J Elast (2020) 139:269–298
https://doi.org/10.1007/s10659-019-09752-w

Identification of Scale-Independent Material Parameters in the Relaxed Micromorphic Model Through Model-Adapted First Order Homogenization

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Dedicated to Prof. Dr. Dr. h. c. Hans-Dieter Alber on the occasion of his 70. birthday with great esteem

Received: 30 August 2019 / Published online: 16 October 2019
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Abstract We rigorously determine the scale-independent short range elastic parameters in the relaxed micromorphic generalized continuum model for a given periodic microstructure. This is done using both classical periodic homogenization and a new procedure involving the concept of apparent material stiffness of a unit-cell under affine Dirichlet boundary conditions and Neumann's principle on the overall representation of anisotropy. We explain our idea of "maximal" stiffness of the unit-cell and use state of the art first order numerical homogenization methods to obtain the needed parameters for a given tetragonal unit-cell. These results are used in the accompanying paper (d'Agostino et al. in J. Elast. 2019. Accepted in this volume) to describe the wave propagation including band-gaps in the same tetragonal metamaterial.

Keywords Anisotropy · Relaxed micromorphic model · Enriched continua · Micro-elasticity · Metamaterial · Size effects · Parameter identification · Periodic homogenization · Effective properties · Unit-cell · Micro-macro transition · Löwner matrix supremum · Effective medium · Tensor harmonic mean · Apparent stiffness tensors · Neumann's principle

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Mathematics Subject Classification (2000) 74A30 · 74A35 · 74A60 · 74B05 · 74M25 · 74Q15

1 Introduction

In this work we aim to partially homogenize a periodic linear elastic Cauchy material towards a relaxed micromorphic continuum. Classical homogenization is a mature subject and delivers rigorously the effective linear elastic stiffness tensor $\mathbb{C}_{\text{macro}}$, which then describes the (very) large scale response of the periodic structure in arbitrary boundary value problems (see Fig. 1). The classical homogenization approaches are valid when the characteristic size of the studied microstructure is orders of magnitudes smaller than the characteristic size of the structure (scale-separation hypothesis). The effective stiffness tensor $\mathbb{C}_{\text{macro}}$ can be conveniently characterized by looking at any possible unit-cell of the periodic Cauchy material and applying periodic boundary conditions (PBC).[1] The ensuing FEM-problem is nowadays routinely solved [74].

However, giving up the scale-separation hypothesis, if one is interested in calculating a finite sized sample of the periodic structure under general non-affine loadings, it becomes apparent that its response is hardly governed by a homogeneous linear elastic surrogate model [69, 76] (see Fig. 2). Here, higher order models [75] or extended continuum models come into play. A key feature of these models is that they are able to naturally describe the appearing size-effects (typically "smaller is stiffer").

This ability is either due to the incorporation of higher gradients of displacement or by amending the kinematics on the homogenized scale with suitable independent additional fields for which new balance equations have to be devised. This leads us to consider the micromorphic framework, introduced by Eringen and Mindlin [21–23, 54] following earlier work of the Cosserat brothers [15]. At present, a rigorous asymptotic expansion or a Γ-convergence result [13, Theorem 5.6] towards a homogenized model for this transition scale seems to be out of reach for periodic materials showing a high stiffness contrast of their phases as in the present case and other technical difficulties. However, heuristically and computationally [43, 44, 69, 71, 76], the micromorphic framework seems to be well suited in these circumstances. In the absence of any comprehensive rigorous result (but see [75]), we therefore postulate for this work that the suitably homogenized model will be of the micromorphic type.

A serious drawback of these models is the increasing number of unknown material parameters, that have to be identified as well as the problematic interpretation of the new kinematical fields.

It is therefore mandatory to devise homogenization rules (analytical or computational) in order to determine relevant material parameters. While the question of homogenization towards an extended continuum model has seen a lot of effort in the last two decades (championed by Samuel Forest and his group [25, 27, 29], recently also Hütter [39, 40]) it is fair to say that no universally valid answer has been found.

The performed heuristic homogenization procedures typically involve the following steps:

[1] As is well-known, under affine loading, the response of a large periodic structure is periodic up to a vanishing boundary layer.

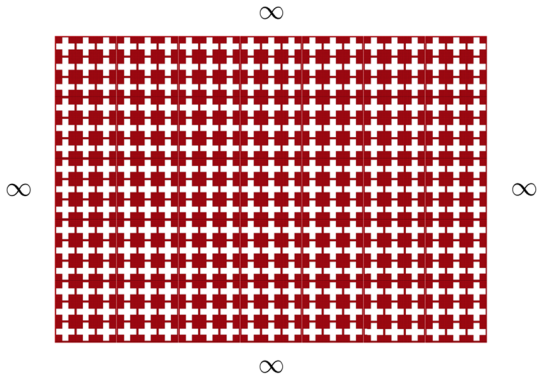


Fig. 1 Infinite periodic structure. The effective properties are given by the effective stiffness tensor $\mathbb{C}_{\text{macro}}$ which can be obtained rigorously by Γ-convergence and asymptotic homogenization

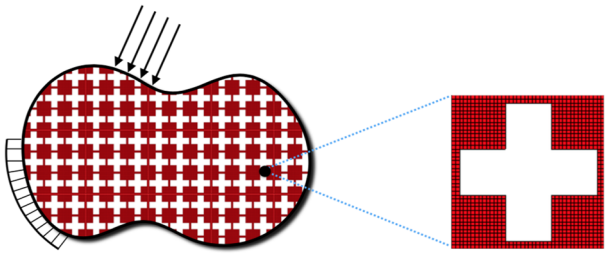


Fig. 2 Finite sized sample of a periodic structure. The scale-separation hypothesis does not anymore apply

a) a postulate on the form of the extended balance equations for generalized macroscopic stresses,
b) a postulate on the relation between microscopic Cauchy stresses and generalized macroscopic stresses, without specifying the precise macroscopic constitutive relations (e.g., a priori restrictions on material symmetry on the micro or macro-scale are not taken into account),
c) a generalized Hill-Mandel postulate expressing energy equivalence between the micro and the macro-scale,
d) a postulate on the connection between microscopic displacements and the additional macroscopic fields,[2]
e) use of higher order boundary conditions on the unit-cell level for triggering inhomogeneous response and activating the higher order effects.

The combination of these 5 steps can involve analytical and computational procedures and leads to some form of "homogenized" material properties. Step a) seems to be uncritical, since most authors working in the field of extended continuum mechanics agree on the same or similar format of balance relations [19, 21, 22, 39, 40, 54]. Step b) already implies some averaging rules [39] and is therefore open to discussion. Step c) is a generalization of the procedure in homogenization towards linear elasticity but involves some arbitrariness, since the proper form of boundary conditions has not yet been found. Step d) seems to be essential but no general agreement has been reached. For example [26, 27, 29] identifies the

[2]For the micromorphic model, postulate d) implies a direct interpretation of what the new degrees of freedom (the non symmetric micro-distortion $P \in \mathbb{R}^{3\times 3}$) is. While we do not discard such a direct micro-macro relation, we rather believe that any simple relation will fall short of the truth for the relaxed micromorphic model.

micro-distortion $P \in \mathbb{R}^{3\times 3}$ (the additional kinematical field) with the first moment of the microscopic displacement field on the unit-cell level. Typically the micro-distortion cannot be prescribed by a boundary condition at the micro-scale in contrast to the classical approach. Hütter [39, 40] follows Forest [25, 27, 29, 79] but introduces in addition a weighting function. In classical first order homogenization, d) is a consequence of c). Finally, step e) seems to be a natural consequence of the procedure to close the argument and has been adopted by Gologanu et al. [32], Kouznetsova et al. [43, 44], Forest and Sab [28], Forest and Trinh [29], Jänicke [41] and Diebels et al. [17].

Let us point out the problematic features of the sketched procedure. There are a number of natural requirements the homogenization process should ideally satisfy:

1.) if the material on the micro-scale is homogeneous and linear elastic, the homogenization should leave invariant the response, i.e., homogeneous Cauchy elasticity on the micro-scale turns into the same Cauchy response on the macro-scale,
2.) there should be a clear separation between (short-range) scale-independent parameters (e.g., a classical shear modulus) and (long-range) scale-dependent parameters (e.g., a characteristic length scale) in the homogenized model,
3.) for very large sample sizes the response should still be governed by linear elasticity with effective stiffness tensor $\mathbb{C}_{\text{macro}}$, and this should induce useful relations between the scale-independent material parameters of the micromorphic model. Here, we call **scale-independent material parameters** those coefficients in the extended continuum model that uniquely determine the large scale response,
4.) the stiffness of the homogenized model should be bounded irrespective of the sample size (as the input micro-scale stiffness is certainly bounded). In other words, the possible storage of elastic energy in arbitrarily small windows on the homogenized scale should be bounded for all non-affine boundary conditions. This avoids unstable parameter identification in a series of size-experiments [61],
5.) the homogenized parameters need to be true material parameters, independent of the applied loading and not influenced by boundary layer effects [48].

At present, it is not known whether requirement 1.) can be satisfied by available homogenization rules, see [40] (for example the approach of Forest seems not to be suitable). Requirement 2.) is usually not addressed but a closer look reveals that it may be violated in the Cosserat framework. Requirement 3.) seems to be mostly ignored and we will indicate that it cannot be met in the general micromorphic setting. Requirement 4.) has led to restrictions in the Cosserat model for bending and torsion [61]. In general, it cannot be satisfied for second gradient continua. Finally, requirement 5.) is difficult to establish. In [30, 48] it is shown that parameters in second gradient formulations may be boundary value dependent due to the presence of nontrivial null-Lagrangians. Thus, 4.) and 5.) seem to rule out second gradient formulations for our purpose.

In this work we want to deliberately dispense with the postulates b), c) and d) and instead place ourselves in a variational context and postulate the extended micromorphic kinematics together with a postulate on the suitable form of the free energy in an effort to reduce the complexity and intrinsic problems of the general Eringen-Mindlin micromorphic model [72]. We have called our model "relaxed" micromorphic model. Our goal is to perform an identification of the (short-range) scale-independent material parameters, consistent with the requirements 1) ... 5) towards homogenization. The rational of the relaxed micromorphic model will guide us and provide some surprising novel routes. Notably, the scale-independent material parameters will be determined solely with standard methods of first order homogenization which is sensible for the described short-range interaction.

1.1 The Static Relaxed Micromorphic Model

The relaxed micromorphic model is a generalized continuum model which includes a characteristic size. For the static case, it can be written in a variational framework. The goal is to find the macroscopic mean displacement $u : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ and the non-symmetric micro-distortion field $P : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ minimizing

$$\int_\Omega W(\nabla u, P, \operatorname{Curl} P) - \langle f, u\rangle \, dx \longrightarrow \min . \quad (u, P) \in H^1(\Omega) \times H(\operatorname{Curl}) \tag{1}$$

where the energy is represented as (see [6, 47, 50, 51, 60])

$$\begin{aligned} W(\nabla u, P, \operatorname{Curl} P) = & \underbrace{\frac{1}{2}\langle \mathbb{C}_{\mathrm{e}} \operatorname{sym}(\nabla u - P), \operatorname{sym}(\nabla u - P)\rangle_{\mathbb{R}^{3\times 3}}}_{\text{anisotropic elastic - energy}} \\ & + \underbrace{\frac{1}{2}\langle \mathbb{C}_{\mathrm{micro}} \operatorname{sym} P, \operatorname{sym} P\rangle_{\mathbb{R}^{3\times 3}}}_{\text{micro - self - energy}} \\ & + \underbrace{\frac{1}{2}\langle \mathbb{C}_{\mathrm{c}} \operatorname{skew}(\nabla u - P), \operatorname{skew}(\nabla u - P)\rangle_{\mathbb{R}^{3\times 3}}}_{\substack{\text{invariant local fourth-order anisotropic} \\ \text{rotational elastic coupling}}} + \underbrace{\frac{\mu L_c^2}{2}\|\operatorname{Curl} P\|^2}_{\text{relaxed curvature}} . \end{aligned} \tag{2}$$

Here, $\mathbb{C}_{\mathrm{e}}$, $\mathbb{C}_{\mathrm{micro}}$ are standard positive definite elasticity tensors with minor and major symmetries mapping symmetric matrices to symmetric matrices, $\mathbb{C}_{\mathrm{c}}$ is a positive semi-definite rotational coupling tensor mapping skew-symmetric matrices to skew-symmetric matrices, $L_c \geq 0$ is a characteristic length scale and μ is a typical effective shear modulus.[3] The Curl-operator $\operatorname{Curl} P$ acts row-wise on P. In all discussed models, linearized frame-indifference dictates that the scale-independent contribution of P can only occur through a dependence on $\operatorname{sym} P$. This framework is based on the additive split

$$\nabla u = e + P, \qquad 0 = \operatorname{Curl} e + \operatorname{Curl} P \tag{3}$$

of the total displacement gradient into meso-scale and micro-scale contributions, e and P respectively. While ∇u is compatible by definition, e and P are in general incompatible, i.e., not a gradient of a vector-field. Note that ∇u and P are still macroscopic variables.[4] The role of P is twofold. On the one hand it describes the collective interaction in a cluster of unit-cells, e.g., 2×2 or 3×3 unit-cells, i.e., the interaction between unit-cells up to a certain range. On the other hand, P also influences the short-range stiffness. Thus P mainly represents a long-range fluctuation field giving rise to higher order moment stresses in contrast to $e = \nabla u - P$, which describes the remaining short-range elastic interaction in and between neighboring unit-cells (the unit-cells represent the macroscopic points of the homogenized continuum only under the scale-separation hypothesis) leading to the force-stresses. Missing is the micro-fluctuation (inside a unit-cell). This micro-fluctuation is conceptionally averaged out in the micromorphic model. In this view, (depending on the characteristic

[3] For the presentation we have chosen throughout the simplest representation of the curvature energy – a one constant isotropic format.

[4] Despite the name micromorphic model.

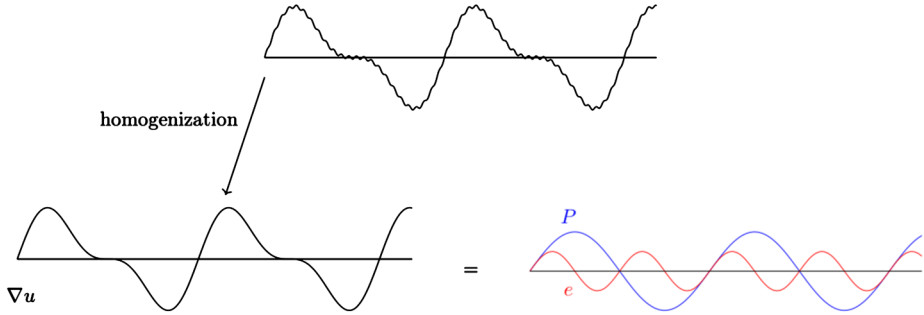


Fig. 3 Long-range fluctuation field P, short-range elastic interaction scale e and averaged out micro-fluctuation

length L_c) we claim that the micro-distortion P is not necessarily related to some average of micro-displacements in the unit-cell (as done by Forest [25, 27, 29], Hütter [39, 40], Biswas and Poh [10] among others) but P is a fully nonlocal object and truly an independent field not slave to the displacement[5] (see Fig. 3).

