)\\
&D_2=W_{\rm curv}\big(\mathcal{K}_{e,s}{\rm B}_{y_0}-2\,{\rm H}\,\mathcal{K}_{e,s}\big).
\end{aligned}
$$

We write the coefficients of $\frac{h^3}{12}$ and $\frac{h^5}{80}$ in (4.68) with the help of $(4.69)_{2,3,4}$ as follows

$$
\begin{aligned}
&(4\,{\rm H}^2-{\rm K})D_0+2\,{\rm H}\,D_1+D_2=-{\rm K}\,W_{\rm curv}\big(\mathcal{K}_{e,s}\big)+W_{\rm curv}\big(\mathcal{K}_{e,s}{\rm B}_{y_0}\big),\\
&({\rm K}^2-12\,{\rm H}^2\,{\rm K}+16\,{\rm H}^4)\,D_0+(8\,{\rm H}^3-4\,{\rm H}\,{\rm K})\,D_1+(4\,{\rm H}^2-{\rm K})\,D_2= \qquad (4.70)\\
&\qquad={\rm K}^2\,W_{\rm curv}\big(\mathcal{K}_{e,s}\big)-4\,{\rm H}\,{\rm K}\,\mathcal{W}_{\rm curv}\big(\mathcal{K}_{e,s},\,\mathcal{K}_{e,s}{\rm B}_{y_0}\big)+(4\,{\rm H}^2-{\rm K})\,W_{\rm curv}\big(\mathcal{K}_{e,s}{\rm B}_{y_0}\big)\\
&\qquad=-{\rm K}\,W_{\rm curv}\big(\mathcal{K}_{e,s}{\rm B}_{y_0}\big)+W_{\rm curv}\big(\mathcal{K}_{e,s}{\rm B}_{y_0}^2\big).
\end{aligned}
$$

Inserting (4.70) into (4.68) (and again neglecting the terms of order $O(h^7)$) we arrive at the following result of this integration

$$\int_{\Omega_h} W_{\rm curv}(\Gamma_s)\det\big[\nabla_x\Theta(x)\big]\,{\rm d}V = \tag{4.71}$$
$$= \int_\omega \Big[\Big(h-{\rm K}\,\frac{h^3}{12}\Big)\,W_{\rm curv}\big(\mathcal{K}_{e,s}\big)+\Big(\frac{h^3}{12}-{\rm K}\,\frac{h^5}{80}\Big)\,W_{\rm curv}\big(\mathcal{K}_{e,s}{\rm B}_{y_0}\big)$$
$$+\,\frac{h^5}{80}\,W_{\rm curv}\big(\mathcal{K}_{e,s}{\rm B}^2_{y_0}\big)\Big]\det(\nabla y_0|n_0)\,{\rm d}a.$$

In order to write the external loads potential in the shell model, we perform next the integration over the thickness of the relations (3.25). Thus, from (3.26) and (4.6) we find

$$\tilde{v}(x_i) = \varphi(x_i)-\Theta(x_i) = \Big(m+x_3\varrho_m\overline{Q}_{e,s}\,n_0+\frac{x_3^2}{2}\varrho_b\overline{Q}_{e,s}\,n_0\Big)-(y_0+x_3n_0)$$
$$= (m-y_0)+x_3(\varrho_m\overline{Q}_{e,s}\,n_0-n_0)+\frac{x_3^2}{2}\varrho_b\overline{Q}_{e,s}\,n_0\,.$$

We insert this into $(3.25)_1$ and use the approximation $\varrho_m\cong\varrho_m^0=1$, $\varrho_b\cong\varrho_b^0=0$ as in (4.26) to obtain the simplified form

$$\int_{\Omega_h}\langle\tilde{f},\tilde{v}\rangle dV = \int_\omega\Big(\Big\langle\int_{-h/2}^{h/2}\tilde{f}\,dx_3,m-y_0\Big\rangle+\Big\langle\int_{-h/2}^{h/2}x_3\tilde{f}\,dx_3,(\overline{Q}_{e,s}-\mathbb{1}_3)\,n_0\Big\rangle\Big)\,da\,. \tag{4.72}$$

Denoting with $\tilde{t}^{\pm}(x_1,x_2):=\tilde{t}(x_1,x_2,\pm\frac{h}{2})$ and taking into account that $\Gamma_t=\Big(\omega\times\Big\{\frac{h}{2}\Big\}\Big)\cup\Big(\omega\times\Big\{-\frac{h}{2}\Big\}\Big)\cup\Big(\gamma_t\times(-\frac{h}{2},\frac{h}{2})\Big)$, we obtain similarly

$$\int_{\Gamma_t}\langle\tilde{t},\tilde{v}\rangle dS = \int_\omega\Big\langle\tilde{t}^{\pm},(m-y_0)\pm\frac{h}{2}(\varrho_m\overline{Q}_{e,s}\,n_0-n_0)+\frac{h^2}{8}\varrho_b\overline{Q}_{e,s}\,n_0\Big\rangle da$$
$$+\int_{\gamma_t}\int_{-h/2}^{h/2}\Big\langle\tilde{t},(m-y_0)+x_3(\varrho_m\overline{Q}_{e,s}\,n_0-n_0)+\frac{x_3^2}{2}\varrho_b\overline{Q}_{e,s}\,n_0\Big\rangle dx_3\,ds\,.$$

Using the same approximation as before ($\varrho_m\cong\varrho_m^0=1$, $\varrho_b\cong\varrho_b^0=0$, see (4.26)) we find

$$\int_{\Gamma_t}\langle\tilde{t},\tilde{v}\rangle dS = \int_\omega\langle\tilde{t}^++\tilde{t}^-,m-y_0\rangle da+\int_\omega\Big\langle\frac{h}{2}(\tilde{t}^+-\tilde{t}^-),(\overline{Q}_{e,s}-\mathbb{1}_3)\,n_0\Big\rangle da \tag{4.73}$$
$$+\int_{\gamma_t}\Big\langle\int_{-h/2}^{h/2}\tilde{t}\,dx_3,m-y_0\Big\rangle ds+\int_{\gamma_t}\Big\langle\int_{-h/2}^{h/2}x_3\tilde{t}\,dx_3,(\overline{Q}_{e,s}-\mathbb{1}_3)\,n_0\Big\rangle ds\,,$$

where ds is the arclength element along the curve γ_t and $da=dx_1dx_2$. With (4.72) and (4.73), the potential of external applied loads $\overline{\Pi}(m,\overline{Q}_{e,s})=\widetilde{\Pi}(\varphi,\overline{R})$ in (3.25) can be written in the form

$$\overline{\Pi}(m,\overline{Q}_{e,s}) = \Pi_\omega(m,\overline{Q}_{e,s})+\Pi_{\gamma_t}(m,\overline{Q}_{e,s})\,, \tag{4.74}$$

with

$$\Pi_\omega(m,\overline{Q}_{e,s}) = \int_\omega\langle\bar{f},\bar{u}\rangle da+\Lambda_\omega(\overline{Q}_{e,s}), \tag{4.75}$$

$$\Pi_{\gamma_t}(m,\overline{Q}_{e,s}) = \int_{\gamma_t} \langle \bar{t}, \bar{u}\rangle ds + \Lambda_{\gamma_t}(\overline{Q}_{e,s}),$$

where $\bar{u}(x_1,x_2) = m(x_1,x_2) - y_0(x_1,x_2)$ is the displacement vector of the midsurface and

$$\bar{f} = \int_{-h/2}^{h/2} \tilde{f}\, dx_3 + (\tilde{t}^+ + \tilde{t}^-), \qquad \bar{t} = \int_{-h/2}^{h/2} \tilde{t}\, dx_3\,, \tag{4.76}$$

$$\Lambda_\omega(\overline{Q}_{e,s}) = \int_\omega \Big\langle \int_{-h/2}^{h/2} x_3\, \tilde{f}\, dx_3 + \frac{h}{2}(\tilde{t}^+ - \tilde{t}^-), (\overline{Q}_{e,s} - \mathbb{1}_3)\, n_0 \Big\rangle da + \overline{\Pi}_\omega(\overline{Q}_{e,s}),$$

$$\Lambda_{\gamma_t}(\overline{Q}_{e,s}) = \int_{\gamma_t} \Big\langle \int_{-h/2}^{h/2} x_3\, \tilde{t}\, dx_3, (\overline{Q}_{e,s} - \mathbb{1}_3)\, n_0 \Big\rangle ds + \overline{\Pi}_{\gamma_t}(\overline{Q}_{e,s})$$

and $\overline{\Pi}_\omega(\overline{Q}_{e,s}) = \widetilde{\Pi}_{\Omega_h}(\overline{R})$, $\overline{\Pi}_{\gamma_t}(\overline{Q}_{e,s}) = \widetilde{\Pi}_{\Gamma_t}(\overline{R})$, since $\overline{R}$ is independent of x_3.

The functions $\overline{\Pi}_\omega$, $\overline{\Pi}_{\gamma_t} : L^2(\omega, \mathrm{SO}(3)) \to \mathbb{R}$ are assumed to be continuous and bounded operators.

5 The New Geometrically Nonlinear Cosserat Shell Model

5.1 Formulation of the Minimization Problem

Gathering our results, see (3.23), (4.65), (4.67) and (4.71), we have obtained the following two-dimensional minimization problem for the deformation of the midsurface $m : \omega \to \mathbb{R}^3$ and the microrotation of the shell $\overline{Q}_{e,s} : \omega \to \mathrm{SO}(3)$ solving on $\omega \subset \mathbb{R}^2$: minimize with respect to $(m, \overline{Q}_{e,s})$ the functional

$$I = \int_\omega \Big[W_{\rm memb}\big(\mathcal{E}_{m,s}\big) + W_{\rm memb,bend}\big(\mathcal{E}_{m,s},\, \mathcal{K}_{e,s}\big) + W_{\rm bend,curv}\big(\mathcal{K}_{e,s}\big)\Big] \det(\nabla y_0|n_0)\, {\rm d}a \\ - \overline{\Pi}(m, \overline{Q}_{e,s})\,, \tag{5.1}$$

where the membrane part $W_{\rm memb}\big(\mathcal{E}_{m,s}\big)$, the membrane–bending part $W_{\rm memb,bend}\big(\mathcal{E}_{m,s},\, \mathcal{K}_{e,s}\big)$ and the bending–curvature part $W_{\rm bend,curv}\big(\mathcal{K}_{e,s}\big)$ of the shell energy density are given by

$$\begin{aligned}
W_{\rm memb}\big(\mathcal{E}_{m,s}\big) &= \Big(h + {\rm K}\,\frac{h^3}{12}\Big)\, W_{\rm shell}\big(\mathcal{E}_{m,s}\big),\\
W_{\rm memb,bend}\big(\mathcal{E}_{m,s},\, \mathcal{K}_{e,s}\big) &= \Big(\frac{h^3}{12} - {\rm K}\,\frac{h^5}{80}\Big)\, W_{\rm shell}\big(\mathcal{E}_{m,s}\,{\rm B}_{y_0} + {\rm C}_{y_0}\mathcal{K}_{e,s}\big)\\
&\quad - \frac{h^3}{3}{\rm H}\,\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s},\, \mathcal{E}_{m,s}{\rm B}_{y_0} + {\rm C}_{y_0}\,\mathcal{K}_{e,s}\big)\\
&\quad + \frac{h^3}{6}\,\mathcal{W}_{\rm shell}\big(\mathcal{E}_{m,s},\, (\mathcal{E}_{m,s}{\rm B}_{y_0} + {\rm C}_{y_0}\,\mathcal{K}_{e,s}){\rm B}_{y_0}\big)\\
&\quad + \frac{h^5}{80}\, W_{\rm mp}\big((\mathcal{E}_{m,s}\,{\rm B}_{y_0} + {\rm C}_{y_0}\mathcal{K}_{e,s}){\rm B}_{y_0}\big),\\
W_{\rm bend,curv}\big(\mathcal{K}_{e,s}\big) &= \Big(h - {\rm K}\,\frac{h^3}{12}\Big)\, W_{\rm curv}\big(\mathcal{K}_{e,s}\big) + \Big(\frac{h^3}{12} - {\rm K}\,\frac{h^5}{80}\Big)\, W_{\rm curv}\big(\mathcal{K}_{e,s}{\rm B}_{y_0}\big)\\
&\quad + \frac{h^5}{80}\, W_{\rm curv}\big(\mathcal{K}_{e,s}{\rm B}^2_{y_0}\big)
\end{aligned} \tag{5.2}$$

and

$$
\begin{aligned}
\mathcal{E}_{m,s} &= \overline{Q}_{e,s}^T \widetilde{F}_m - \mathbb{1}_3, \qquad \widetilde{F}_m = (\nabla m | \overline{Q}_{e,s} \nabla_x \Theta(0)\, e_3)[\nabla_x \Theta(0)]^{-1},\\
\mathcal{K}_{e,s} &= (\operatorname{axl}(\overline{Q}_{e,s}^T\, \partial_{x_1} \overline{Q}_{e,s})\,|\, \operatorname{axl}(\overline{Q}_{e,s}^T\, \partial_{x_2} \overline{Q}_{e,s})\,|0)[\nabla_x \Theta(0)]^{-1},\\
\Theta(x_1,x_2,x_3) &= y_0(x_1,x_2) + x_3\, n_0(x_1,x_2),\\
\nabla_x \Theta(0) &= (\nabla y_0 | n_0), \qquad n_0 = \big(\nabla_x \Theta\big(0\big)\big)\, e_3\\
\mathrm{B}_{y_0} &= -(\nabla n_0 | 0)\, [\nabla_x \Theta(0)]^{-1},\\
\mathrm{C}_{y_0} &= \det(\nabla_x \Theta(0))\, [\nabla_x \Theta(0)]^{-T} \begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix} [\nabla_x \Theta(0)]^{-1} \in \mathfrak{so}(3),\\
\mathrm{K} = \det(\mathrm{L}_{y_0})\,, &\qquad 2\,\mathrm{H} = \operatorname{tr}(\mathrm{L}_{y_0}), \qquad \mathrm{L}_{y_0} = -([\nabla y_0]^T\, \nabla y_0)^{-1}([\nabla n_0]^T\, \nabla y_0),\\
W_{\text{shell}}(X) &= \mu\, \|\operatorname{sym} X\|^2 + \mu_{\text{c}}\, \|\operatorname{skew} X\|^2 + \frac{\lambda\, \mu}{\lambda + 2\mu}\, \big[\operatorname{tr}(X)\big]^2,\\
\mathcal{W}_{\text{shell}}(X,Y) &= \mu\, \big\langle \operatorname{sym} X, \operatorname{sym} Y\big\rangle + \mu_{\text{c}} \big\langle \operatorname{skew} X, \operatorname{skew} Y\big\rangle + \frac{\lambda\, \mu}{\lambda + 2\mu}\operatorname{tr}(X)\operatorname{tr}(Y),\\
W_{\text{mp}}(X) &= \mu\, \|\operatorname{sym} X\|^2 + \mu_{\text{c}}\, \|\operatorname{skew} X\|^2 + \frac{\lambda}{2}\, \big[\operatorname{tr}(X)\big]^2,\\
W_{\text{curv}}(X) &= \mu\, L_{\text{c}}^2 \Big(b_1\, \|\operatorname{dev}\operatorname{sym} X\|^2 + b_2\, \|\operatorname{skew} X\|^2 + 4\, b_3\, \big[\operatorname{tr}(X)\big]^2\Big)\,. \qquad (5.3)
\end{aligned}
$$

In this formulation, all the constitutive coefficients are deduced from the three-dimensional formulation, without using any a posteriori fitting of some two-dimensional constitutive coefficients.

The potential of applied external loads $\overline{\Pi}(m, \overline{Q}_{e,s})$ appearing in (5.1) is expressed by the relations (4.74), (4.76).

