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On the Justification of Viscoelastic Elliptic Membrane Shell Equations

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Abstract We consider a family of linearly viscoelastic shells with thickness 2ε, clamped along their entire lateral face, all having the same middle surface $S = \boldsymbol{\theta}(\bar{\omega}) \subset \mathbb{R}^3$, where $\omega \subset \mathbb{R}^2$ is a bounded and connected open set with a Lipschitz-continuous boundary γ. We make an essential geometrical assumption on the middle surface S, which is satisfied if γ and $\boldsymbol{\theta}$ are smooth enough and S is uniformly elliptic. We show that, if the applied body force density is $O(1)$ with respect to ε and surface tractions density is $O(\varepsilon)$, the solution of the scaled variational problem in curvilinear coordinates, $\boldsymbol{u}(\varepsilon)$, defined over the fixed domain $\Omega = \omega \times (-1, 1)$ for each $t \in [0, T]$, converges to a limit $\boldsymbol{u}$ with $u_\alpha(\varepsilon) \to u_\alpha$ in $W^{1,2}(0, T, H^1(\Omega))$ and $u_3(\varepsilon) \to u_3$ in $W^{1,2}(0, T, L^2(\Omega))$ as $\varepsilon \to 0$. Moreover, we prove that this limit is independent of the transverse variable. Furthermore, the average $\bar{\boldsymbol{u}} = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u} dx_3$, which belongs to the space $W^{1,2}(0, T, V_M(\omega))$, where

$$V_M(\omega) = H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega),$$

satisfies what we have identified as (scaled) two-dimensional equations of a viscoelastic membrane elliptic shell, which includes a long-term memory that takes into account previous deformations. We finally provide convergence results which justify those equations.

Keywords Asymptotic analysis · Viscoelasticity · Shells · Elliptic membranes · Time dependency

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1 Introduction

In solid mechanics, the obtention of models for rods, beams, plates and shells is based on *a priori* hypotheses on the displacement and/or stress fields which, upon substitution in the three-dimensional equilibrium and constitutive equations, lead to useful simplifications. Nevertheless, from both constitutive and geometrical point of views, there is a need to justify the validity of most of the models obtained in this way.

For this reason a considerable effort has been made in the past decades by many authors in order to derive new models and justify the existing ones by using the asymptotic expansion method, whose foundations can be found in [1]. Indeed, the first applied results were obtained with the justification of the linearized theory of plate bending in [2, 3].

The existence and uniqueness of solution of elliptic membrane shell equations, can be found in [4] and in [5]. These two-dimensional models are completely justified with convergence theorems. A complete theory regarding elastic shells can be found in [6], where models for elliptic membranes, generalized membranes and flexural shells are presented. It contains a full description of the asymptotic procedure that leads to the corresponding sets of two-dimensional equations. Also, the dynamic case has been studied in [7–9], concerning the justification of dynamic equations for membrane, flexural and Koiter shells.

A large number of real problems had made it necessary the study of new models which could take into account effects such as hardening and memory of the material. An example of these are the viscoelasticity models (see [10–12]). Many authors have contributed to the nowadays knowledge of this sort of problems, providing justified models and results. Indeed, we can find examples in the literature as [13–18] and in the references therein, a variety of models for problems concerning the viscoelastic behaviour of the material. In particular, there exist studies of the behaviour of viscoelastic plates as in [19, 20], where models for von Kármán plates are analysed. In some of these works, we can find analysis of the influence of short or long term memory in the equations modelling a problem. These terms take into account previous deformations of the body, hence, they are commonly presented in some viscoelastic problems. For instance, on one hand, we can find in [21] models including a short term memory presented by a system of integro-differential and pseudoparabolic equations describing large deflections on a viscoelastic plate. On the other hand, in [22] a long term memory is considered on the study of the asymptotic behaviour of the solution of a von Kármán plate when the time variable tends to infinity. Also, in the reference [23], the authors study the effects of great deflections in thin plates covering both short and long term memory cases. Concerning viscoelastic shell problems, in [24] we can find different kind of studies where the authors also remark the viscoelastic property of the material of a shell. For the problems dealing with the shell-type equations, there exists a very limited amount of results available, for instance, [25] where the authors present a model for a dynamic contact problem where a short memory (Kelvin-Voigt) material is considered. Particularly remarkable is the increasing number of studies of viscoelastic shells problems in order to reproduce the complex behaviour of tissues in the field of biomedicine. For example, in [26] the difficulties of this kind of problems are detailed and even though an one-dimensional model is derived for modelling a vessel wall, the author comments the possibility of considering two-dimensional models with a shell-type description and a viscoelastic constitutive law. In this direction, to our knowledge, in [27] we gave the first steps towards the justification of existing models of viscoelastic shells and the finding of new ones. By using the asymptotic expansion method, we found a rich variety of cases, depending on the geometry of the middle surface, the boundary conditions and the order of the applied forces. The most remarkable feature was that from the asymptotic analysis of the three-dimensional problems which included a short term memory represented by a time derivative, a long term memory arised in the two-dimensional limit problems, represented by an integral with respect to the time variable. This fact, agreed with previous asymptotic analysis of viscoelastic rods in [28, 29] where an analogous behaviour was presented as well.

In this work we justify the two-dimensional equations of a viscoelastic membrane shell where the surface S is elliptic and the boundary condition of place is considered in the whole lateral face of the shell:

Problem 1 Find $\boldsymbol{\xi}^\varepsilon : [0,T] \times \omega \longrightarrow \mathbb{R}^3$ such that,

$$\boldsymbol{\xi}^\varepsilon(t,\cdot) \in V_M(\omega) := H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega), \ \forall t \in [0,T],$$

$$\varepsilon \int_\omega a^{\alpha\beta\sigma\tau,\varepsilon} \gamma_{\sigma\tau}\big(\boldsymbol{\xi}^\varepsilon(t)\big)\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy + \varepsilon \int_\omega b^{\alpha\beta\sigma\tau,\varepsilon} \gamma_{\sigma\tau}\big(\dot{\boldsymbol{\xi}}^\varepsilon(t)\big)\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dy$$

$$- \varepsilon \int_0^t e^{-k(t-s)} \int_\omega c^{\alpha\beta\sigma\tau,\varepsilon} \gamma_{\sigma\tau}\big(\boldsymbol{\xi}^\varepsilon(s)\big)\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dyds$$

$$= \int_\omega p^{i,\varepsilon}(t)\eta_i\sqrt{a}dy \ \forall \boldsymbol{\eta} = (\eta_i) \in V_M(\omega), \quad \text{a.e. in } (0,T),$$

$$\boldsymbol{\xi}^\varepsilon(0,\cdot) = \boldsymbol{\xi}_0^\varepsilon(\cdot),$$

where,

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma_{\alpha\beta}^\sigma \eta_\sigma - b_{\alpha\beta}\eta_3,$$

$$p^{i,\varepsilon}(t) := \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}(t)dx_3^\varepsilon + h_+^{i,\varepsilon}(t) + h_-^{i,\varepsilon}(t) \quad \text{and } h_\pm^{i,\varepsilon}(t) = h^{i,\varepsilon}(t,\cdot,\pm\varepsilon),$$

and where the contravariant components of the fourth order two-dimensional tensors $a^{\alpha\beta\sigma\tau,\varepsilon}$, $b^{\alpha\beta\sigma\tau,\varepsilon}$, $c^{\alpha\beta\sigma\tau,\varepsilon}$ are defined as rescaled versions of two-dimensional fourth order tensors that we shall recall later in (5.1)–(5.3).

In what follows, we shall prove that the scaled three-dimensional unknown, $\boldsymbol{u}(\varepsilon)$, converges as the small parameter ε tends to zero to a limit, $\boldsymbol{u}$, independent of the transversal variable. Moreover, we find that this limit can be identified with transversal average, $\bar{\boldsymbol{u}}$, for all point of the middle surface of the shell. Furthermore, we prove that $\bar{\boldsymbol{u}}$ is the unique solution of Problem 1, hence, the limit of the scaled unknown can be also identified with the solution of the two-dimensional problem, $\boldsymbol{\xi}$, defined over the middle surface of the shell.

We will follow the notation and style of [6], where the linear elastic shells are studied. For this reason, we shall reference auxiliary results which apply in the same manner to the viscoelastic case. One of the major differences with respect to previous works in elasticity, consists on the time dependence, that will lead to ordinary differential equations that need to be solved in order to find the zeroth-order approach of the solution.

The structure of the paper is the following: in Sect. 2 we shall recall the three-dimensional viscoelastic problem in Cartesian coordinates and then, considering the problem for a family of viscoelastic shells of thickness 2ε, we formulate the problem in curvilinear coordinates. In Sect. 3 we will use a projection map into a reference domain independent of the small parameter ε, we will introduce the scaled unknowns and forces and we present the assumptions on coefficients. In Sect. 4 we recall some technical results which will be needed in what follows. In Sect. 5, first we recall the results in [27], where the two-dimensional equations for a viscoelastic membrane shell were studied, in the particular case where the middle surface is elliptic and the boundary condition of place is considered in the whole lateral face of the shell. Then, we present the convergence results when the small parameter ε tends to zero, which is the main result of this paper. After that, we present the convergence results in terms of de-scaled unknowns. In Sect. 6 we shall present some conclusions, including a comparison between the viscoelastic models and the elastic case studied in [6] and announce the convergence results regarding other cases in forthcoming papers.

2 The Three-Dimensional Linearly Viscoelastic Shell Problem

We denote $\mathbb{S}^d$, where $d = 2, 3$ in practice, the space of second-order symmetric tensors on $\mathbb{R}^d$, while "$\cdot$" will represent the inner product and $|\cdot|$ the usual norm in $\mathbb{S}^d$ and $\mathbb{R}^d$. In what follows, unless the contrary is explicitly written, we will use summation convention on repeated indices. Moreover, Latin indices $i, j, k, l, \dots$, take their values in the set $\{1, 2, 3\}$, whereas Greek indices $\alpha, \beta, \sigma, \tau, \dots$, do it in the set $\{1, 2\}$. Also, we use standard notation for the Lebesgue and Sobolev spaces. Also, for a time dependent function u, we denote $\dot{u}$ the first derivative of u with respect to the time variable. Recall that "$\to$" denotes strong convergence, while "$\rightharpoonup$" denotes weak convergence.

Let Ω^* be a domain of $\mathbb{R}^3$, with a Lipschitz-continuous boundary $\Gamma^* = \partial\Omega^*$. Let $\boldsymbol{x}^* = (x_i^*)$ be a generic point of its closure $\bar{\Omega}^*$ and let ∂_i^* denote the partial derivative with respect to x_i^*. Let $d\boldsymbol{x}^*$ denote the volume element in Ω^*, $d\Gamma^*$ denote the area element along Γ^* and $\boldsymbol{n}^*$ denote the unit outer normal vector along Γ^*. Finally, let Γ_0^* and Γ_1^* be subsets of Γ^* such that $meas(\Gamma_0^*) > 0$ and $\Gamma_0^* \cap \Gamma_1^* = \emptyset$.

The set Ω^* is the region occupied by a deformable body in the absence of applied forces. We assume that this body is made of a Kelvin-Voigt viscoelastic material, which is homogeneous and isotropic, so that the material is characterized by its Lamé coefficients $\lambda \geq 0$, $\mu > 0$ and its viscosity coefficients, $\theta \geq 0$, $\rho \geq 0$ (see for instance [10, 11, 30]).

Let $T > 0$ be the time period of observation. Under the effect of applied forces, the body is deformed and we denote by $u_i^* : [0, T] \times \bar{\Omega}^* \to \mathbb{R}^3$ the Cartesian components of the displacements field, defined as $\boldsymbol{u}^* := u_i^* \boldsymbol{e}^i : [0, T] \times \bar{\Omega}^* \to \mathbb{R}^3$, where $\{\boldsymbol{e}^i\}$ denotes the Euclidean canonical basis in $\mathbb{R}^3$. Moreover, we consider that the displacement field vanishes on the set Γ_0^*. Hence, the displacements field $\boldsymbol{u}^* = (u_i^*) : [0, T] \times \Omega^* \longrightarrow \mathbb{R}^3$ is solution of the following three-dimensional problem in Cartesian coordinates.