The static equilibrium equations are the Euler-Lagrange equations to (1) and (2), which read in strong form[6]

$$\begin{aligned} &\operatorname{Div}\big[\mathbb{C}_{\mathrm{e}}\operatorname{sym}(\nabla u - P) + \mathbb{C}_{\mathrm{c}}\operatorname{skew}(\nabla u - P)\big] = f, \\ &\mathbb{C}_{\mathrm{e}}\operatorname{sym}(\nabla u - P) + \mathbb{C}_{\mathrm{c}}\operatorname{skew}(\nabla u - P) - \mathbb{C}_{\mathrm{micro}}\operatorname{sym} P - \mu\, L_c^2 \operatorname{Curl}\operatorname{Curl} P = 0. \end{aligned} \tag{4}$$

While the relaxed micromorphic model allows for balance equations in the classical format of the Eringen-Mindlin approach (with second-order force stresses and third order moment stresses), the generalized moment balance $(4)_2$ is conveniently written in a reduced format with a second-order moment tensor $m = \operatorname{Curl} P$, as in the better known Cosserat model. In fact, one can say that the relaxed micromorphic model has the full micromorphic kinematics but uses the curvature measure of the Cosserat model. The generalized moment balance $(4)_2$ can be seen as a **tensorial Maxwell-problem** due to the Curl Curl operation.

Since only $\operatorname{sym} P$ and $\operatorname{Curl} P$ are controlled in the energy (2) the surprising well-posedness of formulation (4) has been rigorously shown in appropriate Sobolev spaces crucially based on new coercive Korn-type inequalities for incompatible tensor fields P of the form

$$\begin{aligned} &\exists\, c^+ > 0: \quad \forall P \in H(\operatorname{Curl}),\ P \times \vec{n}|_{\Gamma} = 0 \\ &\qquad \|\operatorname{sym} P\|^2_{L^2(\Omega)} + \|\operatorname{Curl} P\|^2_{L^2(\Omega)} \geq c^+\big(\|P\|^2_{L^2(\Omega)} + \|\operatorname{Curl} P\|^2_{L^2(\Omega)}\big). \end{aligned} \tag{5}$$

Here $\vec{n}$ is a unit vector orthogonal to $\Gamma \subset \partial\Omega$, see [8, 57, 63, 65, 66]. Note that $P \in H(\operatorname{Curl})$ is not necessarily a continuous field (see Fig. 4). For $P = \nabla u \in H(\operatorname{Curl})$, inequality (5) turns into a version of Korn's first inequality.

[5]The situation is different when one considers homogenization towards a second gradient continuum (or micromorphic approximations thereof) where there is no independent kinematical field. More precisely, the case $\widehat{\mathbb{C}}_{\mathrm{e}} \gg 1$, $L_c \ll 1$ would be consistent with determining P as some average of the micro-displacements over a unit-cell.

[6]Equation $(4)_1$ and $(4)_2$ together imply $\operatorname{Div}(\mathbb{C}_{\mathrm{micro}}\operatorname{sym} P) = f$. Constraining $\operatorname{sym} P = \operatorname{sym}\nabla u$ gives $\operatorname{Div}(\mathbb{C}_{\mathrm{micro}}\operatorname{sym}\nabla u) = f$.

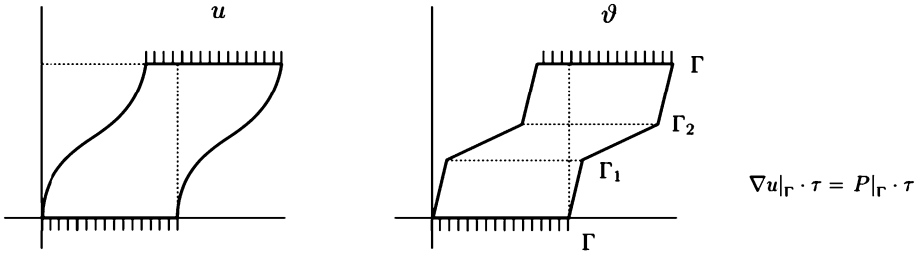


Fig. 4 Shear test on a block. Vertical sides free. Admissible configuration $(u, P) \in H^1(\Omega) \times H(\mathrm{Curl})$ and boundary conditions for the relaxed micromorphic model. Consider $P = \nabla\vartheta \in H(\mathrm{Curl})$, where $\vartheta : \mathbb{R}^3 \to \mathbb{R}^3$ is merely a Lipschitz function whose weak derivatives $\nabla\vartheta$ may have jumps. Along a given interface Γ_1, Γ_2, the micro-distortion P may have normal jumps but is tangentially continuous. The tangential boundary condition for the micro-distortion P applies only at the upper and lower face Γ where the displacement u is prescribed

Using (5), the well-posedness of (4) then only needs that $\mathbb{C}_e$, $\mathbb{C}_{\mathrm{micro}}$ are positive definite, while $\mathbb{C}_c$ may be positive semi-definite (e.g., $\mathbb{C}_c$ may be even absent) and $\mu\, L_c^2 \geq 0$, see [31, 59, 60].

1.2 Comparison to Eringen-Mindlin Micromorphic Models

The status of the relaxed micromorphic model within the framework of Eringen-Mindlin micromorphic approaches or in relation to higher gradient continua is discussed at length in [31, 49, 50, 59, 60]. Here, the following remarks can be made: putting all non-essential differences aside, the energy of a typical Eringen-Mindlin micromorphic approach can be written as

$$W_{\mathrm{EM}}(\nabla u, P, \nabla P) = \frac{1}{2}\underbrace{\langle \widehat{\mathbb{C}}_e(\nabla u - P), \nabla u - P\rangle_{\mathbb{R}^{3\times 3}}}_{\text{relative elastic energy}} + \frac{1}{2}\underbrace{\langle \mathbb{C}_{\mathrm{micro}}\,\mathrm{sym}\,P, \mathrm{sym}\,P\rangle_{\mathbb{R}^{3\times 3}}}_{\text{micro - self - energy}}$$
$$+ \underbrace{\frac{\mu\, L_c^2}{2}\|\nabla P\|^2}_{\text{full curvature}}, \tag{6}$$

where $\widehat{\mathbb{C}}_e$ is a non-standard positive definite fourth-order elasticity tensor, with 45-independent entries, mapping non-symmetric second-order tensors to non-symmetric second-order tensors.[7] Observe that $\nabla P = \nabla e - \nabla\nabla u$ couples derivatives of e and ∇u contrary to the decoupling in (3). The solution (u, P) will be typically in $H^1(\Omega) \times H^1(\Omega)$. The standard micromorphic equilibrium equations in strong form read[8]

$$\begin{aligned} \mathrm{Div}\,\widehat{\mathbb{C}}_e\,(\nabla u - P) &= f, \\ \widehat{\mathbb{C}}_e\,(\nabla u - P) - \mathbb{C}_{\mathrm{micro}}\,\mathrm{sym}\,P + \mu\, L_c^2 \Delta P &= 0. \end{aligned} \tag{7}$$

[7] The curvature expression in the Eringen-Mindlin-model or gradient elasticity model would typically include a sixth-order tensor [5], in contrast to the relaxed micromorphic model, which only needs a fourth-order tensor.

[8] Equation $(7)_1$ and $(7)_2$ together imply that $\mathrm{Div}[\mathbb{C}_{\mathrm{micro}}\,\mathrm{sym}\,P - \mu\, L_c^2 \Delta P] = f$. Here $\mathbb{C}_{\mathrm{micro}}$ is invariably coupled to the characteristic length L_c. Constraining $P = \nabla u$ gives $\mathrm{Div}[\mathbb{C}_{\mathrm{micro}}\,\mathrm{sym}\,\nabla u - \mu\, L_c^2 \Delta\nabla u] = f$. This is the fourth-order equilibrium equation of the second gradient formulation (8).

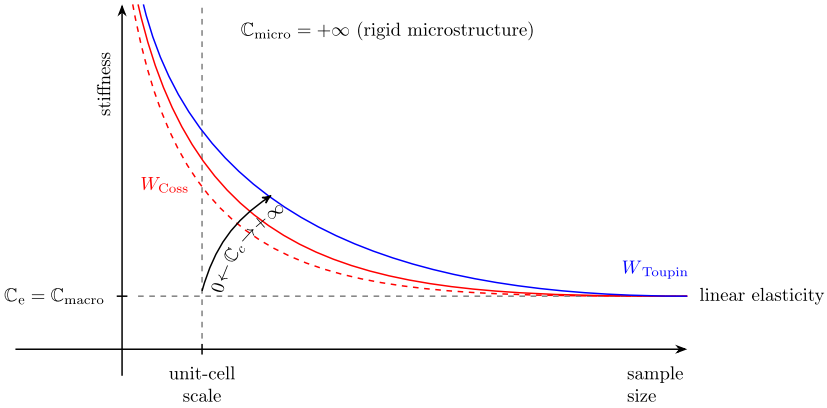


Fig. 5 Qualitative relation between the Cosserat model and Toupin's version of gradient elasticity. Cosserat elasticity can be seen as the formal limit $\mathbb{C}_{\text{micro}} \to +\infty$ of the relaxed micromorphic model. Moreover, Cosserat elasticity is a penalty formulation ($\mathbb{C}_c \to +\infty$) of Toupin's couple-stress model

The Eringen-Mindlin setting contains gradient elasticity W_{GE} by constraining $\nabla u = P$ (i.e., $\widehat{\mathbb{C}}_e \to +\infty$), such that the variational problem is based on

$$W_{\text{GE}}(\nabla u) = \frac{1}{2}\langle \mathbb{C}_{\text{micro}} \operatorname{sym} \nabla u, \operatorname{sym} \nabla u\rangle + \frac{\mu\, L_c^2}{2} \left\| \nabla(\nabla u) \right\|^2. \tag{8}$$

Here, no split in micro- and meso-scale is possible. The solution u will be in $H^2(\Omega)$. On the other hand, W_{EM} can be viewed as a penalty formulation of gradient elasticity (penalty $\mathbb{C}_e \to +\infty$).[9]

Next, the Cosserat model does not live on two scales either; it can be seen as the formal singular limit of the relaxed micromorphic model for $\mathbb{C}_{\text{micro}} \to +\infty$ in which $P = A$ must be skew-symmetric. The corresponding energy is

$$\begin{aligned} W_{\text{Coss}}(\nabla u, A, \operatorname{Curl} A) = {} & \frac{1}{2}\langle \mathbb{C}_e \operatorname{sym}\nabla u, \operatorname{sym}\nabla u\rangle + \frac{1}{2}\big\langle \mathbb{C}_c \operatorname{skew}(\nabla u - A), \operatorname{skew}(\nabla u - A)\big\rangle \\ & + \frac{\mu\, L_c^2}{2} \|\operatorname{Curl} A\|^2, \end{aligned} \tag{9}$$

with equilibrium equations

$$\begin{aligned} \operatorname{Div}\big[\mathbb{C}_e \operatorname{sym}\nabla u + \mathbb{C}_c \operatorname{skew}(\nabla u - A)\big] &= f, \\ \operatorname{skew}\big(\mathbb{C}_c \operatorname{skew}(\nabla u - A) - \mu\, L_c^2 \operatorname{Curl}\operatorname{Curl} A\big) &= 0. \end{aligned} \tag{10}$$

Letting finally $\mathbb{C}_c \to +\infty$, we obtain Toupin's indeterminate couple stress model [30, 48, 55, 62] with energy

$$W_{\text{Toupin}}(\nabla u, \operatorname{Curl} \operatorname{skew} \nabla u) = \frac{1}{2}\langle \mathbb{C}_e \operatorname{sym} \nabla u, \operatorname{sym} \nabla u\rangle + \frac{\mu\, L_c^2}{2} \|\operatorname{Curl} \operatorname{skew} \nabla u\|^2. \tag{11}$$

[9] The relaxed micromorphic model cannot be obtained as penalty formulation of gradient elasticity and in a 1-D setting it reduces to linear elasticity with stiffness $\mathbb{C}_{\text{macro}}$.

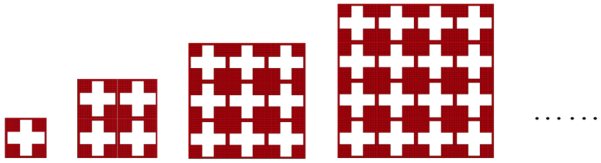


Fig. 6 "Smaller is stiffer": $N \times N$ array of unit cells. The apparent stiffness $\mathbb{C}_{\mathrm{KUBC}}$ (stiffness under applied kinematically uniform boundary conditions) decreases with increasing size to reach in the limit of infinite size $\mathbb{C}_{\mathrm{macro}}$. This stiffness-delta between the smallest discernable size and the infinite body will be essential for the determination of the scale-independent parameters in the relaxed micromorphic model and is, on the other hand, responsible for the size-effect. The classical RVE needs to be so large such that its apparent stiffness approximates already that of the macroscopic specimen. Using periodic boundary conditions (PBC) on a unit cell mimics the large scale response and discards the length-scale effect, but allows to determine the effective stiffness on a unit-cell

Since Curl is isomorphic to ∇ on skew-symmetric matrices [64], the curvature expression in (9) and (11) is fully general, despite appearance (see Fig. 5).

1.3 Scaling Relations and the Relaxed Micromorphic Model

The relation of the characteristic length L_c to the dimensions of the material sample follow from a simple scaling argument. Considering a transformation of an arbitrary domain to a unit domain, the length scale L_c w.r.t. the unit domain is inversely proportional to the domain size such that $L_c \to 0$ for very large samples and $L_c \to \infty$ for small samples. Thus, L_c encodes the relative interaction strength of the microstructure. It is useful to apply a scaling transformation to the sequence in Fig. 6 and to refer every block of cells to the same unit domain $[0, 1] \times [0, 1]$

The scaling to the unit-domain in the relaxed micromorphic model (Fig. 7) modifies only the characteristic length. On the unit-domain we have accordingly

$$\begin{aligned}
&\int_{[0,1]^2} \frac{1}{2}\big\langle \mathbb{C}_{\mathrm{e}}\,\mathrm{sym}(\nabla u - P), \mathrm{sym}(\nabla u - P)\big\rangle \\
&\quad + \frac{1}{2}\big\langle \mathbb{C}_{\mathrm{c}}\,\mathrm{skew}(\nabla u - P), \mathrm{skew}(\nabla u - P)\big\rangle \\
&\quad + \frac{1}{2}\langle \mathbb{C}_{\mathrm{micro}}\,\mathrm{sym}\,P, \mathrm{sym}\,P\rangle + \frac{\mu}{2}\left(\frac{\widehat{L}_c}{N}\right)^2 \|\mathrm{Curl}\,P\|^2\,dx \longrightarrow \min. \qquad (12)
\end{aligned}$$

A very large sample with many crosses $N \to \infty$ corresponds to $L_c = \frac{\widehat{L}_c}{N} \to 0$ and should deliver linear elasticity with $\mathbb{C}_{\mathrm{macro}}$ as response of the relaxed micromorphic model. This will generate a first necessary condition between $\mathbb{C}_{\mathrm{e}}$, $\mathbb{C}_{\mathrm{c}}$, $\mathbb{C}_{\mathrm{micro}}$ and $\mathbb{C}_{\mathrm{macro}}$ for a given microstructure, considered in the next subsection.