We consider the following boundary conditions for the midsurface deformation m and rotation field $\overline{R}_s$ on the Dirichlet part of the lateral boundary $\gamma_0 \subset \partial\omega$:

$$
m\,|_{\gamma_0} = m_0, \qquad \text{simply supported (fixed, welded)} \qquad \overline{R}_s\,|_{\gamma_0} = \hat{R}, \qquad \text{(clamped)}.
$$

It is possible to use the referential fundamental forms I_{y_0}, II_{y_0} and L_{y_0} instead of the matrices A_{y_0}, B_{y_0} and C_{y_0}, and to rewrite all the arguments of the energy terms as

$$
\begin{aligned}
\mathcal{E}_{m,s} &= [\nabla_x \Theta(0)]^{-T} \begin{pmatrix} (\overline{Q}_{e,s} \nabla y_0)^T \nabla m - \mathrm{I}_{y_0} & |0\\ (\overline{Q}_{e,s} n_0)^T \nabla m & |0 \end{pmatrix} [\nabla_x \Theta(0)]^{-1}\\
&= [\nabla_x \Theta(0)]^{-T} \begin{pmatrix} \mathcal{G}\,|\,0\\ \mathcal{T}\,|\,0 \end{pmatrix} [\nabla_x \Theta(0)]^{-1}\\
\mathrm{C}_{y_0} \mathcal{K}_{e,s} &= [\nabla_x \Theta(0)]^{-T} \begin{pmatrix} (\overline{Q}_{e,s} \nabla y_0)^T \nabla(\overline{Q}_{e,s} n_0) + \mathrm{II}_{y_0} & |0\\ 0 & |0 \end{pmatrix} [\nabla_x \Theta(0)]^{-1}\\
&= [\nabla_x \Theta(0)]^{-T} \begin{pmatrix} -\mathcal{R}\,|\,0\\ 0\;\;|\,0 \end{pmatrix} [\nabla_x \Theta(0)]^{-1},\\
\mathcal{E}_{m,s} \mathrm{B}_{y_0} &= [\nabla_x \Theta(0)]^{-T} \begin{pmatrix} \mathcal{G}\,\mathrm{L}_{y_0}\,|\,0\\ \mathcal{T}\,\mathrm{L}_{y_0}\,|\,0 \end{pmatrix} [\nabla_x \Theta(0)]^{-1},
\end{aligned}
$$

$$\begin{aligned}
\mathcal{E}_{m,s}\mathrm{B}_{y_0}^2 &= [\nabla_x\Theta(0)]^{-T}\begin{pmatrix} \mathcal{G}\,\mathrm{L}_{y_0}^2 \,\big|\, 0 \\ \mathcal{T}\,\mathrm{L}_{y_0}^2 \,\big|\, 0 \end{pmatrix}[\nabla_x\Theta(0)]^{-1}, \qquad (5.4)\\
\mathrm{C}_{y_0}\mathcal{K}_{e,s}\mathrm{B}_{y_0} &= -[\nabla_x\Theta(0)]^{-T}\left(\mathcal{R}\,\mathrm{L}_{y_0}\right)^{\flat}[\nabla_x\Theta(0)]^{-1},\\
\mathrm{C}_{y_0}\mathcal{K}_{e,s}\mathrm{B}_{y_0}^2 &= -[\nabla_x\Theta(0)]^{-T}\left(\mathcal{R}\,\mathrm{L}_{y_0}^2\right)^{\flat}[\nabla_x\Theta(0)]^{-1},\\
\mathcal{E}_{m,s}\mathrm{B}_{y_0}+\mathrm{C}_{y_0}\mathcal{K}_{e,s} &= [\nabla_x\Theta(0)]^{-T}\begin{pmatrix} \mathcal{G}\,\mathrm{L}_{y_0}-\mathcal{R} \,\big|\, 0 \\ \mathcal{T}\,\mathrm{L}_{y_0} \,\big|\, 0 \end{pmatrix}[\nabla_x\Theta(0)]^{-1},\\
\mathcal{E}_{m,s}\mathrm{B}_{y_0}^2+\mathrm{C}_{y_0}\mathcal{K}_{e,s}\mathrm{B}_{y_0} &= [\nabla_x\Theta(0)]^{-T}\begin{pmatrix} \mathcal{G}\,\mathrm{L}_{y_0}^2-\mathcal{R}\mathrm{L}_{y_0} \,\big|\, 0 \\ \mathcal{T}\,\mathrm{L}_{y_0}^2 \,\big|\, 0 \end{pmatrix}[\nabla_x\Theta(0)]^{-1},
\end{aligned}$$

where[7]

$$\begin{aligned}
&\mathcal{G} := (\overline{Q}_{e,s}\nabla y_0)^T\nabla m - \mathrm{I}_{y_0} \notin \mathrm{Sym}(2) && \textit{the change of metric tensor (in-plane deformation)},\\
&\mathcal{T} := (\overline{Q}_{e,s}n_0)^T(\nabla m) && \textit{the transverse shear deformation (row) vector},\\
&\mathcal{R} := -(\overline{Q}_{e,s}\nabla y_0)^T\nabla(\overline{Q}_{e,s}n_0) - \mathrm{II}_{y_0} \notin \mathrm{Sym}(2) && \textit{the bending strain tensor}. \qquad (5.5)
\end{aligned}$$

Regarding the arguments of the bending-curvature energy density $W_{\text{bend,curv}}$, we can express the tensor $\mathcal{K}_{e,s}$ in terms of the tensor $\mathrm{C}_{y_0}\,\mathcal{K}_{e,s}$ and the vector $\mathcal{K}_{e,s}^T n_0$, according to Proposition A.3 and to the decomposition

$$\begin{aligned}
\mathcal{K}_{e,s} &= \mathrm{A}_{y_0}\,\mathcal{K}_{e,s} + (0|0|n_0)\,(0|0|n_0)^T\,\mathcal{K}_{e,s}\\
&= \mathrm{C}_{y_0}(-\mathrm{C}_{y_0}\mathcal{K}_{e,s}) + (0|0|n_0)\,(0|0|\mathcal{K}_{e,s}^T\,n_0)^T\,. \qquad (5.6)
\end{aligned}$$

We have already seen that $\mathrm{C}_{y_0}\mathcal{K}_{e,s}$ from the above decomposition can be expressed in terms of the *bending strain* tensor $\mathcal{R}$, see (5.4), while the remaining vector $\mathcal{K}_{e,s}^T\,n_0$ from (5.6) is completely characterized by the row vector

$$\mathcal{N} := n_0^T\big(\mathrm{axl}(\overline{Q}_{e,s}^T\partial_{x_1}\overline{Q}_{e,s})\,|\,\mathrm{axl}(\overline{Q}_{e,s}^T\partial_{x_2}\overline{Q}_{e,s})\big), \qquad (5.7)$$

which is called the row vector of *drilling bendings*.

Remark 5.1 Summarizing, the present shell model is derived:

- under the assumption that $h\,|\kappa_1| < \frac{1}{2},\ h\,|\kappa_2| < \frac{1}{2}$
- considering an approximation of the elastic rotation $\overline{Q}_e : \Omega_h \to \mathrm{SO}(3)$

$$\overline{Q}_e(x_1,x_2,x_3) \cong \overline{Q}_{e,s}(x_1,x_2) = \overline{R}_s(x_1,x_2)\,Q_0^T(x_1,x_2,0)\,; \qquad (5.8)$$

[7]The vector $d_3 = \overline{Q}_{e,s}\,n_0$ represents the classical director, which does not have to be orthogonal to the deformed midsurface. If $\mathcal{T} = (0,0)$, then $\overline{Q}_{e,s}\,n_0$ is orthogonal to the deformed midsurface. An alternative form of the transverse shear deformation row vector is $\mathcal{T} = \big(\langle\overline{Q}_{e,s}n_0, \partial_{x_1}m\rangle, \langle\overline{Q}_{e,s}n_0, \partial_{x_2}m\rangle\big)$.

- choosing an *8-parameter quadratic ansatz* in the thickness direction for the reconstructed total deformation $\varphi_s : \Omega_h \subset \mathbb{R}^3 \to \mathbb{R}^3$ of the shell-like structure

$$\begin{aligned}\varphi_s(x_1, x_2, x_3) \\ &= m(x_1, x_2) + \left(x_3 \varrho_m(x_1, x_2) + \frac{x_3^2}{2} \varrho_b(x_1, x_2) \right) \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0)\, e_3; \quad (5.9)\end{aligned}$$

- taking the exact form of ϱ_m and considering a suitable approximation for ϱ_b (coming from a generalized plane stress condition)

$$\begin{aligned}\varrho_m &= 1 - \frac{\lambda}{\lambda + 2\mu}[\langle \overline{Q}_{e,s}^T(\nabla m|0)[\nabla_x \Theta(0)]^{-1}, \mathbb{1}_3 \rangle - 2] =: \varrho_m^e, \\ \varrho_b &= \frac{\lambda}{\lambda + 2\,\mu} \langle \overline{Q}_{e,s}^T(\nabla m|0)\, [\nabla_x \Theta(0)]^{-1}\, (\nabla n_0|0)\, [\nabla_x \Theta(0)]^{-1}, \mathbb{1}_3 \rangle \qquad (5.10) \\ &\quad - \frac{\lambda}{\lambda + 2\,\mu} \langle \overline{Q}_{e,s}^T(\nabla(\overline{Q}_{e,s} \nabla_x \Theta(0)\, e_3)|0)\, [\nabla_x \Theta(0)]^{-1}, \mathbb{1}_3 \rangle \\ &\quad + \frac{\lambda^2}{(\lambda + 2\,\mu)^2} \left[\langle \overline{Q}_{e,s}^T(\nabla(\overline{Q}_{e,s} \nabla_x \Theta(0)\, e_3)|0)\, [\nabla_x \Theta(0)]^{-1}, \mathbb{1}_3 \rangle \right] \\ &\quad \times \left[\langle \overline{Q}_{e,s}^T(\nabla m|0)[\nabla_x \Theta(0)]^{-1}, \mathbb{1}_3 \rangle - 2 \right] \\ &\cong \varrho_b^e := \frac{\lambda}{\lambda + 2\,\mu} \langle \overline{Q}_{e,s}^T(\nabla m|0)\, [\nabla_x \Theta(0)]^{-1}\, (\nabla n_0|0)\, [\nabla_x \Theta(0)]^{-1}, \mathbb{1}_3 \rangle \\ &\qquad - \frac{\lambda}{\lambda + 2\,\mu} \langle \overline{Q}_{e,s}^T(\nabla(\overline{Q}_{e,s} \nabla_x \Theta(0)\, e_3)|0)\, [\nabla_x \Theta(0)]^{-1}, \mathbb{1}_3 \rangle;\end{aligned}$$

- choosing a further approximation of the deformation gradient (by neglecting space derivatives of ϱ_m^e and ϱ_b^e, respectively)

$$\begin{aligned}F_s &= \nabla_x \varphi_s(x_1, x_2, x_3) \\ &= (\nabla m|\, \varrho_m\, \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0)\, e_3) \\ &\quad + x_3\, (\nabla \left[\varrho_m\, \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0)\, e_3 \right] | \varrho_b\, \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0)\, e_3) \\ &\quad + \frac{x_3^2}{2} (\nabla \left[\varrho_b\, \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0)\, e_3 \right] |0) \qquad (5.11) \\ &\cong \widetilde{F}_s := (\nabla m|\, \varrho_m^e\, \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0)\, e_3) \\ &\qquad + x_3 (\nabla \left[\overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0)\, e_3 \right] | \varrho_b^e\, \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0)\, e_3),\end{aligned}$$

and therefore, the following approximation of the reconstructed gradient

$$\begin{aligned}F_{s,\xi} &= \nabla_x \varphi_s(x_1, x_2, x_3)[\nabla_x \Theta(x_1, x_2, x_3)]^{-1} \\ &\cong \widetilde{F}_{e,s} := (\nabla m|\, \varrho_m^e\, \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0)\, e_3)[\nabla_x \Theta(x_1, x_2, x_3)]^{-1} \\ &\qquad + x_3 (\nabla \left[\overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0)\, e_3 \right] | \varrho_b^e\, \overline{Q}_{e,s}(x_1, x_2) \nabla_x \Theta(x_1, x_2, 0)\, e_3) \\ &\qquad \times [\nabla_x \Theta(x_1, x_2, x_3)]^{-1}. \qquad (5.12)\end{aligned}$$

Moreover, we have used the full expressions of $[\nabla_x \Theta(x_3)]^{-1}$ and $\det(\nabla_x \Theta(x_3))$

$$[\nabla_x \Theta(x_3)]^{-1} = \frac{1}{1-2\,\mathrm{H}\,x_3+\mathrm{K}\,x_3^2} \times \left[\mathbb{1}_3 + x_3(\mathrm{L}^{\flat}_{y_0} - 2\,\mathrm{H}\,\mathbb{1}_3) + x_3^2\,\mathrm{K}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] [\nabla_x \Theta(0)]^{-1}, \qquad (5.13)$$

$$\det(\nabla_x \Theta(x_3)) = \det(\nabla_x \Theta(0)) \Big[1 - 2\,x_3\,\mathrm{H} + x_3^2\,\mathrm{K} \Big].$$

- neglecting the terms of order $O(h^7)$ in the final form of the energy.

After a shell model is proposed, there is a basic requirement: the 2D-shell model must be invariant w.r.t. a reparametrization of the midsurface coordinates. Here, the total elastically stored energy

$$W_{\mathrm{memb}}\big(\mathcal{E}_{m,s}\big) + W_{\mathrm{memb,bend}}\big(\mathcal{E}_{m,s},\, \mathcal{K}_{e,s}\big) + W_{\mathrm{bend,curv}}\big(\mathcal{K}_{e,s}\big)$$

depends on the midsurface deformation gradient ∇m and microrotations $\overline{Q}_{e,s}$ together with their space derivatives only through the frame-indifferent tensors $\mathcal{E}_{m,s}, \mathcal{K}_{e,s}, \mathrm{B}_{y_0}$ and C_{y_0}, which are invariant to the reparametrization.

5.2 Consistency with the Cosserat Plate Model

In the case of Cosserat plates we have $\Theta(x_1, x_2, x_3) = (x_1, x_2, x_3)$ and

$$\begin{aligned}
\nabla_x \Theta(x_3) &= \mathbb{1}_3, \qquad y_0(x_1,x_2) = (x_1,x_2) =: \mathrm{id}(x_1,x_2),\\
Q_0 &= \mathbb{1}_3, \qquad n_0 = e_3, \qquad d_i^0 = e_i,\\
\mathrm{B_{id}} &= 0_3, \qquad \mathrm{C_{id}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3), \qquad (5.14)\\
\mathrm{L_{id}} &= 0_2, \qquad \mathrm{K} = 0, \qquad \mathrm{H} = 0.
\end{aligned}$$

Therefore, for the Cosserat plate model the minimization problem reads: find the deformation of the midsurface $m : \omega \to \mathbb{R}^3$ and the microrotation of the shell $\overline{Q}_{e,s} : \omega \to \mathrm{SO}(3)$ solving on $\omega \subset \mathbb{R}^2$:

$$I = \int_\omega \Big[W_{\mathrm{memb}}\big(\mathcal{E}_{m,s}\big) + W_{\mathrm{memb,bend}}\big(\mathcal{E}_{m,s},\, \mathcal{K}_{e,s}\big) + W_{\mathrm{bend,curv}}\big(\mathcal{K}_{e,s}\big) \Big]\, \mathrm{d}a \;\to\; \min.\ \text{w.r.t. } (m, \overline{Q}_{e,s}) \qquad (5.15)$$

where the membrane part $W_{\mathrm{memb}}\big(\mathcal{E}_{m,s}\big)$, the membrane-bending part $W_{\mathrm{memb,bend}}\big(\mathcal{E}_{m,s},\, \mathcal{K}_{e,s}\big)$ and the bending–curvature part $W_{\mathrm{bend,curv}}\big(\mathcal{K}_{e,s}\big)$ of the shell energy density are given by

$$W_{\mathrm{memb}}\big(\mathcal{E}_{m,s}\big) = h\, W_{\mathrm{shell}}\big(\mathcal{E}_{m,s}\big), \qquad W_{\mathrm{memb,bend}}\big(\mathcal{E}_{m,s},\, \mathcal{K}_{e,s}\big) = \frac{h^3}{12}\, W_{\mathrm{shell}}\big(\mathrm{C_{id}}\mathcal{K}_{e,s}\big),$$
$$W_{\mathrm{bend,curv}}\big(\mathcal{K}_{e,s}\big) = h\, W_{\mathrm{curv}}\big(\mathcal{K}_{e,s}\big), \qquad (5.16)$$

and

$$
\begin{aligned}
\mathcal{E}_{m,s} &= \overline{Q}_{e,s}^T \widetilde{F}_m - \mathbb{1}_3, \qquad \widetilde{F}_m = (\nabla m | \overline{Q}_{e,s}\, e_3),\\
\mathcal{K}_{e,s} &= (\operatorname{axl}(\overline{Q}_{e,s}^T\, \partial_{x_1} \overline{Q}_{e,s}) \,|\, \operatorname{axl}(\overline{Q}_{e,s}^T\, \partial_{x_2} \overline{Q}_{e,s}) \,|0), \quad \mathrm{C}_{\mathrm{id}}\mathcal{K}_{e,s} = \overline{Q}_{e,s}^T (\nabla[\overline{Q}_{e,s}\, e_3] \,|\, 0),\\
W_{\mathrm{shell}}(X) &= \mu \,\| \operatorname{sym} X\|^2 + \mu_{\mathrm{c}} \,\|\operatorname{skew} X\|^2 + \frac{\lambda\, \mu}{\lambda + 2\mu} \big[\operatorname{tr} X\big]^2, \qquad (5.17)\\
W_{\mathrm{curv}}(X) &= \mu\, L_{\mathrm{c}}^2 \big(b_1 \,\|\operatorname{dev}\operatorname{sym} X\|^2 + b_2 \,\|\operatorname{skew} X\|^2 + 4\, b_3\, [\operatorname{tr}(X)]^2\big)\,.
\end{aligned}
$$

In view of Lemma 4.3 the following identity is satisfied

$$
\overline{Q}_{e,s}^T (\nabla[\overline{Q}_{e,s}\, e_3] \,|\, 0) = \mathrm{C}_{\mathrm{id}}\, \mathcal{K}_{e,s}. \qquad (5.18)
$$

Hence, $\mathrm{C}_{\mathrm{id}}\mathcal{K}_{e,s}$ coincides with the second order *non-symmetric bending tensor*

$$
\mathfrak{K}_b := \overline{Q}_{e,s}^T (\nabla[\overline{Q}_{e,s}\, e_3] \,|\, 0) + \mathrm{B}_{y_0} = \overline{Q}_{e,s}^T (\nabla[\overline{Q}_{e,s}\, e_3] \,|\, 0)
$$

considered by Neff in [59]. In consequence, the particular case considered in this subsection is the geometrically nonlinear Cosserat-shell model including size effects introduced in [59] and then used in a numerical approach by Sander et al. [83], but corresponds to another representation of the curvature energy from the three-dimensional formulation.

For purpose of comparison:

Remark 5.2 This Cosserat plate model was derived:

- considering an approximation of the elastic rotation $\overline{Q}_e : \Omega_h \to \mathrm{SO}(3)$

$$
\overline{Q}_e(x_1, x_2, x_3) \cong \overline{Q}_{e,s}(x_1, x_2)\,; \qquad (5.19)
$$

- choosing an *8-parameter quadratic ansatz* in the thickness direction for the reconstructed total deformation $\varphi_s : \Omega_h \subset \mathbb{R}^3 \to \mathbb{R}^3$ of the shell-like structure

$$
\varphi_s(x_1, x_2, x_3) = m(x_1, x_2) + \left(x_3 \varrho_m(x_1, x_2) + \frac{x_3^2}{2} \varrho_b(x_1, x_2)\right) \overline{Q}_{e,s}(x_1, x_2)\, e_3\,; \qquad (5.20)
$$

- taking the exact form of ϱ_m and considering a suitable approximation for ϱ_b (coming from a generalized plane stress condition)

$$
\begin{aligned}
\varrho_m &= 1 - \frac{\lambda}{\lambda + 2\mu} [\langle \overline{Q}_{e,s}^T (\nabla m | 0), \mathbb{1}_3 \rangle - 2] =: \varrho_m^e,\\
\varrho_b &= -\frac{\lambda}{\lambda + 2\,\mu} \langle \overline{Q}_{e,s}^T (\nabla(\overline{Q}_{e,s}\, e_3)|0), \mathbb{1}_3 \rangle\\
&\quad + \frac{\lambda^2}{(\lambda + 2\,\mu)^2} \Big[\langle \overline{Q}_{e,s}^T (\nabla(\overline{Q}_{e,s}\, e_3)|0), \mathbb{1}_3 \rangle\Big]\Big[\langle \overline{Q}_{e,s}^T (\nabla m|0), \mathbb{1}_3 \rangle - 2\Big]\\
&\cong \varrho_b^e := -\frac{\lambda}{\lambda + 2\,\mu} \langle \overline{Q}_{e,s}^T (\nabla(\overline{Q}_{e,s}\, e_3)|0), \mathbb{1}_3 \rangle;
\end{aligned}
$$

- choosing a further approximation of the deformation gradient (by neglecting space derivatives of ϱ_m^e and ϱ_b^e, respectively)

$$\begin{aligned} F_s &= (\nabla m \,|\, \varrho_m \,\overline{Q}_{e,s}(x_1,x_2)\, e_3) + x_3\,(\nabla \left[\varrho_m \,\overline{Q}_{e,s}(x_1,x_2)\, e_3\right] |\, \varrho_b \,\overline{Q}_{e,s}(x_1,x_2)\, e_3) \\ &\qquad + \frac{x_3^2}{2}(\nabla \left[\varrho_b \,\overline{Q}_{e,s}(x_1,x_2)\, e_3\right] | 0) \\ &\cong \widetilde{F}_s := (\nabla m \,|\, \varrho_m^e \,\overline{Q}_{e,s}(x_1,x_2)\, e_3) + x_3(\nabla \left[\,\overline{Q}_{e,s}(x_1,x_2)\, e_3\right] |\, \varrho_b^e \,\overline{Q}_{e,s}(x_1,x_2)\, e_3); \end{aligned} \tag{5.21}$$

- neglecting the terms of order $O(h^5)$ in the final form of the energy.