Problem 2 Find $\boldsymbol{u}^* = (u_i^*) : [0, T] \times \Omega^* \longrightarrow \mathbb{R}^3$ such that,

$$-\partial_j^* \sigma^{ij,*}(\boldsymbol{u}^*) = f^{i,*} \quad \text{in } \Omega^*, \tag{2.1}$$

$$u_i^* = 0 \quad \text{on } \Gamma_0^*, \tag{2.2}$$

$$\sigma^{ij,*}(\boldsymbol{u}^*) n_j^* = h^{i,*} \quad \text{on } \Gamma_1^*, \tag{2.3}$$

$$\boldsymbol{u}^*(0, \cdot) = \boldsymbol{u}_0^* \quad \text{in } \Omega^*, \tag{2.4}$$

where the functions

$$\sigma^{ij,*}(\boldsymbol{u}^*) := A^{ijkl,*} e_{kl}^*(\boldsymbol{u}^*) + B^{ijkl,*} e_{kl}^*(\dot{\boldsymbol{u}}^*),$$

are the components of the linearized stress tensor field and where the functions

$$A^{ijkl,*} := \lambda\delta^{ij}\delta^{kl} + \mu\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right),$$
$$B^{ijkl,*} := \theta\delta^{ij}\delta^{kl} + \frac{\rho}{2}\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right),$$

are the components of the three-dimensional elasticity and viscosity fourth order tensors, respectively, and

$$e^*_{ij}\left(\boldsymbol{u}^*\right) := \frac{1}{2}\left(\partial^*_j u^*_i + \partial^*_i u^*_j\right),$$

designates the components of the linearized strain tensor associated with the displacement field $\boldsymbol{u}^*$of the set $\bar{\Omega}^*$.

We now proceed to describe the equations in Problem 2. Expression (2.1) is the equilibrium equation, where $f^{i,*}$ are the components of the volumic force densities. The equality (2.2) is the Dirichlet condition of place, (2.3) is the Neumann condition, where $h^{i,*}$ are the components of surface force densities and (2.4) is the initial condition, where $\boldsymbol{u}^*_0$ denotes the initial displacements.

Note that, for the sake of briefness, we omit the explicit dependence on the space and time variables when there is no ambiguity. Let us define the space of admissible unknowns,

$$V\left(\Omega^*\right) = \left\{\boldsymbol{v}^* = \left(v^*_i\right) \in \left[H^1\left(\Omega^*\right)\right]^3; \boldsymbol{v}^* = \boldsymbol{0} \text{ on } \Gamma^*_0\right\}.$$

Therefore, assuming enough regularity, the unknown $\boldsymbol{u}^* = (u^*_i)$ satisfies the following variational problem in Cartesian coordinates:

Problem 3 Find $\boldsymbol{u}^* = (u^*_i) : [0, T] \times \Omega^* \to \mathbb{R}^3$ such that,

$$\boldsymbol{u}^*(t, \cdot) \in V\left(\Omega^*\right) \quad \forall t \in [0, T],$$
$$\int_{\Omega^*} A^{ijkl,*} e^*_{kl}\left(\boldsymbol{u}^*\right) e^*_{ij}\left(\boldsymbol{v}^*\right) dx^* + \int_{\Omega^*} B^{ijkl,*} e^*_{kl}\left(\dot{\boldsymbol{u}}^*\right) e^*_{ij}\left(\boldsymbol{v}^*\right) dx^*$$
$$= \int_{\Omega^*} f^{i,*} v^*_i dx^* + \int_{\Gamma^*_1} h^{i,*} v^*_i d\Gamma^* \quad \forall \boldsymbol{v}^* \in V\left(\Omega^*\right), \quad \text{a.e. in } (0, T),$$
$$\boldsymbol{u}^*(0, \cdot) = \boldsymbol{u}^*_0(\cdot).$$

Let us consider that Ω^* is a viscoelastic shell of thickness 2ε. Now, we shall express the equations of Problem 3 in terms of curvilinear coordinates. Let ω be a domain of $\mathbb{R}^2$, with a Lipschitz-continuous boundary $\gamma = \partial\omega$. Let $\boldsymbol{y} = (y_\alpha)$ be a generic point of its closure $\bar{\omega}$ and let ∂_α denote the partial derivative with respect to y_α.

Let $\boldsymbol{\theta} \in \mathcal{C}^2(\bar{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha(\boldsymbol{y}) := \partial_\alpha\boldsymbol{\theta}(\boldsymbol{y})$ are linearly independent. These vectors form the covariant basis of the tangent plane to the surface $S := \boldsymbol{\theta}(\bar{\omega})$ at the point $\boldsymbol{\theta}(\boldsymbol{y}) = \boldsymbol{y}^*$. The surface S is uniformly elliptic, in the sense that the two principal radius of curvature are either both positive at all points of S, or both negative at all points of S. We can consider the two vectors $\boldsymbol{a}^\alpha(\boldsymbol{y})$ of the same tangent plane defined by the relations $\boldsymbol{a}^\alpha(\boldsymbol{y}) \cdot \boldsymbol{a}_\beta(\boldsymbol{y}) = \delta^\alpha_\beta$, that constitute the contravariant basis. We define the unit vector,

$$\boldsymbol{a}_3(\boldsymbol{y}) = \boldsymbol{a}^3(\boldsymbol{y}) := \frac{\boldsymbol{a}_1(\boldsymbol{y}) \wedge \boldsymbol{a}_2(\boldsymbol{y})}{|\boldsymbol{a}_1(\boldsymbol{y}) \wedge \boldsymbol{a}_2(\boldsymbol{y})|}, \tag{2.5}$$

normal vector to S at the point $\boldsymbol{\theta}(\boldsymbol{y}) = \boldsymbol{y}^*$, where $\wedge$ denotes vector product in $\mathbb{R}^3$.

We can define the first fundamental form, given as metric tensor, in covariant or contravariant components, respectively, by

$$a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \qquad a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta,$$

the second fundamental form, given as curvature tensor, in covariant or mixed components, respectively, by

$$b_{\alpha\beta} := \boldsymbol{a}^3 \cdot \partial_\beta \boldsymbol{a}_\alpha, \qquad b_\alpha^\beta := a^{\beta\sigma} b_{\sigma\alpha},$$

and the Christoffel symbols of the surface S by

$$\Gamma^\sigma_{\alpha\beta} := \boldsymbol{a}^\sigma \cdot \partial_\beta \boldsymbol{a}_\alpha.$$

The area element along S is $\sqrt{a}dy = dy^*$ where

$$a := \det(a_{\alpha\beta}). \tag{2.6}$$

For each $\varepsilon > 0$, we define the three-dimensional domain $\Omega^\varepsilon := \omega \times (-\varepsilon, \varepsilon)$ and its boundary $\Gamma^\varepsilon = \partial\Omega^\varepsilon$. We also define the following parts of the boundary,

$$\Gamma_+^\varepsilon := \omega \times \{\varepsilon\}, \qquad \Gamma_-^\varepsilon := \omega \times \{-\varepsilon\}, \qquad \Gamma_0^\varepsilon := \gamma \times [-\varepsilon, \varepsilon].$$

Let $\boldsymbol{x}^\varepsilon = (x_i^\varepsilon)$ be a generic point of $\bar{\Omega}^\varepsilon$ and let ∂_i^ε denote the partial derivative with respect to x_i^ε. Note that $x_\alpha^\varepsilon = y_\alpha$ and $\partial_\alpha^\varepsilon = \partial_\alpha$. Let $\boldsymbol{\Theta} : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ be the mapping defined by

$$\boldsymbol{\Theta}\left(\boldsymbol{x}^\varepsilon\right) := \boldsymbol{\theta}(\boldsymbol{y}) + x_3^\varepsilon \boldsymbol{a}_3(\boldsymbol{y}) \quad \forall \boldsymbol{x}^\varepsilon = \left(\boldsymbol{y}, x_3^\varepsilon\right) = \left(y_1, y_2, x_3^\varepsilon\right) \in \bar{\Omega}^\varepsilon. \tag{2.7}$$

The next theorem shows that if the injective mapping $\boldsymbol{\theta} : \bar{\omega} \to \mathbb{R}^3$ is smooth enough, the mapping $\boldsymbol{\Theta} : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ is also injective for $\varepsilon > 0$ small enough (see Theorem 3.1-1, [6]).

Theorem 1 *Let ω be a domain in $\mathbb{R}^2$. Let $\boldsymbol{\theta} \in \mathcal{C}^2(\bar{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are linearly independent at all points of $\bar{\omega}$ and let $\boldsymbol{a}_3$, defined in (2.5). Then there exists $\varepsilon_0 > 0$ such that the mapping $\boldsymbol{\Theta} : \bar{\Omega}_0 \to \mathbb{R}^3$ defined by*

$$\boldsymbol{\Theta}(\boldsymbol{y}, x_3) := \boldsymbol{\theta}(\boldsymbol{y}) + x_3 \boldsymbol{a}_3(\boldsymbol{y}) \quad \forall (\boldsymbol{y}, x_3) \in \bar{\Omega}_0, \quad \text{where } \Omega_0 := \omega \times (-\varepsilon_0, \varepsilon_0),$$

is a $\mathcal{C}^1-$ diffeomorphism from $\bar{\Omega}_0$ onto $\boldsymbol{\Theta}(\bar{\Omega}_0)$ and $\det(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3) > 0$ in $\bar{\Omega}_0$, where $\boldsymbol{g}_i := \partial_i \boldsymbol{\Theta}$.

For each ε, $0 < \varepsilon \leq \varepsilon_0$, the set $\boldsymbol{\Theta}(\bar{\Omega}^\varepsilon) = \bar{\Omega}^*$ is the reference configuration of a viscoelastic shell, with middle surface $S = \boldsymbol{\theta}(\bar{\omega})$ and thickness $2\varepsilon > 0$. Furthermore for $\varepsilon > 0$, $\boldsymbol{g}_i^\varepsilon(\boldsymbol{x}^\varepsilon) := \partial_i^\varepsilon \boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$ are linearly independent and the mapping $\boldsymbol{\Theta} : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ is injective for all ε, $0 < \varepsilon \leq \varepsilon_0$, as a consequence of injectivity of the mapping $\boldsymbol{\theta}$. Hence, the three vectors $\boldsymbol{g}_i^\varepsilon(\boldsymbol{x}^\varepsilon)$ form the covariant basis of the tangent space at the point $\boldsymbol{x}^* = \boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$ and $\boldsymbol{g}^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)$

defined by the relations $\boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}_j^{\varepsilon} = \delta_j^i$ form the contravariant basis at the point $\boldsymbol{x}^* = \boldsymbol{\Theta}(\boldsymbol{x}^{\varepsilon})$. We define the metric tensor, in covariant or contravariant components, respectively, by

$$g_{ij}^{\varepsilon} := \boldsymbol{g}_i^{\varepsilon} \cdot \boldsymbol{g}_j^{\varepsilon}, \qquad g^{ij,\varepsilon} := \boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^{j,\varepsilon},$$

and Christoffel symbols by

$$\Gamma_{ij}^{p,\varepsilon} := \boldsymbol{g}^{p,\varepsilon} \cdot \partial_i^{\varepsilon} \boldsymbol{g}_j^{\varepsilon}. \tag{2.8}$$

The volume element in the set $\boldsymbol{\Theta}(\bar{\Omega}^{\varepsilon}) = \bar{\Omega}^*$ is $\sqrt{g^{\varepsilon}} dx^{\varepsilon} = dx^*$ and the surface element in $\boldsymbol{\Theta}(\Gamma^{\varepsilon}) = \Gamma^*$ is $\sqrt{g^{\varepsilon}} d\Gamma^{\varepsilon} = d\Gamma^*$ where

$$g^{\varepsilon} := \det\bigl(g_{ij}^{\varepsilon}\bigr). \tag{2.9}$$

Therefore, for a field $\boldsymbol{v}^*$ defined in $\boldsymbol{\Theta}(\bar{\Omega}^{\varepsilon}) = \bar{\Omega}^*$, we define its covariant curvilinear coordinates v_i^{ε} by

$$\boldsymbol{v}^*\bigl(\boldsymbol{x}^*\bigr) = v_i^*\bigl(\boldsymbol{x}^*\bigr)\boldsymbol{e}^i =: v_i^{\varepsilon}\bigl(\boldsymbol{x}^{\varepsilon}\bigr)\boldsymbol{g}^i\bigl(\boldsymbol{x}^{\varepsilon}\bigr), \quad \text{with } \boldsymbol{x}^* = \boldsymbol{\Theta}\bigl(\boldsymbol{x}^{\varepsilon}\bigr).$$

Besides, we denote by $u_i^{\varepsilon} : [0, T] \times \bar{\Omega}^{\varepsilon} \to \mathbb{R}^3$ the covariant components of the displacements field, that is $\boldsymbol{\mathcal{U}}^{\varepsilon} := u_i^{\varepsilon} \boldsymbol{g}^{i,\varepsilon} : [0, T] \times \bar{\Omega}^{\varepsilon} \to \mathbb{R}^3$. For simplicity, we define the vector field $\boldsymbol{u}^{\varepsilon} = (u_i^{\varepsilon}) : [0, T] \times \Omega^{\varepsilon} \to \mathbb{R}^3$ which will be denoted vector of unknowns.

Recall that we assumed that the shell is subjected to a boundary condition of place; in particular that the displacements field vanishes in $\boldsymbol{\Theta}(\Gamma_0^{\varepsilon}) = \Gamma_0^*$, this is, on the whole lateral face of the shell.