1.4 Linear Elastic Effective Macroscopic Response and Tensor Harmonic Mean

For characteristic length $L_c \to 0$, the relaxed micromorphic equilibrium equations (4) turn into

$$\begin{aligned}
\mathrm{Div}\big[\mathbb{C}_{\mathrm{e}}\,\mathrm{sym}(\nabla u - P) + \mathbb{C}_{\mathrm{c}}\,\mathrm{skew}(\nabla u - P)\big] &= f, \\
\mathbb{C}_{\mathrm{e}}\,\mathrm{sym}(\nabla u - P) + \mathbb{C}_{\mathrm{c}}\,\mathrm{skew}(\nabla u - P) - \mathbb{C}_{\mathrm{micro}}\,\mathrm{sym}\,P &= 0. \qquad (13)
\end{aligned}$$

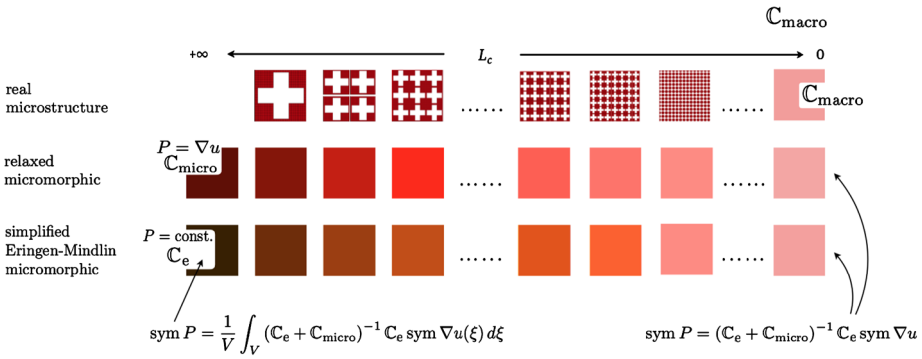


Fig. 7 (Stiffness in affine loading w.r.t. displacement u, no boundary condition on P) First line: $N \times N$ arrays with crosses after transformation to the unit domain. Here, volume fractions of air (white) and aluminium (red) are constant. Note that the formerly smallest block (on the left) still has bounded stiffness under affine Dirichlet boundary conditions. There is no given morphology corresponding to $L_c = +\infty$. For arbitrary many crosses (right) the affine stiffness approximates the effective stiffness $\mathbb{C}_{\text{macro}}$. Second line: the relaxed micromorphic model is intended to model the response of an $N \times N$ array of crosses, down to perhaps an 3×3 or 4×4 array and to improve upon a modelling solely with $\mathbb{C}_{\text{macro}}$ under arbitrary boundary conditions. The characteristic length L_c w.r.t. the unit-domain scales inversely with the size. $L_c \to \infty$ corresponds to using $P = \nabla u$ and yields linear elasticity with stiffness $\mathbb{C}_{\text{micro}}$, while letting $L_c \to 0$ yields linear elasticity with stiffness $\mathbb{C}_{\text{micro}}(\mathbb{C}_{\text{micro}} + \mathbb{C}_{\text{e}})^{-1}\mathbb{C}_{\text{e}}$, which must be identified with $\mathbb{C}_{\text{macro}}$. Third line: for comparison, we depicted a simplified Eringen-Mindlin micromorphic model with scale-independent constitutive law coinciding with the relaxed micromorphic model. For $L_c \to \infty$ the micro-distortion P is necessarily constant and the generated stiffness of the simplified Eringen-Mindlin model in affine loading approximates $\mathbb{C}_{\text{e}}$. The constant micro-distortion P gets coupled to a volume average of ∇u (Color figure online)

Observing that $(13)_2$ implies $\mathbb{C}_{\text{c}} \operatorname{skew}(\nabla u - P) = 0$ we may solve for $\operatorname{sym} P$ in $(13)_2$ and reinserting the result in $(13)_1$ we obtain [6] the *scale-independent large scale linear elastic response*[10]

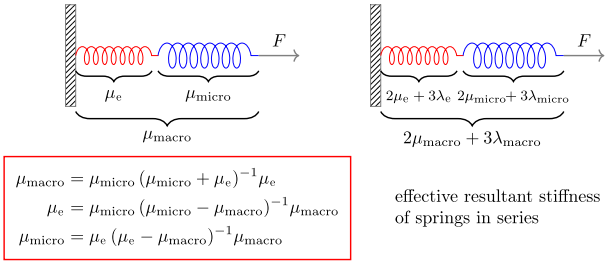
$$\operatorname{Div} \mathbb{C}_{\text{macro}} \operatorname{sym} \nabla u = f. \tag{14}$$

In [6] it is shown that the effective macroscopic elasticity tensor $\mathbb{C}_{\text{macro}}$ is exactly given by the Reuß lower bound [45, 70] of the meso-scale stiffness $\mathbb{C}_{\text{e}}$ and the micro-scale stiffness $\mathbb{C}_{\text{micro}}$

$$\begin{aligned} \mathbb{C}_{\text{macro}} &= \mathbb{C}_{\text{micro}}(\mathbb{C}_{\text{micro}} + \mathbb{C}_{\text{e}})^{-1}\mathbb{C}_{\text{e}} = \left(\mathbb{C}_{\text{e}}^{-1} + \mathbb{C}_{\text{micro}}^{-1}\right)^{-1}, \\ \mathbb{C}_{\text{e}} &= \mathbb{C}_{\text{micro}}(\mathbb{C}_{\text{micro}} - \mathbb{C}_{\text{macro}})^{-1}\mathbb{C}_{\text{macro}}. \end{aligned} \tag{15}$$

Remark that $\mathbb{C}_{\text{c}}$ does not intervene in (15) and therefore the scale-independent material parameters in the relaxed micromorphic model are only $\mathbb{C}_{\text{e}}$ and $\mathbb{C}_{\text{micro}}$. If we specialize formula (15) to the isotropic relaxed micromorphic model, the macroscopic coefficients of the

[10] Letting $L_c \to 0$ in (7) leads to the algebraic side condition $\widehat{\mathbb{C}}_{\text{e}}\,(\nabla u - P) = \mathbb{C}_{\text{micro}} \operatorname{sym} P$. Due to the more general format of $\widehat{\mathbb{C}}_{\text{e}}$ as compared to $\mathbb{C}_{\text{e}}$ and $\mathbb{C}_{\text{c}}$ in (13), it is not possible to analytically solve for $\operatorname{sym} P$ and no transparent formula connecting $\widehat{\mathbb{C}}_{\text{e}}$ and $\mathbb{C}_{\text{micro}}$ to $\mathbb{C}_{\text{macro}}$ like (15) results. The formally scale-independent material parameters of the classical Eringen-Mindlin-model are $\widehat{\mathbb{C}}_{\text{e}}$ and $\mathbb{C}_{\text{micro}}$ and the scale-independent parameters of W_{GE} are $\mathbb{C}_{\text{micro}} = \mathbb{C}_{\text{macro}}$. For the Cosserat model, the respective scale-independent stiffness is $\mathbb{C}_{\text{e}} = \mathbb{C}_{\text{macro}}$. However, considering (footnote 8) $\operatorname{Div}[\mathbb{C}_{\text{micro}} \operatorname{sym} P - \mu\, L_c^2 \Delta P] = f$, it is not strictly possible to say that $\mathbb{C}_{\text{micro}}$ is scale-independent in the Eringen-Mindlin model. The identification of $\mathbb{C}_{\text{micro}}$ (and therefore also $\widehat{\mathbb{C}}_{\text{e}}$) in the Eringen-Mindlin model may be length-scale dependent after all.



$$\mu_{macro} = \mu_{micro}\,(\mu_{micro} + \mu_e)^{-1}\mu_e$$
$$\mu_e = \mu_{micro}\,(\mu_{micro} - \mu_{macro})^{-1}\mu_{macro}$$
$$\mu_{micro} = \mu_e\,(\mu_e - \mu_{macro})^{-1}\mu_{macro}$$

effective resultant stiffness of springs in series

Fig. 8 Scale-independent response governed by two springs in series. If $\mu_{micro} = \mu_{macro}$ then $\mu_e = +\infty$

equivalent macroscopic continuum are related to the parameters of the relaxed micromorphic model through the homogenization formulas (see [6, 56, 58])

$$\mu_{macro} = \frac{\mu_e\,\mu_{micro}}{\mu_e + \mu_{micro}}, \qquad 2\mu_{macro} + 3\lambda_{macro} = \frac{(2\mu_e + 3\lambda_e)(2\mu_{micro} + 3\lambda_{micro})}{(2\mu_e + 3\lambda_e) + (2\mu_{micro} + 3\lambda_{micro})}. \tag{16}$$

These formulas are each identical to calculating the equivalent stiffness of two linear springs in series: for $(16)_1$ it is the equivalent stiffness μ_{macro} from the springs with stiffness μ_e and μ_{micro}, for $(16)_2$ the equivalent stiffness $2\mu_{macro} + 3\lambda_{macro}$ is analogously obtained from the springs in series of stiffness $2\mu_e + 3\lambda_e$ and of stiffness $2\mu_{micro} + 3\lambda_{micro}$ (see Fig. 8). Such formulas, as we will show in this paper for the anisotropic case, are essential to characterize the mechanical behavior of heterogeneous metamaterials on different scales.

As it will turn out, the modeling perspective of the relaxed micromorphic model endows the macro-scale as well as the micro-scale with sufficient physical characteristics to define the setting how to compute the parameters of each of these scales by first order homogenization. Put different and with reference to the stiffness laws for springs in series, the equivalent stiffness and the stiffness of one spring in series can be identified. The unknown stiffness of the second spring representing the transition scale however, is obtained from $(16)_1$ and $(16)_2$ by solving for μ_e and λ_e.

1.5 Microscopic Response – Zoom into the Microstructure

Next, we consider the limit $L_c \to \infty$ for the characteristic length (see Fig. 7 and Fig. 9). Looking at the variational formulation (1), (2), we see that in simply connected domains Ω this generates the constraint $P = \nabla\vartheta$ for some function $\vartheta : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ due to the presence of the Curl-curvature measure. Together with the appropriate tangential boundary condition [31, 60] $P|_{\partial\Omega} \cdot \tau = \nabla u|_{\partial\Omega} \cdot \tau$ we obtain the new minimization problem

$$\int_\Omega W(\nabla u, \nabla\vartheta, 0)\,dx \longrightarrow \min. \qquad (u, \vartheta) \in H^1(\Omega) \times H^1(\Omega), \tag{17}$$

the solution of which necessitates $u = \vartheta$ and the remaining macroscopic displacement field u is obtained from the minimization problem

$$\int_\Omega \frac{1}{2}\langle \mathbb{C}_{micro}\,\mathrm{sym}\,\nabla u, \mathrm{sym}\,\nabla u\rangle\,dx \longrightarrow \min. \qquad u \in H^1(\Omega). \tag{18}$$

This reduction feature depends critically on using Curl P as curvature measure and clearly distinguishes the relaxed micromorphic model from the standard Eringen-Mindlin approaches which use ∇P as curvature measure.

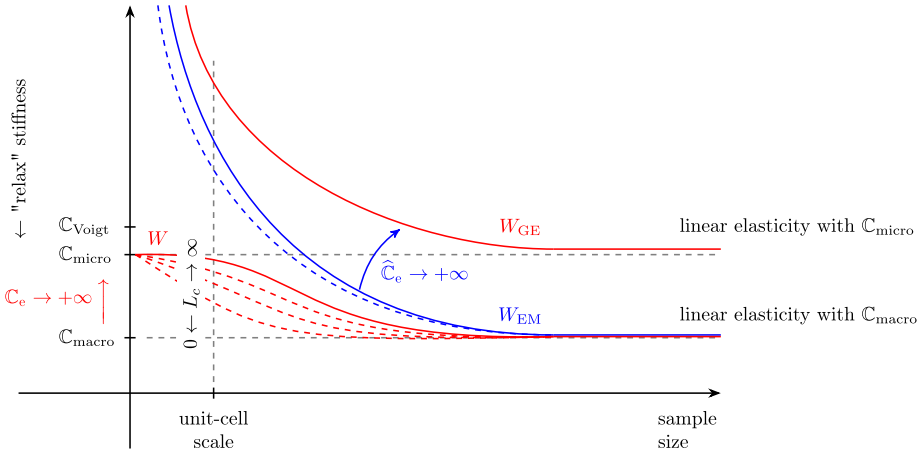


Fig. 9 (Stiffness in non-affine loading, like torsion) Qualitative difference of the relaxed micromorphic model W, the standard Eringen-Mindlin model W_{EM} and gradient elasticity W_{GE}. In all three cases "smaller is stiffer". Typically, the Eringen-Mindlin model and gradient elasticity model dramatically overestimate the stiffness and both show unbounded stiffness for small sample sizes. Only the relaxed micromorphic model has bounded stiffness for all sample sizes (including below a unit-cell level) as well as a clear separation of microscopic and macroscopic scales, $\mathbb{C}_{\text{micro}}$ and $\mathbb{C}_{\text{macro}}$, respectively. For a microscopically homogeneous material, $\mathbb{C}_{\text{micro}} = \mathbb{C}_{\text{macro}}$ induces $\mathbb{C}_{\text{e}} = +\infty$ which enforces $\operatorname{sym} P = \operatorname{sym} \nabla u$ and yields linear elastic response with $\mathbb{C}_{\text{macro}}$ for the relaxed micromorphic model, while the constraint $\widehat{\mathbb{C}}_{\text{e}} = +\infty$ enforces $P = \nabla u$ in W_{EM} and generates the second gradient model W_{GE} with unbounded stiffness. The stiffness of the relaxed micromorphic model W is "relaxed" as compared to W_{EM}. Note that the characteristic length scale $L_c > 0$ always interferes with the actual stiffness at the unit-cell level

The main task which has to be accomplished to successfully apply constitutive laws to the real material world is the identification of their parameters. For the present anisotropic relaxed micromorphic model there are essentially three sets of material parameters $\mathbb{C}_{\text{micro}}$, $\mathbb{C}_{\text{e}}$ and $\mathbb{C}_{\text{macro}}$ describing the scale-independent static response. By virtue of the recently established harmonic mean-type micro-macro homogenization rule (15), the elasticities of the three scales are connected such that the determination of two sets is enough to infer on the third one. Only the micro-scale $\mathbb{C}_{\text{micro}}$ as well as mesoscale-elastic parameters $\mathbb{C}_{\text{e}}$ appear directly in the relaxed micromorphic energy (2), while $\mathbb{C}_{\text{macro}}$ refers to a macroscopic, energetically equivalent, linear elastic surrogate model. In view of these characteristics,[11] the coefficients of $\mathbb{C}_{\text{macro}}$ can be identified by standard homogenization on the periodic unit-cell level. However, the identification of the micro-scale parameters $\mathbb{C}_{\text{micro}}$ through homogenization is completely non-standard, since novel criteria have to be established to identify from all possible unit-cell variants (notice that for periodicity the unit-cell is unique but its

[11] It is indeed well known in the field of homogenization techniques (see, e.g., [20, 68]) that the homogenization of a unit-cell on which one imposes periodic boundary conditions mimics the behavior of a very large specimen of the associated equivalent Cauchy continuum. Usually, homogenization techniques only provide a direct transition from the micro to the macro-scale without considering the intermediate (transition) scale in which all relevant microstructure-related phenomena are manifest. Some attempts to introduce a transition scale via the homogenization towards a micromorphic continuum are made in [39, 79], even if it is clear that a definitive answer is far from being provided (see [39, 79] and references cited there). Our relaxed micromorphic model naturally provides the bridge between the micro and macro behavior of the considered homogenized material with the simple and transparent tensor homogenization formulas (15).

representation is not) those which are consistent with the present modeling. Similarly, the boundary conditions have to be consistently chosen.