6 A Comparison with the General 6-Parameter Shell Model

In this section we give an overview of the quantities appearing in the general 6-parameter shell model presented in [36]. Eremeyev and Pietraskiewicz have considered the classical multiplicative decomposition $F = F^e F^0$ of the (reconstructed) total deformation gradient into elastic and initial ("plastic") parts [58, 63], i.e., F^e represents the (reconstructed) elastic shell deformation gradient, while $F^0 = P$ is the initial deformation gradient. In the general 6-parameter shell model, the following form for the elastic strain tensor E^e is used (written in matrix notation)

$$\begin{aligned} E^e &= \overline{Q}_e^T (\,\partial_{x_1} m \,|\, \partial_{x_2} m \,|\, \overline{Q}_e n_0\,)\, P^{-1} - \mathbb{1}_3, \quad \text{or} \\ E^e &= \overline{U}^e - \mathbb{1}_3 = \overline{Q}_e^T\, F^e - \mathbb{1}_3 = \overline{Q}_e^T\, F\, P^{-1} - \mathbb{1}_3, \end{aligned} \tag{6.1}$$

with

$$\begin{aligned} F &= F^e\, F^0, \qquad F = (\,\partial_{x_1} m \,|\, \partial_{x_2} m \,|\, \overline{Q}_e n_0\,) = (\,\nabla m \,|\, \overline{Q}_e n_0\,), \\ F^e &= (\,\partial_{x_1} m \,|\, \partial_{x_2} m \,|\, \overline{Q}_e n_0\,)\, P^{-1}, \qquad F^0 = P = (\,\partial_{x_1} y_0 \,|\, \partial_{x_2} y_0 \,|\, n_0\,) = (\,\nabla y_0 \,|\, n_0\,), \\ \overline{U}^e &= \overline{Q}_e^T (\,\partial_{x_1} m \,|\, \partial_{x_2} m \,|\, \overline{Q}^e n_0\,)\, P^{-1} = \overline{Q}_e^T (\,\nabla m \,|\, \overline{Q}_e n_0\,)\, P^{-1}. \end{aligned} \tag{6.2}$$

Since, $\nabla_x \Theta(0) = (\,\nabla y_0 \,|\, n_0\,) = P$ and $n_0 = Q_0\, e_3 = [\nabla_x \Theta(0)]\, e_3$, we remark that $E^e = \mathcal{E}_{m,s}$. Hence, in the general 6-parameter shell model the same elastic shell strain tensor $\mathcal{E}_{m,s}$ as in our shell model is used.

Regarding the bending curvature tensor, in the general 6-parameter shell model the tensor $\mathcal{K}$ is the total bending–curvature tensor, while $\mathcal{K}^0$ is the initial bending-curvature (or structure curvature tensor of Ω_h). The matrix $\mathcal{K}^e = \mathcal{K} - \mathcal{K}_0$ is given by

$$\mathcal{K}^e = (\,\mathrm{axl}(\overline{Q}_e^T\, \partial_{x_1} \overline{Q}_e) \,|\, \mathrm{axl}(\overline{Q}_e^T\, \partial_{x_2} \overline{Q}_e) \,|\, 0\,)\, P^{-1}, \tag{6.3}$$

or

$$\begin{aligned} \mathcal{K}^e &= Q_0\, L\, P^{-1} = \mathcal{K} - \mathcal{K}^0 \qquad \text{with} \qquad R = \overline{Q}_e Q_0, \\ L &= (\,\mathrm{axl}(R^T \partial_{x_1} R) - \mathrm{axl}(Q_0^T \partial_{x_1} Q_0) \,|\, \mathrm{axl}(R^T \partial_{x_2} R) - \mathrm{axl}(Q_0^T \partial_{x_2} Q_0) \,|\, 0\,)_{3\times 3}, \\ \mathcal{K} &= Q_0(\,\mathrm{axl}(R^T \partial_{x_1} R) \,|\, \mathrm{axl}(R^T \partial_{x_2} R) \,|\, 0\,) P^{-1}, \\ \mathcal{K}^0 &= Q_0(\,\mathrm{axl}(Q_0^T \partial_{x_1} Q_0) \,|\, \mathrm{axl}(Q_0^T \partial_{x_2} Q_0) \,|\, 0\,) P^{-1}. \end{aligned} \tag{6.4}$$

Using again that $P = \nabla_x \Theta(0)$, we have that the total bending-curvature tensor from Pietraskiewicz and his collaborators coincides with the elastic shell-bending-curvature tensor from our model, i.e., $\mathcal{K}^e = \mathcal{K}_{e,s}$. Therefore, we conclude that:

Remark 6.1

1) A direct comparison with our model shows that the strain tensors E^e and $\mathcal{K}^e$ from the general 6-parameter shell model corresponds to the tensor $\mathcal{E}_{m,s}$ and $\mathcal{K}_{e,s}$, respectively, from our model;
2) While the general 6-parameter shell model is not deduced from a three dimensional energy, the strain tensors E^e and $\mathcal{K}^e$ are directly introduced in the model as work-conjugate strain measures [76], without any explanation about how a reconstructed (three-dimensional) ansatz, which minimizes (approximatively) a three-dimensional variational problem, leads to the form of the constitutive tensors E^e and $\mathcal{K}^e$. However, discussions of these two-dimensional strain measures and their three-dimensional counterparts one can find in the book by Libai and Simmonds [52] as well as in the book by Chroscielewski, Makowski and Pietraszkiewicz [20].
3) Contrary to the general 6-parameter shell model, in our description, the roles and the deduction of the strain tensors $\mathcal{E}_{m,s}$ and $\mathcal{K}_{e,s}$ is explained by the dimensional reduction method:
   - the elastic shell bending-curvature tensor $\mathcal{K}_{e,s}$ is appearing in the model from the form of the three-dimensional curvature energy (see (2.2)), after using Nye's formula (3.11), the chain rule (see (3.22)) and using the ansatz (4.1) (see (4.49)).
   - the elastic shell strain tensor $\mathcal{E}_{m,s}$ is appearing in the modelling process after the ansatz for the (reconstructed) deformation gradient (4.32) is proposed and it is suggested by the expressions of ϱ_m^e and ϱ_b^e (see (4.46)), in order to satisfy (approximatively) the Neumann plane-stress boundary conditions in the reference configuration.
   - while ϱ_m^e depends only on $\mathcal{E}_{m,s}$, ϱ_b^e depends on both tensors $\mathcal{E}_{m,s}$ and $\mathcal{K}_{e,s}$. This means that the symmetric thickness stretch about the midsurface is influenced only by the elastic shell strain tensor $\mathcal{E}_{m,s}$, while the asymmetric thickness stretch about the midsurface is influenced by both the elastic shell strain tensor $\mathcal{E}_{m,s}$ and the elastic shell bending-curvature tensor $\mathcal{K}_{e,s}$.
4) The complete description and role of the involved tensors in the dimensional deduction process is the effect of three factors:
   - in the deduction of our model we start with a three-dimensional variational problem for an (three-dimensional) elastic body. A shell is actually a three-dimensional body.
   - we use the matrix formulation in the entire modelling process.
   - we propose a specific isotropic form for the three-dimensional curvature energy in the parent three-dimensional variational problem.

In the resultant 6-parameter theory of shells, the strain energy density for isotropic shells has been presented in various forms. The simplest expression $W_{\rm P}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s})$ has been proposed in the papers [20, 21] in the form

$$\begin{aligned} 2W_{\rm P}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) =& C\big[\nu(\mathrm{tr}\mathcal{E}^{\parallel}_{m,s})^2 + (1-\nu)\mathrm{tr}((\mathcal{E}^{\parallel}_{m,s})^T\mathcal{E}^{\parallel}_{m,s})\big] + \alpha_s C(1-\nu)\|\mathcal{E}^T_{m,s}n_0\|^2 \qquad (6.5)\\ &+ D\big[\nu\,(\mathrm{tr}\,\mathcal{K}^{\parallel}_{e,s})^2 + (1-\nu)\,\mathrm{tr}((\mathcal{K}^{\parallel}_{e,s})^T\mathcal{K}^{\parallel}_{e,s})\big] + \alpha_t\, D(1-\nu)\,\|\mathcal{K}^T_{e,s}n_0\|^2, \end{aligned}$$

where the decompositions of $\mathcal{E}_{m,s}$ and $\mathcal{K}_{e,s}$ into two orthogonal directions (in the tangential plane and in the normal direction)[8] are considered

$$\mathcal{E}^{\|}_{m,s} = \mathrm{A}_{y_0}\mathcal{E}_{m,s} = (\mathbb{1}_3 - n_0 \otimes n_0)\mathcal{E}_{m,s}, \qquad \mathcal{K}^{\|}_{e,s} = \mathrm{A}_{y_0}\,\mathcal{K}_{e,s} = (\mathbb{1}_3 - n_0 \otimes n_0)\,\mathcal{K}_{e,s}, \tag{6.6}$$
$$\mathcal{E}^{\perp}_{m,s} = (\mathbb{1}_3 - \mathrm{A}_{y_0})\,\mathcal{E}_{m,s} = n_0 \otimes n_0\,\mathcal{E}_{m,s}, \qquad \mathcal{K}^{\perp}_{e,s} = (\mathbb{1}_3 - \mathrm{A}_{y_0})\,\mathcal{K}_{e,s} = n_0 \otimes n_0\,\mathcal{K}_{e,s}.$$

Here, we have used that, since $\mathrm{A}_{y_0} = \mathbb{1}_3 - (0|0|n_0)\,(0|0|n_0)^T = \mathbb{1}_3 - n_0 \otimes n_0$, for all $X \in \mathbb{R}^{3\times3}$ the following equalities holds

$$\begin{aligned}\|X^{\perp}\|^2 &= \|(\mathbb{1}_3 - \mathrm{A}_{y_0})\,X\|^2 = \big\langle X, (\mathbb{1}_3 - \mathrm{A}_{y_0})^2\,X\big\rangle = \big\langle X, (\mathbb{1}_3 - \mathrm{A}_{y_0})\,X\big\rangle \\ &= \big\langle X, (0|0|n_0)\,(0|0|n_0)^T\,X\big\rangle = \big\langle (0|0|n_0)^T X, (0|0|n_0)^T\,X\big\rangle \\ &= \|X\,(0|0|n_0)^T\|^2 = \|X^T\,(0|0|n_0)\|^2 = \|X^T\,n_0\|^2.\end{aligned} \tag{6.7}$$

The constitutive coefficient $C = \frac{E\,h}{1-\nu^2}$ is the stretching (in-plane) stiffness of the shell, $D = \frac{E\,h^3}{12(1-\nu^2)}$ is the bending stiffness, and α_s, α_t are two shear correction factors. Also, $E = \frac{\mu\,(3\,\lambda+2\,\mu)}{\lambda+\mu}$ and $\nu = \frac{\lambda}{2\,(\lambda+\mu)}$ denote the Young modulus and Poisson ratio of the isotropic and homogeneous material. In the numerical treatment of non-linear shell problems, the values of the shear correction factors have been set to $\alpha_s = 5/6$, $\alpha_t = 7/10$ in [21]. The value $\alpha_s = 5/6$ is a classical suggestion, which has been previously deduced analytically by Reissner in the case of plates [56, 79]. Also, the value $\alpha_t = 7/10$ was proposed earlier in [74, see p. 78] and has been suggested in the work [73]. However, the discussion concerning the possible values of shear correction factors for shells is long and controversial in the literature [56, 57].

We write the strain energy density (6.5) in the equivalent form

$$\begin{aligned}2\,W_{\mathrm{P}}(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}) = C\,(1-\nu)\,\big[&\|\mathrm{sym}(\mathrm{A}_{y_0}\mathcal{E}_{m,s})\|^2 + \|\mathrm{skew}(\mathrm{A}_{y_0}\mathcal{E}_{m,s})\|^2\big] \\ &+ C\,\nu\,[\mathrm{tr}(\mathrm{A}_{y_0}\mathcal{E}_{m,s})]^2 + \alpha_s\,C(1-\nu)\,\|\mathcal{E}^T_{m,s} n_0\|^2 \\ &+ D\,(1-\nu)\,\big[\|\mathrm{sym}(\mathrm{A}_{y_0}\mathcal{K}_{e,s})\|^2 + \|\mathrm{skew}(\mathrm{A}_{y_0}\mathcal{K}_{e,s})\|^2\big] \\ &+ D\,\nu\,[\mathrm{tr}(\mathrm{A}_{y_0}\mathcal{K}_{e,s})]^2 + \alpha_t\,D(1-\nu)\,\|\mathcal{K}^T_{e,s} n_0\|^2\,.\end{aligned} \tag{6.8}$$

The coefficients in (6.8) are expressed in terms of the Lamé constants of the material λ and μ now by the relations

$$C\,\nu = \frac{4\,\mu\,(\lambda+\mu)}{\lambda + 2\,\mu}\,h\,, \qquad C(1-\nu) = 2\,\mu\,h,$$
$$D\,\nu = \frac{4\,\mu\,(\lambda+\mu)}{\lambda + 2\,\mu}\,\frac{h^3}{12}\,, \qquad D(1-\nu) = \mu\,\frac{h^3}{6}\,.$$

In [36], Eremeyev and Pietraszkiewicz have proposed a more general form of the strain energy density, namely

[8] Here we have used that $\langle \mathrm{A}_{y_0}, \mathbb{1}_3 - \mathrm{A}_{y_0}\rangle = \langle \mathrm{A}_{y_0}, \mathbb{1}_3\rangle - \langle \mathrm{A}_{y_0}\mathrm{A}^T_{y_0}, \mathbb{1}_3\rangle = \langle \mathrm{A}_{y_0}, \mathbb{1}_3\rangle - \langle \mathrm{A}^2_{y_0}, \mathbb{1}_3\rangle = \langle \mathrm{A}_{y_0}, \mathbb{1}_3\rangle - \langle \mathrm{A}_{y_0}, \mathbb{1}_3\rangle = 0$.

$$2W_{\rm EP}(\mathcal{E}_{m,s},\mathcal{K}_{e,s})=\alpha_1\big(\mathrm{tr}\mathcal{E}^{\parallel}_{m,s}\big)^2+\alpha_2\mathrm{tr}\big(\mathcal{E}^{\parallel}_{m,s}\big)^2+\alpha_3\mathrm{tr}\big((\mathcal{E}^{\parallel}_{m,s})^T\mathcal{E}^{\parallel}_{m,s}\big)+\alpha_4\|\mathcal{E}^T_{m,s}n_0\|^2 \qquad (6.9)$$
$$+\beta_1\,\big(\mathrm{tr}\,\mathcal{K}^{\parallel}_{e,s}\big)^2+\beta_2\,\mathrm{tr}\big(\mathcal{K}^{\parallel}_{e,s}\big)^2+\beta_3\,\mathrm{tr}\big((\mathcal{K}^{\parallel}_{e,s})^T\mathcal{K}^{\parallel}_{e,s}\big)+\beta_4\,\|\mathcal{K}^T_{e,s}n_0\|^2.$$

Already, note the absence of coupling terms involving $\mathcal{K}^{\parallel}_{e,s}$ and $\mathcal{E}^{\parallel}_{m,s}$. The eight coefficients $\alpha_k,\ \beta_k$ $(k=1,2,3,4)$ can depend in general on the structure curvature tensor $\mathcal{K}^0$ of the reference configuration. We can decompose the strain energy density (6.9) in the in-plane part $W_{\rm plane-EP}(\mathcal{E}_{m,s})$ and the curvature part $W_{\rm curv-EP}(\mathcal{K}_{e,s})$ and write their expressions in the form

$$\begin{aligned}W_{\rm EP}(\mathcal{E}_{m,s},\mathcal{K}_{e,s})&=W_{\rm plane-EP}(\mathcal{E}_{m,s})+W_{\rm curv-EP}(\mathcal{K}_{e,s})\,, \qquad (6.10)\\ 2\,W_{\rm plane-EP}(\mathcal{E}_{m,s})&=(\alpha_2+\alpha_3)\,\|\mathrm{sym}\,\mathcal{E}^{\parallel}_{m,s}\|^2+(\alpha_3-\alpha_2)\,\|\mathrm{skew}\,\mathcal{E}^{\parallel}_{m,s}\|^2+\alpha_1\,\big(\mathrm{tr}(\mathcal{E}^{\parallel}_{m,s})\big)^2\\ &\quad+\alpha_4\,\|\mathcal{E}^T_{m,s}n^0\|^2,\\ 2\,W_{\rm curv-EP}(\mathcal{K}_{e,s})&=(\beta_2+\beta_3)\,\|\mathrm{sym}\,\mathcal{K}^{\parallel}_{e,s}\|^2+(\beta_3-\beta_2)\,\|\mathrm{skew}\,\mathcal{K}^{\parallel}_{e,s}\|^2+\beta_1\,\big(\mathrm{tr}(\mathcal{K}^{\parallel}_{e,s})\big)^2\\ &\quad+\beta_4\,\|\mathcal{K}^T_{e,s}n^0\|^2.\end{aligned}$$