Accordingly, let us define the space of admissible unknowns,

$$V\bigl(\Omega^{\varepsilon}\bigr) = \bigl\{\boldsymbol{v}^{\varepsilon} = \bigl(v_i^{\varepsilon}\bigr) \in \bigl[H^1\bigl(\Omega^{\varepsilon}\bigr)\bigr]^3; \boldsymbol{v}^{\varepsilon} = \boldsymbol{0} \text{ on } \Gamma_0^{\varepsilon}\bigr\}.$$

This is a real Hilbert space with the induced inner product of $[H^1(\Omega^{\varepsilon})]^3$. The corresponding norm is denoted by $\|\cdot\|_{1,\Omega^{\varepsilon}}$.

Therefore, we can find the expression of Problem 3 in curvilinear coordinates (see [6] for details). Hence, the "displacements" field $\boldsymbol{u}^{\varepsilon} = (u_i^{\varepsilon})$ verifies the following variational problem of a three-dimensional viscoelastic shell in curvilinear coordinates:

Problem 4 Find $\boldsymbol{u}^{\varepsilon} = (u_i^{\varepsilon}) : [0, T] \times \Omega^{\varepsilon} \to \mathbb{R}^3$ such that,

$$\begin{aligned}
&\boldsymbol{u}^{\varepsilon}(t, \cdot) \in V\bigl(\Omega^{\varepsilon}\bigr) \quad \forall t \in [0, T], \\
&\int_{\Omega^{\varepsilon}} A^{ijkl,\varepsilon} e_{k\|l}^{\varepsilon}\bigl(\boldsymbol{u}^{\varepsilon}\bigr) e_{i\|j}^{\varepsilon}\bigl(\boldsymbol{v}^{\varepsilon}\bigr)\sqrt{g^{\varepsilon}} dx^{\varepsilon} + \int_{\Omega^{\varepsilon}} B^{ijkl,\varepsilon} e_{k\|l}^{\varepsilon}\bigl(\dot{\boldsymbol{u}}^{\varepsilon}\bigr) e_{i\|j}^{\varepsilon}\bigl(\boldsymbol{v}^{\varepsilon}\bigr)\sqrt{g^{\varepsilon}} dx^{\varepsilon} \\
&\quad = \int_{\Omega^{\varepsilon}} f^{i,\varepsilon} v_i^{\varepsilon} \sqrt{g^{\varepsilon}} dx^{\varepsilon} + \int_{\Gamma_+^{\varepsilon} \cup \Gamma_-^{\varepsilon}} h^{i,\varepsilon} v_i^{\varepsilon} \sqrt{g^{\varepsilon}} d\Gamma^{\varepsilon} \quad \forall \boldsymbol{v}^{\varepsilon} \in V\bigl(\Omega^{\varepsilon}\bigr), \quad \text{a.e. in } (0, T), \\
&\boldsymbol{u}^{\varepsilon}(0, \cdot) = \boldsymbol{u}_0^{\varepsilon}(\cdot),
\end{aligned}$$

where the functions

$$A^{ijkl,\varepsilon} := \lambda g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu\bigl(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}\bigr), \tag{2.10}$$

$$B^{ijkl,\varepsilon} := \theta g^{ij,\varepsilon} g^{kl,\varepsilon} + \frac{\rho}{2}\bigl(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}\bigr), \tag{2.11}$$

are the contravariant components of the three-dimensional elasticity and viscosity tensors, respectively. We assume that the Lamé coefficients $\lambda \geq 0$, $\mu > 0$ and the viscosity coefficients $\theta \geq 0$, $\rho \geq 0$ are all independent of ε. Moreover, the terms

$$e^{\varepsilon}_{i\|j}(\boldsymbol{u}^{\varepsilon}) := \frac{1}{2}\left(u^{\varepsilon}_{i\|j} + u^{\varepsilon}_{j\|i}\right) = \frac{1}{2}\left(\partial^{\varepsilon}_j u^{\varepsilon}_i + \partial^{\varepsilon}_i u^{\varepsilon}_j\right) - \Gamma^{p,\varepsilon}_{ij} u^{\varepsilon}_p,$$

designate the covariant components of the linearized strain tensor associated with the displacement field $\boldsymbol{\mathcal{U}}^{\varepsilon}$ of the set $\boldsymbol{\Theta}(\bar{\Omega}^{\varepsilon})$. Moreover, $f^{i,\varepsilon}$ denotes the contravariant components of the volumic force densities, $h^{i,\varepsilon}$ denotes contravariant components of surface force densities and $\boldsymbol{u}^{\varepsilon}_0$ denotes the initial "displacements" (actually, the initial displacement is $\boldsymbol{\mathcal{U}}^{\varepsilon}_0 := (u^{\varepsilon}_0)_i \boldsymbol{g}^{i,\varepsilon}$).

Note that the following additional relations are satisfied,

$$\begin{aligned} &\Gamma^{3,\varepsilon}_{\alpha 3} = \Gamma^{p,\varepsilon}_{33} = 0 \quad \text{in } \bar{\Omega}^{\varepsilon}, \\ &A^{\alpha\beta\sigma 3,\varepsilon} = A^{\alpha 333,\varepsilon} = B^{\alpha\beta\sigma 3,\varepsilon} = B^{\alpha 333,\varepsilon} = 0 \quad \text{in } \bar{\Omega}^{\varepsilon}, \end{aligned} \tag{2.12}$$

as a consequence of the definition of $\boldsymbol{\Theta}$ in (2.7).

The existence and uniqueness of solution of Problem 4 for $\varepsilon > 0$ small enough, established in the following theorem, was proved in [27].

Theorem 2 *Let Ω^{ε} be a domain in $\mathbb{R}^3$ defined previously in this section and let $\boldsymbol{\Theta}$ be a $\mathcal{C}^2$-diffeomorphism of $\bar{\Omega}^{\varepsilon}$ in its image $\boldsymbol{\Theta}(\bar{\Omega}^{\varepsilon})$, such that the three vectors $\boldsymbol{g}^{\varepsilon}_i(\boldsymbol{x}) = \partial^{\varepsilon}_i \boldsymbol{\Theta}(\boldsymbol{x}^{\varepsilon})$ are linearly independent for all $\boldsymbol{x}^{\varepsilon} \in \bar{\Omega}^{\varepsilon}$. Let Γ^{ε}_0 be a $d\Gamma^{\varepsilon}$-measurable subset of $\Gamma^{\varepsilon} = \partial\Omega^{\varepsilon}$ such that $meas(\Gamma^{\varepsilon}_0) > 0$. Let $f^{i,\varepsilon} \in L^2(0,T; L^2(\Omega^{\varepsilon}))$, $h^{i,\varepsilon} \in L^2(0,T; L^2(\Gamma^{\varepsilon}_1))$, where $\Gamma^{\varepsilon}_1 := \Gamma^{\varepsilon}_+ \cup \Gamma^{\varepsilon}_-$. Let $\boldsymbol{u}^{\varepsilon}_0 \in V(\Omega^{\varepsilon})$. Then, there exists a unique solution $\boldsymbol{u}^{\varepsilon} = (u^{\varepsilon}_i) : [0,T] \times \Omega^{\varepsilon} \to \mathbb{R}^3$ satisfying Problem 4. Moreover, $\boldsymbol{u}^{\varepsilon} \in W^{1,2}(0,T; V(\Omega^{\varepsilon}))$. In addition to that, if $\dot{f}^{i,\varepsilon} \in L^2(0,T; L^2(\Omega^{\varepsilon}))$, $\dot{h}^{i,\varepsilon} \in L^2(0,T; L^2(\Gamma^{\varepsilon}_1))$, then $\boldsymbol{u}^{\varepsilon} \in W^{2,2}(0,T; V(\Omega^{\varepsilon}))$.*

3 The Scaled Three-Dimensional Shell Problem

For convenience, we consider a reference domain independent of the small parameter ε. Hence, let us define the three-dimensional domain $\Omega := \omega \times (-1,1)$ and its boundary $\Gamma = \partial\Omega$. We also define the following parts of the boundary,

$$\Gamma_+ := \omega \times \{1\}, \qquad \Gamma_- := \omega \times \{-1\}, \qquad \Gamma_0 := \gamma \times [-1,1].$$

Let $\boldsymbol{x} = (x_1, x_2, x_3)$ be a generic point in $\bar{\Omega}$ and we consider the notation ∂_i for the partial derivative with respect to x_i. We define the following projection map,

$$\pi^{\varepsilon} : \boldsymbol{x} = (x_1, x_2, x_3) \in \bar{\Omega} \to \pi^{\varepsilon}(\boldsymbol{x}) = \boldsymbol{x}^{\varepsilon} = \left(x^{\varepsilon}_i\right) = \left(x^{\varepsilon}_1, x^{\varepsilon}_2, x^{\varepsilon}_3\right) = (x_1, x_2, \varepsilon x_3) \in \bar{\Omega}^{\varepsilon},$$

hence, $\partial^{\varepsilon}_{\alpha} = \partial_{\alpha}$ and $\partial^{\varepsilon}_3 = \frac{1}{\varepsilon}\partial_3$. We consider the scaled unknown $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) : [0,T] \times \bar{\Omega} \to \mathbb{R}^3$ and the scaled vector fields $\boldsymbol{v} = (v_i) : \bar{\Omega} \to \mathbb{R}^3$ defined as

$$u^{\varepsilon}_i\left(t, \boldsymbol{x}^{\varepsilon}\right) =: u_i(\varepsilon)(t, \boldsymbol{x}) \quad \text{and} \quad v^{\varepsilon}_i\left(\boldsymbol{x}^{\varepsilon}\right) =: v_i(\boldsymbol{x}) \; \forall \boldsymbol{x}^{\varepsilon} = \pi^{\varepsilon}(\boldsymbol{x}) \in \bar{\Omega}^{\varepsilon}, \quad \forall t \in [0,T].$$

Also, let the functions, $\Gamma_{ij}^{p,\varepsilon}$, g^ε, $A^{ijkl,\varepsilon}$, $B^{ijkl,\varepsilon}$ defined in (2.8), (2.9), (2.10) and (2.11), be associated with the functions $\Gamma_{ij}^p(\varepsilon)$, $g(\varepsilon)$, $A^{ijkl}(\varepsilon)$, $B^{ijkl}(\varepsilon)$ defined by

$$\Gamma_{ij}^p(\varepsilon)(\boldsymbol{x}) := \Gamma_{ij}^{p,\varepsilon}\big(\boldsymbol{x}^\varepsilon\big), \tag{3.1}$$

$$g(\varepsilon)(\boldsymbol{x}) := g^\varepsilon\big(\boldsymbol{x}^\varepsilon\big), \tag{3.2}$$

$$A^{ijkl}(\varepsilon)(\boldsymbol{x}) := A^{ijkl,\varepsilon}\big(\boldsymbol{x}^\varepsilon\big), \tag{3.3}$$

$$B^{ijkl}(\varepsilon)(\boldsymbol{x}) := B^{ijkl,\varepsilon}\big(\boldsymbol{x}^\varepsilon\big), \tag{3.4}$$

for all $\boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \bar{\Omega}^\varepsilon$. For all $\boldsymbol{v} = (v_i) \in [H^1(\Omega)]^3$, let there be associated the scaled linearized strains components $e_{i\|j}(\varepsilon)(\boldsymbol{v}) \in L^2(\Omega)$, defined by

$$e_{\alpha\|\beta}(\varepsilon;\boldsymbol{v}) := \frac{1}{2}(\partial_\beta v_\alpha + \partial_\alpha v_\beta) - \Gamma_{\alpha\beta}^p(\varepsilon)v_p, \tag{3.5}$$

$$e_{\alpha\|3}(\varepsilon;\boldsymbol{v}) := \frac{1}{2}\left(\frac{1}{\varepsilon}\partial_3 v_\alpha + \partial_\alpha v_3\right) - \Gamma_{\alpha 3}^p(\varepsilon)v_p, \tag{3.6}$$

$$e_{3\|3}(\varepsilon;\boldsymbol{v}) := \frac{1}{\varepsilon}\partial_3 v_3. \tag{3.7}$$

Note that with these definitions it is verified that

$$e_{i\|j}^\varepsilon\big(\boldsymbol{v}^\varepsilon\big)\big(\pi^\varepsilon(\boldsymbol{x})\big) = e_{i\|j}(\varepsilon;\boldsymbol{v})(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega.$$

Remark 1 The functions $\Gamma_{ij}^p(\varepsilon)$, $g(\varepsilon)$, $A^{ijkl}(\varepsilon)$, $B^{ijkl}(\varepsilon)$ converge in $\mathcal{C}^0(\bar{\Omega})$ when ε tends to zero.

Remark 2 When we consider $\varepsilon = 0$ the functions will be defined with respect to $\boldsymbol{y} \in \bar{\omega}$. We shall distinguish the three-dimensional Christoffel symbols from the two-dimensional ones by using $\Gamma_{\alpha\beta}^\sigma(\varepsilon)$ and $\Gamma_{\alpha\beta}^\sigma$, respectively.