1.6 Outline

In this paper, then, we will propose a method to quantify $\mathbb{C}_{\text{micro}}$ (and a fortiori $\mathbb{C}_{\text{e}}$) for a given microstructure. The conceptual idea is simple. Instead of subjecting the relaxed micromorphic model to large specimen size ($L_c \to 0$), we consider arbitrarily small specimen sizes ($L_c \to +\infty$). For this formal limit, the relaxed micromorphic model turns as well into a linear elastic boundary value problem with constant stiffness tensor $\mathbb{C}_{\text{micro}}$. It is important to note at this point that it is meaningless to follow this idea in the context of standard Eringen-Mindlin or higher gradient and Cosserat continua since they have unbounded stiffness for $L_c \to +\infty$ (see Fig. 9).

Having a precise formal limit for $L_c \to +\infty$ at hand we realize, that the smallest resolution we may actually consider is the size of a unit-cell of the material which is to be described. Moreover, using affine Dirichlet boundary conditions, the resulting stiffness depends on the size and shape of the unit-cell where, in general, a smaller unit-cell is "stiffer" than a larger unit-cell. In order to reasonably match the actual response of the given microstructure for small specimen size to $\mathbb{C}_{\text{micro}}$, we need to observe that i) there are several valid variants of the unit-cell (different sizes and geometries generate the same macroscopic specimens), and ii) among these unit-cells there is in general not a single stiffest one. Therefore, we will be satisfied with a quantitative estimate relating $\mathbb{C}_{\text{micro}}$ and calculated stiffnesses for several unit-cells under affine Dirichlet boundary conditions.

For consistency in the scale transition process it is a necessity to claim that the micro-scale reflects and preserves the existing material symmetries of the macro-scale. For that reason we have to invoke Neumann's principle on the representation of anisotropy. As a consequence of this requirement the number of candidate unit-cells is drastically reduced.

We will be left with an estimate of the sort that $\mathbb{C}_{\text{micro}}$ of the relaxed micromorphic model should bound all obtained stiffnesses of competing unit-cells in the energy norm. Insisting on an optimal choice for $\mathbb{C}_{\text{micro}}$ we try to determine one stiffness tensor, which is the least stiff, satisfying the foregoing estimates. This leads us formally to the so called Löwner matrix supremum problem, the solution of which is in general not unique. In the application to tetragonal planar metamaterials we are considering, however, it will be shown that this tensor is uniquely determined.

The only short-range elastic parameter which is not yet determined by the presented arguments is the generalized Cosserat couple modulus $\mathbb{C}_{\text{c}}$, but it can be evaluated when considering the dynamical analysis of the proposed metamaterial [16].

2 Rigorous Determination of $\mathbb{C}_{\text{micro}}$

2.1 Maximal Stiffness on the Micro-Scale

In this section we describe the mathematical underpinning towards determining the stiffness $\mathbb{C}_{\text{micro}}$ in the relaxed micromorphic model. We do this in the static case, in which the equilibrium problem (4) can be obtained as the energy minimization problem

$$I(u, P) := \int_{\Omega} W(\nabla u, P, \operatorname{Curl} P)\, dx \longrightarrow \min. \quad (u, P) \in H^1(\Omega) \times H(\operatorname{Curl}), \tag{19}$$

under suitable boundary conditions, with W given in (2). For the displacement field u we apply overall affine Dirichlet boundary conditions

$$u|_{\partial\Omega}(x) = \overline{B} \cdot x, \qquad \overline{B} \in \mathbb{R}^{3\times 3} \tag{20}$$

and the micro-distortion tensor P has to satisfy the compatible boundary condition[12]

$$\nabla u|_{\partial\Omega}(x) \cdot \tau_{1,2} = P|_{\partial\Omega}(x) \cdot \tau_{1,2}\,, \tag{21}$$

where $\tau_{1,2}$ are linear independent tangent vectors to $\partial\Omega$. One then observes that the minimal energy content of a solution (u, P) to the minimization problem (19), (2), (20), (21) is easily bounded above by choosing the macroscopic fields (u, P) such that

$$\nabla u(x) = P(x), \qquad x \in \Omega. \tag{22}$$

This gives the estimate

$$\begin{aligned}\inf_{(u,P)} \int_{x\in\Omega} W(\nabla u, P, \operatorname{Curl} P)\, dx &\le \inf_{u} \int_{x\in\Omega} W(\nabla u, \nabla u, 0)\, dx \\ &= \inf_{u} \int_{x\in\Omega} \frac{1}{2}\big\langle \mathbb{C}_{\text{micro}} \operatorname{sym}\nabla u(x), \operatorname{sym}\nabla u(x)\big\rangle\, dx. \end{aligned} \tag{23}$$

Therefore, the maximal possible stored elastic energy of the relaxed micromorphic model over an arbitrary window $\widetilde{\Omega} \subset \Omega$ is

$$\inf_{u} \int_{x\in\widetilde{\Omega}} \frac{1}{2}\big\langle \mathbb{C}_{\text{micro}} \operatorname{sym}\nabla u(x), \operatorname{sym}\nabla u(x)\big\rangle\, dx, \qquad u|_{\partial\widetilde{\Omega}}(x) = \overline{B} \cdot x, \tag{24}$$

and this value is attained for $\nabla u(x) = P(x)$ for all $x \in \Omega$. Below, we will evaluate the latter condition over a given unit-cell $V(x) = \widetilde{\Omega}$ attached at the macroscopic point $x \in \Omega$. Following classical analysis, the average displacement gradient over the unit-cell satisfies [82, 3.1, $u \in C^\infty$]

$$\begin{aligned}\frac{1}{|V(x)|}\int_{\xi\in V(x)} \nabla_\xi u(\xi)\, d\xi &= \frac{1}{|V(x)|}\int_{\xi\in\partial V(x)} u(\xi)\otimes n(\xi)\, dS \\ &= \frac{1}{|V(x)|}\int_{\xi\in\partial V(x)} (\overline{B}\cdot\xi)\otimes n(\xi)\, dS \\ &= \frac{1}{|V(x)|}\int_{\xi\in V(x)} \nabla_\xi[\overline{B}\cdot\xi]\, d\xi = \frac{1}{|V(x)|}\int_{\xi\in V(x)} \overline{B}\, d\xi = \overline{B}. \end{aligned} \tag{25}$$

Symmetrization yields as well for the averaged strain tensor

$$\overline{\varepsilon} := \frac{1}{|V(x)|}\int_{\xi\in V(x)} \operatorname{sym}\nabla_\xi u(\xi)\, d\xi = \operatorname{sym}\overline{B} = \overline{E}. \tag{26}$$

[12] In this way, artificial boundary layer effects are avoided.

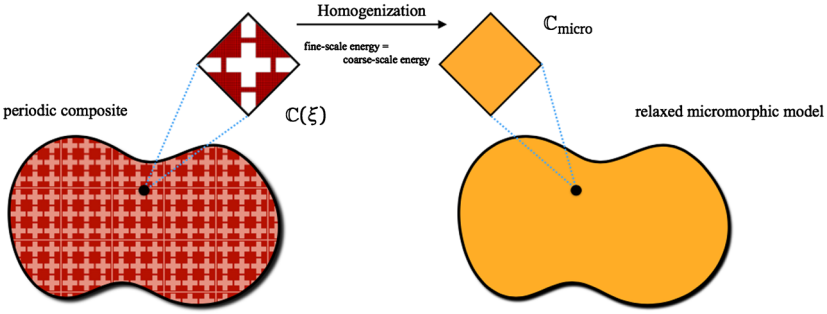


Fig. 10 The process of homogenization. Following the classical Hill-Mandel lemma, we demand energy equivalence of the fine-scale linear elastic energy (left) with the coarse-scale relaxed micromorphic energy (right) over the same domain $V(x)$, under the same affine Dirichlet-boundary conditions and for the same material that is to be described

Since the integrand in (24) is convex (quasiconvex) and $\mathbb{C}_{\text{micro}}$ is constant by assumption, the maximal storage of elastic energy in $V(x)$, according to the relaxed micromorphic model is realized already by the homogeneous displacement $\overline{B} \cdot \xi$ which yields

$$\begin{aligned}
&\inf\left\{ \int_{\xi \in V(x)} \frac{1}{2} \left\langle \mathbb{C}_{\text{micro}} \operatorname{sym} \nabla_\xi \widetilde{v}(\xi), \operatorname{sym} \nabla_\xi \widetilde{v}(\xi) \right\rangle d\xi \mid \widetilde{v} : V(x) \to \mathbb{R}^3,\ \widetilde{v}|_{\partial V(x)}(\xi) = \overline{B} \cdot \xi \right\} \\
&= \frac{1}{2} \langle \mathbb{C}_{\text{micro}} \operatorname{sym} \overline{B}, \operatorname{sym} \overline{B} \rangle \left| V(x) \right| = \frac{1}{2} \langle \mathbb{C}_{\text{micro}} \overline{E}, \overline{E} \rangle \left| V(x) \right|, \quad \overline{E} = \operatorname{sym} \overline{B},
\end{aligned} \tag{27}$$

for the relaxed micromorphic model.

Now we switch to considering the unit-cell as described by classical linear elasticity with inhomogeneous material properties. Macroscopic variables are conceptionally some "averages" over the micro-scale. Hence the attached unit-cell $V(x)$ must be considered to be loaded such that it produces the given superposed macroscopic average $\nabla u(x)$. There are several choices satisfying this requirement; prominently affine Dirichlet boundary conditions (or KUBC – kinematically uniform boundary conditions) or periodic boundary conditions (PBC) (see Fig. 13). It is well known that affine Dirichlet conditions generate stiffer response than PBC [36, 42, 68]. Let $\mathbb{C}(\xi)$ be the geometry dependent inhomogeneous elasticity tensor of the given metamaterial. Under affine Dirichlet conditions (KUBC) (see Fig. 11) the classical linear elastic stored energy of the unit-cell is given by

$$\begin{aligned}
&\inf\left\{ \int_{\xi \in V(x)} \frac{1}{2} \left\langle \mathbb{C}(\xi) \operatorname{sym} \nabla_\xi \widetilde{v}(\xi), \operatorname{sym} \nabla_\xi \widetilde{v}(\xi) \right\rangle d\xi \mid \widetilde{v} : V(x) \to \mathbb{R}^3,\ \widetilde{v}|_{\partial V(x)}(\xi) = \overline{B} \cdot \xi \right\} \\
&= \inf\left\{ \int_{\xi \in V(x)} \frac{1}{2} \left\langle \mathbb{C}(\xi) \operatorname{sym}\left(\nabla_\xi \left[v(\xi) + \overline{B} \cdot \xi \right]\right), \operatorname{sym}\left(\nabla_\xi \left[v(\xi) + \overline{B} \cdot \xi \right]\right) \right\rangle d\xi \mid v \in C_0^\infty\left(V(x), \mathbb{R}^3\right) \right\} \\
&= \inf\left\{ \int_{\xi \in V(x)} \frac{1}{2} \left\langle \mathbb{C}(\xi) \left(\operatorname{sym} \nabla_\xi v(\xi) + \overline{E} \right), \operatorname{sym} \nabla_\xi v(\xi) + \overline{E} \right\rangle d\xi \mid v \in C_0^\infty\left(V(x), \mathbb{R}^3\right) \right\} \\
&= \int_{\xi \in V(x)} \frac{1}{2} \left\langle \mathbb{C}(\xi) \left(\operatorname{sym} \nabla_\xi \widehat{v}_{\overline{E}}(\xi) + \overline{E} \right), \operatorname{sym} \nabla_\xi \widehat{v}_{\overline{E}}(\xi) + \overline{E} \right\rangle d\xi,
\end{aligned} \tag{28}$$

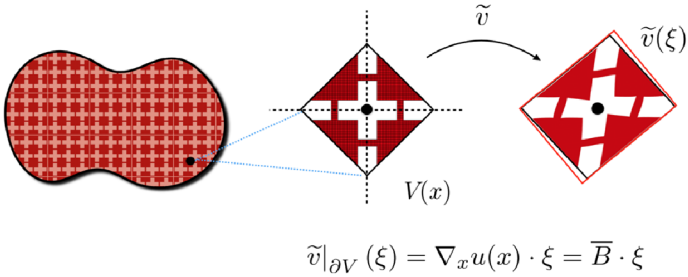


Fig. 11 Affine Dirichlet loading (KUBC) of the unit-cell $V(x)$. The macroscopic deformation state $\nabla_\xi u(x) = \overline{B}$ defines a microscopic Dirichlet boundary value problem on the boundary of the microvolume $V(x)$ attached to each macroscopic material point $x \in \Omega$. Here, we show the superposition of simple shear and elongation

where (the corrector) $\widehat{v}_{\overline{E}} \in C_0^\infty(V(x), \mathbb{R}^3)$ is the realizing minimizer (which is not known a priori).