Since in all the energies presented until now in this section, there exists no coupling terms in $\mathcal{E}_{m,s}$ and $\mathcal{K}_{e,s}$, in the rest of this section, we compare them with a particular form of the energy proposed in our new model, i.e.,

$$W_{\rm our}\big(\mathcal{E}_{m,s},\mathcal{K}_{e,s}\big)=\Big(h+\mathrm{K}\,\frac{h^3}{12}\Big)\,W_{\rm shell}\big(\mathcal{E}_{m,s}\big)+\Big(h-\mathrm{K}\,\frac{h^3}{12}\Big)\,W_{\rm curv}\big(\mathcal{K}_{e,s}\big), \qquad (6.11)$$

where

$$\begin{aligned}W_{\rm shell}(X)&=\mu\,\|\mathrm{sym}\,X\|^2+\mu_{\rm c}\,\|\mathrm{skew}\,X\|^2+\frac{\lambda\,\mu}{\lambda+2\mu}\,\big[\mathrm{tr}\,X\big]^2,\\ W_{\rm curv}(X)&=\mu\,L_{\rm c}^2\,\big(b_1\,\|\mathrm{dev\ sym}\,X\|^2+b_2\,\|\mathrm{skew}\,X\|^2+4\,b_3\,[\mathrm{tr}(X)]^2\big)\,.\end{aligned}$$

To this aim, we consider the decompositions (6.6) and an arbitrary matrix $X=(*|*|0)\,[\nabla_x\Theta(0)]^{-1}$. Since $\mathrm{A}_{y_0}^2=\mathrm{A}_{y_0}\in\mathrm{Sym}(3)$ and $X\mathrm{A}_{y_0}=X$ we have

$$\big\langle(\mathbb{1}_3-\mathrm{A}_{y_0})\,X,\mathrm{A}_{y_0}\,X\big\rangle=\big\langle(\mathrm{A}_{y_0}-\mathrm{A}_{y_0}^2)\,X,\ X\big\rangle=0,$$

but also

$$(\mathbb{1}_3-\mathrm{A}_{y_0})\,X^T=\big(X(\mathbb{1}_3-\mathrm{A}_{y_0})\big)^T=\big(X-X\mathrm{A}_{y_0}\big)^T=0, \qquad (6.12)$$

and consequently

$$\big\langle X^T(\mathbb{1}_3-\mathrm{A}_{y_0}),\mathrm{A}_{y_0}\,X\big\rangle=0 \qquad \text{as well as} \qquad \big\langle X^T(\mathbb{1}_3-\mathrm{A}_{y_0}),(\mathbb{1}_3-\mathrm{A}_{y_0})\,X\big\rangle=0.$$

Hence, we deduce that for all $X=(*|*|0)\cdot[\nabla_x\Theta(0)]^{-1}$ we have the following split in the expression of the considered quadratic forms

$$W_{\rm shell}(X) = \mu\,\|{\rm sym}\,X^{\|}\|^2 + \mu_{\rm c}\,\|{\rm skew}\,X^{\|}\|^2 + \frac{\mu+\mu_{\rm c}}{2}\,\|X^{\perp}\|^2 + \frac{\lambda\,\mu}{\lambda+2\mu}\big[{\rm tr}(X)\big]^2,$$

$$W_{\rm curv}(X) = \mu\,L_{\rm c}^2\bigg(b_1\,\|{\rm sym}\,X^{\|}\|^2 + b_2\,\|{\rm skew}\,X^{\|}\|^2 + \frac{b_1+b_2}{2}\,\|X^{\perp}\|^2 + \frac{12\,b_3-b_1}{3}\,[{\rm tr}(X)]^2\bigg),$$

where we have set $X^{\|} := {\rm A}_{y_0}\,X$ and $X^{\perp} := (\mathbb{1}_3 - {\rm A}_{y_0})\,X$. Moreover, using that for all $X = (*|*|0)\,[\nabla_x\Theta(0)]^{-1}$ it holds true that

$$\mathrm{tr}(X^{\perp}) = \mathrm{tr}\big((\mathbb{1}_3 - {\rm A}_{y_0})X\big) = \mathrm{tr}(X) - \mathrm{tr}({\rm A}_{y_0}X) = \mathrm{tr}(X) - \mathrm{tr}(X\,{\rm A}_{y_0}) = 0, \tag{6.13}$$

we obtain for our model

$$\begin{aligned} W_{\rm shell}\big(\mathcal{E}_{m,s}\big) &= \mu\|{\rm sym}\,\mathcal{E}^{\|}_{m,s}\|^2 + \mu_{\rm c}\|{\rm skew}\,\mathcal{E}^{\|}_{m,s}\|^2 + \frac{\lambda\,\mu}{\lambda+2\mu}\big[{\rm tr}(\mathcal{E}^{\|}_{m,s})\big]^2 + \frac{\mu+\mu_{\rm c}}{2}\|\mathcal{E}^{\perp}_{m,s}\|^2 \\ &= \mu\,\|{\rm sym}\,\mathcal{E}^{\|}_{m,s}\|^2 + \mu_{\rm c}\,\|{\rm skew}\,\mathcal{E}^{\|}_{m,s}\|^2 + \frac{\lambda\,\mu}{\lambda+2\mu}\big[{\rm tr}(\mathcal{E}^{\|}_{m,s})\big]^2 \\ &\quad + \frac{\mu+\mu_{\rm c}}{2}\,\|\mathcal{E}^{T}_{m,s}\,n_0\|^2, \end{aligned} \tag{6.14}$$

and

$$W_{\rm curv}(\mathcal{K}_{e,s}) = \mu\,L_{\rm c}^2\bigg(b_1\,\|{\rm sym}\,\mathcal{K}^{\|}_{e,s}\|^2 + b_2\,\|{\rm skew}\,\mathcal{K}^{\|}_{e,s}\|^2 + \frac{12\,b_3-b_1}{3}\,[{\rm tr}(\mathcal{K}^{\|}_{e,s})]^2 + \frac{b_1+b_2}{2}\,\|\mathcal{K}^{\perp}_{e,s}\|^2\bigg). \tag{6.15}$$

For the final comparison between the models, we rewrite our particular energy in the form

$$\begin{aligned} W_{\rm our}\big(\mathcal{E}_{m,s},\mathcal{K}_{e,s}\big) = \Big(h + {\rm K}\,\frac{h^3}{12}\Big)\Big[&\mu\,\|{\rm sym}\,\mathcal{E}^{\|}_{m,s}\|^2 + \mu_{\rm c}\,\|{\rm skew}\,\mathcal{E}^{\|}_{m,s}\|^2 \\ &+ \frac{\lambda\,\mu}{\lambda+2\mu}\big[{\rm tr}\,(\mathcal{E}^{\|}_{m,s})\big]^2 + \frac{\mu+\mu_{\rm c}}{2}\,\|\mathcal{E}^{T}_{m,s}\,n_0\|^2\Big] \\ &+ \Big(h - {\rm K}\,\frac{h^3}{12}\Big)\,\mu\,L_{\rm c}^2\Big[b_1\,\|{\rm sym}\,\mathcal{K}^{\|}_{e,s}\|^2 + b_2\,\|{\rm skew}\,\mathcal{K}^{\|}_{e,s}\|^2 \\ &\qquad + \frac{12\,b_3-b_1}{3}\,[{\rm tr}(\mathcal{K}_{e,s})]^2 + \frac{b_1+b_2}{2}\,\|\mathcal{K}^{\perp}_{e,s}\|^2\Big]. \end{aligned} \tag{6.16}$$

This allows us to conclude

Remark 6.2 i) By comparing our $W_{\rm our}\big(\mathcal{E}_{m,s},\mathcal{K}_{e,s}\big)$ with $W_{\rm EP}\big(\mathcal{E}_{m,s},\mathcal{K}_{e,s}\big)$ we deduce the following identification of the constitutive coefficients $\alpha_1,\ldots,\alpha_4,\beta_1,\ldots,\beta_4$

$$\alpha_1 = \Big(h + {\rm K}\,\frac{h^3}{12}\Big)\,\frac{2\mu\lambda}{2\mu+\lambda}, \qquad \alpha_2 = \Big(h + {\rm K}\,\frac{h^3}{12}\Big)\,(\mu-\mu_{\rm c}),$$

$$\alpha_3 = \Big(h + \mathrm{K}\,\frac{h^3}{12}\Big)\,(\mu + \mu_c), \qquad \alpha_4 = \Big(h + \mathrm{K}\,\frac{h^3}{12}\Big)(\mu + \mu_c), \tag{6.17}$$

$$\beta_1 = 2\,\Big(h - \mathrm{K}\,\frac{h^3}{12}\Big)\,\mu\, L_c^2\,\frac{12\,b_3 - b_1}{3}, \quad \beta_2 = \Big(h - \mathrm{K}\,\frac{h^3}{12}\Big)\,\mu\, L_c^2(b_1 - b_2),$$

$$\beta_3 = \Big(h - \mathrm{K}\,\frac{h^3}{12}\Big)\,\mu\, L_c^2(b_1 + b_2), \qquad \beta_4 = \Big(h - \mathrm{K}\,\frac{h^3}{12}\Big)\,\mu\, L_c^2(b_1 + b_2).$$

ii) We observe that

$$\mu_c^{\mathrm{drill}} := \alpha_3 - \alpha_2 \,=\, 2\Big(h + \mathrm{K}\,\frac{h^3}{12}\Big)\,\mu_c\,, \tag{6.18}$$

which means that the in-plane rotational couple modulus μ_c^{drill} of the Cosserat shell model is determined by the Cosserat couple modulus μ_c of the 3D Cosserat material. An analogous conclusion is given in [1] where linear deformations are considered.

iii) In our shell model, the constitutive coefficients are those from the three-dimensional formulation, while the influence of the curved initial shell configuration appears explicitly in the expression of the coefficients of the energies for the reduced two-dimensional variational problem.

iv) The major difference between our model and the previously considered general 6-parameter shell model is that we include terms up to order $O(h^5)$ and that, even in the case of a simplified theory of order $O(h^3)$, additional mixed terms like the membrane–bending part $W_{\mathrm{memb,bend}}\big(\mathcal{E}_{m,s},\, \mathcal{K}_{e,s}\big)$ and $W_{\mathrm{curv}}\big(\mathcal{K}_{e,s}\mathrm{B}_{y_0}\big)$ are included, which are otherwise difficult to guess.

v) In this section we have considered only a particular form $W_{\mathrm{our}}\big(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}\big)$ of the density energy considered in our Cosserat-shell model (5.2). However, beside the fact that mixed membrane–bending terms are included, the constitutive coefficients in our shell model depend on both the Gauß curvature K and the mean curvature H, see item i) and compare to (5.2). Moreover, due to the bilinearity of the density energy, if the final form of the energy density is expressed as a quadratic form in terms of $\big(\mathcal{E}_{m,s},\, \mathcal{K}_{e,s}\big)$, as in the $W_{\mathrm{EP}}\big(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}\big)$, then we remark that the dependence on the mean curvature is not only the effect of the presence of the mixed terms, due to the Cayley-Hamilton equation $\mathrm{B}_{y_0}^2 = 2\,\mathrm{H}\,\mathrm{B}_{y_0} - \mathrm{K}\,\mathrm{A}_{y_0}$. See for instance the energy term $W_{\mathrm{curv}}\big(\mathcal{K}_{e,s}\mathrm{B}_{y_0}^2\big)$ or even $W_{\mathrm{mp}}\big((\mathcal{E}_{m,s}\,\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\mathcal{K}_{e,s})\mathrm{B}_{y_0}\big)$ from (5.2).

7 Final Comments

In this article, using a step by step transparent method, we have extended the modelling from flat Cosserat shells (plate) to the most general case of initially curved isotropic Cosserat shells. For flat shells, in a numerical approach Sander et al. [83] have shown that the new shell model offers a very good concordance with available experiments in the framework of nonlinear shell modelling. Our ansatz allows for a consistent shell model up to order $O(h^5)$ in the shell thickness. Interestingly, all $O(h^5)$ terms in the shell energy depend on the initial curvature of the shell and vanish for a flat shell. The $O(h^5)$ terms do not come up with a definite sign, such that the additional terms can be stabilizing as well as destabilizing, depending on the local shell geometry. However, all occurring material coefficients of the shell model are uniquely determined from the isotropic three-dimensional Cosserat model and the given initial geometry of the shell. Hence, in contrast to other Cosserat shell models, we give an explicit form of the curvature energy, and therefore, we fill a certain gap in the general 6-parameter shell theories, which all leave the precise structure of the constitutive equations wide open.

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Appendix

A.1 Notation

We denote by $\mathbb{R}^{n\times n}$, $n\in\mathbb{N}$, the set of real $n\times n$ second order tensors, written with capital letters. We adopt the usual abbreviations of Lie-group theory, i.e., $\mathrm{GL}(n)=\{X\in\mathbb{R}^{n\times n}\,|\det(X)\neq 0\}$ the general linear group, $\mathrm{SL}(n)=\{X\in\mathrm{GL}(n)\,|\det(X)=1\}$, $\mathrm{O}(n)=\{X\in\mathrm{GL}(n)\mid X^TX=\mathbb{1}_n\}$, $\mathrm{SO}(n)=\{X\in\mathrm{GL}(n)|X^TX=\mathbb{1}_n,\det(X)=1\}$ with corresponding Lie-algebras $\mathfrak{so}(n)=\{X\in\mathbb{R}^{n\times n}\,|X^T=-X\}$ of skew symmetric tensors and $\mathfrak{sl}(n)=\{X\in\mathbb{R}^{n\times n}\,|\operatorname{tr}(X)=0\}$ of traceless tensors. Here, for $a,b\in\mathbb{R}^n$ we let $\langle a,b\rangle_{\mathbb{R}^n}$ denote the scalar product on $\mathbb{R}^n$ with associated (squared) vector norm $\|a\|^2_{\mathbb{R}^n}=\langle a,a\rangle_{\mathbb{R}^n}$. The standard Euclidean scalar product on $\mathbb{R}^{n\times n}$ is given by $\langle X,Y\rangle_{\mathbb{R}^{n\times n}}=\operatorname{tr}(XY^T)$, and thus the (squared) Frobenius tensor norm is $\|X\|^2=\langle X,X\rangle_{\mathbb{R}^{n\times n}}$. In the following we omit the index $\mathbb{R}^n,\mathbb{R}^{n\times n}$. The identity tensor on $\mathbb{R}^{n\times n}$ will be denoted by $\mathbb{1}_n$, so that $\operatorname{tr}(X)=\langle X,\mathbb{1}_n\rangle$. We let $\mathrm{Sym}(n)$ and $\mathrm{Sym}^+(n)$ denote the symmetric and positive definite symmetric tensors, respectively. For all $X\in\mathbb{R}^{3\times3}$ we set $\operatorname{sym}X=\frac12(X^T+X)\in\mathrm{Sym}(3)$, $\operatorname{skew}X=\frac12(X-X^T)\in\mathfrak{so}(3)$ and the deviatoric part $\operatorname{dev}X=X-\frac1n\operatorname{tr}(X)\,\mathbb{1}_n\in\mathfrak{sl}(n)$ and we have the *orthogonal Cartan-decomposition of the Lie-algebra* $\mathfrak{gl}(3)$

$$\mathfrak{gl}(3)=\{\mathfrak{sl}(3)\cap\mathrm{Sym}(3)\}\oplus\mathfrak{so}(3)\oplus\mathbb{R}\cdot\mathbb{1}_3, \tag{A.1}$$
$$X=\operatorname{dev}\operatorname{sym}X+\operatorname{skew}X+\frac13\operatorname{tr}(X)\,\mathbb{1}_3\,.$$

We make use of the operator $\operatorname{axl}:\mathfrak{so}(3)\to\mathbb{R}^3$ associating with a matrix $A\in\mathfrak{so}(3)$ the vector $\operatorname{axl}A:=(-A_{23},A_{13},-A_{12})^T$.

For $X\in\mathrm{GL}(n)$, $\mathrm{Adj}(X)$ denotes the tensor of transposed cofactors, while the (i,j) entry of the cofactor is the (i,j)-minor times a sign factor. For vectors $\xi,\eta\in\mathbb{R}^n$, we have the tensor product $(\xi\otimes\eta)_{ij}=\xi_i\,\eta_j$. A matrix having the three column vectors A_1,A_2,A_3 will be written as $(A_1\,|\,A_2\,|\,A_3)$. For a given matrix $M\in\mathbb{R}^{2\times2}$ we define the lifted quantity $M^\flat=\begin{pmatrix}M_{11}&M_{12}&0\\M_{21}&M_{22}&0\\0&0&0\end{pmatrix}\in\mathbb{R}^{3\times3}$.

Let Ω be an open domain of $\mathbb{R}^3$. The usual Lebesgue spaces of square integrable functions, vector or tensor fields on Ω with values in $\mathbb{R}$, $\mathbb{R}^3$ or $\mathbb{R}^{3\times3}$, respectively will be denoted by $\mathrm{L}^2(\Omega)$. Moreover, we introduce the standard Sobolev spaces $\mathrm{H}^1(\Omega)=\{u\in\mathrm{L}^2(\Omega)\,|\,\nabla u\in\mathrm{L}^2(\Omega)\}$, $\mathrm{H}(\mathrm{curl};\Omega)=\{v\in\mathrm{L}^2(\Omega)\,|\,\operatorname{curl}v\in\mathrm{L}^2(\Omega)\}$ of functions u or vector fields v, respectively. For vector fields $u=(u_1,u_2,u_3)$ with $u_i\in\mathrm{H}^1(\Omega)$, $i=1,2,3$, we define

$$\nabla u:=(\nabla u_1\,|\,\nabla u_2\,|\,\nabla u_3)^T\,,$$

while for tensor fields P with rows in $\mathrm{H}(\mathrm{curl}\,;\Omega)$, i.e., $P = \left(P^T.e_1\,|\,P^T.e_2\,|\,P^T e_3\right)^T$ with $(P^T.e_i)^T \in \mathrm{H}(\mathrm{curl}\,;\Omega)$, $i = 1,2,3$, we define

$$\mathrm{Curl}\,P := \left(\mathrm{curl}\,(P^T.e_1)^T\,|\,\mathrm{curl}\,(P^T.e_2)^T\,|\,\mathrm{curl}\,(P^T e_3)^T\right)^T.$$

The corresponding Sobolev-spaces will be denoted by $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^1(\mathrm{Curl};\Omega)$, respectively. We will use the notations: ∇_ξ, ∇_x, Curl_ξ, Curl_x etc. to indicate the variables for which these quantities are calculated.