The next result is an adaptation of (*b*) in Theorem 3.3-2, [6] to the viscoelastic case. We will study the asymptotic behaviour of the scaled contravariant components $A^{ijkl}(\varepsilon)$, $B^{ijkl}(\varepsilon)$ of the three-dimensional elasticity and viscosity tensors defined in (3.3)–(3.4), as $\varepsilon \to 0$. We show their uniform positive definiteness not only with respect to $\boldsymbol{x} \in \bar{\Omega}$, but also with respect to ε, $0 < \varepsilon \le \varepsilon_0$. Finally, their limits are functions of $\boldsymbol{y} \in \bar{\omega}$ only, that is, independent of the transversal variable x_3.

Theorem 3 *Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^2(\bar{\omega};\mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha\boldsymbol{\theta}$ are linearly independent at all points of $\bar{\omega}$, let $a^{\alpha\beta}$ denote the contravariant components of the metric tensor of $S = \boldsymbol{\theta}(\bar{\omega})$. In addition to that, let the other assumptions on the mapping $\boldsymbol{\theta}$ and the definition of ε_0 be as in Theorem 1. The contravariant components $A^{ijkl}(\varepsilon)$, $B^{ijkl}(\varepsilon)$ of the scaled three-dimensional elasticity and viscosity tensors, respectively, defined in* (3.3)–(3.4) *satisfy*

$$A^{ijkl}(\varepsilon) = A^{ijkl}(0) + O(\varepsilon) \quad \textit{and} \quad A^{\alpha\beta\sigma 3}(\varepsilon) = A^{\alpha 333}(\varepsilon) = 0,$$

$$B^{ijkl}(\varepsilon) = B^{ijkl}(0) + O(\varepsilon) \quad \textit{and} \quad B^{\alpha\beta\sigma 3}(\varepsilon) = B^{\alpha 333}(\varepsilon) = 0,$$

for all ε, $0 < \varepsilon \leq \varepsilon_0$, *and*

$$A^{\alpha\beta\sigma\tau}(0) = \lambda a^{\alpha\beta} a^{\sigma\tau} + \mu\big(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}\big), \qquad A^{\alpha\beta 33}(0) = \lambda a^{\alpha\beta},$$
$$A^{\alpha 3\sigma 3}(0) = \mu a^{\alpha\sigma}, \qquad A^{3333}(0) = \lambda + 2\mu,$$
$$A^{\alpha\beta\sigma 3}(0) = A^{\alpha 333}(0) = 0,$$
$$B^{\alpha\beta\sigma\tau}(0) = \theta a^{\alpha\beta} a^{\sigma\tau} + \frac{\rho}{2}\big(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}\big), \qquad B^{\alpha\beta 33}(0) = \theta a^{\alpha\beta},$$
$$B^{\alpha 3\sigma 3}(0) = \frac{\rho}{2} a^{\alpha\sigma}, \qquad B^{3333}(0) = \theta + \rho,$$
$$B^{\alpha\beta\sigma 3}(0) = B^{\alpha 333}(0) = 0.$$

Moreover, there exist two constants $C_e > 0$ *and* $C_v > 0$, *independent of the variables and* ε, *such that*

$$\sum_{i,j} |t_{ij}|^2 \leq C_e A^{ijkl}(\varepsilon)(\boldsymbol{x}) t_{kl} t_{ij}, \tag{3.8}$$

$$\sum_{i,j} |t_{ij}|^2 \leq C_v B^{ijkl}(\varepsilon)(\boldsymbol{x}) t_{kl} t_{ij}, \tag{3.9}$$

for all ε, $0 < \varepsilon \leq \varepsilon_0$, *for all* $\boldsymbol{x} \in \bar{\Omega}$ *and all* $\boldsymbol{t} = (t_{ij}) \in \mathbb{S}^2$.

Remark 3 Note that the proof for the scaled viscosity tensor $(B^{ijkl}(\varepsilon))$ would follow the steps of the proof for the elasticity tensor $(A^{ijkl}(\varepsilon))$ in Theorem 3.3-2, [6], since from a quality point of view their expressions differ in replacing the Lamé constants by the two viscosity coefficients.

Let the scaled applied forces $\boldsymbol{f}^i(\varepsilon) : [0, T] \times \Omega \to \mathbb{R}^3$ and $\boldsymbol{h}^i(\varepsilon) : [0, T] \times (\Gamma_+ \cup \Gamma_-) \to \mathbb{R}^3$ be defined by

$$\boldsymbol{f}^\varepsilon = \big(f^{i,\varepsilon}\big)\big(t, \boldsymbol{x}^\varepsilon\big) =: \boldsymbol{f}(\varepsilon) = \big(f^i(\varepsilon)\big)(t, \boldsymbol{x})$$
$$\forall \boldsymbol{x} \in \Omega, \quad \text{where } \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \Omega^\varepsilon \quad \text{and} \quad \forall t \in [0, T],$$
$$\boldsymbol{h}^\varepsilon = \big(h^{i,\varepsilon}\big)\big(t, \boldsymbol{x}^\varepsilon\big) =: \boldsymbol{h}(\varepsilon) = \big(h^i(\varepsilon)\big)(t, \boldsymbol{x})$$
$$\forall \boldsymbol{x} \in \Gamma_+ \cup \Gamma_-, \quad \text{where } \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon \quad \text{and} \quad \forall t \in [0, T].$$

Also, we introduce $\boldsymbol{u}_0(\varepsilon) : \Omega \to \mathbb{R}^3$ by

$$\boldsymbol{u}_0(\varepsilon)(\boldsymbol{x}) := \boldsymbol{u}_0^\varepsilon\big(\boldsymbol{x}^\varepsilon\big) \quad \forall \boldsymbol{x} \in \Omega, \quad \text{where } \boldsymbol{x}^\varepsilon = \pi^\varepsilon(\boldsymbol{x}) \in \Omega^\varepsilon,$$

and define the space

$$V(\Omega) := \big\{\boldsymbol{v} = (v_i) \in \big[H^1(\Omega)\big]^3; \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0\big\},$$

which is a Hilbert space, with associated norm denoted by $\|\cdot\|_{1,\Omega}$.

We assume that the scaled applied forces are given by

$$\boldsymbol{f}(\varepsilon)(t, \boldsymbol{x}) = \varepsilon^p \boldsymbol{f}^p(t, \boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega \quad \text{and} \quad \forall t \in [0, T],$$

$$\boldsymbol{h}(\varepsilon)(t,\boldsymbol{x}) = \varepsilon^{p+1}\boldsymbol{h}^{p+1}(t,\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Gamma_+ \cup \Gamma_- \quad \text{and} \quad \forall t \in [0,T],$$

where $\boldsymbol{f}^p$ and $\boldsymbol{h}^{p+1}$ are functions independent of ε and where p is a natural number that will show the order of the volume and surface forces, respectively. Then, the scaled variational problem can be written as follows:

Problem 5 Find $\boldsymbol{u}(\varepsilon) : [0,T] \times \Omega \to \mathbb{R}^3$ such that,

$$\begin{aligned}
&\boldsymbol{u}(\varepsilon)(t,\cdot) \in V(\Omega) \quad \forall t \in [0,T],\\
&\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}\big(\varepsilon, \boldsymbol{u}(\varepsilon)\big) e_{i\|j}(\varepsilon,\boldsymbol{v})\sqrt{g(\varepsilon)}dx + \int_\Omega B^{ijkl}(\varepsilon) e_{k\|l}\big(\varepsilon, \dot{\boldsymbol{u}}(\varepsilon)\big) e_{i\|j}(\varepsilon,\boldsymbol{v})\sqrt{g(\varepsilon)}dx\\
&\quad = \int_\Omega \varepsilon^p f^{i,p} v_i \sqrt{g(\varepsilon)}dx + \int_{\Gamma_+\cup\Gamma_-} \varepsilon^p h^{i,p+1} v_i \sqrt{g(\varepsilon)}d\Gamma \quad \forall \boldsymbol{v} \in V(\Omega), \quad \text{a.e. in } (0,T),\\
&\boldsymbol{u}(\varepsilon)(0,\cdot) = \boldsymbol{u}_0(\varepsilon)(\cdot).
\end{aligned}$$

From now on, for each $\varepsilon > 0$, we shall use the shorter notation $e_{i\|j}(\varepsilon) \equiv e_{i\|j}(\varepsilon; \boldsymbol{u}(\varepsilon))$ and $\dot{e}_{i\|j}(\varepsilon) \equiv e_{i\|j}(\varepsilon; \dot{\boldsymbol{u}}(\varepsilon))$, for its time derivative. We recall the existence and uniqueness of Problem 5 in the following theorem whose proof can be found in [27]:

Theorem 4 *Let Ω be a domain in $\mathbb{R}^3$ defined previously in this section and let $\boldsymbol{\Theta}$ be a $\mathcal{C}^2$-diffeomorphism of $\bar{\Omega}$ onto its image $\boldsymbol{\Theta}(\bar{\Omega})$, such that the three vectors $\boldsymbol{g}_i = \partial_i \boldsymbol{\Theta}(\boldsymbol{x})$ are linearly independent for all $\boldsymbol{x} \in \bar{\Omega}$. Let $f^i(\varepsilon) \in L^2(0,T; L^2(\Omega))$, $h^i(\varepsilon) \in L^2(0,T; L^2(\Gamma_1))$, where $\Gamma_1 := \Gamma_+ \cup \Gamma_-$. Let $\boldsymbol{u}_0(\varepsilon) \in V(\Omega)$. Then, there exists a unique solution $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) : [0,T] \times \Omega \to \mathbb{R}^3$ satisfying Problem 5. Moreover $\boldsymbol{u}(\varepsilon) \in W^{1,2}(0,T; V(\Omega))$. In addition to that, if $\dot{f}^i(\varepsilon) \in L^2(0,T; L^2(\Omega))$, $\dot{h}^i(\varepsilon) \in L^2(0,T; L^2(\Gamma_1))$, then $\boldsymbol{u}(\varepsilon) \in W^{2,2}(0,T; V(\Omega))$.*

4 Technical Preliminaries

Concerning geometrical and mechanical preliminaries, we shall present some theorems, which will be used in the following sections. First, we recall Theorem 3.3-1, [6].

Theorem 5 *Let ω be a domain in $\mathbb{R}^2$, let $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathcal{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are linearly independent at all points of $\bar{\omega}$ and let $\varepsilon_0 > 0$ be as in Theorem 1. The functions $\Gamma_{ij}^p(\varepsilon) = \Gamma_{ji}^p(\varepsilon)$ and $g(\varepsilon)$ are defined in (3.1)–(3.2), the functions $b_{\alpha\beta}, b_\alpha^\sigma, \Gamma_{\alpha\beta}^\sigma, a$, are defined in Sect. 2 and the covariant derivatives $b_\beta^\sigma|_\alpha$ are defined by*

$$b_\beta^\sigma\big|_\alpha := \partial_\alpha b_\beta^\sigma + \Gamma_{\alpha\tau}^\sigma b_\beta^\tau - \Gamma_{\alpha\beta}^\tau b_\tau^\sigma.$$

The functions $b_{\alpha\beta}, b_\alpha^\sigma, \Gamma_{\alpha\beta}^\sigma, b_\beta^\sigma|_\alpha$ and a are identified with functions in $\mathcal{C}^0(\bar{\Omega})$. Then

$$\begin{aligned}
\Gamma_{\alpha\beta}^\sigma(\varepsilon) &= \Gamma_{\alpha\beta}^\sigma - \varepsilon x_3 b_\beta^\sigma\big|_\alpha + O\big(\varepsilon^2\big), & \Gamma_{\alpha\beta}^3(\varepsilon) &= b_{\alpha\beta} - \varepsilon x_3 b_\alpha^\sigma b_{\sigma\beta},\\
\partial_3 \Gamma_{\alpha\beta}^p(\varepsilon) &= O(\varepsilon), & \Gamma_{\alpha 3}^\sigma(\varepsilon) &= -b_\alpha^\sigma - \varepsilon x_3 b_\alpha^\tau b_\tau^\sigma + O\big(\varepsilon^2\big),\\
\Gamma_{\alpha 3}^3(\varepsilon) &= \Gamma_{33}^p(\varepsilon) = 0, & g(\varepsilon) &= a + O(\varepsilon),
\end{aligned}$$

for all ε, $0 < \varepsilon \leq \varepsilon_0$, *where the order symbols* $O(\varepsilon)$ *and* $O(\varepsilon^2)$ *are meant with respect to the norm* $\|\cdot\|_{0,\infty,\bar{\Omega}}$ *defined by*

$$\|w\|_{0,\infty,\bar{\Omega}} = \sup\{|w(\boldsymbol{x})|; \boldsymbol{x} \in \bar{\Omega}\}.$$

Finally, there exist constants a_0, g_0 *and* g_1 *such that*

$$\begin{aligned} &0 < a_0 \leq a(\boldsymbol{y}) \quad \forall \boldsymbol{y} \in \bar{\omega}, \\ &0 < g_0 \leq g(\varepsilon)(\boldsymbol{x}) \leq g_1 \quad \forall \boldsymbol{x} \in \bar{\Omega} \quad \text{and} \quad \forall \varepsilon,\ 0 < \varepsilon \leq \varepsilon_0. \end{aligned} \tag{4.1}$$

Remark 4 The asymptotic behaviour of $g(\varepsilon)$ and the contravariant components of elasticity and viscosity tensors, $A^{ijkl}(\varepsilon)$, $B^{ijkl}(\varepsilon)$ also implies that

$$A^{ijkl}(\varepsilon)\sqrt{g(\varepsilon)} = A^{ijkl}(0)\sqrt{a} + \varepsilon \tilde{A}^{ijkl,1} + \varepsilon^2 \tilde{A}^{ijkl,2} + o(\varepsilon^2), \tag{4.2}$$

$$B^{ijkl}(\varepsilon)\sqrt{g(\varepsilon)} = B^{ijkl}(0)\sqrt{a} + \varepsilon \tilde{B}^{ijkl,1} + \varepsilon^2 \tilde{B}^{ijkl,2} + o(\varepsilon^2), \tag{4.3}$$

for certain regular contravariant components $\tilde{A}^{ijkl,\alpha}$, $\tilde{B}^{ijkl,\alpha}$ of certain tensors.