In the context of homogenization, we now demand that the (fully resolved linear elastic) fine-scale energy (28) should equal the (relaxed micromorphic) coarse-scale energy $(23)_{\text{left}}$ over the same domain $V(x)$, under the same affine Dirichlet-boundary conditions $\overline{B} \cdot \xi$ and for the same material (see Fig. 10). This means we require that for all affine loadings $\overline{B} \in \mathbb{R}^{3\times 3}$ we have:

$$\inf_{(u,P)} \left\{ \int_{\xi \in V(x)} W\big(\nabla_\xi u(\xi), P(\xi), \operatorname{Curl} P(\xi)\big)\, d\xi \mid u|_{\xi\in\partial V}(\xi) = \overline{B}\cdot\xi,\ \overline{B}\cdot\tau_{1,2} = P|_{\xi\in\partial V}(\xi)\cdot\tau_{1,2} \right\}$$
$$\stackrel{!}{=} \inf_{v} \left\{ \int_{\xi\in V(x)} \frac{1}{2}\big\langle \mathbb{C}(\xi)(\operatorname{sym}\nabla_\xi v + \overline{E}), \operatorname{sym}\nabla_\xi v + \overline{E}\big\rangle\, d\xi \mid v \in C_0^\infty\big(V(x), \mathbb{R}^3\big), \overline{E} = \operatorname{sym}\overline{B} \right\}. \quad (29)$$

With estimates (23) and (27), taking (29) into account, we obtain the inequality

$$\underbrace{\frac{1}{2}\langle \mathbb{C}_{\text{micro}}\, \overline{E}, \overline{E}\rangle \big|V(x)\big|}_{\substack{\text{coarse scale micromorphic}\\ \text{upper energy bound}}} \quad (30)$$
$$\geq \underbrace{\inf\left\{ \int_{\xi\in V(x)} \frac{1}{2}\big\langle \mathbb{C}(\xi)\big(\operatorname{sym}\nabla_\xi v(\xi) + \overline{E}\big), \operatorname{sym}\nabla_\xi v(\xi) + \overline{E}\big\rangle\, d\xi \mid v \in C_0^\infty\big(V(x), \mathbb{R}^3\big) \right\}}_{\text{fine-scale linear elastic energy}},$$

which must be satisfied for all $\overline{E} \in \operatorname{Sym}(3)$. On the other hand, according to the classical[13] Hill-Mandel lemma [33–35, 52, 82] we can define a unique apparent [36] constant stiffness tensor $\mathbb{C}^V_{\text{KUBC}}$, independent of $\overline{E}$, but depending on the chosen unit-cell V, by set-

[13] And not any of the ambiguous extended versions for generalized continua [26, 27, 29, 40].

ting[14]

$$\frac{1}{2}\langle \mathbb{C}^V_{\mathrm{KUBC}}\,\overline{E},\overline{E}\rangle |V(x)| \tag{32}$$
$$= \inf\left\{\int_{\xi\in V(x)} \frac{1}{2}\langle \mathbb{C}(\xi)\big(\mathrm{sym}\nabla_\xi v(\xi)+\overline{E}\big), \mathrm{sym}\nabla_\xi v(\xi)+\overline{E}\rangle\, d\xi \mid v\in C_0^\infty\big(V(x),\mathbb{R}^3\big)\right\}.$$

The details for this construction will be explained in the next subsection. Combining this with inequality (31) we must have for all applied loadings $\overline{E}\in \mathrm{Sym}\,(3)$:

$$\langle \mathbb{C}_{\mathrm{micro}}\,\overline{E},\overline{E}\rangle \geq \langle \mathbb{C}^V_{\mathrm{KUBC}}\,\overline{E},\overline{E}\rangle. \tag{33}$$

2.2 The Hill-Mandel Energy Equivalence for Affine Dirichlet Conditions

The Hill-Mandel energy equivalence for KUBC can be obtained easily. We provide it for the convenience of the reader. On the one hand we have in mechanical equilibrium for linear elasticity

$$\begin{aligned}\int_{\xi\in\partial V}\langle \widetilde{v}(\xi),\sigma(\xi)\cdot\vec{n}\rangle_{\mathbb{R}^3}\,dS &= \int_{\xi\in\partial V}\langle \sigma^T(\xi)\cdot\widetilde{v}(\xi),\vec{n}\rangle_{\mathbb{R}^3}\,dS = \int_{\xi\in V}\mathrm{div}\big(\sigma^T\cdot\widetilde{v}\big)\,d\xi\\ &= \int_{\xi\in V}\mathrm{div}(\sigma\cdot\widetilde{v})\,d\xi = \int_{\xi\in V}\langle\nabla\widetilde{v},\sigma\rangle_{\mathbb{R}^{3\times3}} + \langle\widetilde{v},\underbrace{\mathrm{Div}\,\sigma}_{=0}\rangle\,d\xi\\ &= \int_{\xi\in V}\big\langle\sigma(\xi),\varepsilon(\xi)\big\rangle\,d\xi.\end{aligned} \tag{34}$$

On the other hand, for KUBC, we have $\widetilde{v}|_{\partial V} = \overline{B}\cdot\xi$ and

$$\begin{aligned}\int_{\xi\in\partial V}\langle \widetilde{v}(\xi),\sigma(\xi)\cdot\vec{n}\rangle_{\mathbb{R}^3}\,dS &= \int_{\xi\in\partial V}\langle \overline{B}\cdot\xi,\sigma(\xi)\cdot\vec{n}\rangle_{\mathbb{R}^3}\,dS = \int_{\xi\in V}\mathrm{div}\big(\sigma^T\cdot(\overline{B}\cdot\xi)\big)\,d\xi\\ &= \int_{\xi\in V}\mathrm{div}\big(\sigma\cdot(\overline{B}\cdot\xi)\big)\,d\xi = \int_{\xi\in V}\big\langle\nabla[\overline{B}\cdot\xi],\sigma\big\rangle + \langle\overline{B}\cdot\xi,\underbrace{\mathrm{Div}\,\sigma}_{=0}\rangle\,d\xi\\ &= \int_{\xi\in V}\langle\overline{B},\sigma\rangle\,d\xi = \left\langle\overline{B},\int_{\xi\in V}\sigma\,d\xi\right\rangle = |V|\left\langle\overline{B},\frac{1}{|V|}\int_{\xi\in V}\sigma\,d\xi\right\rangle = |V|\langle\mathrm{sym}\,\overline{B},\overline{\sigma}\rangle.\end{aligned} \tag{35}$$

[14] Since

$$\frac{1}{2}\langle \mathbb{C}^V_{\mathrm{KUBC}}\,\overline{E},\overline{E}\rangle |V(x)| \tag{31}$$
$$= \inf\left\{\int_{\xi\in V(x)} \frac{1}{2}\langle \mathbb{C}(\xi)\big(\mathrm{sym}\nabla_\xi v(\xi)+\overline{E}\big), \mathrm{sym}\nabla_\xi v(\xi)+\overline{E}\rangle\, d\xi \mid v\in C_0^\infty\big(V(x),\mathbb{R}^3\big)\right\}$$

$v\equiv 0$ (constant strain assumption: Taylor/Voigt)

$$\leq \int_V \frac{1}{2}\langle \mathbb{C}(\xi)\overline{E},\overline{E}\rangle\,d\xi = \frac{1}{2}\left\langle\overline{E},\int_V \mathbb{C}(\xi)\,d\xi\,\overline{E}\right\rangle = \frac{1}{2}|V|\left\langle\overline{E},\frac{1}{|V|}\int_V \mathbb{C}(\xi)\,d\xi\,\overline{E}\right\rangle = \frac{1}{2}|V|\langle\overline{E},\mathbb{C}_{\mathrm{Voigt}}\,\overline{E}\rangle$$

it is clear that $\langle \mathbb{C}^V_{\mathrm{KUBC}}\,\overline{E},\overline{E}\rangle \leq \langle \mathbb{C}_{\mathrm{Voigt}}\,\overline{E},\overline{E}\rangle$ for all applied loadings $\overline{E}\in\mathrm{Sym}\,(3)$. On the other hand, it is natural to require as well $\langle \mathbb{C}_{\mathrm{micro}}\,\overline{E},\overline{E}\rangle \leq \langle \mathbb{C}_{\mathrm{Voigt}}\,\overline{E},\overline{E}\rangle$, where equality will be obtained if and only if the material on the micro-scale is homogeneous, i.e., $\mathbb{C}(\xi)=\mathrm{const}$.

Since $\operatorname{sym}\overline{B} = \overline{E} = \overline{\varepsilon}$, see (25), taking (34) and (35) together we obtain

$$\langle \overline{\sigma}, \overline{\varepsilon} \rangle = \frac{1}{|V|} \int_{\xi \in V} \big\langle \sigma(\xi), \varepsilon(\xi) \big\rangle \, d\xi. \tag{36}$$

Summarizing, the Hill-Mandel lemma (36) implies that for KUBC (among other boundary conditions) it holds that,

$$\langle \overline{\sigma}, \overline{\varepsilon} \rangle = \frac{1}{|V|} \int_{\xi \in V} \big\langle \sigma(\xi), \varepsilon(\xi) \big\rangle \, d\xi, \quad \operatorname{Div} \sigma(\xi) = 0, \quad \sigma^T(\xi) = \sigma(\xi), \quad \widetilde{v}|_{\partial V} = \overline{B} \cdot \xi, \tag{37}$$

where $\overline{\varepsilon}$, $\overline{\sigma}$ are the mean strain and mean stress, respectively.

Next, let us assume that on the fine scale we have the linear elastic constitutive law $\sigma(\xi) = \mathbb{C}(\xi)\,\varepsilon(\xi)$, where $\mathbb{C}(\xi)$ is uniformly positive definite. Then the equilibrium equation $\operatorname{Div}(\mathbb{C}(\xi)\,\varepsilon(\xi)) = 0$, $\widetilde{v}|_{\partial V}(\xi) = \overline{B} \cdot \xi$ has a unique (inhomogeneous) solution $\widehat{v}(\xi)$, such that $\varepsilon(\xi) = \operatorname{sym} \nabla \widehat{v}(\xi)$ depends linearly on $\overline{E} = \operatorname{sym} \overline{B}$. Thus, the micro-scale Cauchy stress $\widehat{\sigma}(\xi) = \mathbb{C}(\xi)\,\varepsilon(\xi)$ depends also linearly on $\overline{E}$. On the other hand, it follows by partial integration that the mean strain tensor satisfies $\overline{\varepsilon} = \frac{1}{|V|} \int_{\xi \in V} \varepsilon(\xi)\, d\xi = \overline{E}$, see (26), and moreover, that the mean Cauchy stress tensor $\overline{\sigma} = \frac{1}{|V|} \int_{\xi \in V} \sigma(\xi)\, d\xi$ depend also linearly on $\overline{E}$. This implies that there exists a unique linear mapping with constant coefficients $\mathbb{C}^V_{\text{KUBC}}$ such that $\overline{\sigma} = \mathbb{C}^V_{\text{KUBC}}\, \overline{E}$. Therefore, using (37), we must have[15]

$$\begin{aligned} &\frac{1}{2} \big\langle \mathbb{C}^V_{\text{KUBC}}\, \overline{E}, \overline{E} \big\rangle \\ &\quad = \inf \bigg(\frac{1}{|V|} \int_{\xi \in V} \frac{1}{2} \big\langle \mathbb{C}(\xi) \big(\operatorname{sym} \nabla_\xi v(\xi) + \overline{E} \big), \operatorname{sym} \nabla_\xi v(\xi) + \overline{E} \big\rangle \, d\xi \mid v \in C_0^\infty\big(V(x), \mathbb{R}^3\big) \bigg). \end{aligned} \tag{39}$$

The reader is warned that for classical periodic homogenization, the Hill-Mandel lemma is used with periodic boundary conditions PBC. In this case a unique constant linear mapping $\mathbb{C}^V_{\text{PBC}}$ can be obtained accordingly. However, $\mathbb{C}^V_{\text{PBC}}$ turns out to be independent of the size and shape of the chosen unit-cell V such that

$$\mathbb{C}^V_{\text{PBC}} =: \mathbb{C}_{\text{macro}} \tag{40}$$

defines the unique macroscopic homogenized stiffness. It is well known that for large unit-cells V, the homogenized apparent stiffness tensor $\mathbb{C}^V_{\text{KUBC}}$ approximates $\mathbb{C}_{\text{macro}}$ [35, 66], see

[15] An equivalent, more algorithmic procedure to determine $\mathbb{C}^V_{\text{KUBC}}$ is obtained as follows. Consider again (37)

$$\langle \overline{\sigma}, \overline{\varepsilon} \rangle = \frac{1}{|V|} \int_V \big\langle \sigma(\xi), \varepsilon(\xi) \big\rangle \, d\xi = \frac{1}{|V|} \int_V \big\langle \mathbb{C}(\xi)\,\varepsilon(\xi), \varepsilon(\xi) \big\rangle \, d\xi, \quad \operatorname{Div} \sigma(\xi) = 0, \quad \sigma^T(\xi) = \sigma(\xi), \tag{38}$$

and $\widetilde{v} = \overline{\varepsilon} \cdot \xi$ at the boundary. Let us define the corresponding linear solution operator of the linear elastic problem at the micro-scale $\mathscr{L}(\xi) \cdot \overline{\varepsilon} = \varepsilon(\xi)$, ("localization tensor") and insert this back into (37). This gives

$$\begin{aligned} \langle \overline{\sigma}, \overline{\varepsilon} \rangle &= \frac{1}{|V|} \int_V \big\langle \mathbb{C}(\xi)\, \mathscr{L}(\xi) \cdot \overline{\varepsilon}, \mathscr{L}(\xi) \cdot \overline{\varepsilon} \big\rangle \, d\xi = \frac{1}{|V|} \int_V \big\langle \mathscr{L}(\xi)^T \mathbb{C}(\xi)\, \mathscr{L}(\xi) \cdot \underbrace{\overline{\varepsilon}}_{\text{const. in } \xi}, \overline{\varepsilon} \big\rangle \, d\xi \\ &= \Big\langle \overline{\varepsilon}, \underbrace{\bigg(\frac{1}{|V|} \int_V \mathscr{L}(\xi)^T \mathbb{C}(\xi)\, \mathscr{L}(\xi)\, d\xi \bigg)}_{=: \mathbb{C}^V_{\text{KUBC}}} \cdot \overline{\varepsilon} \Big\rangle = \big\langle \overline{\varepsilon}, \mathbb{C}^V_{\text{KUBC}}\, \overline{\varepsilon} \big\rangle. \end{aligned}$$

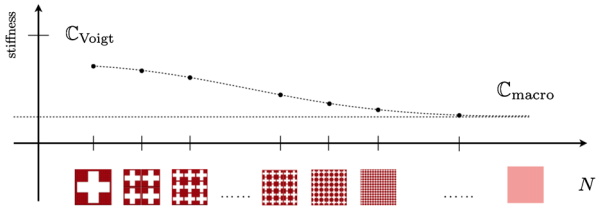


Fig. 12 Qualitative result for the apparent stiffness $\mathbb{C}^V_{\text{KUBC}}$, depending on the number of arrays

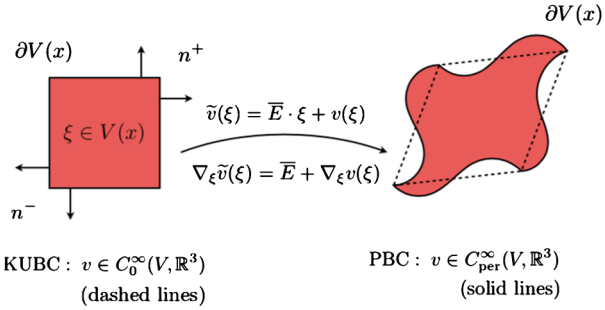


Fig. 13 Difference between affine Dirichlet boundary conditions (KUBC) and periodic Dirichlet boundary conditions (PBC). The fluctuation v is either zero at the boundary (KUBC) or periodic (PBC). The apparent stiffness $\mathbb{C}^V_{\text{KUBC}}$ determines $\mathbb{C}_{\text{micro}}$ via the Löwner matrix supremum and $\mathbb{C}^V_{\text{PBC}}$ determines directly the effective stiffness $\mathbb{C}_{\text{macro}}$

also [81]. In general, $\mathbb{C}^V_{\text{KUBC}}$ is stiffer than $\mathbb{C}_{\text{macro}}$ measured in the energy norm [42]. More precisely,

$$\forall \overline{E} \in \text{Sym}(3): \quad \langle \mathbb{C}^{V_\delta}_{\text{KUBC}} \overline{E}, \overline{E} \rangle \geqslant \langle \mathbb{C}^{V_{2\delta}}_{\text{KUBC}} \overline{E}, \overline{E} \rangle \geqslant \overset{\delta \to +\infty}{\longrightarrow \cdots} \geqslant \langle \mathbb{C}_{\text{macro}} \overline{E}, \overline{E} \rangle, \tag{41}$$

where $\delta > 0$ is a typical size of the cell V_δ. These hierarchies were first established by [36], see also [37, 38, 73] (see Fig. 12).