A.2 Prerequisites from Classical Differential Geometry

Let $\omega \subset \mathbb{R}^2$ be an open domain. A given regular mapping $y_0 : \omega \to \mathbb{R}^3$, describing a *surface imbedded in the three-dimensional space* is called *regular* whenever $\mathrm{rank}(\nabla y_0) = 2$. The column vector

$$n_0 := \frac{\partial_{x_1} y_0 \times \partial_{x_2} y_0}{\|\partial_{x_1} y_0 \times \partial_{x_2} y_0\|} \tag{A.2}$$

is the *Gauß unit normal field* on the surface. We need to compute

$$\mathrm{Adj}[(\nabla y_0|0)] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \begin{vmatrix} \partial_{x_1} y_{02} & \partial_{x_2} y_{02} \\ \partial_{x_1} y_{03} & \partial_{x_2} y_{03} \end{vmatrix} & -\begin{vmatrix} \partial_{x_1} y_{01} & \partial_{x_2} y_{01} \\ \partial_{x_1} y_{03} & \partial_{x_2} y_{03} \end{vmatrix} & \begin{vmatrix} \partial_{x_1} y_{01} & \partial_{x_2} y_{01} \\ \partial_{x_1} y_{02} & \partial_{x_2} y_{02} \end{vmatrix} \end{pmatrix}. \tag{A.3}$$

Hence, it follows

$$n_0 := \frac{\mathrm{Cof}(\nabla y_0|0)\, e_3}{\|\mathrm{Cof}(\nabla y_0|0)\, e_3\|}, \qquad \mathrm{Cof}(X) = [\mathrm{Adj}(X)]^T \quad \forall\, X \in \mathbb{R}^{3\times 3}. \tag{A.4}$$

The map $n_0 : \omega \to S^2$ is called the *Gauß map* (where S^2 is the unit sphere in $\mathbb{R}^3$) and the moving 3-frame $(\partial_{x_1} y_0|\partial_{x_2} y_0|n_0)$ is called the *Gauß frame* of the surface $y_0(\omega)$, which in general is not orthonormal. The matrix representation of the *first fundamental form (metric)* on $y_0(\omega)$ is given through

$$\mathrm{I}_{y_0} := [\nabla y_0]^T\, \nabla y_0 = \begin{pmatrix} \|\partial_{x_1} y_0\|^2 & \langle \partial_{x_1} y_0, \partial_{x_2} y_0 \rangle \\ \langle \partial_{x_1} y_0, \partial_{x_2} y_0 \rangle & \|\partial_{x_2} y_0\|^2 \end{pmatrix} \in \mathbb{R}^{2\times 2}. \tag{A.5}$$

Because $\mathrm{rank}(\nabla y_0) = 2$, the tensor $[\nabla y_0]^T \nabla y_0$ is positive definite.

The metric alone is not sufficient to describe the shape of a surface in the ambient three-dimensional Euclidean space, the curvature is also needed. However, in the case $(\nabla y_0|n_0) \in \mathrm{SO}(3)$, the metric is indeed enough. With the metric, the length and angles (and changes of length and angles) of a surface can be completely described.

The matrix representation of the *second fundamental form* on $y_0(\omega)$ providing a measure for curvature of the surface is given by

$$\begin{aligned} \mathrm{II}_{y_0} &:= -[\nabla y_0]^T\, \nabla n_0 = -(\partial_{x_1} y_0|\partial_{x_2} y_0)^T (\partial_{x_1} n_0|\partial_{x_2} n_0) \\ &= -\begin{pmatrix} \langle \partial_{x_1} y_0, \partial_{x_1} n_0 \rangle & \langle \partial_{x_1} y_0, \partial_{x_2} n_0 \rangle \\ \langle \partial_{x_2} y_0, \partial_{x_1} n_0 \rangle & \langle \partial_{x_2} y_0, \partial_{x_2} n_0 \rangle \end{pmatrix} \in \mathbb{R}^{2\times 2}. \end{aligned} \tag{A.6}$$

Since n_0 is orthogonal to the tangent space $T_x y_0$ of the surface y_0, the relation $0 = \partial_{x_1}\langle \partial_{x_2} y_0, n_0\rangle = \partial_{x_2}\langle \partial_{x_1} y_0, n_0\rangle$ shows easily that $\mathrm{II}_{y_0} \in \mathrm{Sym}(2)$.

The *third fundamental form* of the surface $y_0(\omega)$ in matrix representation is defined as

$$\mathrm{III}_{y_0} := \nabla n_0{}^T \nabla n_0 = \begin{pmatrix} \|\partial_{x_1} n_0\|^2 & \langle \partial_{x_1} n_0, \partial_{x_2} n_0\rangle \\ \langle \partial_{x_2} n_0, \partial_{x_1} n_0\rangle & \|\partial_{x_2} n_0\|^2 \end{pmatrix} \in \mathbb{R}^{2\times 2}. \tag{A.7}$$

Since $[\nabla y_0]^T \nabla y_0$ is positive definite, the first fundamental form induces a scalar product $g(\xi_1, \xi_2) := \langle \mathrm{I}_{y_0}\xi_1, \xi_2\rangle_{\mathbb{R}^2}$, while the second fundamental form induces a symmetric bilinear form $\widetilde{g}(\xi_1, \xi_2) := \langle \mathrm{II}_{y_0}\xi_1, \xi_2\rangle_{\mathbb{R}^2}$.

The first fundamental form and the second fundamental form are connected by *the Weingarten map* (or shape operator) which we again identify with the associated matrix $\mathrm{L}_{y_0} \in \mathbb{R}^{2\times 2}$, i.e., $\forall\, \xi_1, \xi_2 \in \mathbb{R}^2$ we have $\langle \mathrm{I}_{y_0}\xi_1, \mathrm{L}_{y_0}\,\xi_2\rangle_{\mathbb{R}^2} = \langle \mathrm{II}_{y_0}\xi_1, \xi_2\rangle_{\mathbb{R}^2}$. This is an implicit definition of the Weingarten map L_{y_0}. Using the definitions of the first and the second fundamental form, we have

$$\begin{aligned} &\langle [\nabla y_0]^T \nabla y_0\, \xi_1, \mathrm{L}_{y_0}\xi_2\rangle_{\mathbb{R}^2} = \langle -[\nabla y_0]^T \nabla n_0\, \xi_1, \xi_2\rangle_{\mathbb{R}^2} \\ \Leftrightarrow\quad &\langle (\mathrm{L}_{y_0}^T [\nabla y_0]^T \nabla y_0 + [\nabla y_0]^T \nabla n_0)\, \xi_1, \xi_2\rangle_{\mathbb{R}^2} = 0, \quad \forall\, \xi_1, \xi_2 \in \mathbb{R}^2. \end{aligned} \tag{A.8}$$

Thus, we have

$$\mathrm{L}_{y_0}^T [\nabla y_0]^T \nabla y_0 + [\nabla y_0]^T \nabla n_0 = 0 \quad \Leftrightarrow \quad \mathrm{L}_{y_0}^T [\nabla y_0]^T \nabla y_0 = -[\nabla y_0]^T \nabla n_0. \tag{A.9}$$

Hence, we obtain the following alternative expression for the Weingarten map via the so called *Weingarten equations*:

$$\mathrm{L}_{y_0} = -([\nabla y_0]^T \nabla y_0)^{-1} (\nabla n_0{}^T \nabla y_0) \qquad \text{or} \qquad \mathrm{L}_{y_0} = \mathrm{I}_{y_0}^{-1} \mathrm{II}_{y_0}. \tag{A.10}$$

Moreover, using the symmetry of the second fundamental form we see that the Weingarten map satisfies:

$$\nabla y_0\, \mathrm{L}_{y_0} = -\nabla n_0. \tag{A.11}$$

We have also

$$\mathrm{III}_{y_0} = \mathrm{I}_{y_0} \mathrm{L}_{y_0}^2 = \mathrm{II}_{y_0} \mathrm{I}_{y_0}^{-1} \mathrm{II}_{y_0}. \tag{A.12}$$

The *Gauß curvature* K of the surface $y_0(\omega)$ is determined by

$$\mathrm{K} := \det(\mathrm{II}_{y_0}\, \mathrm{I}_{y_0}^{-1}) = \det(\mathrm{L}_{y_0}), \tag{A.13}$$

and the *mean curvature* H through

$$2\,\mathrm{H} := \mathrm{tr}(\mathrm{L}_{y_0}). \tag{A.14}$$

The following classification is standard. The surface y_0 is locally

$$\left\{\begin{array}{l} \text{elliptic} \\ \text{parabolic} \\ \text{hyperbolic} \\ \text{planar} \end{array}\right. \quad \text{at} \quad (x_1, x_2) \in \omega \quad \text{if} \quad \mathrm{K} \quad \text{is} \quad \left\{\begin{array}{ll} > 0 & \\ = 0 & \text{and } \mathrm{H} \neq 0 \\ < 0 & \\ = 0 & \text{and } \mathrm{H} = 0 \end{array}\right. . \tag{A.15}$$

It is well known that $\mathrm{H} = 0$ is satisfied for all sufficiently regular stationary points of the minimal surface area functional. The Caley-Hamilton theorem implies

$$\mathrm{L}_{y_0}^2 - 2\,\mathrm{H}\,\mathrm{L}_{y_0} + \mathrm{K}\,\mathbb{1}_2 = 0.$$

Thus, the relation

$$\mathrm{III}_{y_0} - 2\,\mathrm{H}\,\mathrm{II}_{y_0} + \mathrm{K}\,\mathrm{I}_{y_0} = 0$$

([48, Prop. 3.5.6]) is a consequence of the Caley-Hamilton theorem and shows that III_{y_0} is symmetric and is not independent of $\mathrm{I}_{y_0}, \mathrm{II}_{y_0}$. The principal curvatures κ_1, κ_2 are the solutions of the characteristic equation of L_{y_0}, i.e.,

$$\kappa^2 - \mathrm{tr}(\mathrm{L}_{y_0})\,\kappa + \det(\mathrm{L}_{y_0}) = \kappa^2 - 2\,\mathrm{H}\,\kappa + \mathrm{K} = 0.$$

We define the lifted quantity $\widehat{\mathrm{I}}_{y_0} \in \mathbb{R}^{3\times 3}$ by

$$\begin{aligned}\widehat{\mathrm{I}}_{y_0} &= (\nabla y_0|n_0)^T(\nabla y_0|n_0) = \begin{pmatrix} \|\partial_{x_1} y_0\|^2 & \langle \partial_{x_1} y_0, \partial_{x_2} y_0\rangle & 0 \\ \langle \partial_{x_1} y_0, \partial_{x_2} y_0\rangle & \|\partial_{x_2} y_0\|^2 & 0 \\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix} \multicolumn{2}{c}{\mathrm{I}_{y_0}} & 0 \\ & & 0 \\ 0 & 0 & 1\end{pmatrix}\\ &= \mathrm{I}_{y_0}^\flat + \widehat{0}_3\,, \end{aligned} \tag{A.16}$$

where $\mathrm{I}_{y_0}^\flat = \begin{pmatrix} \multicolumn{2}{c}{\mathrm{I}_{y_0}} & 0 \\ & & 0 \\ 0 & 0 & 0\end{pmatrix}$ and $\widehat{0}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix}$. Hence, $\widehat{\mathrm{I}}_{y_0}$ has the properties

$$\det(\mathrm{I}_{y_0}) = \det(\widehat{\mathrm{I}}_{y_0}) = \det(\nabla y_0|n_0)^2, \quad \mathrm{tr}(\mathrm{I}_{y_0}) + 1 = \mathrm{tr}(\widehat{\mathrm{I}}_{y_0})\,.$$

Corresponding to the second fundamental form we define the lifted quantity $\widehat{\mathrm{II}}_{y_0} \in \mathbb{R}^{3\times 3}$ by

$$\begin{aligned}\widehat{\mathrm{II}}_{y_0} &= -(\nabla y_0|n_0)^T(\nabla n_0|n_0) = -\begin{pmatrix} \langle \partial_{x_1} y_0, \partial_{x_1} n_0\rangle & \langle \partial_{x_1} y_0, \partial_{x_2} n_0\rangle & 0 \\ \langle \partial_{x_2} y_0, \partial_{x_1} n_0\rangle & \langle \partial_{x_2} y_0, \partial_{x_2} n_0\rangle & 0 \\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix} \multicolumn{2}{c}{\mathrm{II}_{y_0}} & 0 \\ & & 0 \\ 0 & 0 & -1\end{pmatrix}\\ &= \mathrm{II}_{y_0}^\flat - \widehat{0}_3\,, \end{aligned} \tag{A.17}$$

where $\mathrm{II}_{y_0}^\flat = \begin{pmatrix} \multicolumn{2}{c}{\mathrm{II}_{y_0}} & 0 \\ & & 0 \\ 0 & 0 & 0\end{pmatrix}$. It has the properties

$$\det(\mathrm{II}_{y_0}) = -\det(\widehat{\mathrm{II}}_{y_0}), \qquad \mathrm{tr}(\mathrm{II}_{y_0}) = \mathrm{tr}(\widehat{\mathrm{II}}_{y_0}) + 1\,. \tag{A.18}$$

Let us consider as well the lifted Weingarten map $\widehat{\mathrm{L}}_{y_0} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\widehat{\mathrm{L}}_{y_0}^T = \widehat{\mathrm{II}}_{y_0}\widehat{\mathrm{I}}_{y_0}^{-1}\,. \tag{A.19}$$

Thus, we have

$$\begin{aligned}\widehat{\mathrm{L}}_{y_0}^T &= -\,(\nabla y_0|n_0)^T(\nabla n_0|n_0)[(\nabla y_0|n_0)^T(\nabla y_0|n_0)]^{-1}\\ &= \begin{pmatrix} \multicolumn{2}{c}{\mathrm{II}_{y_0}} & 0 \\ & & 0 \\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix} \multicolumn{2}{c}{\mathrm{I}_{y_0}^{-1}} & 0 \\ & & 0 \\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix} \multicolumn{2}{c}{\mathrm{II}_{y_0}\mathrm{I}_{y_0}^{-1}} & 0 \\ & & 0 \\ 0 & 0 & -1\end{pmatrix} = \begin{pmatrix} \multicolumn{2}{c}{\mathrm{L}_{y_0}^T} & 0 \\ & & 0 \\ 0 & 0 & -1\end{pmatrix}. \end{aligned} \tag{A.20}$$

The lifted Weingarten map $\widehat{\mathrm{L}}$ has the following properties

$$\det(\widehat{\mathrm{L}}_{y_0}) = -\det(\mathrm{L}_{y_0}), \quad \mathrm{tr}(\widehat{\mathrm{L}}_{y_0}) = \mathrm{tr}(\mathrm{L}_{y_0}) - 1\,. \tag{A.21}$$

A.3 Properties of the Diffeomorphism Θ

Lemma A.1 *For all $A \in \mathbb{R}^{2\times 2}$, there exists $a > 0$, such that for all $x_3 \in (-a, a)$, the formula*

$$(\mathbb{1}_2 - x_3\, A)^{-1} = \frac{1}{1 - x_3\,\mathrm{tr}A + x_3^2 \det A}[(1 - x_3\,\mathrm{tr}A)\mathbb{1}_2 + x_3 A] \tag{A.22}$$

holds true.

Proof Consider an arbitrary matrix $A \in \mathbb{R}^{2\times 2}$. From the continuity of the mapping $x_3 \mapsto 1 - x_3\,\mathrm{tr}(A) + x_3^2 \det(A)$, it exists an $a > 0$ such that $1 - x_3\,\mathrm{tr}(A) + x_3^2 \det(A) > 0$ for all $x_3 \in (-a, a)$. Taking $x_3 \in (-a, a)$, we compute

$$\begin{aligned}
&(\mathbb{1}_2 - x_3\, A)^{-1}\,(\mathbb{1}_2 - x_3\, A)\\
&\quad= \frac{1}{1 - x_3\,\mathrm{tr}(A) + x_3^2 \det(A)}[(1 - x_3\,\mathrm{tr}(A))\mathbb{1}_2 + x_3 A]\,[\mathbb{1}_2 - x_3\, A] \qquad (\mathrm{A.23})\\
&\quad= \frac{1}{1 - x_3\,\mathrm{tr}(A) + x_3^2 \det(A)}[(1 - x_3\,\mathrm{tr}(A))\mathbb{1}_2 - x_3\, A + x_3^2\,\mathrm{tr}(A)\, A + x_3 A - x_3^2\, A^2]\\
&\quad= \frac{1}{1 - x_3\,\mathrm{tr}\,A + x_3^2 \det A}[(1 - x_3\,\mathrm{tr}(A))\mathbb{1}_2 + x_3^2\,\underbrace{(\mathrm{tr}(A)\, A - A^2)}_{\det(A)\,\mathbb{1}_2}] = \mathbb{1}_2.
\end{aligned}$$

□

With the help of the above lemma, we prove the following proposition.

Proposition A.2 *The diffeomorphism Θ has the following properties for all x_3:*

i) $\det(\nabla_x \Theta(x_3)) = \det(\nabla y_0|n_0)\Big[1 - 2\,x_3\,\mathrm{H} + x_3^2\,\mathrm{K}\Big]$;
ii) $\nabla_x \Theta(x_3)$ *belongs to* $\mathrm{GKC} := \{X \in \mathrm{GL}^+(3) \mid X^T X\, e_3 = \varrho^2 e_3,\ \varrho \in \mathbb{R}^+\}$;
iii) *if* $h\,|\kappa_1| < \frac{1}{2}$, $h\,|\kappa_2| < \frac{1}{2}$, *then for all* $x_3 \in \left(-\frac{h}{2}, \frac{h}{2}\right)$:

$$\begin{aligned}
[\nabla_x \Theta(x_3)]^{-1} = &\frac{1}{1 - 2\,\mathrm{H}\,x_3 + \mathrm{K}\,x_3^2}\left[\mathbb{1}_3 + x_3(\mathrm{L}^\flat_{y_0} - 2\,\mathrm{H}\,\mathbb{1}_3) + x_3^2\,\mathrm{K}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right]\\
&\times [\nabla_x \Theta(0)]^{-1};
\end{aligned}$$

Proof Since $\|n_0\|^2 = 1$ we have $\langle n_0, \partial_{x_\alpha} n_0\rangle = 0$, $\langle n_0, \partial_{x_\alpha} y_0\rangle = 0$, $\alpha = 1, 2$. Using the Weingarten map (or shape operator) $\mathrm{L}_{y_0} \in \mathbb{R}^{2\times 2}$ defined in Appendix A.2 by relation (A.11) we deduce the following form of $\nabla_x \Theta$:

$$\nabla_x \Theta(x_3) = (\nabla y_0|\,n_0) + x_3(\nabla n_0|0) = (\nabla y_0|\,n_0) - x_3(\nabla y_0\,\mathrm{L}_{y_0}|0), \tag{A.24}$$

which implies

$$\nabla_x \Theta(x_3) = \nabla_x \Theta(0)\begin{pmatrix} \mathbb{1}_2 - x_3\,\mathrm{L}_{y_0} & \begin{matrix}0\\0\end{matrix}\\ \begin{matrix}0 & 0\end{matrix} & 1\end{pmatrix}. \tag{A.25}$$

Then, we have $\det(\nabla_x\Theta(x_3)) = \det(\nabla_x\Theta(0))\det(\mathbb{1}_2 - x_3\mathrm{L}_{y_0})$. Using the two-dimensional expansion of the determinant $\det(\mathbb{1}_2 - x_3\mathrm{L}_{y_0}) = 1 - x_3\,\mathrm{tr}(\mathrm{L}_{y_0}) + x_3^2\det(\mathrm{L}_{y_0})$, we deduce $\det(\nabla_x\Theta(x_3)) = \det(\nabla_x\Theta(0))\Big[1 - x_3\,\mathrm{tr}(\mathrm{L}_{y_0}) + x_3^2\det(\mathrm{L}_{y_0})\Big]$. In terms of the mean curvature H and the Gauss curvature K, we have therefore the well known formula i).