We now include the following result that will be used repeatedly in what follows (see Theorem 3.4-1, [6], for details).

Theorem 6 *Let* ω *be a domain in* $\mathbb{R}^2$ *with boundary* γ, *let* $\Omega = \omega \times (-1, 1)$, *and let* $g \in L^p(\Omega)$, $p > 1$, *be a function such that*

$$\int_\Omega g \partial_3 v dx = 0, \quad \textit{for all } v \in \mathcal{C}^\infty(\bar{\Omega}) \textit{ with } v = 0 \textit{ on } \gamma \times [-1, 1].$$

Then $g = 0$.

Remark 5 This result holds if $\int_\Omega g \partial_3 v dx = 0$ for all $v \in H^1(\Omega)$ such that $v = 0$ in Γ_0. It is in this way that we will use this result in the following.

We now introduce the average with respect to the transversal variable, which plays a major role in this study. To that end, let $\boldsymbol{v}$ represent real or vectorial functions defined almost everywhere over $\Omega = \omega \times (-1, 1)$. We define the transversal average as

$$\bar{\boldsymbol{v}}(\boldsymbol{y}) = \frac{1}{2} \int_{-1}^{1} \boldsymbol{v}(\boldsymbol{y}, x_3) dx_3$$

for almost all $\boldsymbol{y} \in \omega$. Given $\boldsymbol{\eta} = (\eta_i) \in [H^1(\omega)]^3$, let

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma^\sigma_{\alpha\beta} \eta_\sigma - b_{\alpha\beta} \eta_3, \tag{4.4}$$

denote the covariant components of the linearized change of metric tensor associated with a displacement field $\eta_i \boldsymbol{a}^i$ of the surface S. Next theorem will show some results related with the transversal averages that will be useful in the next section.

Theorem 7 *Let* ω *be a domain in* $\mathbb{R}^2$, *let* $\Omega = \omega \times (-1, 1)$ *and* $T > 0$.

(a) *Let* $v \in W^{1,2}(0,T,L^2(\Omega))$. *Then* $\bar{v}(\boldsymbol{y})$ *is finite for almost all* $\boldsymbol{y} \in \omega$, *belongs to* $W^{1,2}(0,T,L^2(\omega))$, *and*

$$|\bar{v}|_{W^{1,2}(0,T,L^2(\omega))} \leq \frac{1}{\sqrt{2}} |v|_{W^{1,2}(0,T,L^2(\Omega))}.$$

If $\partial_3 v = 0$ *in the distributions sense* ($\int_\Omega v \partial_3 \varphi dx = 0 \ \forall \varphi \in \mathcal{D}(\Omega)$) *then* v *does not depend on* x_3 *and*

$$v(\boldsymbol{y}, x_3) = \bar{v}(\boldsymbol{y}) \quad \text{for almost all } (\boldsymbol{y}, x_3) \in \Omega.$$

(b) *Let* $v \in W^{1,2}(0,T,H^1(\Omega))$. *Then* $\bar{v} \in W^{1,2}(0,T,H^1(\omega))$, $\partial_\alpha \bar{v} = \overline{\partial_\alpha v}$ *and*

$$\|\bar{v}\|_{W^{1,2}(0,T,H^1(\omega))} \leq \frac{1}{\sqrt{2}} \|v\|_{W^{1,2}(0,T,H^1(\Omega))}.$$

Let γ_0 *be a subset* $\partial\gamma$*-measurable of* γ. *If* $v = 0$ *on* $\gamma_0 \times [-1,1]$ *then* $\bar{v} = 0$ *on* γ_0; *in particular,* $\bar{v} \in W^{1,2}(0,T,H_0^1(\omega))$ *if* $v = 0$ *on* $\gamma \times [-1,1]$.

(c) *Let* $(v(\varepsilon))_{\varepsilon>0}$ *be a sequence of functions* $v(\varepsilon) \in W^{1,2}(0,T,H^1(\Omega))$ *and let* $\bar{v} \in W^{1,2}(0,T,L^2(\omega))$ *such that*

$$\partial_3 v(\varepsilon) \to 0 \quad \text{in } W^{1,2}\big(0,T,L^2(\Omega)\big) \quad \text{and}$$
$$\bar{v}(\varepsilon) \to \bar{v} \quad \text{in } W^{1,2}\big(0,T,L^2(\omega)\big) \text{ when } \varepsilon \to 0.$$

Then, $v(\varepsilon) \to \bar{v}$ *in* $W^{1,2}(0,T,L^2(\Omega))$ *when* $\varepsilon \to 0$, *where the function* $\bar{v} \in W^{1,2}(0,T,L^2(\omega))$ *is identified with a function in* $W^{1,2}(0,T,L^2(\Omega))$ *taking* $\bar{v}(\boldsymbol{y}, x_3) := \bar{v}(\boldsymbol{y})$ *for almost all* $(\boldsymbol{y}, x_3) \in \Omega$.

(d) *Let* $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{R}^3)$. *We consider a sequence* $(\boldsymbol{v}(\varepsilon))_{\varepsilon>0}$ *of vector fields* $\boldsymbol{v}(\varepsilon) = (v_i(\varepsilon)) \in W^{1,2}(0,T,[H^1(\Omega)]^3)$ *that is bounded in* $W^{1,2}(0,T,[L^2(\Omega)]^3)$. *Then,*

$$\big(\overline{e_{\alpha\|\beta}\big(\varepsilon; \boldsymbol{v}(\varepsilon)\big)} - \gamma_{\alpha\beta}\big(\overline{\boldsymbol{v}(\varepsilon)}\big)\big) \to 0 \quad \text{in } W^{1,2}\big(0,T,L^2(\omega)\big) \text{ when } \varepsilon \to 0.$$

Remark 6 This theorem is a corollary of Theorem 4.2-1, [6] and its proof follows straightforward from the result presented there. The main difference is that we are interested in obtaining the corresponding conclusions in the Bochner spaces, namely $W^{1,2}(0,T,L^2(\Omega))$, $W^{1,2}(0,T,L^2(\omega))$, $W^{1,2}(0,T,H^1(\Omega))$, $W^{1,2}(0,T,H^1(\omega))$. Therefore, most of the changes of the proof consist in adding an additional integral with respect to the time variable and proving the statements for the functions and their time derivatives, alternately, over the spaces $L^2(0,T,L^2(\Omega))$, $L^2(0,T,L^2(\omega))$, $L^2(0,T,H^1(\Omega))$, $L^2(0,T,H^1(\omega))$.

In the next theorem we recall a three-dimensional inequality of Korn's type for a family of elliptic membrane shells, that can also be found in Theorem 4.3-1, [6].

Theorem 8 *Assume that* $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{R}^3)$ *and consider* ε_0 *defined as in Theorem* 1. *We consider a family of elliptic membrane shells with thickness* 2ε *with each having the same middle surface* $S = \boldsymbol{\theta}(\bar{\omega})$. *Then there exist a constant* ε_1 *verifying* $0 < \varepsilon_1 < \varepsilon_0$ *and a constant* $C > 0$ *such that, for all* ε, $0 < \varepsilon \leq \varepsilon_1$, *the following three-dimensional inequality of Korn's type holds,*

$$\Big(\sum_\alpha \|v_\alpha\|_{1,\Omega}^2 + |v_3|_{0,\Omega}^2\Big)^{1/2} \leq C \Big(\sum_{i,j} \big|e_{i\|j}(\varepsilon; \boldsymbol{v})\big|_{0,\Omega}^2\Big)^{1/2} \quad \forall \boldsymbol{v} = (v_i) \in V(\Omega). \tag{4.5}$$

Let us define the space

$$V(\omega) := \{\boldsymbol{v} = (v_i) \in [H^1(\omega)]^3;\, v_i = 0 \text{ on } \gamma\}.$$

Theorem 9 *Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^{2,1}(\bar{\omega};\mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are linearly independent at all points of $\bar{\omega}$ and such that the surface $S = \boldsymbol{\theta}(\bar{\omega})$ is elliptic. Then the following inequality is verified*

$$\left(\sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + |\eta_3|_{0,\omega}^2\right)^{1/2} \leq C_M \left(\sum_{\alpha,\beta} \left|\gamma_{\alpha\beta}(\boldsymbol{\eta})\right|_{0,\omega}^2\right)^{1/2} \quad \forall \boldsymbol{\eta} \in V_M(\omega). \tag{4.6}$$

The proof of this result can be found in Theorem 2.7-3, [6]. Note that this implies that the completion of $V(\omega)$ with the norm

$$|\cdot|_\omega^M := \left(\sum_{\alpha,\beta} \left|\gamma_{\alpha\beta}(\boldsymbol{\eta})\right|_{0,\omega}^2\right)^{1/2}$$

is precisely $V_M(\omega)$.

5 Asymptotic Analysis. Convergence Results as $\varepsilon \to 0$

Firstly, we recall the two-dimensional equations obtained for a viscoelastic membrane shell as a consequence of the formal asymptotic study made in [27]. For the case of elliptic membranes, the right space where the problem is well posed is $V_M(\omega)$. Moreover, the space defined by

$$V_0(\omega) := \{\boldsymbol{\eta} \in [H^1(\omega)]^3;\, \boldsymbol{\eta} = \boldsymbol{0} \text{ on } \gamma,\, \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ on } \omega\},$$

is such that only contains the element $\boldsymbol{\eta} = \boldsymbol{0}$ (see (4.6)).

From the asymptotic analysis made in [27], we show that, if the applied body force density is $O(1)$ with respect to ε and surface tractions density is $O(\varepsilon)$ in Problem 5, we obtain the two-dimensional variational problem for a viscoelastic membrane shell. Let us remind the definition of the two-dimensional fourth-order tensors that appeared naturally in that study,

$$a^{\alpha\beta\sigma\tau} := \frac{2\lambda\rho^2 + 4\mu\theta^2}{(\theta+\rho)^2} a^{\alpha\beta}a^{\sigma\tau} + 2\mu\left(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\right), \tag{5.1}$$

$$b^{\alpha\beta\sigma\tau} := \frac{2\theta\rho}{\theta+\rho} a^{\alpha\beta}a^{\sigma\tau} + \rho\left(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\right), \tag{5.2}$$

$$c^{\alpha\beta\sigma\tau} := \frac{2(\theta\Lambda)^2}{\theta+\rho} a^{\alpha\beta}a^{\sigma\tau}, \tag{5.3}$$

where

$$\Lambda := \left(\frac{\lambda}{\theta} - \frac{\lambda+2\mu}{\theta+\rho}\right). \tag{5.4}$$

Therefore, we can enunciate the two-dimensional variational problem for a linear viscoelastic elliptic membrane shell:

Problem 6 Find $\boldsymbol{\xi} : [0, T] \times \omega \to \mathbb{R}^3$ such that,

$$\boldsymbol{\xi}(t, \cdot) \in V_M(\omega) \quad \forall t \in [0, T],$$

$$\int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\xi}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} dy + \int_\omega b^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\dot{\boldsymbol{\xi}}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} dy$$

$$- \int_0^t e^{-k(t-s)} \int_\omega c^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}\big(\boldsymbol{\xi}(s)\big) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} dy ds$$

$$= \int_\omega p^{i,0} \eta_i \sqrt{a} dy \quad \forall \boldsymbol{\eta} = (\eta_i) \in V_M(\omega), \quad \text{a.e. } t \in (0, T)$$

$$\boldsymbol{\xi}(0, \cdot) = \boldsymbol{\xi}_0(\cdot),$$

where we introduced the constant k defined by

$$k := \frac{\lambda + 2\mu}{\theta + \rho}, \tag{5.5}$$

and

$$p^{i,0}(t) := \int_{-1}^{1} f^{i,0}(t) dx_3 + h_+^{i,1}(t) + h_-^{i,1}(t), \quad \text{with } h_\pm^{i,1}(t) = h^{i,1}(t, \cdot, \pm 1).$$

The Problem 6 is well posed and it has existence and uniqueness of solution. Furthermore, we obtained the following result (see [27] for details of the proof of the de-scaled version):

Theorem 10 *Let ω be a domain in $\mathbb{R}^2$, let $\boldsymbol{\theta} \in \mathcal{C}^2(\bar{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are linearly independent at all points of $\bar{\omega}$. Let $f^{i,0} \in L^2(0, T; L^2(\Omega))$, $h^{i,1} \in L^2(0, T; L^2(\Gamma_1))$, where $\Gamma_1 := \Gamma_+ \cup \Gamma_-$. Let $\boldsymbol{\xi}_0 \in V_M(\omega)$. Then Problem 6, has a unique solution $\boldsymbol{\xi} \in W^{1,2}(0, T; V_M(\omega))$. In addition to that, if $\dot{f}^{i,0} \in L^2(0, T; L^2(\Omega))$, $\dot{h}^{i,1} \in L^2(0, T; L^2(\Gamma_1))$, then $\boldsymbol{\xi} \in W^{2,2}(0, T; V_M(\omega))$.*

For each $\varepsilon > 0$, we assume that the initial condition for the scaled linear strains is

$$e_{i\|j}(\varepsilon)(0, \cdot) = 0, \tag{5.6}$$

this is, the domain is on its natural state with no strains on it at the beginning of the period of observation.