Summarizing, both stiffness tensors $\mathbb{C}^V_{\text{KUBC}}$ and $\mathbb{C}_{\text{macro}}$ involve a homogenization step but they are clearly distinguished and reflect the two-scale nature of the relaxed micromorphic model.

2.3 Neumann's Principle and the Löwner Bound

In our given periodic arrangement there are many different possibilities to choose unit-cells V, see Fig. 15 in the next section. In the following we use an **extended Neumann's principle** [67, 80] and [46, p.155], suitably adapted to our setting:

extended Neumann's principle
The invariance group of every stiffness tensor of the relaxed micromorphic model $(\mathbb{C}_e, \mathbb{C}_{\text{micro}}, \mathbb{C}_c, \mathbb{L})$ must contain the maximal invariance group of the periodic metamaterial.[16]

Here, the maximal invariance group generated by the periodic metamaterial is the tetragonal group. Therefore the effective stiffness tensor $\mathbb{C}_{\text{macro}}$ already has tetragonal symmetry.

[16] Here, $\mu L_c^2 \langle \mathbb{L}\,\text{Curl}\,P, \text{Curl}\,P \rangle$ would represent the most general quadratic anisotropic curvature energy in the relaxed micromorphic model, where $\mathbb{L}$ is a fourth-order tensor mapping non-symmetric second-order tensors to non-symmetric second-order tensors.

Neumann's principle now requires that $\mathbb{C}_{\text{micro}}$ must have tetragonal symmetry, too.[17] Formula (15) then shows that $\mathbb{C}_{\text{e}}$ will also be tetragonal. Moreover, we only consider those apparent stiffness tensors $\mathbb{C}^V_{\text{KUBC}}$ which are themselves tetragonal. This reduces the number of candidate unit-cells V in (33) considerably. Finally, we determine a tensor $\mathbb{C}^0_{\text{micro}}$ in (33) by requiring

$$\forall\, \overline{E} \in \operatorname{Sym}(3): \quad \left\langle \mathbb{C}^0_{\text{micro}}\, \overline{E}, \overline{E} \right\rangle \geq \left\langle \mathbb{C}^V_{\text{KUBC}}\, \overline{E}, \overline{E} \right\rangle \tag{42}$$

for all remaining candidate unit-cell variants V and in addition, any other possible tensor $\widetilde{\mathbb{C}}_{\text{micro}}$, verifying estimate (42) should satisfy

$$\forall\, \overline{E} \in \operatorname{Sym}(3): \quad \langle \widetilde{\mathbb{C}}_{\text{micro}}\, \overline{E}, \overline{E} \rangle \geq \left\langle \mathbb{C}^0_{\text{micro}}\, \overline{E}, \overline{E} \right\rangle.$$

In this sense $\mathbb{C}^0_{\text{micro}}$ is optimal. It turns out that $\mathbb{C}^0_{\text{micro}}$ is a matrix supremum in the so called Löwner-half-order [14].

Löwner matrix supremum problem

Given a family of positive definite symmetric stiffness tensors $\mathbb{C}_k$, $k = 1, \ldots, n$, find a positive definite stiffness tensor $\overline{\mathbb{C}}$ such that,

1. $\forall \overline{E} \in \operatorname{Sym}(3): \quad \langle \overline{\mathbb{C}}\, \overline{E}, \overline{E} \rangle \geq \langle \mathbb{C}_k\, \overline{E}, \overline{E} \rangle$, $k = 1, \ldots, n$, "$\overline{\mathbb{C}}$ is upper bound in the Löwner order",
2. If $\langle \widetilde{\mathbb{C}}\, \overline{E}, \overline{E} \rangle \geq \langle \mathbb{C}_k\, \overline{E}, \overline{E} \rangle$, $k = 1, \ldots, n$ then $\langle \widetilde{\mathbb{C}}\, \overline{E}, \overline{E} \rangle \geq \langle \overline{\mathbb{C}}\, \overline{E}, \overline{E} \rangle$ "$\overline{\mathbb{C}}$ is least upper bound".

This is the obtained requirement on $\mathbb{C}_{\text{micro}}$.[18]

Gathering our findings, we have obtained the following characterization

Theorem 1 *Let $\mathbb{C}_{\text{macro}}$ be the classical effective elasticity tensor obtained by periodic homogenization for a given periodic microstructure. Assume that the relaxed micromorphic model (2) is chosen as an effective medium to describe the given periodic microstructure. Then, the meso-scale elasticity tensor $\mathbb{C}_{\text{e}}$ and the micro-scale elasticity tensor $\mathbb{C}_{\text{micro}}$ in the relaxed micromorphic model are related by the formula*

$$\mathbb{C}_{\text{macro}} = \mathbb{C}_{\text{micro}} (\mathbb{C}_{\text{micro}} + \mathbb{C}_{\text{e}})^{-1} \mathbb{C}_{\text{e}}. \tag{43}$$

Moreover, the microscopic stiffness tensor $\mathbb{C}_{\text{micro}}$ satisfies the bound

$$\forall\, \overline{E} \in \operatorname{Sym}(3): \quad \langle \mathbb{C}_{\text{micro}}\, \overline{E}, \overline{E} \rangle \geq \left\langle \mathbb{C}^0_{\text{micro}}\, \overline{E}, \overline{E} \right\rangle, \tag{44}$$

where $\mathbb{C}^0_{\text{micro}}$ is the Löwner matrix supremum of the family of apparent stiffness tensors $\mathbb{C}^V_{\text{KUBC}}$ under affine Dirichlet boundary conditions which are obtained from admissible unit-cells V satisfying the Neumann's principle.

Remark 1 In the applications [16], a first choice is to set $\mathbb{C}_{\text{micro}} = \mathbb{C}^0_{\text{micro}}$. However, a fit for the dynamic range may eventually be improved by taking any other positive definite

[17] $\mathbb{C}_{\text{micro}}$ could be isotropic nevertheless, since isotropy is a subclass of the tetragonal symmetry.

[18] Considering the Voigt upper bound $\mathbb{C}_{\text{Voigt}} := \frac{1}{|V|} \int_V \mathbb{C}(\xi)\, d\xi$ as representing the maximal microscopic stiffness is not useful for two reasons: First, $\mathbb{C}_{\text{Voigt}}$ will be isotropic and lose the information of the geometry of the microstructure. Second, the actual deformation in any unit-cell will never exhibit constant strain.

elasticity tensor $\mathbb{C}_{\text{micro}}$ satisfying estimate (44). Identifying $\mathbb{C}_{\text{micro}}$ as the Löwner supremum is an entirely new approach unique to the relaxed micromorphic model.

Remark 2 In the classical Eringen-Mindlin W_{EM}-model (6), a transparent relation like (43) is impossible due to the missing proper split of the scale-independent constitutive tensors. On the other hand, the bound (44) is impossible due to the presence of $\|\nabla P\|^2$ instead of $\|\text{Curl}\, P\|^2$, see inequality (23), which fails for W_{EM}. This observation highlights the appropriate constitutive assumptions made in the relaxed micromorphic model which allows a rational a priori separation of large and small scale response.

2.4 The Löwner Matrix Supremum in Plane Strain for Tetragonal Symmetry

In the next section, the apparent stiffness tensors $\mathbb{C}^{V}_{\text{KUBC}}$ will be determined for the extended Neumann's principle unit-cells V_i, $i = 1, \dots, n$. In order to distill the missing information on $\mathbb{C}_{\text{micro}}$ in (42), we proceed as follows. Our goal is to find a positive definite tensor $\mathbb{C}_{\text{micro}}$ which is the least upper bound of the apparent stiffness of the underlying microstructure measured in the energy norm. Since the fitting will be done in the 2D-case, we turn to the planar Voigt representation and the inequality condition (42) can be restated as

$$\begin{aligned}\forall (x,y,z)^T \in \mathbb{R}^3 : \quad & \left\langle \underbrace{\begin{pmatrix} 2\widehat{\mu}+\widehat{\lambda} & \widehat{\lambda} & 0 \\ \widehat{\lambda} & 2\widehat{\mu}+\widehat{\lambda} & 0 \\ 0 & 0 & \widehat{\mu}^* \end{pmatrix}}_{\mathbb{C}_{\textbf{micro}}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle \\ & \geq \left\langle \underbrace{\begin{pmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu^* \end{pmatrix}}_{\mathbb{C}^{V_i}_{\textbf{KUBC}}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle, \; i = 1 \dots n \end{aligned} \tag{45}$$

where (x, y, z) represents the planar strain tensor entries $(\overline{E}_{11}, \overline{E}_{22}, \overline{E}_{12})$. We denote the extended Neumann's principle admissible KUBC-entries as $\mu_i, \lambda_i, \mu_i^*$, $i = 1, \dots, n$, respectively.

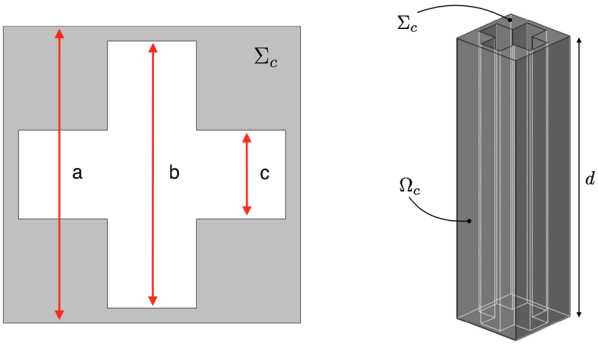
It remains to obtain numerical values $\widehat{\mu}, \widehat{\lambda}, \widehat{\mu}^*$ such that (45) is always verified. Since μ^* sits on the diagonal, we must necessarily have that $\mu^* \geq \mu_i^*$ for all $i = 1, 2, 3, 4$. Therefore, the problem is reduced to the 2×2 block

$$\begin{aligned}\forall (x,y)^T \in \mathbb{R}^2 : \quad & \left\langle \begin{pmatrix} 2\widehat{\mu}+\widehat{\lambda} & \widehat{\lambda} \\ \widehat{\lambda} & 2\widehat{\mu}+\widehat{\lambda} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle \\ & \geq \left\langle \begin{pmatrix} 2\mu_i+\lambda_i & \lambda_i \\ \lambda_i & 2\mu_i+\lambda_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle, \quad i = 1, 2, 3, 4. \end{aligned} \tag{46}$$

According to the Sylvester-criterion for the difference of positive definite tensors, we must have $\widehat{\mu} \geq \mu_i$ and $\widehat{\mu} + \widehat{\lambda} \geq \mu_i + \lambda_i$ for $i = 1, 2, 3, 4$. We choose

$$\begin{aligned} \widehat{\mu} &:= \max_i \{\mu_i\}, \qquad \widehat{\mu}^* := \max_i \{\mu_i^*\}, \\ \widehat{\mu} + \widehat{\lambda} &:= \max_i \{\mu_i + \lambda_i\}, \qquad \widehat{\lambda} := \max_i \{\mu_i + \lambda_i\} - \widehat{\mu}. \end{aligned} \tag{47}$$

This determines μ, μ^*, λ uniquely. In the next section, we have picked the highlighted values in Table 2 for $\mathbb{C}_{\text{micro}}$ in (50).



a	b	c	d
[mm]	[mm]	[mm]	[m]
1	0.9	0.3	1
E	ν	μ	λ
[GPa]	–	[GPa]	[GPa]
70	0.33	26.32	51.08

Fig. 14 Geometry of the unit-cell and elastic parameters for Aluminum

3 Elastic Parameter Identification by Numerical Homogenization

For analyses of the considered tetragonal metamaterial in the plane, the number of independent material constants in linear elasticity is three for each of the scales. Both the macro as well as the microscopic elasticity parameters are computed by numerical homogenization on the unit-cell level, see [53, 77, 78]. To that aim we employ the Finite-Element Heterogeneous Multiscale Method FE-HMM [20, 24], a two-level finite element method, which is based on asymptotic homogenization and on the most general Heterogeneous Multiscale Method HMM [18]. A mathematical analysis of FE-HMM for linear elasticity is provided in [1].

3.1 Determination of $\mathbb{C}_{\text{macro}}$ – Classical Periodic Homogenization

In mathematical terms, the macroscopic effective stiffness $\mathbb{C}_{\text{macro}}$ is obtained by using the classical result of periodic homogenization (see, e.g., [9, 11, 12]):

$$\frac{1}{2}\langle \mathbb{C}_{\text{macro}}\,\overline{E},\overline{E}\rangle \big|V(x)\big| \tag{48}$$
$$:= \inf\left\{ \int_{\xi\in V(x)} \frac{1}{2}\big\langle \mathbb{C}(\xi)\,\mathrm{sym}\big(\nabla_\xi v(\xi)+\overline{E}\big), \mathrm{sym}\big(\nabla_\xi v(\xi)+\overline{E}\big)\big\rangle\, d\xi \mid v\in C^\infty_{\text{per}}\big(V(x),\mathbb{R}^3\big)\right\},$$

where $\mathbb{C}(\xi)$ is the elasticity tensor of the aluminum phase or air depending on the position of ξ in the unit-cell[19] and $\overline{E}=\mathrm{sym}\nabla u(x)$ is the applied straining at the macroscopic point x, where the unit-cell $V(x)$ is centered at x. For the computation of these macroscopic elasticity coefficients we use the two-scale finite element method FE-HMM [20] (see also [53, 77, 78]) and we assume that the microproblem is driven under macroscopic plane strain conditions.

The Lamé constants for the converged solution are obtained for mesh-size $h = 1/2560$ mm (for the geometry of the unit-cell see Fig. 14); they are displayed in Table 1.

[19] Here, x is the macro space variable of the continuum, while ξ is the micro-variable spanning inside the unit-cell.

Table 1 Homogenized macroscopic Lamé constants identified under plane strain and PBC

$\mathbb{C}_{\text{macro}}$	geometry Fig. 16	boundary conditions plane strain	elasticity parameters λ_{macro} [GPa]	μ_{macro} [GPa]	μ^*_{macro} [GPa]
	(a)–(d)	periodic	1.738	5.895	0.620

3.2 Determination of $\mathbb{C}_{\text{micro}}$ – Apparent Stiffness for Affine Dirichlet Boundary Conditions

Similar to the macroscopic parameters, the micro set $\mathbb{C}_{\text{micro}}$ shall be identified by numerical homogenization. In contrast to the macroscopic scale, the relaxed micromorphic model imposes conditions at the micro-scale, which are non-standard, and in particular, in their combination, very selective as far as the choice of the unit-cells is concerned. The conditions which are imposed by the relaxed micromorphic model on choice of the unit-cells and on the boundary conditions read:

(i) they correspond to the case $L_c \to \infty$, a maximal zoom into the material,
(ii) they represent the maximal stiffness response of the (meta)material at the micro-scale,
(iii) they reflect the material's overall (tetragonal) symmetry.

The first condition $L_c \to \infty$ is rather vague and not very selective. The zoom into the single solid phase of the material however can be ruled out, since the resultant isotropy of aluminum violates condition (iii).