Regarding ii), we have already seen in (A.16) that $\nabla_x\Theta(0) = (\nabla y_0|\,n_0) \in \mathrm{GKC}$, so that the conclusion $\nabla_x\Theta(x_3) \in \mathrm{GKC}$ then follows from the decomposition (A.25).

In order to prove iii), we prove that the conditions $h\,|\kappa_1| < \frac{1}{2}$, $h\,|\kappa_2| < \frac{1}{2}$, ensure that

$$1 - 2\,\mathrm{H}\,x_3 + \mathrm{K}\,x_3^2 \neq 0 \qquad \text{for all} \qquad x_3 \in \left(-\frac{h}{2}, \frac{h}{2}\right).$$

It follows that $h^2|K| = h^2\,|\kappa_1|\,|\kappa_2| < \frac{1}{4}$, $2\,h\,|H| = h\,|\kappa_1 + \kappa_2| < 1$. Hence, $1 - 2\,\mathrm{H}\,x_3 + \mathrm{K}\,x_3^2 \geq 1 - 2\,|\mathrm{H}|\,|x_3| - |\mathrm{K}|\,|x_3^2| > 0$. Thus, we use the representation (A.25) and we apply Lemma A.1 and iii) is proven. □

A.4 Properties of the Tensors A_{y_0}, B_{y_0} and C_{y_0}

Proposition A.3 *The tensors* $\mathrm{A}_{y_0}, \mathrm{B}_{y_0}$ *and* C_{y_0} *have the following properties*:

i) $\mathrm{A}_{y_0} = [\nabla_x\Theta(0)]^{-T}\mathrm{I}^\flat_{y_0}[\nabla_x\Theta(0)]^{-1} \in \mathrm{Sym}(3)$, $\mathrm{tr}(\mathrm{A}_{y_0}) = 2$, $\det(\mathrm{A}_{y_0}) = 0$,
$\mathrm{A}_{y_0} = \mathbb{1}_3 - (0|0|\nabla_x\Theta(0)\,e_3)\,[\nabla_x\Theta(0)]^{-1} = \mathbb{1}_3 - (0|0|n_0)\,(0|0|n_0)^T$;

ii) $\mathrm{B}_{y_0} = [\nabla_x\Theta(0)]^{-T}\mathrm{II}^\flat_{y_0}[\nabla_x\Theta(0)]^{-1} = \nabla_x\Theta(0)\mathrm{L}^\flat_{y_0}[\nabla_x\Theta(0)]^{-1} \in \mathrm{Sym}(3)$,
$\mathrm{tr}(\mathrm{B}_{y_0}) = 2\,\mathrm{H}$, $\det[\mathrm{B}_{y_0}] = 0$, $\mathrm{tr}(\mathrm{Cof}\,\mathrm{B}_{y_0}) = \mathrm{K}$,
$$\mathrm{Cof}\,\mathrm{B}_{y_0} = \nabla_x\Theta(0)\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \mathrm{K}\end{pmatrix}[\nabla_x\Theta(0)]^{-1},$$
$\mathrm{B}^2_{y_0} = [\nabla_x\Theta(0)]^{-T}\mathrm{III}^\flat_{y_0}[\nabla_x\Theta(0)]^{-1} = [\nabla_x\Theta(0)]\big(\mathrm{L}^\flat_{y_0}\big)^2[\nabla_x\Theta(0)]^{-1}$;

iii) B_{y_0} *satisfies the equation of Cayley-Hamilton type* $\mathrm{B}^2_{y_0} - 2\,\mathrm{H}\,\mathrm{B}_{y_0} + \mathrm{K}\,\mathrm{A}_{y_0} = 0_3$;

iv) $\mathrm{A}_{y_0}\mathrm{B}_{y_0} = \mathrm{B}_{y_0}\mathrm{A}_{y_0} = \mathrm{B}_{y_0}$, $\mathrm{A}^2_{y_0} = \mathrm{A}_{y_0}$;

v) $(u_1|u_2|0)\,[\nabla_x\Theta(0)]^{-1}\,\mathrm{A}_{y_0} = (u_1|u_2|0)\,[\nabla_x\Theta(0)]^{-1}$ *for all* $u_1, u_2 \in \mathbb{R}^3$;

vi) $\mathrm{C}_{y_0} \in \mathfrak{so}(3)$, $\mathrm{C}^2_{y_0} = -\mathrm{A}_{y_0}$ *and it has the simplified form*
$$\mathrm{C}_{y_0} := Q_0(0)\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}Q_0^T(0);$$

vii) $\mathrm{B}_{y_0} = -\mathrm{C}_{y_0}\,Q_0(0)\Big(\mathrm{axl}(Q_0^T(0)\,\partial_{x_1}Q_0(0))\,|\,\mathrm{axl}(Q_0^T(0)\,\partial_{x_2}Q_0(0))\,|\,0\Big)[\nabla_x\Theta(0)]^{-1}$.

Proof i) We deduce

$$\begin{aligned}\mathrm{A}_{y_0} &= (\nabla y_0|0)\,[\nabla_x\Theta(0)]^{-1} = (\nabla y_0|n_0)\,\mathbb{1}_2^\flat\,[\nabla_x\Theta(0)]^{-1}\\ &= [\nabla_x\Theta(0)]^{-T}\,[\nabla_x\Theta(0)]^T\,[\nabla_x\Theta(0)]\,\mathbb{1}_2^\flat\,[\nabla_x\Theta(0)]^{-1}\\ &= [\nabla_x\Theta(0)]^{-T}\widehat{\mathrm{I}}_{y_0}\,\mathbb{1}_2^\flat\,[\nabla_x\Theta(0)]^{-1} = [\nabla_x\Theta(0)]^{-T}\,\mathrm{I}^\flat_{y_0}\,[\nabla_x\Theta(0)]^{-1}.\end{aligned} \tag{A.26}$$

Therefore, the first identity of i) is proven and it also follows that

$$\begin{aligned}\mathrm{tr}(\mathrm{A}_{y_0}) &= \big\langle[\nabla_x\Theta(0)]^{-T}\,\mathrm{I}^\flat_{y_0}\,[\nabla_x\Theta(0)]^{-1}, \mathbb{1}_3\big\rangle = \big\langle\mathrm{I}^\flat_{y_0}, [\nabla_x\Theta(0)]^{-1}\,[\nabla_x\Theta(0)]^{-T}\big\rangle\\ &= \big\langle\mathrm{I}^\flat_{y_0}, (\mathrm{I}^{-1}_{y_0})^\flat\big\rangle = 2.\end{aligned} \tag{A.27}$$

We use that $\nabla_x\Theta(x_3)\in$ GKC and calculate

$$
\begin{aligned}
(\nabla y_0|0)(\nabla y_0|n_0)^{-1} &= (\nabla y_0|n_0)(\nabla y_0|n_0)^{-1} - (0|0|n_0)(\nabla y_0|n_0)^{-1} \\
&= \mathbb{1}_3 - (0|0|n_0)(\nabla y_0|n_0)^{-1} = \mathbb{1}_3 - (0|0|n_0)U_0^{-1}(0)\, Q_0^T(0) \qquad \text{(A.28)} \\
&= \mathbb{1}_3 - (0|0|n_0)\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} Q_0^T(0) = \mathbb{1}_3 - (0|0|n_0)\, Q_0^T(0) \\
&= \mathbb{1}_3 - (0|0|n_0)\,(d_1^0(0)|d_2^0(0)|n_0)^T = \mathbb{1}_3 - (0|0|n_0)\,(0|0|n_0)^T .
\end{aligned}
$$

The last identity of i) follows directly from (3.29).

ii) In terms of the second fundamental form the tensor B_{y_0} has the form

$$
\mathrm{B}_{y_0} = -[\nabla_x\Theta(0)]^{-T}(\nabla y_0|n_0)^T(\nabla n_0|0)[\nabla_x\Theta(0)]^{-1} = [\nabla_x\Theta(0)]^{-T}\mathrm{II}_{y_0}^{\flat}[\nabla_x\Theta(0)]^{-1}, \qquad \text{(A.29)}
$$

from which symmetry follows, since $\mathrm{II}_{y_0}^{\flat}$ is symmetric. Moreover, we have

$$
\begin{aligned}
[\nabla_x\Theta(0)]^{-1}\mathrm{B}_{y_0} &= [\nabla_x\Theta(0)]^{-1}[\nabla_x\Theta(0)]^{-T}\mathrm{II}_{y_0}^{\flat}[\nabla_x\Theta(0)]^{-1} \\
&= \begin{pmatrix} \mathrm{I}_{y_0}^{-1} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \mathrm{II}_{y_0} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}[\nabla_x\Theta(0)]^{-1} = \mathrm{L}_{y_0}^{\flat}[\nabla_x\Theta(0)]^{-1}. \qquad \text{(A.30)}
\end{aligned}
$$

The identity $\det[\mathrm{B}_{y_0}]=0$ follows directly from (3.29). A direct consequence of the above relation is

$$
\mathrm{tr}(\mathrm{B}_{y_0}) = \mathrm{tr}[(\nabla_x\Theta(0))\mathrm{L}_{y_0}^{\flat}[\nabla_x\Theta(0)]^{-1}] = \mathrm{tr}(\mathrm{L}_{y_0}^{\flat}) = \mathrm{tr}(\mathrm{L}_{y_0}) = 2\,\mathrm{H}. \qquad \text{(A.31)}
$$

To compute $\mathrm{Cof}\,\mathrm{B}_{y_0}$ we use that $\mathrm{Cof}\,(XY)=\mathrm{Cof}\,(X)\,\mathrm{Cof}\,(Y)$ for any $X, Y\in\mathbb{R}^{3\times3}$. It follows

$$
\begin{aligned}
\mathrm{Cof}\,\mathrm{B}_{y_0} &= \mathrm{Cof}\,[\nabla_x\Theta(0)]\,\mathrm{Cof}\,(\mathrm{L}_{y_0}^{\flat})\,\mathrm{Cof}\,[\nabla_x\Theta(0)]^{-1} \qquad \text{(A.32)} \\
&= \det[\nabla_x\Theta(0)]\cdot[\nabla_x\Theta(0)]^{-T}\,\mathrm{Cof}\begin{pmatrix} \mathrm{L}_{y_0} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}[\nabla_x\Theta(0)]^T\det[\nabla_x\Theta(0)]^{-1} \\
&= [\nabla_x\Theta(0)]^{-T}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{K} \end{pmatrix}[\nabla_x\Theta(0)]^T = (0|0|n_0)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{K} \end{pmatrix}(0|0|n_0)^T \\
&= [\nabla_x\Theta(0)]\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{K} \end{pmatrix}[\nabla_x\Theta(0)]^{-1}.
\end{aligned}
$$

The relation for $\mathrm{B}_{y_0}^2$ can be proved similarly without difficulties.

iii) In the following we prove the Cayley-Hamilton type equation. Using ii), we deduce

$$
\begin{aligned}
\mathrm{B}_{y_0}^2 - 2\,\mathrm{H}\,\mathrm{B}_{y_0} + \mathrm{K}\,\mathrm{A}_{y_0} &= \nabla_x\Theta(0)(\mathrm{L}_{y_0}^{\flat})^2[\nabla_x\Theta(0)]^{-1} - 2\,\mathrm{H}\,\nabla_x\Theta(0)\mathrm{L}_{y_0}^{\flat}[\nabla_x\Theta(0)]^{-1} \\
&\quad + \mathrm{K}\,(\nabla y_0|n_0)\,\mathbb{1}_2^{\flat}\,[\nabla_x\Theta(0)]^{-1} \\
&= \nabla_x\Theta(0)(\mathrm{L}_{y_0}^2 - 2\,\mathrm{H}\,\mathrm{L}_{y_0} + \mathrm{K}\,\mathbb{1}_2)^{\flat}\,[\nabla_x\Theta(0)]^{-1} = 0_3. \qquad \text{(A.33)}
\end{aligned}
$$

iv) In order to prove iv), we deduce from i) and ii)

$$\begin{aligned} \mathrm{A}_{y_0}\mathrm{B}_{y_0} &= [\nabla_x\Theta(0)]^{-T}\mathrm{I}^\flat_{y_0}[\nabla_x\Theta(0)]^{-1}\,\nabla_x\Theta(0)\mathrm{L}^\flat_{y_0}[\nabla_x\Theta(0)]^{-1} \\ &= [\nabla_x\Theta(0)]^{-T}\mathrm{I}^\flat_{y_0}\mathrm{L}^\flat_{y_0}[\nabla_x\Theta(0)]^{-1} = [\nabla_x\Theta(0)]^{-T}\mathrm{II}^\flat_{y_0}[\nabla_x\Theta(0)]^{-1} = \mathrm{B}_{y_0}\,. \end{aligned} \tag{A.34}$$

Moreover, we have

$$\begin{aligned} \mathrm{B}_{y_0}\mathrm{A}_{y_0} &= \mathrm{B}_{y_0}(\mathbb{1}_3 - (0|0|n_0)\,(0|0|n_0)^T) \\ &= \mathrm{B}_{y_0} - \nabla_x\Theta(0)\mathrm{L}^\flat_{y_0}[\nabla_x\Theta(0)]^{-1}(0|0|n_0)\,(0|0|n_0)^T. \end{aligned} \tag{A.35}$$

We notice that

$$\begin{aligned} &[\nabla_x\Theta(0)]^{-1}(0|0|n_0) = (0|0|e_3) \\ &\quad\Rightarrow\quad [\nabla_x\Theta(0)]^{-1}(0|0|n_0)\,(0|0|n_0)^T = (0|0|e_3)\,(0|0|n_0)^T = (0|0|n_0)^T \\ &\quad\Rightarrow\quad \mathrm{L}^\flat_{y_0}[\nabla_x\Theta(0)]^{-1}(0|0|n_0)\,(0|0|n_0)^T = \mathrm{L}^\flat_{y_0}(0|0|n_0)^T = (0|0|0) = 0_3\,. \end{aligned} \tag{A.36}$$

Using (A.36) in (A.35) we obtain $\mathrm{B}_{y_0}\mathrm{A}_{y_0} = \mathrm{B}_{y_0}$. Similarly, from (A.36) we find

$$\begin{aligned} \mathrm{A}^2_{y_0} &= \mathrm{A}_{y_0}(\mathbb{1}_3 - (0|0|n_0)\,(0|0|n_0)^T) = \mathrm{A}_{y_0} - [\nabla_x\Theta(0)]^{-T}\mathrm{I}^\flat_{y_0}[\nabla_x\Theta(0)]^{-1}(0|0|n_0)\,(0|0|n_0)^T \\ &= \mathrm{A}_{y_0} - [\nabla_x\Theta(0)]^{-T}\mathrm{I}^\flat_{y_0}(0|0|n_0)^T = \mathrm{A}_{y_0}\,. \end{aligned}$$

v) We consider two vectors $u_1, u_2 \in \mathbb{R}^3$ and we compute

$$\begin{aligned} (u_1|u_2|0)\,[\nabla_x\Theta(0)]^{-1}\,\mathrm{A}_{y_0} &= (u_1|u_2|0)\,[\nabla_x\Theta(0)]^{-1} \\ &\quad - (u_1|u_2|0)\,[\nabla_x\Theta(0)]^{-1}(0|0|\nabla_x\Theta(0)\,e_3)\,[\nabla_x\Theta(0)]^{-1} \\ &= (u_1|u_2|0)\,[\nabla_x\Theta(0)]^{-1} - (u_1|u_2|0)\,(0|0|e_3)\,[\nabla_x\Theta(0)]^{-1} \\ &= (u_1|u_2|0)\,[\nabla_x\Theta(0)]^{-1}. \end{aligned} \tag{A.37}$$

vi) Regarding item vi), remark that

$$\begin{aligned} \mathrm{C}_{y_0} &= \mathrm{Cof}(\nabla_x\Theta(0))\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}[\nabla_x\Theta(0)]^{-1} \\ &= Q_0(0)\,(\det U_0(0))\,U_0^{-1}(0)\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}U_0^{-1}(0)\,Q_0^T(0) \\ &= Q_0(0)\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}Q_0(0)^T, \end{aligned} \tag{A.38}$$

since $\det Q_0 = 1$ and

$$(\det U_0(0))\,U_0^{-1}(0)\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}U_0^{-1}(0) = \begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{A.39}$$

The alternator tensor has the representation Proposition A.3 v), which shows that C_{y_0} is antisymmetric.