Now, we present here the main result of this paper, namely that the scaled three-dimensional unknown $\boldsymbol{u}(\varepsilon)$ converges, as ε tends to zero, towards a limit $\boldsymbol{u}$ independent of the transversal variable. Moreover, this limit can be identified with the solution $\boldsymbol{\xi} = \bar{\boldsymbol{u}}$ of Problem 6, posed over the set ω.

Theorem 11 *Assume that $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{R}^3)$. Consider a family of viscoelastic elliptic membrane shells with thickness 2ε approaching zero and with each having the same elliptic middle surface $S = \boldsymbol{\theta}(\bar{\omega})$, and let the assumptions on the data be as in Theorem 10. For all ε, $0 < \varepsilon \leq \varepsilon_0$ let $\boldsymbol{u}(\varepsilon)$ be the solution of the associated three-dimensional scaled Problem 5 for $p = 0$. Then, there exist functions $u_\alpha \in W^{1,2}(0, T, H^1(\Omega))$ satisfying $u_\alpha = 0$ on $\gamma \times [-1, 1]$ and a function $u_3 \in W^{1,2}(0, T, L^2(\Omega))$, such that*

(i) *$u_\alpha(\varepsilon) \to u_\alpha$ in $W^{1,2}(0, T, H^1(\Omega))$ and $u_3(\varepsilon) \to u_3$ in $W^{1,2}(0, T, L^2(\Omega))$ when $\varepsilon \to 0$,*

(ii) $\boldsymbol{u} := (u_i)$ *is independent of the transversal variable* x_3.

Furthermore, the average $\bar{\boldsymbol{u}} := \frac{1}{2}\int_{-1}^{1} \boldsymbol{u} dx_3$ *verifies Problem* 6.

Proof We follow the same structure of the proof in Theorem 4.4-1, [6]. Hence, we shall reference some steps which apply in the same manner. We start by considering that the proposed problem is subjected only to volume forces in order to simplify the exposition. We will add the surface forces in latter steps. Therefore, we suppose that the scaled unknown $\boldsymbol{u}(\varepsilon)$ satisfies the following variational problem:

Find $\boldsymbol{u}(\varepsilon)(t, \cdot) \in V(\Omega)\ \forall t \in [0, T]$ such that

$$\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon) e_{i\|j}(\varepsilon, \boldsymbol{v})\sqrt{g(\varepsilon)}dx + \int_\Omega B^{ijkl}(\varepsilon)\dot{e}_{k\|l}(\varepsilon) e_{i\|j}(\varepsilon, \boldsymbol{v})\sqrt{g(\varepsilon)}dx$$
$$= \int_\Omega f^i v_i \sqrt{g(\varepsilon)}dx \quad \forall \boldsymbol{v} \in V(\Omega), \quad \text{a.e. } t \in (0, T) \tag{5.7}$$
$$\boldsymbol{u}(\varepsilon)(0, \cdot) = \boldsymbol{u}_0(\varepsilon)(\cdot),$$

where we identified $f^i \equiv f^{i,0}$, for notational brevity. The proof is divided into several parts, numbered from (i) to (x).

(i) *A priori boundedness and extraction of weak convergent sequences.*
The norms $|e_{i\|j}(\varepsilon)|_{W^{1,2}(0,T,L^2(\Omega))}, \|u_\alpha(\varepsilon)\|_{W^{1,2}(0,T,H^1(\Omega))}$, *and* $|u_3(\varepsilon)|_{W^{1,2}(0,T,L^2(\Omega))}$ *are bounded independently of* $\varepsilon, 0 < \varepsilon \le \varepsilon_1$, *where* $\varepsilon_1 > 0$ *is given in Theorem* 8. *Consequently, there exists a subsequence, also denoted* $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$, *and functions* $e_{i\|j} \in W^{1,2}(0, T, L^2(\Omega)), u_\alpha \in W^{1,2}(0, T, H^1(\Omega))$, *satisfying* $u_\alpha = 0$ *on* Γ_0, *and* $u_3 \in W^{1,2}(0, T, L^2(\Omega))$, *such that*

$$e_{i\|j}(\varepsilon) \rightharpoonup e_{i\|j} \quad \text{in } W^{1,2}\big(0, T, L^2(\Omega)\big), \tag{5.8}$$
$$u_\alpha(\varepsilon) \rightharpoonup u_\alpha \quad \text{in } W^{1,2}\big(0, T, H^1(\Omega)\big) \quad \text{and hence} \tag{5.9}$$
$$u_\alpha(\varepsilon) \to u_\alpha \quad \text{in } W^{1,2}\big(0, T, L^2(\Omega)\big), \tag{5.10}$$
$$u_3(\varepsilon) \rightharpoonup u_3 \quad \text{in } W^{1,2}\big(0, T, L^2(\Omega)\big). \tag{5.11}$$

For the proof of this step we take $\boldsymbol{v} = \boldsymbol{u}(\varepsilon)(t, \cdot)$ in (5.7) and find

$$\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon) e_{i\|j}(\varepsilon)\sqrt{g(\varepsilon)}dx + \int_\Omega B^{ijkl}(\varepsilon)\dot{e}_{k\|l}(\varepsilon) e_{i\|j}(\varepsilon)\sqrt{g(\varepsilon)}dx$$
$$= \int_\Omega f^i u_i(\varepsilon)\sqrt{g(\varepsilon)}dx, \quad \text{a.e. in } (0, T),$$

which is equivalent to,

$$\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon) e_{i\|j}(\varepsilon)\sqrt{g(\varepsilon)}dx + \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega B^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon) e_{i\|j}(\varepsilon)\sqrt{g(\varepsilon)}dx$$
$$= \int_\Omega f^i u_i(\varepsilon)\sqrt{g(\varepsilon)}dx, \quad \text{a.e. in } (0, T).$$

Now, integrating over $[0, T]$ and using (3.9) and (5.6), we find that

$$\int_0^T \left(\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon) e_{i\|j}(\varepsilon)\sqrt{g(\varepsilon)}dx\right)dt \le \int_0^T \left(\int_\Omega f^i u_i(\varepsilon)\sqrt{g(\varepsilon)}dx\right)dt, \tag{5.12}$$

Now, using the Cauchy-Schwartz inequality and (4.1),

$$\int_\Omega f^i u_i(\varepsilon)\sqrt{g(\varepsilon)}dx \le g_1^{1/2}\left(\sum_i \left|f^i\right|_{0,\Omega}^2\right)^{1/2}\left(\sum_i \left|u_i(\varepsilon)\right|_{0,\Omega}^2\right)^{1/2}. \tag{5.13}$$

On the other hand, by using (3.8), (4.1) and (4.5) we obtain

$$\begin{aligned}\int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}(\varepsilon)e_{i\|j}(\varepsilon)\sqrt{g(\varepsilon)}dx &\ge g_0^{1/2}C_e\left|e_{k\|l}(\varepsilon)\right|_{0,\Omega}^2\\ &\ge g_0^{1/2}C_eC^{-2}\left(\sum_\alpha\left\|u_\alpha(\varepsilon)\right\|_{1,\Omega}^2+\left|u_3(\varepsilon)\right|_{0,\Omega}^2\right),\end{aligned} \tag{5.14}$$

Now, (5.12)–(5.14) together and the Cauchy-Schwartz inequality imply that

$$\begin{aligned}&g_0^{1/2}C_eC^{-2}\int_0^T\left(\sum_\alpha\left\|u_\alpha(\varepsilon)(t)\right\|_{1,\Omega}^2+\left|u_3(\varepsilon)(t)\right|_{0,\Omega}^2\right)dt\\ &\le g_1^{1/2}\int_0^T\left(\left|\boldsymbol{f}(t)\right|_{0,\Omega}\left|\boldsymbol{u}(\varepsilon)(t)\right|_{0,\Omega}\right)dt\\ &\le g_1^{1/2}\left(\int_0^T\left|\boldsymbol{f}(t)\right|_{0,\Omega}^2dt\right)^{1/2}\left(\int_0^T\left|\boldsymbol{u}(\varepsilon)(t)\right|_{0,\Omega}^2dt\right)^{1/2}.\end{aligned}$$

Hence, since $|\boldsymbol{u}(\varepsilon)(t)|_{0,\Omega}^2 \le \sum_\alpha \|u_\alpha(\varepsilon)(t)\|_{1,\Omega}^2 + |u_3(\varepsilon)(t)|_{0,\Omega}^2$, $\forall t \in [0,T]$, we conclude that there exists a constant $\tilde{k}_1 > 0$ independent of ε such that

$$\int_0^T\left(\sum_\alpha\left\|u_\alpha(\varepsilon)(t)\right\|_{1,\Omega}^2+\left|u_3(\varepsilon)(t)\right|_{0,\Omega}^2\right)dt \le \tilde{k}_1.$$

Now, if we take $\boldsymbol{v} = \dot{\boldsymbol{u}}(\varepsilon)(t,\cdot)$ in (5.7), we find that

$$\begin{aligned}&\int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}(\varepsilon)\dot{e}_{i\|j}(\varepsilon)\sqrt{g(\varepsilon)}dx+\int_\Omega B^{ijkl}(\varepsilon)\dot{e}_{k\|l}(\varepsilon)\dot{e}_{i\|j}(\varepsilon)\sqrt{g(\varepsilon)}dx\\ &=\int_\Omega f^i\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}dx, \quad \text{a.e. in } (0,T).\end{aligned}$$

Integrating over $[0,T]$ and using (3.8) and (5.6), we find that

$$\int_0^T\left(\int_\Omega B^{ijkl}(\varepsilon)\dot{e}_{k\|l}(\varepsilon)\dot{e}_{i\|j}(\varepsilon)\sqrt{g(\varepsilon)}dx\right)dt \le \int_0^T\left(\int_\Omega f^i\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}dx\right)dt,$$

that is analogous to (5.12) with the contravariant components of the viscoelasticity tensor instead. Hence, using similar arguments and (3.9), we find that there exists a constant $\tilde{k}_2 > 0$ independent of ε such that

$$\int_0^T\left(\sum_\alpha\|\dot{u}_\alpha(\varepsilon)(t)\|_{1,\Omega}^2+|\dot{u}_3(\varepsilon)(t)|_{0,\Omega}^2\right)dt \le \tilde{k}_2.$$

Therefore, there exists $u_\alpha \in W^{1,2}(0,T,H^1(\Omega))$, $u_3 \in W^{1,2}(0,T,L^2(\Omega))$ and $e_{i\|j} \in W^{1,2}(0,T,L^2(\Omega))$, such that the convergences considered in (5.8)–(5.11) are verified.

Remark 7 Note that from a functional analysis result which can be found, for example, in Lemma 2.55, [31] convergences (5.8)–(5.11) imply that $u_\alpha(\varepsilon)(t,\cdot) \rightharpoonup u_\alpha(t,\cdot)$ in $H^1(\Omega)$, $\forall t \in [0,T]$, $u_3(\varepsilon)(t,\cdot) \rightharpoonup u_3(t,\cdot)$ in $L^2(\Omega)$, $\forall t \in [0,T]$ and $e_{i\|j}(\varepsilon)(t,\cdot) \rightharpoonup e_{i\|j}(t,\cdot)$ in $L^2(\Omega)$, $\forall t \in [0,T]$.