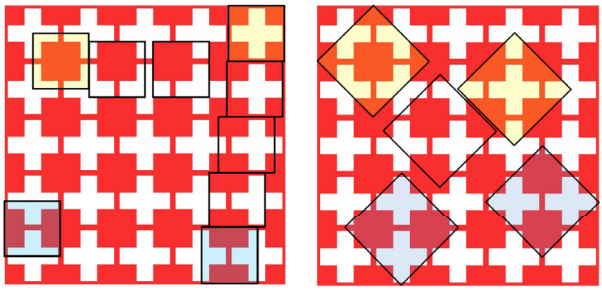
Condition (ii) alone suggests to consider constant strain conditions, since they yield the upper bound of stiffness, the Voigt-bound. Under constant strain assumption however, symmetry information of the microstructure is lost; numerical homogenization results in an isotropic material response, $\mu = \mu^*$, see Table 2, which again violates condition (iii). The conclusion is, that conditions (ii) and (iii) cannot be fulfilled by the constant strain assumption except for the trivial case of isotropy.

Among the boundary and loading conditions fulfilling the Hill-Mandel postulate (see Sect. 2.2), affine Dirichlet boundary conditions are the candidate to estimate the maximal stiffness while preserving material symmetries. It is well known, that periodic boundary conditions (PBC) yield less stiff results, and the constant stress assumption defines the lower bound of stiffness, the Reuss-bound.

There is an infinite number of valid, hence "equivalent" unit-cell variants for the homogenization of periodic media, if PBC are applied.[20] Figure 15 shows some of them for the case of periodic tessellation based on squares of side length a and based on rotated squares of sidelength $a\sqrt{2}$. Additionally, other quadrilaterals like rectangles and parallelograms can be used for valid periodic tessellation as well. They all result in the same material macroscopic stiffness and they all preserve the tetragonal symmetry, if PBC are applied to the unit-cell.

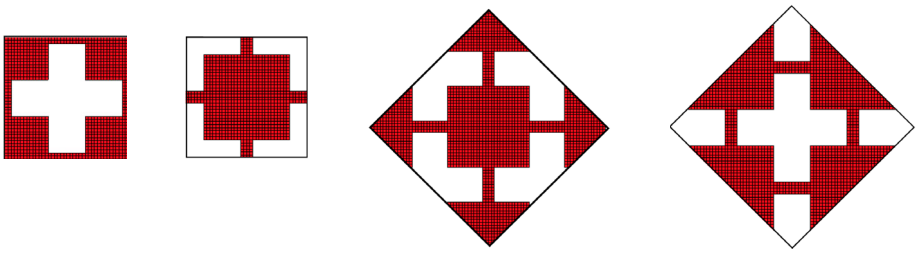
The application of affine Dirichlet BCs (KUBC) drastically reduces the above set of periodically "equivalent" unit-cells of sidelenghts a and $a\sqrt{2}$, since only four of them capture the tetragonal symmetry under these boundary conditions. These cells are highlighted in yellow color in Fig. 15, those with two symmetry axes of orthorhombic materials appear in blue shading. The rest of the displayed unit-cells exhibit only one symmetry axis. The KUBC render the symmetry criterion (iii) very selective for unit-cells.

[20] For a discussion of the non-uniqueness of the unit-cell, see [74].



(a) (b)

Fig. 15 Identification of candidate unit-cell variants that fulfill the extended Neumann's principle for (a) standard square cells of sidelength a and (b) for rotated square cells of sidelength $a\sqrt{2}$. Yellow/blue shading indicates tetragonal/orthorhombic symmetries. No shading for cells with only one symmetry axis. Note that the metamaterial obtained by the infinite repetition of all these different unit-cells is the same tetragonal metamaterial (Color figure online)



(a) (b) (c) (d)

Fig. 16 The four regular square unit-cells respecting tetragonal symmetry

The requirement for maximal stiffness on top of that condition, however, does not determine without ambiguity any variant in Fig. 16 as the single stiffest unit-cell.

Nevertheless, from Table 2 we will distill an upper estimate for the stiffness of the microstructural response. This is done as follows. For each unit-cell (a), (b), (c), (d) in Fig. 16, we have calculated the corresponding apparent stiffness tensor $\mathbb{C}^V_{\text{KUBC}}$. We will then determine that positive definite tensor $\mathbb{C}_{\text{micro}}$ which has tetragonal symmetry and which satisfies

$$\forall\, \overline{E} \in \operatorname{Sym}(3): \quad \langle \mathbb{C}_{\text{micro}}\, \overline{E}, \overline{E} \rangle \geq \left\langle \mathbb{C}^V_{\text{KUBC}}\, \overline{E}, \overline{E} \right\rangle \tag{49}$$

for all 4 unit-cells according to Theorem 1.

In Table 2, we also report the computed values of the unit-cell stiffnesses using constant-strain conditions, as well as the stiffness of a unit-cell completely filled by aluminum computed both with KUBC and PBC.

Finally, we remark that in the case of an isotropic unit-cell (full aluminum) the homogenization results for KUBC and PBC coincide.

The entries of the Löwner matrix supremum $\mathbb{C}_{\text{micro}}$ are obtained from Table 2 as follows[21]

$$\lambda_{\text{micro}} = \mathbf{5.270}, \qquad \mu_{\text{micro}} = \mathbf{8.927}, \qquad \mu^*_{\text{micro}} = \mathbf{8.332}. \tag{50}$$

[21] N.B. the value of λ_{micro} is correctly 5.270 and not 5.981 because of the definition of $\widehat{\lambda}$ in (47).

Table 2 Elastic parameter values in Voigt notation and restricted to the planar case. Homogenized material parameters [GPa] for tetragonal unit-cell variants (a)–(d) in Fig. 16 for KUBC and PBC, for constant strain assumption (Voigt-bound), and the case of single phase aluminum. Note that the PBC-values are the homogenized effective macroscopic stiffnesses, while the KUBC-values are the homogenized apparent stiffnesses of chosen unit-cells

geometry Fig. 16	boundary conditions in x-y-dir./loading	elasticity parameters λ_{hom}	μ_{hom}	μ^*_{hom}	
(a)–(d)	PBC	1.738	5.895	0.620	$\mathbb{C}_{\mathbf{macro}}$
(a)	KUBC	4.37	6.242	**8.332**	$\mathbb{C}_{\mathbf{micro}}$
(b)		2.125	5.899	2.264	
(c)		**5.270**	**8.927**	4.042	
(d)		5.981	6.254	4.96	
(a)–(d)	constant strain – Voigt	28.10	14.47	14.47	$\mathbb{C}_{\mathbf{Voigt}}$
aluminum full	KUBC and PBC	51.08	26.32	26.32	

In conclusion, the homogenization requirements (i)–(iii) for the set of microparameters along with the Hill-Mandel condition lead without ambiguity to KUBC and uniquely identify the stiffest microscopic response in terms of a function of the Lamé-parameters. With $\mathbb{C}_{\text{micro}}$ and $\mathbb{C}_{\text{macro}}$ in hand, we are able to compute $\mathbb{C}_{\text{e}}$ with formula (15).

We have used these values in a diversity of scenarios [2–4, 7, 16] to good avail.

4 Conclusion

The relaxed micromorphic model is a "macroscopic continuum" homogenized model which is able to reproduce the response of the selected metamaterial including band-gaps with only few material parameters which do not depend on frequency [16]. Using the Curl-curvature measure $\operatorname{Curl} P$ instead of the full gradient ∇P conveys $\mathbb{C}_{\text{micro}}$ (and a fortiori $\mathbb{C}_{\text{e}}$) a scale-independent meaning. We have mathematically justified that the tensor $\mathbb{C}_{\text{micro}}$ can be identified with the Löwner-half-order matrix-supremum of suitable apparent stiffness tensors on the micro-scale. To our understanding this identification is entirely new. In contrast, $\mathbb{C}_{\text{macro}}$ follows from standard periodic homogenization and determines the meso-scale elasticity tensor $\mathbb{C}_{\text{e}}$ via the exact micro-macro homogenization formula $\mathbb{C}_{\text{e}} = \mathbb{C}_{\text{micro}}(\mathbb{C}_{\text{micro}} - \mathbb{C}_{\text{macro}})^{-1}\mathbb{C}_{\text{macro}}$. Summarizing, the salient features of our novel approach of parameter identification are:

- $\mathbb{C}_{\text{micro}}$ represents the stiffest possible estimate of the linear elastic response of any admissible unit-cell under affine Dirichlet boundary conditions.
- Both $\mathbb{C}_{\text{micro}}$ and $\mathbb{C}_{\text{macro}}$ can be determined independently of the characteristic length-scale L_c of the relaxed micromorphic model.
- Both $\mathbb{C}_{\text{micro}}$ and $\mathbb{C}_{\text{macro}}$ are readily available by first order numerical homogenization on the unit-cell level.
- If the unit-cell is homogeneous, then $\mathbb{C}_{\text{micro}} = \mathbb{C}_{\text{macro}}$ implies $\mathbb{C}_{\text{e}} = +\infty$ and the relaxed micromorphic model automatically turns into classical linear elasticity with stiffness $\mathbb{C}_{\text{macro}}$, while the classical Eringen-Mindlin model would turn into a second gradient formulation with unbounded stiffness.

- For large rigid inclusions in the unit-cell, we have in the limit of infinite rigidity that $\mathbb{C}_{\text{micro}} \to +\infty$ ($\mathbb{C}_{\text{macro}} = \mathbb{C}_{\text{e}}$), reducing the relaxed micromorphic model effectively to a Cosserat model (a model with "rigid microstructure"), which is sensible.

5 Open Problems

We have mathematically justified that the tensor $\mathbb{C}_{\text{micro}}$ can be identified with the Löwner-half-order matrix-supremum of suitable apparent stiffness tensors on the micro-scale. On the other hand, $\mathbb{C}_{\text{macro}}$ follows from standard periodic homogenization and determines the mesoscale elasticity tensor $\mathbb{C}_{\text{e}}$ via the micro-macro homogenization formula (15)

$$\mathbb{C}_{\text{e}} = \mathbb{C}_{\text{micro}}(\mathbb{C}_{\text{micro}} - \mathbb{C}_{\text{macro}})^{-1}\mathbb{C}_{\text{macro}}.$$

For this micro-macro homogenization formula to make sense we need to have that $\mathbb{C}_{\text{micro}} - \mathbb{C}_{\text{macro}}$ is positive definite (and therefore invertible). For this positive definiteness, consider the difference

$$\begin{aligned}
&\big|V(x)\big|\frac{1}{2}\big\langle(\mathbb{C}_{\text{micro}} - \mathbb{C}_{\text{macro}})\overline{E}, \overline{E}\big\rangle \geq \big|V(x)\big|\frac{1}{2}\big\langle\big(\mathbb{C}^{V}_{\text{KUBC}} - \mathbb{C}_{\text{macro}}\big)\overline{E}, \overline{E}\big\rangle \\
&\quad = \inf\left\{\int_{\xi\in V(x)} \frac{1}{2}\big\langle\mathbb{C}(\xi)\big(\operatorname{sym}\nabla_\xi v(\xi) + \overline{E}\big), \operatorname{sym}\nabla_\xi v(\xi) + \overline{E}\big\rangle d\xi \mid v \in C_0^\infty\big(V(x), \mathbb{R}^3\big)\right\} \\
&\qquad - \inf\left\{\int_{\xi\in V(x)} \frac{1}{2}\big\langle\mathbb{C}(\xi)\big(\operatorname{sym}\nabla_\xi v(\xi) + \overline{E}\big), \operatorname{sym}\nabla_\xi v(\xi) + \overline{E}\big\rangle d\xi \mid v \in C_{\text{per}}^\infty\big(V(x), \mathbb{R}^3\big)\right\} \\
&\quad =: Q(\overline{E}, \overline{E}).
\end{aligned} \tag{51}$$

By compactness it would be sufficient for strict positive definiteness of $\mathbb{C}_{\text{micro}} - \mathbb{C}_{\text{macro}}$ that $Q(\overline{E}, \overline{E}) > 0\ \forall\, \overline{E} \in \operatorname{Sym}(3)$. Although it is easy to see that $Q(\overline{E}, \overline{E}) \geq 0$ in general, it remains to investigate under which assumptions on the geometry and material of the unit-cell the strict positivity of $Q(\overline{E}, \overline{E})$ can be established for other metamaterials. We believe that this is true for microstructures with sufficient contrast in material properties in all directions. This will be subject of further research.

We did not yet approach the determination of the static curvature parameters, i.e., the characteristic length scale L_c (or the curvature parameters induced by the more general anisotropic quadratic expression $\frac{\mu L_c^2}{2}\langle\mathbb{L}\operatorname{Curl} P, \operatorname{Curl} P\rangle$), where $\mathbb{L}$ is a positive definite fourth-order tensor mapping non-symmetric second-order tensors to non-symmetric second-order tensors). Knowledge of L_c (or $\mathbb{L}$) determines the possible long-range interaction of the microstructure. This task, however, will be greatly facilitated in future works since we already know the scale-independent short-range material parameters $\mathbb{C}_{\text{micro}}$ and $\mathbb{C}_{\text{macro}}$ and $\mathbb{C}_{\text{e}}$. It suffices then, in principle, to perform a range of inhomogeneous boundary value problems on different sized clusters of unit-cells (mimicking size-experiments) which activate the curvature terms of the relaxed micromorphic model in order to fit L_c (or $\mathbb{L}$), now via a suitably generalized Hill-Mandel energy equivalence condition. This will be the subject of further work.

Acknowledgements Patrizio Neff thanks Samuel Forest (Ecole des Mines, Paris), Geralf Hütter (TU Freiberg) and Jörg Schröder (University of Duisburg-Essen) for helpful discussions. The authors are also indebted to Lev Truskinovsky (ESPCI, Paris) for pertinent remarks which helped improve the paper.