Moreover, we deduce

$$
\begin{aligned}
\mathrm{C}_{y_0}^2 &= Q_0(0)\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}Q_0^T(0)\\
&= -Q_0(0)\,U_0(0)\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}U_0^{-1}(0)\,Q_0^T(0) \qquad\qquad (A.40)\\
&= -(\nabla y_0|\,n_0)\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}[\nabla_x\Theta(0)]^{-1} = -(\nabla y_0|0)\,[\nabla_x\Theta(0)]^{-1} = -\mathrm{A}_{y_0}.
\end{aligned}
$$

vii) Let us compute

$$
\begin{aligned}
Q_0^T(0)\,\partial_{x_\alpha}Q_0(0) &= \Big(d_1^0(0)\,|\,d_2^0(0)\,|\,d_3^0(0)\Big)^T\Big(\partial_{x_\alpha}d_1^0(0)\,|\,\partial_{x_\alpha}d_2^0(0)\,|\,\partial_{x_\alpha}d_3^0(0)\Big)\\
&= \begin{pmatrix}0 & \langle d_1^0(0),\partial_{x_\alpha}d_2^0(0)\rangle & \langle d_1^0(0),\partial_{x_\alpha}d_3^0(0)\rangle\\ \langle d_2^0(0),\partial_{x_\alpha}d_1^0(0)\rangle & 0 & \langle d_2(0),\partial_{x_\alpha}d_3(0)\rangle\\ \langle d_3^0(0),\partial_{x_\alpha}d_1^0(0)\rangle & \langle d_3^0(0),\partial_{x_\alpha}d_2^0(0)\rangle & 0\end{pmatrix}, \qquad (A.41)\\
\mathrm{axl}(Q_0^T(0)\,\partial_{x_\alpha}Q_0(0)) &= \Big(-\langle d_2^0(0),\partial_{x_\alpha}d_3^0(0)\rangle\,|\,\langle d_1^0(0),\partial_{x_\alpha}d_3^0(0)\rangle\,|\,-\langle d_1^0(0),\partial_{x_\alpha}d_2^0(0)\rangle\Big)^T,\\
&\qquad \alpha = 1,2.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\mathrm{C}_{y_0}\,Q_0(0)\Big(\mathrm{axl}(Q_0^T(0)\,\partial_{x_1}Q_0(0))\,|\,\mathrm{axl}(Q_0^T(0)\,\partial_{x_2}Q_0(0))\,|\,0\Big)[\nabla_x\Theta(0)]^{-1} \qquad (A.42)\\
&= Q_0(0)\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}-\langle d_2^0(0),\partial_{x_1}d_3^0(0)\rangle & -\langle d_2^0(0),\partial_{x_2}d_3^0(0)\rangle & 0\\ \langle d_1^0(0),\partial_{x_1}d_3^0(0)\rangle & \langle d_1^0(0),\partial_{x_2}d_3^0(0)\rangle & 0\\ -\langle d_1^0(0),\partial_{x_1}d_2^0(0)\rangle & -\langle d_1^0(0),\partial_{x_2}d_2^0(0)\rangle & 0\end{pmatrix}[\nabla_x\Theta(0)]^{-1}\\
&= Q_0(0)\begin{pmatrix}\langle d_1^0(0),\partial_{x_1}d_3^0(0)\rangle & \langle d_1^0(0),\partial_{x_2}d_3^0(0)\rangle & 0\\ \langle d_2^0(0),\partial_{x_1}d_3^0(0)\rangle & \langle d_2^0(0),\partial_{x_2}d_3^0(0)\rangle & 0\\ 0&0&0\end{pmatrix}[\nabla_x\Theta(0)]^{-1}\\
&= Q_0(0)\,Q_0^T(0)\Big(\partial_{x_1}d_3^0(0)\,|\,\partial_{x_2}d_3^0(0)\,|\,0\Big)[\nabla_x\Theta(0)]^{-1}\\
&= (\nabla n_0|0)\,[\nabla_x\Theta(0)]^{-1} = -\mathrm{B}_{y_0}. \qquad \square
\end{aligned}
$$

A.5 Neumann Condition on the Transverse Boundary: An Alternative Approach

One can also assume that on the transverse boundary (upper and lower face of the fictitious Cartesian configuration Ω_h) the Neumann condition

$$
S_1\Big(\nabla_x\varphi\big(x_1,x_2,\pm\frac{h}{2}\big),\overline{R}_s\big(x_1,x_2,\pm\frac{h}{2}\big)\Big)(\pm e_3) = 0 \qquad (A.43)
$$

holds. Using (3.31), this implies

$$
\Big\langle T_{\mathrm{Biot}}\Big(\overline{U}_{e,s}\big(x_1,x_2,\pm\frac{h}{2}\big)\Big)n_0,n_0\Big\rangle = 0\,. \qquad (A.44)
$$

Hence, for these boundary conditions, we obtain the following linear algebraic system of equations for the two unknown functions ϱ_m^e and ϱ_b^e

$$\begin{aligned}\varrho_m^e&\Big[(\lambda+2\mu)\pm\lambda\frac{h}{2}\langle\overline{Q}_e^T(\nabla(\overline{Q}_e\nabla_x\Theta(0)\,e_3)|0),\Big[\nabla_x\Theta(\pm\frac{h}{2})\Big]^{-T}\rangle\Big]\\&+\varrho_b^e\Big[\pm\frac{h}{2}(\lambda+2\mu)+\lambda\frac{h^2}{8}\langle\overline{Q}_e^T(\nabla(\overline{Q}_e\nabla_x\Theta(0)\,e_3)|0),\Big[\nabla_x\Theta(\pm\frac{h}{2})\Big]^{-T}\rangle\Big]\\&=\lambda+2\mu-\lambda[\langle\overline{Q}_e^T(\nabla m|0),\Big[\nabla_x\Theta(\pm\frac{h}{2})\Big]^{-T}\rangle-2]\,.\end{aligned}\tag{A.45}$$

Further, the dependence of $\nabla_x\Theta(x_1,x_2,x_3)$ on x_3 will be denoted by $\nabla_x\Theta(x_3)$. The above linear algebraic system is equivalent with the system

$$\begin{aligned}&\varrho_m^e\Big[2(\lambda+2\mu)+\lambda\frac{h}{2}\langle\overline{Q}_e^T(\nabla(\overline{Q}_e\nabla_x\Theta(0)\,e_3)|0),\Big[\nabla_x\Theta(\frac{h}{2})\Big]^{-T}-\Big[\nabla_x\Theta(-\frac{h}{2})\Big]^{-T}\rangle\Big]\\&\quad+\varrho_b^e\Big[\lambda\frac{h^2}{8}\langle\overline{Q}_e^T(\nabla(\overline{Q}_e\nabla_x\Theta(0)\,e_3)|0),\Big[\nabla_x\Theta(\frac{h}{2})\Big]^{-1}+\Big[\nabla_x\Theta(-\frac{h}{2})\Big]^{-T}\rangle\Big]\\&\quad=2(\lambda+2\mu)-\lambda[\langle\overline{Q}_e^T(\nabla m|0),\Big[\nabla_x\Theta(\frac{h}{2})\Big]^{-T}+\Big[\nabla_x\Theta(-\frac{h}{2})\Big]^{-T}-4],\qquad\text{(A.46)}\\&\varrho_m^e\lambda\frac{h}{2}\langle\overline{Q}_e^T(\nabla(\overline{Q}_e\nabla_x\Theta(0)\,e_3)|0),\Big[\nabla_x\Theta(\frac{h}{2})\Big]^{-T}+\Big[\nabla_x\Theta(-\frac{h}{2})\Big]^{-T}\rangle\Big]\\&\quad+\varrho_b^e\Big[h(\lambda+2\mu)+\lambda\frac{h^2}{8}\langle\overline{Q}_e^T(\nabla(\overline{Q}_e\nabla_x\Theta(0)\,e_3)|0),\Big[\nabla_x\Theta(\frac{h}{2})\Big]^{-T}-\Big[\nabla_x\Theta(-\frac{h}{2})\Big]^{-T}\rangle\Big]\\&\quad=-\lambda\langle\overline{Q}_e^T(\nabla m|0),\Big[\nabla_x\Theta(\frac{h}{2})\Big]^{-T}-\Big[\nabla_x\Theta(-\frac{h}{2})\Big]^{-T}\rangle.\end{aligned}$$

The determinant of this system is $\delta_1(h)=h\,\delta_2(h)$, where

$$\begin{aligned}\delta_2(h)&=2\,(\lambda+2\mu)^2\\&\quad+\frac{3h}{4}\lambda(\lambda+2\mu)\langle\overline{Q}_e^T(\nabla(\overline{Q}_e\nabla_x\Theta(0)\,e_3)|0),\Big[\nabla_x\Theta(\frac{h}{2})\Big]^{-T}-\Big[\nabla_x\Theta(-\frac{h}{2})\Big]^{-T}\rangle\\&\quad-\frac{h^2\lambda^2}{4}\langle\overline{Q}_e^T(\nabla(\overline{Q}_e\nabla_x\Theta(0)\,e_3)|0),\Big[\nabla_x\Theta(\frac{h}{2})\Big]^{-T}\rangle\\&\quad\times\langle\overline{Q}_e^T(\nabla(\overline{Q}_e\nabla_x\Theta(0)\,e_3)|0),\Big[\nabla_x\Theta(-\frac{h}{2})\Big]^{-T}\rangle>0\quad\text{for }h\ll1.\end{aligned}\tag{A.47}$$

We define the quantities

$$
\begin{aligned}
M^{h,+} &= \overline{Q}_e^T (\nabla(\overline{Q}_e \nabla_x \Theta(0)\, e_3)|0) \left\{ \left[\nabla_x \Theta(\frac{h}{2})\right]^{-1} + \left[\nabla_x \Theta(-\frac{h}{2})\right]^{-1} \right\}, \\
M^{h,-} &= \overline{Q}_e^T (\nabla(\overline{Q}_e \nabla_x \Theta(0)\, e_3)|0) \left\{ \left[\nabla_x \Theta(\frac{h}{2})\right]^{-1} - \left[\nabla_x \Theta(-\frac{h}{2})\right]^{-1} \right\}, \\
N^{h,+} &= \overline{Q}_e^T (\nabla m|0) \left\{ \left[\nabla_x \Theta(\frac{h}{2})\right]^{-1} + \left[\nabla_x \Theta(-\frac{h}{2})\right]^{-1} \right\}, \\
N^{h,-} &= \overline{Q}_e^T (\nabla m|0) \left\{ \left[\nabla_x \Theta(\frac{h}{2})\right]^{-1} - \left[\nabla_x \Theta(-\frac{h}{2})\right]^{-1} \right\}.
\end{aligned} \tag{A.48}
$$

Then, the exact solution is given by

$$
\begin{pmatrix} \varrho_m^e \\ \varrho_b^e \end{pmatrix} = \delta_1(h)^{-1} \begin{pmatrix} (\lambda+2\mu) + \lambda \frac{h}{8}\langle M^{h,-}, \mathbb{1}_3\rangle & -\lambda \frac{h}{8}\langle M^{h,+}, \mathbb{1}_3\rangle \\ -\frac{\lambda}{2}\langle M^{h,+}, \mathbb{1}_3\rangle & (\lambda+2\mu)\frac{2}{h} + \frac{\lambda}{2}\langle M^{h,-}, \mathbb{1}_3\rangle \end{pmatrix}
\times \begin{pmatrix} 2(\lambda+2\mu) - \lambda[\langle N^{h,+}, \mathbb{1}_3\rangle - 4] \\ -\lambda\langle N^{h,-}, \mathbb{1}_3\rangle \end{pmatrix}.
$$

Consider the quantities $C_1 = (\nabla y_0|n_0), C_2 = (\nabla n_0|0)$. Then $\nabla_x \Theta(x_3) = C_1 + C_2\, x_3$. Moreover

$$
\begin{aligned}
[\nabla_x \Theta(x_3)]^{-1} &= [\mathbb{1}_3 + x_3 C_1^{-1} C_2]^{-1} C_1^{-1} \\
&= [\mathbb{1}_3 - x_3 C_1^{-1} C_2 + x_3^2 (C_1^{-1} C_2)^2 - x_3^3 (C_1^{-1} C_2)^3 + \cdots] C_1^{-1}.
\end{aligned} \tag{A.49}
$$

Due to the fact that C_1 and C_2 do not depend on x_3, we have

$$
\begin{aligned}
&\left[\nabla_x \Theta(\frac{h}{2})\right]^{-1} - \left[\nabla_x \Theta(-\frac{h}{2})\right]^{-1} = -2 \left[\frac{h}{2} C_1^{-1} C_2 + \left(\frac{h}{2}\right)^3 (C_1^{-1} C_2)^3 + \cdots \right] C_1^{-1}, \\
&\left[\nabla_x \Theta(\frac{h}{2})\right]^{-1} + \left[\nabla_x \Theta(-\frac{h}{2})\right]^{-1} \\
&\quad = 2 \left[\mathbb{1}_3 + \left(\frac{h}{2}\right)^2 (C_1^{-1} C_2)^2 + \left(\frac{h}{2}\right)^4 (C_1^{-1} C_2)^4 + \cdots \right] C_1^{-1}.
\end{aligned} \tag{A.50}
$$

Hence, we deduce

$$
\begin{aligned}
M^{h,+} &= 2\, \overline{Q}_e^T (\nabla(\overline{Q}_e \nabla_x \Theta(0)\, e_3)|0)[\nabla_x \Theta(0)]^{-1} + o(h^2), \\
N^{h,+} &= 2\, \overline{Q}_e^T (\nabla m|0)[\nabla_x \Theta(0)]^{-1} + o(h^2), \\
M^{h,-} &= o(h), \quad N^{h,-} = -h\, \overline{Q}_e^T (\nabla m|0)[\nabla_x \Theta(0)]^{-1} (\nabla n_0|0)\, [\nabla_x \Theta(0)]^{-1} + o(h^2), \\
\delta_1 &= 2(\lambda+2\mu)^2 + o(h).
\end{aligned} \tag{A.51}
$$

and

$$\begin{aligned}\varrho_m^e &= \frac{1}{\delta_1(h)}\Big\{[(\lambda+2\mu)+\lambda\frac{h}{8}\langle M^{h,-},\mathbb{1}_3\rangle][2(\lambda+2\mu)-\lambda[\langle N^{h,+},\mathbb{1}_3\rangle-4]\\ &\quad+[\lambda\frac{h}{8}\langle M^{h,+},\mathbb{1}_3\rangle][\lambda\langle N^{h,-},\mathbb{1}_3\rangle]\Big\},\\ \varrho_b^e &= \frac{1}{\delta_1(h)}\Big\{[-\frac{\lambda}{2}\langle M^{h,+},\mathbb{1}_3\rangle][2(\lambda+2\mu)-\lambda[\langle N^{h,+},\mathbb{1}_3\rangle-4]-[(\lambda+2\mu)\frac{2}{h}\\ &\quad+\frac{\lambda}{2}\langle M^{h,-},\mathbb{1}_3\rangle][\lambda\langle N^{h,-},\mathbb{1}_3\rangle]\Big\}.\end{aligned} \tag{A.52}$$

If we take $h\to 0$ (as appropriate for a very thin shell) in the expressions (A.52) of ϱ_m^e and ϱ_b^e, then we obtain

$$\begin{aligned}\varrho_m^e &= \frac{1}{2(\lambda+2\mu)^2}\Big\{(\lambda+2\mu)[2(\lambda+2\mu)-\lambda[2\langle\overline{Q}_e^T(\nabla m|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle-4]\Big\},\\ &= 1-\frac{\lambda}{\lambda+2\mu}[\langle\overline{Q}_e^T(\nabla m|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle-2],\\ \varrho_b^e &= \frac{1}{2(\lambda+2\mu)^2}\Big\{-\frac{\lambda}{2}\langle 2\,\overline{Q}_e^T(\nabla(\overline{Q}_e\nabla_x\Theta(0)\,e_3)|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle\\ &\qquad\times\Big[2(\lambda+2\mu)-\lambda[\langle 2\,\overline{Q}_e^T(\nabla m|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle-4]\Big]\\ &\qquad+2\,(\lambda+2\mu)\lambda\langle\overline{Q}_e^T(\nabla m|0)[\nabla_x\Theta(0)]^{-1}\,(\nabla n_0|0)\,[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle]\Big\},\\ &= -\frac{\lambda}{\lambda+2\mu}\langle\overline{Q}_e^T(\nabla(\overline{Q}_e\nabla_x\Theta(0)\,e_3)|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle\\ &\quad+\frac{\lambda}{\lambda+2\mu}\langle\overline{Q}_e^T(\nabla m|0)[\nabla_x\Theta(0)]^{-1}\,(\nabla n_0|0)\,[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle]\\ &\quad+\frac{\lambda^2}{(\lambda+2\mu)^2}\langle\overline{Q}_e^T(\nabla(\overline{Q}_e\nabla_x\Theta(0)\,e_3)|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle\\ &\qquad\times[\langle\overline{Q}_e^T(\nabla m|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle-2]\Big].\end{aligned} \tag{A.53}$$

Ignoring the term $\frac{\lambda^2}{(\lambda+2\mu)^2}\langle\overline{Q}_e^T(\nabla(\overline{Q}_e\nabla_x\Theta(0)\,e_3)|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle[\langle\overline{Q}_e^T(\nabla m|0)[\nabla_x\Theta(0)]^{-1},\mathbb{1}_3\rangle-2]$ we obtain the same expressions for ϱ_m^e and ϱ_b^e as in (4.26), but here considering the Neumann condition (A.43) instead of the approximated boundary conditions (3.40). Therefore, our result in (4.26) is correct.