(ii) *The limits of the scaled unknown found in step* (i) *are independent of* x_3.
We adapt the proof in step (ii) in Theorem 4.4-1 [6] for this case. By the step (i) we have that

$$\partial_3 u_\alpha(\varepsilon) + \varepsilon\partial_\alpha u_3(\varepsilon) = 2\varepsilon\big(e_{\alpha\|3}(\varepsilon) + \Gamma^\sigma_{\alpha 3}(\varepsilon)u_\sigma(\varepsilon)\big) \to 0,\ in\ W^{1,2}\big(0,T,L^2(\Omega)\big),$$

since the functions $\Gamma^\sigma_{\alpha 3}(\varepsilon)$ converge in $\mathcal{C}^0(\bar{\Omega})$ (see Theorem 5). Let there be given $\varphi \in \mathcal{D}(\Omega)$ and $\psi \in \mathcal{D}(0,T)$. Since $u_\alpha(\varepsilon) \rightharpoonup u_\alpha$ in $W^{1,2}(0,T,H^1(\Omega))$ and $(u_3(\varepsilon))_{\varepsilon>0}$ is bounded in $W^{1,2}(0,T,L^2(\Omega))$ by the step (i) we find that

$$\int_0^T\left(\int_\Omega \partial_3 u_\alpha \varphi dx\right)\psi dt = \lim_{\varepsilon\to 0}\left(\int_0^T\left(\int_\Omega \partial_3 u_\alpha(\varepsilon)\varphi dx\right)\psi dt\right),$$

$$\lim_{\varepsilon\to 0}\left(\int_0^T\left(\int_\Omega \varepsilon\partial_\alpha u_3(\varepsilon)\varphi dx\right)\psi dt\right) = -\lim_{\varepsilon\to 0}\left(\int_0^T\left(\int_\Omega \varepsilon u_3(\varepsilon)\partial_\alpha\varphi dx\right)\psi dt\right) = 0$$

hence,

$$0 = \lim_{\varepsilon\to 0}\left(\int_0^T\left(\int_\Omega \big(\partial_3 u_\alpha(\varepsilon) + \varepsilon\partial_\alpha u_3(\varepsilon)\big)\varphi dx\right)\psi dt\right) = \int_0^T\left(\int_\Omega \partial_3 u_\alpha\varphi dx\right)\psi dt,$$

which means that $\partial_3 u_\alpha = 0$ in $L^2(0,T,L^2(\Omega))$. Analogously, we use the same arguments for the respective time derivatives, hence, we have that $\partial_3 u_\alpha = 0$ in $W^{1,2}(0,T,L^2(\Omega))$. Now, by the step (i) we also have that

$$\partial_3 u_3(\varepsilon) = \varepsilon e_{3\|3}(\varepsilon) \to 0 \quad \text{in } W^{1,2}\big(0,T,L^2(\Omega)\big).$$

Let $\varphi \in \mathcal{D}(\Omega)$ and $\psi \in \mathcal{D}(0,T)$. Since $u_3(\varepsilon) \rightharpoonup u_3$ in $W^{1,2}(0,T,L^2(\Omega))$ by the step (i) then,

$$\begin{aligned}\int_0^T\left(\int_\Omega u_3\partial_3\varphi dx\right)\psi dt &= \lim_{\varepsilon\to 0}\left(\int_0^T\left(\int_\Omega u_3(\varepsilon)\partial_3\varphi dx\right)\psi dt\right)\\ &= -\lim_{\varepsilon\to 0}\left(\int_0^T\left(\int_\Omega \partial_3 u_3(\varepsilon)\varphi dx\right)\psi dt\right) = 0\end{aligned}$$

and correspondingly for its time derivative, which means that $\partial_3 u_3 = 0$ in the sense of distributions. Applying Theorem 7 (a), the conclusion follows.

(iii) The limits $e_{i\|j}$ found in (i) are independent of the variable x_3. Moreover, they are related with the limits $\boldsymbol{u} := (u_i)$ by

$$e_{\alpha\|\beta} = \gamma_{\alpha\beta}(\boldsymbol{u}) := \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \Gamma^\sigma_{\alpha\beta}u_\sigma - b_{\alpha\beta}u_3, \quad e_{\alpha\|3} = 0,$$

$$e_{3\|3}(t) = -\frac{\theta}{\theta+\rho}\left(a^{\alpha\beta}e_{\alpha\|\beta}(t) + \Lambda\int_0^t e^{-k(t-s)}a^{\alpha\beta}e_{\alpha\|\beta}(s)ds\right), \quad \text{in } \Omega, \quad \forall t \in [0,T],$$

where the constants Λ and k are defined in (5.4) and (5.5), respectively. Moreover,

$$\dot{e}_{3\|3}(t) = -\frac{\lambda}{\theta+\rho} a^{\alpha\beta} e_{\alpha\|\beta}(t) - \frac{\lambda+2\mu}{\theta+\rho} e_{3\|3}(t) - \frac{\theta}{\theta+\rho} a^{\alpha\beta} \dot{e}_{\alpha\|\beta}(t),$$

in Ω, a.e. $t \in (0,T)$.

Considering $\boldsymbol{v} = \boldsymbol{u}(\varepsilon)$ in (3.5) and $\boldsymbol{\eta} = \boldsymbol{u}$ in (4.4) (*par abus de langage*, since $\boldsymbol{u} \in V(\Omega)$), taking into account step (i) and the convergences $\Gamma^{\sigma}_{\alpha\beta}(\varepsilon) \to \Gamma^{\sigma}_{\alpha\beta}$ and $\Gamma^{3}_{\alpha\beta}(\varepsilon) \to b_{\alpha\beta}$ in $\mathcal{C}^0(\bar{\Omega})$ given by Theorem 5, we have that

$$e_{\alpha\|\beta}(\varepsilon) = \frac{1}{2}\big(\partial_\beta u_\alpha(\varepsilon) + \partial_\alpha u_\beta(\varepsilon)\big) - \Gamma^{p}_{\alpha\beta}(\varepsilon) u_p(\varepsilon) \rightharpoonup e_{\alpha\|\beta} = \gamma_{\alpha\beta}(\boldsymbol{u}) \quad \text{in } W^{1,2}\big(0,T,L^2(\Omega)\big).$$

Moreover, $e_{\alpha\|\beta}$ are independent of x_3, as a straightforward consequence of the independence on x_3 of u_i (step (ii)). In addition, let $\boldsymbol{v} \in V(\Omega)$. As a consequence of the definition of the scaled strains in (3.5)–(3.7), we find

$$\begin{aligned}
&\varepsilon e_{\alpha\|\beta}(\varepsilon;\boldsymbol{v}) \to 0 \quad \text{in } L^2(\Omega),\\
&\varepsilon e_{\alpha\|3}(\varepsilon;\boldsymbol{v}) \to \frac{1}{2}\partial_3 v_\alpha \quad \text{in } L^2(\Omega),\\
&\varepsilon e_{3\|3}(\varepsilon;\boldsymbol{v}) = \partial_3 v_3 \quad \text{for all } \varepsilon > 0.
\end{aligned}$$

Using the variational formulation (5.7) for $\boldsymbol{v} = \varepsilon\boldsymbol{v}$ and taking into account (2.12), (3.3) and (3.4), we have

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\varepsilon)\big(\varepsilon e_{k\|l}(\varepsilon) e_{i\|j}(\varepsilon,\boldsymbol{v})\big)\sqrt{g(\varepsilon)}dx + \int_\Omega B^{ijkl}(\varepsilon)\big(\varepsilon \dot{e}_{k\|l}(\varepsilon) e_{i\|j}(\varepsilon,\boldsymbol{v})\big)\sqrt{g(\varepsilon)}dx\\
&\quad= \int_\Omega \big(A^{\alpha\beta\sigma\tau}(\varepsilon)e_{\sigma\|\tau}(\varepsilon) + A^{\alpha\beta33}(\varepsilon)e_{3\|3}(\varepsilon)\big)\big(\varepsilon e_{\alpha\|\beta}(\varepsilon;\boldsymbol{v})\big)\sqrt{g(\varepsilon)}dx\\
&\qquad+ \int_\Omega 4A^{\alpha3\sigma3}(\varepsilon)e_{\sigma\|3}(\varepsilon)\big(\varepsilon e_{\alpha\|3}(\varepsilon;\boldsymbol{v})\big)\sqrt{g(\varepsilon)}dx\\
&\qquad+ \int_\Omega \big(A^{33\sigma\tau}(\varepsilon)e_{\sigma\|\tau}(\varepsilon) + A^{3333}(\varepsilon)e_{3\|3}(\varepsilon)\big)\big(\varepsilon e_{3\|3}(\varepsilon;\boldsymbol{v})\big)\sqrt{g(\varepsilon)}dx\\
&\qquad+ \int_\Omega \big(B^{\alpha\beta\sigma\tau}(\varepsilon)\dot{e}_{\sigma\|\tau}(\varepsilon) + B^{\alpha\beta33}(\varepsilon)\dot{e}_{3\|3}(\varepsilon)\big)\big(\varepsilon e_{\alpha\|\beta}(\varepsilon;\boldsymbol{v})\big)\sqrt{g(\varepsilon)}dx\\
&\qquad+ \int_\Omega 4B^{\alpha3\sigma3}(\varepsilon)\dot{e}_{\sigma\|3}(\varepsilon)\big(\varepsilon e_{\alpha\|3}(\varepsilon;\boldsymbol{v})\big)\sqrt{g(\varepsilon)}dx\\
&\qquad+ \int_\Omega \big(B^{33\sigma\tau}(\varepsilon)\dot{e}_{\sigma\|\tau}(\varepsilon) + B^{3333}(\varepsilon)\dot{e}_{3\|3}(\varepsilon)\big)\big(\varepsilon e_{3\|3}(\varepsilon;\boldsymbol{v})\big)\sqrt{g(\varepsilon)}dx\\
&\quad= \varepsilon\int_\Omega f^i v_i\sqrt{g(\varepsilon)}dx \quad \forall \boldsymbol{v} \in V(\Omega), \quad \text{a.e. in } (0,T).
\end{aligned}$$

We pass to the limit as $\varepsilon \to 0$ and by taking into account the asymptotic behaviour of the contravariant components of the fourth order tensors $A^{ijkl}(\varepsilon)$, $B^{ijkl}(\varepsilon)$ (see Theorem 3 and (4.2)–(4.3)), $g(\varepsilon)$ (see Theorem 5) and the convergences above, we obtain the following integral equation

$$\int_{\Omega}\big(2\mu a^{\alpha\sigma}e_{\alpha\|3}\partial_3 v_\sigma+(\lambda+2\mu)e_{3\|3}\partial_3 v_3\big)\sqrt{a}dx+\int_{\Omega}\lambda a^{\alpha\beta}e_{\alpha\|\beta}\partial_3 v_3\sqrt{a}dx$$
$$+\int_{\Omega}\big(\rho a^{\alpha\sigma}\dot e_{\alpha\|3}\partial_3 v_\sigma+(\theta+\rho)\dot e_{3\|3}\partial_3 v_3\big)\sqrt{a}dx+\int_{\Omega}\theta a^{\alpha\beta}\dot e_{\alpha\|\beta}\partial_3 v_3\sqrt{a}dx=0, \quad (5.15)$$

in Ω, a.e. in $(0,T)$. On one hand, if we take $\boldsymbol{v}\in V(\Omega)$ such that $v_2=v_3=0$ and using Theorem 6, we have

$$2\mu a^{\alpha 1}e_{\alpha\|3}+\rho a^{\alpha 1}\dot e_{\alpha\|3}=0, \quad \text{a.e. in } (0,T). \quad (5.16)$$

On the other hand, if we take $\boldsymbol{v}\in V(\Omega)$ such that $v_1=v_3=0$ and using Theorem 6, we have

$$2\mu a^{\alpha 2}e_{\alpha\|3}+\rho a^{\alpha 2}\dot e_{\alpha\|3}=0, \quad \text{a.e. in } (0,T). \quad (5.17)$$

Multiplying (5.16) by a^{22} and (5.17) by $-a^{21}$ and adding both expressions we have

$$2\mu\big(a^{22}a^{11}-a^{21}a^{12}\big)e_{1\|3}+\rho\big(a^{22}a^{11}-a^{21}a^{12}\big)\dot e_{1\|3}=2\mu a e_{1\|3}+\rho a\dot e_{1\|3}=0,$$

a.e. in $(0,T)$, by (2.6). Now, by the initial condition in (5.6) we conclude

$$e_{1\|3}(t)=0 \quad \text{in } \Omega, \text{ for all } t\in(0,T).$$

Multiplying (5.16) by a^{12} and (5.17) by $-a^{11}$ and adding both expressions we have

$$2\mu a e_{2\|3}+\rho a\dot e_{2\|3}=0, \quad \text{a.e. in } (0,T).$$

Now, by the initial condition in (5.6) we conclude

$$e_{2\|3}(t)=0 \quad \text{in } \Omega, \text{ for all } t\in(0,T).$$

Taking $\boldsymbol{v}\in V(\Omega)$ such that $v_\alpha=0$ in (5.15), we obtain

$$\int_{\Omega}\big(\lambda a^{\alpha\beta}e_{\alpha\|\beta}+(\lambda+2\mu)e_{3\|3}\big)\partial_3 v_3\sqrt{a}dx+\int_{\Omega}\big(\theta a^{\alpha\beta}\dot e_{\alpha\|\beta}+(\theta+\rho)\dot e_{3\|3}\big)\partial_3 v_3\sqrt{a}dx=0,$$

for all $v_3\in H^1(\Omega)$ with $v_3=0$ in Γ_0 and a.e. in $(0,T)$. By Theorem 6, we obtain the following differential equation

$$\lambda a^{\alpha\beta}e_{\alpha\|\beta}+(\lambda+2\mu)e_{3\|3}+\theta a^{\alpha\beta}\dot e_{\alpha\|\beta}+(\theta+\rho)\dot e_{3\|3}=0. \quad (5.18)$$

Remark 8 Note that removing time dependency and viscosity, that is taking $\theta=\rho=0$, the equation leads to the one studied in [6], that is, the elastic case.