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References

1. Abdulle, A.: Analysis of a heterogeneous multiscale FEM for problems in elasticity. Math. Models Methods Appl. Sci. **16**(04), 615–635 (2006)
2. Aivaliotis, A., Daouadji, A., Barbagallo, G., Tallarico, D., Neff, P., Madeo, A.: Low-and high-frequency Stoneley waves, reflection and transmission at a Cauchy/relaxed micromorphic interface (2018). arXiv preprint. arXiv:1810.12578
3. Aivaliotis, A., Daouadji, A., Barbagallo, G., Tallarico, D., Neff, P., Madeo, A.: Microstructure-related Stoneley waves and their effect on the scattering properties of a 2d Cauchy/relaxed-micromorphic interface. Wave Motion **90**, 99–120 (2019)
4. Aivaliotis, A., Tallarico, D., Daouadji, A., Neff, P., Madeo, A.: Scattering of finite-size anisotropic metastructures via the relaxed micromorphic model (2019). arXiv preprint. arXiv:1905.12297
5. Auffray, N., Bouchet, R., Brechet, Y.: Derivation of anisotropic matrix for bi-dimensional strain-gradient elasticity behavior. Int. J. Solids Struct. **46**(2), 440–454 (2009)
6. Barbagallo, G., Madeo, A., d'Agostino, M.V., Abreu, R., Ghiba, I.-D., Neff, P.: Transparent anisotropy for the relaxed micromorphic model: macroscopic consistency conditions and long wave length asymptotics. Int. J. Solids Struct. **120**, 7–30 (2017)
7. Barbagallo, G., Tallarico, D., d'Agostino, M.V., Aivaliotis, A., Neff, P., Madeo, A.: Relaxed micromorphic model of transient wave propagation in anisotropic band-gap metastructures. Int. J. Solids Struct. **162**, 148–163 (2019)
8. Bauer, S., Neff, P., Pauly, D., Starke, G.: Dev-Div- and DevSym-DevCurl-inequalities for incompatible square tensor fields with mixed boundary conditions. ESAIM Control Optim. Calc. Var. **22**(1), 112–133 (2016)
9. Bensoussan, A., Lions, J.-L., Papanicolaou, G.: Asymptotic Analysis for Periodic Structures, vol. 5. North-Holland Publishing Company, Amsterdam (1978)
10. Biswas, R., Poh, L.H.: A micromorphic computational homogenization framework for heterogeneous materials. J. Mech. Phys. Solids **102**, 187–208 (2017)
11. Boutin, C., Rallu, A., Hans, S.: Large scale modulation of high frequency waves in periodic elastic composites. J. Mech. Phys. Solids **70**, 362–381 (2014)
12. Bouyge, F., Jasiuk, I., Boccara, S., Ostoja-Starzewski, M.: A micromechanically based couple-stress model of an elastic orthotropic two-phase composite. Eur. J. Mech. A, Solids **21**(3), 465–481 (2002)
13. Braides, A.: A handbook of Γ-convergence. Handb. Differ. Equ. **3**, 101–213 (2006)
14. Burgeth, B., Welk, M., Feddern, C., Weickert, J.: Mathematical morphology on tensor data using the Löwner ordering. In: Visualization and Processing of Tensor Fields, pp. 357–368. Springer, Berlin (2006)
15. Cosserat, E., Cosserat, F.: Théorie des corps déformables (1909). (engl. translation by D. Delphenich, pdf available at http://www.uni-due.de/%7ehm0014/Cosserat_files/Cosserat09_eng.pdf)
16. d'Agostino, M.V., Barbagallo, G., Ghiba, I.-D., Eidel, B., Neff, P., Madeo, A.: Effective description of anisotropic wave dispersion in mechanical metamaterials via the relaxed micromorphic model. J. Elast. (2019). Accepted in this volume
17. Diebels, S., Steeb, H.: Stress and couple stress in foams. Comput. Mater. Sci. **28**(3–4), 714–722 (2003)
18. Weinan, E., Engquist, B.: The heterogeneous multiscale methods. Commun. Math. Sci. **1**(1), 87–132 (2003)
19. Ehlers, W., Bidier, S.: From particle mechanics to micromorphic media. Part I: Homogenisation of discrete interactions towards stress quantities. Int. J. Solids Struct. (2018). https://doi.org/10.1016/j.ijsolstr.2018.08.013
20. Eidel, B., Fischer, A.: The heterogeneous multiscale finite element method for the homogenization of linear elastic solids and a comparison with the FE^2 method. Comput. Methods Appl. Mech. Eng. **329**, 332–368 (2018)
21. Cemal, A.: Eringen. Mechanics of micromorphic materials. In: Applied Mechanics, pp. 131–138. Springer, Berlin (1966)
22. Eringen, A.C.: Microcontinuum Field Theories. Springer, New York (1999)
23. Eringen, A.C., Suhubi, E.S.: Nonlinear theory of simple micro-elastic solids – I. Int. J. Eng. Sci. **2**(2), 189–203 (1964)
24. Fischer, A., Eidel, B.: Convergence and error analysis of FE-HMM/FE^2 for energetically consistent micro-coupling conditions in linear elastic solids. Eur. J. Mech. A, Solids **77**, 103735 (2019)

25. Forest, S.: Mechanics of generalized continua: construction by homogenizaton. J. Phys. IV **08**(4), 39–48 (1998)
26. Forest, S.: Aufbau und Identifikation von Stoffgleichungen für höhere Kontinua mittels Homogenisierungsmethoden. Tech. Mech. **19**(4), 297–306 (1999)
27. Forest, S.: Homogenization methods and mechanics of generalized continua – Part 2. Theor. Appl. Mech. **28–29**, 113–144 (2002)
28. Forest, S., Sab, K.: Cosserat overall modeling of heterogeneous materials. Mech. Res. Commun. **25**(4), 449–454 (1998)
29. Forest, S., Trinh, D.K.: Generalized continua and non-homogeneous boundary conditions in homogenisation methods. Z. Angew. Math. Mech. **91**(2), 90–109 (2011)
30. Ghiba, I.-D., Neff, P., Madeo, A., Münch, I.: A variant of the linear isotropic indeterminate couple-stress model with symmetric local force-stress, symmetric nonlocal force-stress, symmetric couple-stresses and orthogonal boundary conditions. Math. Mech. Solids **22**, 1221–1266 (2016)
31. Ghiba, I.-D., Neff, P., Madeo, A., Placidi, L., Rosi, G.: The relaxed linear micromorphic continuum: existence, uniqueness and continuous dependence in dynamics. Math. Mech. Solids **20**(10), 1171–1197 (2014)
32. Gologanu, M., Leblond, J.-B., Perrin, G., Devaux, J.: Recent extensions of Gurson's model for porous ductile metals Part II: a Gurson-like model including the effect of strong gradients of the macroscopic field. Contin. Micromech. **377**, 97–130 (1997)
33. Hill, R.: Elastic properties of reinforced solids: some theoretical principles. J. Mech. Phys. Solids **11**(5), 357–372 (1963)
34. Hill, R.: On constitutive macro-variables for heterogeneous solids at finite strain. Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. **326**(1565), 131–147 (1972)
35. Hori, M., Nemat-Nasser, S.: Micromechanics: Overall Properties of Heterogeneous Materials, vol. 37. Elsevier, Amsterdam (2013)
36. Huet, C.: Application of variational concepts to size effects in elastic heterogeneous bodies. J. Mech. Phys. Solids **38**(6), 813–841 (1990)
37. Huet, C.: An integrated micromechanics and statistical continuum thermodynamics approach for studying the fracture behaviour of microcracked heterogeneous materials with delayed response. Eng. Fract. Mech. **58**(5–6), 459–556 (1997)
38. Huet, C.: Coupled size and boundary-condition effects in viscoelastic heterogeneous and composite bodies. Mech. Mater. **31**(12), 787–829 (1999)
39. Hütter, G.: Homogenization of a Cauchy continuum towards a micromorphic continuum. J. Mech. Phys. Solids **99**, 394–408 (2017)
40. Hütter, G.: On the micro-macro relation for the microdeformation in the homogenization towards micromorphic and micropolar continua. J. Mech. Phys. Solids **127**, 62–79 (2019)
41. Jänicke, R., Diebels, S., Sehlhorst, H.-G., Düster, A.: Two-scale modelling of micromorphic continua. Contin. Mech. Thermodyn. **21**(4), 297–315 (2009)
42. Kanit, T., Forest, S., Galliet, I., Mounoury, V., Jeulin, D.: Determination of the size of the representative volume element for random composites: statistical and numerical approach. Int. J. Solids Struct. **40**(13–14), 3647–3679 (2003)
43. Kouznetsova, V., Geers, M.G.D., Brekelmans, M.W.A.: Multi-scale constitutive modelling of heterogeneous materials with a gradient-enhanced computational homogenization scheme. Int. J. Numer. Methods Eng. **54**(8), 1235–1260 (2002)
44. Kouznetsova, V., Geers, M.G.D., Brekelmans, M.W.A.: Multi-scale second-order computational homogenization of multi-phase materials: a nested finite element solution strategy. Comput. Methods Appl. Mech. Eng. **193**(48–51), 5525–5550 (2004)
45. Lobos, M., Yuzbasioglu, T., Böhlke, T.: Homogenization and materials design of anisotropic multiphase linear elastic materials using central model functions. J. Elast. **128**(1), 17–60 (2017)
46. Love, A.E.H.: A Treatise on the Mathematical Theory of Elasticity. Dover, New York (1944)
47. Madeo, A., Collet, M., Miniaci, M., Billon, K., Ouisse, M., Neff, P.: Modeling phononic crystals via the weighted relaxed micromorphic model with free and gradient micro-inertia. J. Elast. **130**, 1–25 (2017)
48. Madeo, A., Ghiba, I.-D., Neff, P., Münch, I.: A new view on boundary conditions in the Grioli–Koiter–Mindlin–Toupin indeterminate couple stress model. Eur. J. Mech. A, Solids **59**, 294–322 (2016)
49. Madeo, A., Neff, P., d'Agostino, M.V., Barbagallo, G.: Complete band gaps including non-local effects occur only in the relaxed micromorphic model. C. R., Méc. **344**(11), 784–796 (2016)
50. Madeo, A., Neff, P., Ghiba, I.-D., Placidi, L., Rosi, G.: Band gaps in the relaxed linear micromorphic continuum. Z. Angew. Math. Mech. **95**(9), 880–887 (2014)
51. Madeo, A., Neff, P., Ghiba, I.-D., Rosi, G.: Reflection and transmission of elastic waves in non-local band-gap metamaterials: a comprehensive study via the relaxed micromorphic model. J. Mech. Phys. Solids **95**, 441–479 (2016)

52. Mandel, J.: Plasticité classique et viscoplasticité. International Centre for Mechanical Sciences. Courses and Lectures (1971)
53. Michel, J.-C., Moulinec, H., Suquet, P.M.: Effective properties of composite materials with periodic microstructure: a computational approach. Comput. Methods Appl. Mech. Eng. **172**(1–4), 109–143 (1999)
54. Mindlin, R.D.: Micro-structure in linear elasticity. Arch. Ration. Mech. Anal. **16**(1), 51–78 (1964)
55. Münch, I., Neff, P., Madeo, A., Ghiba, I.-D.: The modified indeterminate couple stress model: why Yang et al.'s arguments motivating a symmetric couple stress tensor contain a gap and why the couple stress tensor may be chosen symmetric nevertheless. Z. Angew. Math. Mech. **97**(12), 1524–1554 (2017)
56. Neff, P.: On material constants for micromorphic continua. In: Trends in Applications of Mathematics to Mechanics, STAMM Proceedings, Seeheim, pp. 337–348. Shaker–Verlag, Aachen (2004)
57. Neff, P.: Existence of minimizers for a finite-strain micromorphic elastic solid. Proc. R. Soc. Edinb., Sect. A, Math. **136**(05), 997–1012 (2006)
58. Neff, P., Forest, S.: A geometrically exact micromorphic model for elastic metallic foams accounting for affine microstructure. Modelling, existence of minimizers, identification of moduli and computational results. J. Elast. **87**(2–3), 239–276 (2007)
59. Neff, P., Ghiba, I.-D., Lazar, M., Madeo, A.: The relaxed linear micromorphic continuum: well-posedness of the static problem and relations to the gauge theory of dislocations. Q. J. Mech. Appl. Math. **68**(1), 53–84 (2014)
60. Neff, P., Ghiba, I.-D., Madeo, A., Placidi, L., Rosi, G.: A unifying perspective: the relaxed linear micromorphic continuum. Contin. Mech. Thermodyn. **26**(5), 639–681 (2014)
61. Neff, P., Jeong, J., Fischle, A.: Stable identification of linear isotropic Cosserat parameters: bounded stiffness in bending and torsion implies conformal invariance of curvature. Acta Mech. **211**(3–4), 237–249 (2010)
62. Neff, P., Jeong, J., Ramézani, H.: Subgrid interaction and micro-randomness – Novel invariance requirements in infinitesimal gradient elasticity. Int. J. Solids Struct. **46**(25–26), 4261–4276 (2009)
63. Neff, P., Madeo, A., Barbagallo, G., d'Agostino, M.V., Abreu, R., Ghiba, I.-D.: Real wave propagation in the isotropic-relaxed micromorphic model. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **473**(2197) (2017)
64. Neff, P., Münch, I.: Curl bounds Grad on SO(3). ESAIM Control Optim. Calc. Var. **14**(1), 148–159 (2008)
65. Neff, P., Pauly, D., Witsch, K.-J.: Maxwell meets Korn: a new coercive inequality for tensor fields in $\mathbb{R}^{N\times N}$ with square-integrable exterior derivative. Math. Methods Appl. Sci. **35**(1), 65–71 (2012)
66. Neff, P., Pauly, D., Witsch, K.-J.: Poincaré meets Korn via Maxwell: extending Korn's first inequality to incompatible tensor fields. J. Differ. Equ. **258**(4), 1267–1302 (2015)
67. Neumann, F.E.: Vorlesungen über die Theorie der Elasticität der festen Körper und des Lichtäthers. B.G. Teubner, Leipzig (1885)
68. Pecullan, S., Gibiansky, L., Torquato, S.: Scale effects on the elastic behavior of periodic and hierarchical two-dimensional composites. J. Mech. Phys. Solids **47**(7), 1509–1542 (1999)
69. Pham, K., Kouznetsova, V.G., Geers, M.G.D.: Transient computational homogenization for heterogeneous materials under dynamic excitation. J. Mech. Phys. Solids **61**(11), 2125–2146 (2013)
70. Reuß, A.: Berechnung der Fließgrenze von Mischkristallen auf Grund der Plastizitätsbedingung für Einkristalle. Z. Angew. Math. Mech. **9**(1), 49–58 (1929)
71. Rokoš, O., Ameen, M.M., Peerlings, R.H.J., Geers, M.G.D.: Micromorphic computational homogenization for mechanical metamaterials with patterning fluctuation fields. J. Mech. Phys. Solids **123**, 119–137 (2019)
72. Romano, G., Barretta, R., Diaco, M.: Micromorphic continua: non-redundant formulations. Contin. Mech. Thermodyn. **28**(6), 1659–1670 (2016)
73. Karam, S.: On the homogenization and the simulation of random materials. Eur. J. Mech. A, Solids **5**, 585–607 (1992)
74. Schröder, J.: A numerical two-scale homogenization scheme: the FE^2-method. Schröder, J., Hackl, K. (eds.) Plasticity and Beyond: Microstructures, Crystal-Plasticity and Phase Transitions, vol. 550, pp. 1–64. Springer, Berlin (2014)
75. Smyshlyaev, V.P., Cherednichenko, K.D.: On rigorous derivation of strain gradient effects in the overall behaviour of periodic heterogeneous media. J. Mech. Phys. Solids **48**(6–7), 1325–1357 (2000)
76. Sridhar, A., Kouznetsova, V.G., Geers, M.G.D.: Homogenization of locally resonant acoustic metamaterials towards an emergent enriched continuum. Comput. Mech. **57**(3), 423–435 (2016)
77. Suquet, P.M.: Local and global aspects in the mathematical theory of plasticity. Plasticity today, 279–309 (1985)
78. Suquet, P.M.: Effective properties of nonlinear composites. In: Continuum Micromechanics, pp. 197–264. Springer, Berlin (1997)

79. Trinh D.K., Janicke, R., Auffray, N., Diebels, S., Forest, S.: Evaluation of generalized continuum substitution models for heterogeneous materials. Int. J. Multiscale Comput. Eng. **10**(6), 527–549 (2012)
80. Voigt, W.: Lehrbuch der Krystallphysik (mit Ausschluss der Krystalloptik). B.G. Teubner, Leipzig (1910)
81. Wang, C., Feng, L., Jasiuk, I.: Scale and boundary conditions effects on the apparent elastic moduli of trabecular bone modeled as a periodic cellular solid. J. Biomech. Eng. **131**(12), 121008 (2009)
82. Zohdi, T.I.: Homogenization Methods and Multiscale Modeling. Encyclopedia of Computational Mechanics (2004)