References

1. Altenbach, H., Eremeyev, V.A.: On the linear theory of micropolar plates. Z. Angew. Math. Mech. **89**(4), 242–256 (2009)
2. Altenbach, H., Eremeyev, V.A.: Cosserat–type shells. In: Altenbach, H., Eremeyev, V.A. (eds.) Generalized Continua – from the Theory to Engineering Applications. CISM Courses and Lectures, vol. 541, pp. 131–178. Springer, Wien (2013)

3. Altenbach, H., Zhilin, P.A.: The theory of simple elastic shells. In: Kienzler, R., Altenbach, H., Ott, I. (eds.) Theories of Plates and Shells. Critical Review and New Applications. Euromech Colloquium, vol. 444, pp. 1–12. Springer, Heidelberg (2004)
4. Ball, J.M.: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Ration. Mech. Anal. **63**, 337–403 (1977)
5. Bîrsan, M.: Derivation of a refined 6-parameter shell model: Descent from the three-dimensional Cosserat elasticity using a method of classical shell theory. Math. Mech. Solids **25**(6), 1318–1339 (2020)
6. Bîrsan, M., Altenbach, H.: Analysis of the deformation of multi-layered orthotropic cylindrical elastic shells using the direct approach. In: Altenbach, H., Eremeyev, V.A. (eds.) Shell-Like Structures: Non-Classical Theories and Applications, pp. 29–52. Springer, Berlin (2011)
7. Bîrsan, M., Neff, P.: Existence of minimizers in the geometrically non-linear 6-parameter resultant shell theory with drilling rotations. Math. Mech. Solids **19**(4), 376–397 (2014)
8. Bîrsan, M., Neff, P.: Shells without drilling rotations: a representation theorem in the framework of the geometrically nonlinear 6-parameter resultant shell theory. Int. J. Eng. Sci. **80**, 32–42 (2014)
9. Bîrsan, M., Ghiba, I.D., Martin, R.J., Neff, P.: Refined dimensional reduction for isotropic elastic Cosserat shells with initial curvature. Math. Mech. Solids **24**(12), 4000–4019 (2019)
10. Bischoff, M., Ramm, E.: Shear deformable shell elements for large strains and rotations. Int. J. Numer. Methods Eng. **40**, 4427–4449 (1997)
11. Bischoff, M., Ramm, E.: On the physical significance of higher order kinematic and static variables in a three-dimensional shell formulation. Int. J. Solids Struct. **37**, 6933–6960 (2000)
12. Borisov, L., Fischle, A., Neff, P.: Optimality of the relaxed polar factors by a characterization of the set of real square roots of real symmetric matrices. Z. Angew. Math. Mech. **99**(6), e201800120 (2019)
13. Bose, D.K., Kienzler, R.: On material conservation laws for a consistent plate theory. Arch. Appl. Mech. **75**(10–12), 607–617 (2006)
14. Braun, M., Bischoff, M., Ramm, E.: Nonlinear shell formulations for complete three-dimensional constitutive laws including composites and laminates. Comput. Mech. **15**, 1–18 (1994)
15. Büchter, N., Ramm, E.: Shell theory versus degeneration-a comparison in large rotation finite element analysis. Int. J. Numer. Methods Eng. **34**, 39–59 (1992)
16. Bunoiu, R., Ciarlet, Ph.G., Mardare, C.: Existence theorem for a nonlinear elliptic shell model. J. Elliptic Parabolic Equ. **1**(1), 31–48 (2015)
17. Burzyński, S., Chróścielewski, J., Witkowski, W.: Geometrically nonlinear FEM analysis of 6-parameter resultant shell theory based on 2-D Cosserat constitutive model. Z. Angew. Math. Mech. **96**, 191–204 (2016)
18. Chapelle, D., Collin, A.: Strong convergence results for the asymptotic behavior of the 3D-shell model. J. Elast. **115**, 173–192 (2014)
19. Chernykh, K.: Nonlinear theory of isotropically elastic thin shells. Mech. Solids **15**(2), 118–127 (1980). Transl. of Meh. Tverd. Tela
20. Chróścielewski, J., Makowski, J., Pietraszkiewicz, W.: Statics and Dynamics of Multifold Shells: Nonlinear Theory and Finite Element Method. Wydawnictwo IPPT PAN, Warsaw (2004) (in Polish)
21. Chróścielewski, J., Pietraszkiewicz, W., Witkowski, W.: On shear correction factors in the non-linear theory of elastic shells. Int. J. Solids Struct. **47**, 3537–3545 (2010)
22. Chróścielewski, J., Kreja, I., Sabik, A., Witkowski, W.: Modeling of composite shells in 6-parameter nonlinear theory with drilling degree of freedom. Mech. Adv. Mat. Struct. **18**, 403–419 (2011)
23. Ciarlet, Ph.G.: Mathematical Elasticity, Vol. III: Theory of Shells, 1st edn. North-Holland, Amsterdam (2000)
24. Ciarlet, Ph.G., Geymonat, G.: Sur les lois de comportement en élasticité non linéaire compressible. C. R. Acad. Sci. Paris, Ser. II **295**, 423–426 (1982)
25. Ciarlet, Ph.G., Lods, V.: Asymptotic analysis of linearly elastic shells. III. Justification of Koiter's shell equations. Arch. Ration. Mech. Anal. **136**, 191–200 (1996)
26. Ciarlet, Ph.G., Mardare, C.: Nonlinear Korn inequalities. J. Math. Pures Appl. **104**(6), 1119–1134 (2015)
27. Ciarlet, Ph.G., Mardare, C.: An existence theorem for a two-dimensional nonlinear shell model of Koiter's type. Math. Models Methods Appl. Sci. **28**(14), 2833–2861 (2018)
28. Ciarlet, Ph.G., Lods, V., Miara, B.: Asymptotic analysis of linearly elastic shells. II. Justification of flexural shell equations. Arch. Ration. Mech. Anal. **136**, 163–1190 (1996)
29. Ciarlet, Ph.G., Gogu, R., Mardare, C.: Orientation-preserving condition and polyconvexity on a surface: application to nonlinear shell theory. J. Math. Pures Appl. **99**, 704–725 (2013)
30. Ciarlet, Ph.G., Mălin, M., Mardare, C.: New estimates of the distance between two surfaces in terms of the distance between their fundamental forms. Anal. Appl. **17**(03), 363–392 (2019)

31. Cirak, F., Cisternas, J.E., Cuitino, A.M., Ertl, G., Holmes, P., Kevrekidis, I.G., Ortiz, M., Rotermund, H.H., Schunack, M., Wolff, J.: Oscillatory thermomechanical instability of an ultrathin catalyst. Science **300**, 1932–1936 (2003)
32. Cosserat, E., Cosserat, F.: Sur la théorie des corps minces. C. R. Acad. Sci. Paris **146**, 169–172 (1908)
33. Cosserat, E., Cosserat, F.: Théorie des corps déformables. Hermann et Fils, Paris (1909) (Reprint 2009)
34. Danescu, A., Ionescu, I.R.: Shell design from planar pre-stressed structures. Math. Mech. Solids **25**, 1247–1266 (2020)
35. Eremeyev, V.A., Pietraszkiewicz, W.: The nonlinear theory of elastic shells with phase transitions. J. Elast. **74**, 67–86 (2004)
36. Eremeyev, V.A., Pietraszkiewicz, W.: Local symmetry group in the general theory of elastic shells. J. Elast. **85**, 125–152 (2006)
37. Eremeyev, V.A., Pietraszkiewicz, W.: Thermomechanics of shells undergoing phase transition. J. Mech. Phys. Solids **59**, 1395–1412 (2011)
38. Fischle, A., Neff, P.: The geometrically nonlinear Cosserat micropolar shear–stretch energy. Part I: general parameter reduction formula and energy-minimizing microrotations in 2D. Z. Angew. Math. Mech. **97**(7), 828–842 (2017)
39. Fischle, A., Neff, P.: The geometrically nonlinear Cosserat micropolar shear–stretch energy. Part II: non-classical energy-minimizing microrotations in 3D and their computational validation. Z. Angew. Math. Mech. **97**(7), 843–871 (2017)
40. Fox, D.D., Raoult, A., Simo, J.C.: A justification of nonlinear properly invariant plate theories. Arch. Ration. Mech. Anal. **124**, 157–199 (1993)
41. Friesecke, G., James, R.D., Müller, S.: A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. Commun. Pure Appl. Math. **55**(11), 1461–1506 (2002)
42. Ge, Z., Kruse, H.P., Marsden, J.E.: The limits of Hamiltonian structures in three-dimensional elasticity, shells, and rods. J. Nonlinear Sci. **6**, 19–57 (1996)
43. Ghiba, I.D., Bîrsan, M., Lewintan, P., Neff, P.: The isotropic Cosserat shell model including terms up to $O(h^5)$. Part II: Existence of minimizers. J. Elast. (2020). https://doi.org/10.1007/s10659-020-09795-4. arXiv:2003.00549
44. John, F.: Rotation and strain. Commun. Pure Appl. Math. **14**, 391–413 (1961)
45. John, F.: Estimates for the derivatives of the stresses in a thin shell and interior shell equations. Commun. Pure Appl. Math. **18**, 235–267 (1965)
46. Kienzler, R.: On consistent plate theories. Arch. Appl. Mech. **72**(4–5), 229–247 (2002)
47. Kienzler, R., Altenbach, H., Ott, I.: Theories of Plates and Shells: Critical Review and New Applications, vol. 16. Springer, Berlin (2013)
48. Klingenberg, W.: A Course in Differential Geometry. Graduate Texts in Mathematics, vol. 51. Springer, New York (1978)
49. Koiter, W.T.: A consistent first approximation in the general theory of thin elastic shells. In: Koiter, W.T. (ed.) The Theory of Thin Elastic Shells, IUTAM Symposium, Delft, 1960, pp. 12–33. North-Holland, Amsterdam (1960)
50. Koiter, W.T.: Foundations and basic equations of shell theory. A survey of recent progress. In: Niordson, F.I. (ed.) Theory of Thin Shells IUTAM Symposium, Copenhagen, 1967 pp. 93–105. Springer, Heidelberg (1969)
51. Le Dret, H., Raoult, A.: From three-dimensional elasticity to nonlinear membranes. In: Ciarlet, Ph.G., Trabucho, L., Viano, J.M. (eds.) Asymptotic Methods for Elastic Structures. Proceedings of the International Conference, pp. 89–102. de Gruyter, Berlin (1995)
52. Libai, A., Simmonds, J.G.: The Nonlinear Theory of Elastic Shells, 2nd edn. Cambridge University Press, Cambridge (1998)
53. Lods, V., Miara, B.: Nonlinearly elastic shell models: a formal asymptotic approach. II. The flexural model. Arch. Ration. Mech. Anal. **142**, 355–374 (1998)
54. Marohnić, M., Tambača, J.: On a model of a flexural prestressed shell. Math. Methods Appl. Sci. **38**(18), 5231–5241 (2015)
55. Miara, B.: Nonlinearly elastic shell models: a formal asymptotic approach. I. The membrane model. Arch. Ration. Mech. Anal. **142**, 331–353 (1998)
56. Naghdi, P.M.: The theory of shells and plates. In: Flügge, S. (ed.) Handbuch der Physik, Mechanics of Solids, vol. VI a/2, pp. 425–640. Springer, Berlin (1972)
57. Naghdi, P.M., Rubin, M.B.: Restrictions on nonlinear constitutive equations for elastic shells. J. Elast. **39**, 133–163 (1995)
58. Neff, P.: Some results concerning the mathematical treatment of finite multiplicative elasto-plasticity. In: Hutter, K., Baaser, H. (eds.) SFB298: Deformation and Failure in Metallic and Granular Structures-Abschlussbericht. Lecture Notes in Applied and Computational Mechanics, vol. 10, pp. 251–274. Springer, Berlin (2003)

59. Neff, P.: A geometrically exact Cosserat-shell model including size effects, avoiding degeneracy in the thin shell limit. Part I: formal dimensional reduction for elastic plates and existence of minimizers for positive Cosserat couple modulus. Contin. Mech. Thermodyn. **16**, 577–628 (2004)
60. Neff, P.: Geometrically exact Cosserat theory for bulk behaviour and thin structures. Modelling and mathematical analysis. Universitäts- und Landesbibliothek, Technische Universität Darmstadt, Darmstadt (2004)
61. Neff, P.: A geometrically exact viscoplastic membrane-shell with viscoelastic transverse shear resistance avoiding degeneracy in the thin-shell limit. Part I: the viscoelastic membrane-plate. Z. Angew. Math. Phys. **56**(1), 148–182 (2005)
62. Neff, P.: Local existence and uniqueness for a geometrically exact membrane-plate with viscoelastic transverse shear resistance. Math. Methods Appl. Sci. **28**, 1031–1060 (2005)
63. Neff, P.: A finite-strain elastic-plastic Cosserat theory for polycrystals with grain rotations. Int. J. Eng. Sci. **44**, 574–594 (2006)
64. Neff, P.: A geometrically exact planar Cosserat shell-model with microstructure: existence of minimizers for zero Cosserat couple modulus. Math. Models Methods Appl. Sci. **17**, 363–392 (2007)
65. Neff, P., Chełmiński, K.: A geometrically exact Cosserat shell-model for defective elastic crystals. Justification via Γ-convergence. Interfaces Free Bound. **9**, 455–492 (2007)
66. Neff, P., Hong, K.-I.: The Reissner-Mindlin plate is the Γ-limit of linear Cosserat elasticity. In: Pietraszkiewiecz, W., Kreja, I. (eds.) Shell-Structures: Theory and Applications, vol. 2, pp. 91–94. Taylor & Francis, London (2009)
67. Neff, P., Münch, I.: Curl bounds Grad on SO(3). ESAIM Control Optim. Calc. Var. **14**, 148–159 (2008)
68. Neff, P., Hong, K.-I., Jeong, J.: The Reissner-Mindlin plate is the Γ-limit of Cosserat elasticity. Math. Models Methods Appl. Sci. **20**, 1553–1590 (2010)
69. Neff, P., Lankeit, J., Madeo, A.: On Grioli's minimum property and its relation to Cauchy's polar decomposition. Int. J. Eng. Sci. **80**, 207–217 (2014)
70. Neff, P., Bîrsan, M., Osterbrink, F.: Existence theorem for the classical nonlinear Cosserat elastic model. J. Elast. **121**(1), 119–141 (2015)
71. Neff, P., Bîrsan, M., Ghiba, I.D.: A higher order geometrically nonlinear Cosserat-shell model with initial curvature effects. Proc. Appl. Math. Mech. **19**(1), e201900351 (2019)
72. Neff, P., Fischle, A., Borisov, L.: Explicit global minimization of the symmetrized Euclidean distance by a characterization of real matrices with symmetric square. SIAM J. Appl. Algebra Geom. **3**(1), 31–43 (2019)
73. Pietraszkiewicz, W.: Consistent second approximation to the elastic strain energy of a shell. Z. Angew. Math. Mech. **59**, 206–208 (1979)
74. Pietraszkiewicz, W.: Finite Rotations and Lagrangian Description in the Non-linear Theory of Shells. Polish Sci., Warsaw (1979)
75. Pietraszkiewicz, W.: Finite Rotations in Structural Mechanics. Lectures Notes in Engineering, vol. 19. Springer, Berlin (1985)
76. Pietraszkiewicz, W., Eremeyev, V.A.: On natural strain measures of the non-linear micropolar continuum. Int. J. Solids Struct. **46**, 774–787 (2009)
77. Pietraszkiewicz, W., Konopińska, V.: Drilling couples and refined constitutive equations in the resultant geometrically non-linear theory of elastic shells. Int. J. Solids Struct. **51**, 2133–2143 (2014)
78. Pimenta, P.M., Edgard, E.S.A., Campello, E.M.B.: A fully nonlinear thin shell model of Kirchhoff-Love type. In: New Trends in Thin Structures: Formulation, Optimization and Coupled Problems, pp. 29–58. Springer, Berlin (2010)
79. Reissner, E.: The effect of transverse shear deformation on the bending of elastic plates. J. Appl. Mech. **12**, A69–A77 (1945)
80. Reissner, E.: Linear and nonlinear theory of shells. In: Fung, Y.C., Sechler, E.E. (eds.) Thin Shell Structures, pp. 29–44. Prentice Hall, Englewood Cliffs (1974)
81. Roehl, D., Ramm, E.: Large elasto-plastic finite element analysis of solids and shells with the enhanced assumed strain concept. Int. J. Solids Struct. **33**, 3215–3237 (1996)
82. Roychowdhury, A., Gupta, A.: Growth and non-metricity in Föppl-von Kármán shells. J. Elast. **140**, 337–348 (2020). https://doi.org/10.1007/s10659-020-09766-9
83. Sander, O., Neff, P., Bîrsan, M.: Numerical treatment of a geometrically nonlinear planar Cosserat shell model. Comput. Mech. **57**, 817–841 (2016)
84. Sansour, C., Bocko, J.: On hybrid stress, hybrid strain and enhanced strain finite element formulations for a geometrically exact shell theory with drilling degrees of freedom. Int. J. Numer. Methods Eng. **43**, 175–192 (1998)
85. Sauer, R.A., Ghaffari, R., Gupta, A.: The multiplicative deformation split for shells with application to growth, chemical swelling, thermoelasticity, viscoelasticity and elastoplasticity. Int. J. Solids Struct. **174**, 53–68 (2019)

86. Schmidt, R.: Polar decomposition and finite rotation vector in first order finite elastic strain shell theory. In: Pietraszkiewicz, W. (ed.) Finite Rotations in Structural Mechanics. Lecture Notes in Engineering., vol. 19. Springer, Berlin (1985)
87. Shirani, M., Luo, C., Steigmann, D.J.: Cosserat elasticity of lattice shells with kinematically independent flexure and twist. Contin. Mech. Thermodyn. **31**(4), 1087–1097 (2019)
88. Simo, J.C., Fox, D.D., Hughes, T.J.R.: Formulations of finite elasticity with independent rotations. Comput. Methods Appl. Mech. Eng. **95**, 277–288 (1992)
89. Steigmann, D.J.: Koiter's shell theory from the perspective of three-dimensional nonlinear elasticity. J. Elast. **111**, 91–107 (2013)
90. Tambaca, J.: A new linear shell model for shells with little regularity. J. Elast. **117**, 163–188 (2014)
91. Tambaća, J., Ljulj, M.: A Naghdi type nonlinear model for shells with little regularity. J. Elast. (2020, to appear)
92. Tambača, J., Tutek, Z.: A new linear Naghdi type shell model for shells with little regularity. Appl. Math. Model. **40**(23–24), 10549–10562 (2016)
93. Tambača, J., Velčić, I.: Semicontinuity theorem in the micropolar elasticity. ESAIM Control Optim. Calc. Var., **16**(2) 337–355 (2010)
94. Tambača, J., Velčić, I.: Existence theorem for nonlinear micropolar elasticity. ESAIM Control Optim. Calc. Var. **16**, 92–110 (2010)
95. Taylor, M., Shirani, M., Dabiri, Y., Guccione, J.M., Steigmann, D.J.: Finite elastic wrinkling deformations of incompressible fiber-reinforced plates. Int. J. Eng. Sci. **144**, 103138 (2019)
96. Wang, F.F., Steigmann, D.J., Dai, H.H.: On a uniformly-valid asymptotic plate theory. Int. J. Non-Linear Mech. **112**, 117–125 (2019)
97. Weinberg, K., Neff, P.: A geometrically exact thin membrane model-investigation of large deformations and wrinkling. Int. J. Numer. Methods Eng. **74**(6), 871–893 (2008)
98. Wisniewski, K.: Finite Rotation Shells. Basic Equations and Finite Elements for Reissner Kinematics. Springer, Berlin (2010)
99. Zhilin, P.A.: Mechanics of deformable directed surfaces. Int. J. Solids Struct. **12**, 635–648 (1976)
100. Zhilin, P.A.: Applied Mechanics – Foundations of Shell Theory. State Polytechnical University Publisher, Sankt Petersburg (2006) (In Russian)