In order to solve (5.18) in the more general case, we assume that the viscosity coefficient θ is strictly positive. Thus, we can prove that this equation is equivalent to

$$\theta e^{-\frac{\lambda}{\theta}t}\frac{\partial}{\partial t}\big(a^{\alpha\beta}e_{\alpha\|\beta}e^{\frac{\lambda}{\theta}t}\big)=-(\theta+\rho)e^{-\frac{\lambda+2\mu}{\theta+\rho}t}\frac{\partial}{\partial t}\big(e_{3\|3}e^{\frac{\lambda+2\mu}{\theta+\rho}t}\big).$$

Integrating with respect to the time variable and using (5.6) we find that,

$$e_{3\|3}e^{\frac{\lambda+2\mu}{\theta+\rho}t}=-\frac{\theta}{\theta+\rho}\int_0^t e^{(\frac{\lambda+2\mu}{\theta+\rho}-\frac{\lambda}{\theta})s}\frac{\partial}{\partial s}\big(a^{\alpha\beta}e_{\alpha\|\beta}(s)e^{\frac{\lambda}{\theta}s}\big)ds,$$

now integrating by parts and simplifying we conclude that,

$$e_{3\|3}(t)=-\frac{\theta}{\theta+\rho}\left(a^{\alpha\beta}e_{\alpha\|\beta}(t)+\Lambda\int_0^t e^{-k(t-s)}a^{\alpha\beta}e_{\alpha\|\beta}(s)ds\right),$$

in Ω, $\forall t\in[0,T]$, and where Λ and k are defined in (5.4) and (5.5), respectively. Moreover, from (5.18) we obtain that,

$$\dot{e}_{3\|3}(t)=-\frac{\lambda}{\theta+\rho}a^{\alpha\beta}e_{\alpha\|\beta}(t)-\frac{\lambda+2\mu}{\theta+\rho}e_{3\|3}(t)-\frac{\theta}{\theta+\rho}a^{\alpha\beta}\dot{e}_{\alpha\|\beta}(t),$$

in Ω, a.e. $t\in(0,T)$.

(iv) The function $\bar{\boldsymbol{u}}=(\bar{\boldsymbol{u}}_i)$ satisfies the two-dimensional variational Problem 6 with $p^{i,0}:=\int_{-1}^1 f^i dx_3$. In particular, since the solution of this problem is unique (by Theorem 4), the convergences on (i) are verified for all the family $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$. We have that $\bar{\boldsymbol{u}}(t,\cdot)=(\bar{u}_i(t,\cdot))\in V_M(\omega)$, $\forall t\in[0,T]$.

Let $\boldsymbol{v}=(v_i)\in V(\Omega)$ be independent of the variable x_3, then, the asymptotic behaviour of the functions $\Gamma^p_{\alpha\beta}(\varepsilon)$ and $\Gamma^\sigma_{\alpha 3}(\varepsilon)$ in Theorem 5 implies the following convergences when $\varepsilon\to 0$ (see (3.5)–(3.7)):

$$\begin{aligned}
&e_{\alpha\|\beta}(\varepsilon;\boldsymbol{v})\to\gamma_{\alpha\beta}(\boldsymbol{v}):=\frac{1}{2}(\partial_\alpha v_\beta+\partial_\beta v_\alpha)-\Gamma^\sigma_{\alpha\beta}v_\sigma-b_{\alpha\beta}v_3 \text{ in } L^2(\Omega),\\
&e_{\alpha\|3}(\varepsilon;\boldsymbol{v})\to\frac{1}{2}\partial_\alpha v_3+b^\sigma_\alpha v_\sigma \text{ in } L^2(\Omega),\\
&e_{3\|3}(\varepsilon;\boldsymbol{v})=0.
\end{aligned}$$

Let $\boldsymbol{v}=(v_i)\in V(\Omega)$ be independent of x_3 in (5.7) and take the limit when $\varepsilon\to 0$. Then, the asymptotic behaviour of the functions $e_{i\|j}(\varepsilon;\boldsymbol{v})$, $A^{ijkl}(\varepsilon)$, $B^{ijkl}(\varepsilon)$ (see Theorem 3 and (4.2)–(4.3)) and $g(\varepsilon)$ (see Theorem 5) and the weak convergences $e_{i\|j}(\varepsilon)\rightharpoonup e_{i\|j}$ in $W^{1,2}(0,T,L^2(\Omega))$ from step (i), lead to

$$\begin{aligned}
&\int_\Omega A^{ijkl}(0)e_{k\|l}e_{i\|j}(\boldsymbol{v})\sqrt{a}dx+\int_\Omega B^{ijkl}(0)\dot{e}_{k\|l}e_{i\|j}(\boldsymbol{v})\sqrt{a}dx\\
&=\int_\Omega\big(\lambda a^{\alpha\beta}a^{\sigma\tau}+\mu\big(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}\big)\big)e_{\sigma\|\tau}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx\\
&\quad+\int_\Omega\lambda a^{\alpha\beta}e_{3\|3}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx\\
&\quad+\int_\Omega\left(\theta a^{\alpha\beta}a^{\sigma\tau}+\frac{\rho}{2}\big(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}\big)\right)\dot{e}_{\sigma\|\tau}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx\\
&\quad+\int_\Omega\theta a^{\alpha\beta}\dot{e}_{3\|3}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx\\
&=\int_\Omega f^i v_i\sqrt{a}dx,
\end{aligned}\tag{5.19}$$

a.e. $t \in (0, T)$. Using the findings presented in step (iii), we have that the left-hand side can be simplified as follows:

$$\int_\Omega \big(\lambda a^{\alpha\beta}a^{\sigma\tau} + \mu\big(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\big)\big)e_{\sigma\|\tau}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx + \int_\Omega \lambda a^{\alpha\beta}e_{3\|3}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx$$
$$+ \int_\Omega \left(\theta a^{\alpha\beta}a^{\sigma\tau} + \frac{\rho}{2}\big(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\big)\right)\dot{e}_{\sigma\|\tau}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx + \int_\Omega \theta a^{\alpha\beta}\dot{e}_{3\|3}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx$$
$$= \int_\Omega \big(\lambda a^{\alpha\beta}a^{\sigma\tau} + \mu\big(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\big)\big)e_{\sigma\|\tau}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx$$
$$+ \int_\Omega \left(\theta a^{\alpha\beta}a^{\sigma\tau} + \frac{\rho}{2}\big(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\big)\right)\dot{e}_{\sigma\|\tau}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx$$
$$+ \int_\Omega \left(\lambda - \theta\frac{\lambda+2\mu}{\theta+\rho}\right)\left(-\frac{\theta}{\theta+\rho}\left(a^{\sigma\tau}e_{\sigma\|\tau}\right.\right.$$
$$\left.\left. + \Lambda\int_0^t e^{-k(t-s)}a^{\sigma\tau}e_{\sigma\|\tau}(s)ds\right)\right)a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx$$
$$- \int_\Omega \frac{\theta}{\theta+\rho}\big(\lambda a^{\sigma\tau}e_{\sigma\|\tau} + \theta a^{\sigma\tau}\dot{e}_{\sigma\|\tau}\big)a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx,$$

a.e. $t \in (0, T)$, which is equivalent to,

$$\int_\Omega \left(\left(\lambda - \frac{\theta}{\theta+\rho}(\theta\Lambda+\lambda)\right)a^{\alpha\beta}a^{\sigma\tau} + \mu\big(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\big)\right)e_{\sigma\|\tau}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx$$
$$+ \int_\Omega \left(\frac{\theta\rho}{\theta+\rho}a^{\alpha\beta}a^{\sigma\tau} + \frac{\rho}{2}\big(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\big)\right)\dot{e}_{\sigma\|\tau}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx$$
$$- \int_\Omega \frac{(\theta\Lambda)^2}{\theta+\rho}\int_0^t e^{-k(t-s)}a^{\sigma\tau}e_{\sigma\|\tau}(s)ds\, a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}dx.$$

Since both $\boldsymbol{u}$ and $\boldsymbol{v}$ are independent of x_3 (see step (i)), we obtain from (5.19) that

$$\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\bar{\boldsymbol{u}})\gamma_{\alpha\beta}(\bar{\boldsymbol{v}})\sqrt{a}dy + \int_\omega b^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\dot{\bar{\boldsymbol{u}}})\gamma_{\alpha\beta}(\bar{\boldsymbol{v}})\sqrt{a}dy$$
$$- \int_0^t e^{-k(t-s)}\int_\omega c^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}\big(\bar{\boldsymbol{u}}(s)\big)\gamma_{\alpha\beta}(\bar{\boldsymbol{v}})\sqrt{a}dyds$$
$$= \int_\omega \left(\int_{-1}^1 f^i dx_3\right)\bar{v}_i\sqrt{a}dy, \quad \text{a.e. in } (0, T), \tag{5.20}$$

where $a^{\alpha\beta\sigma\tau}$, $b^{\alpha\beta\sigma\tau}$ and $c^{\alpha\beta\sigma\tau}$ denote the contravariant components of the fourth order two-dimensional tensors, defined in (5.1)–(5.3).

Now, given $\boldsymbol{\eta} = (\eta_i) \in [H_0^1(\omega)]^3$ we can define $\boldsymbol{v} = (v_i)$ such that $\boldsymbol{v}(\boldsymbol{y}, x_3) = \boldsymbol{\eta}(\boldsymbol{y})$ for all $(\boldsymbol{y}, x_3) \in \Omega$. Then $\boldsymbol{v} \in V(\Omega)$ and it is independent of x_3; hence, as a consequence of Theorem 7 (b), the variational problems above are satisfied for $\bar{\boldsymbol{v}} = \boldsymbol{\eta}$.

Since both members of equations are continuous linear forms with respect to $\bar{v}_3 = \eta_3 \in L^2(\omega)$ for any given $\bar{v}_\alpha \in H_0^1(\omega)$, these equations are valid for all $\boldsymbol{\eta} = (\eta_i) \in V_M(\omega)$, since $H_0^1(\omega)$ is dense in $L^2(\omega)$.

(v) The weak convergences $e_{i\|j}(\varepsilon)(t,\cdot) \rightharpoonup e_{i\|j}(t,\cdot)$ in $W^{1,2}(0, T, L^2(\Omega))$ are, in fact, strong.

Indeed, we define

$$\begin{aligned}\Psi(\varepsilon) &:= \int_\Omega A^{ijkl}(\varepsilon)\big(e_{k\|l}(\varepsilon) - e_{k\|l}\big)\big(e_{i\|j}(\varepsilon) - e_{i\|j}\big)\sqrt{g(\varepsilon)}dx \\ &\quad + \int_\Omega B^{ijkl}(\varepsilon)\big(\dot{e}_{k\|l}(\varepsilon) - \dot{e}_{k\|l}\big)\big(e_{i\|j}(\varepsilon) - e_{i\|j}\big)\sqrt{g(\varepsilon)}dx \\ &= \int_\Omega f^i u_i(\varepsilon)\sqrt{g(\varepsilon)}dx - \int_\Omega A^{ijkl}(\varepsilon)\big(2e_{k\|l}(\varepsilon) - e_{k\|l}\big)e_{i\|j}\sqrt{g(\varepsilon)}dx \\ &\quad + \int_\Omega B^{ijkl}(\varepsilon)\left(\dot{e}_{k\|l}e_{i\|j} - \frac{\partial}{\partial t}\big(e_{k\|l}(\varepsilon)e_{i\|j}\big)\right)\sqrt{g(\varepsilon)}dx.\end{aligned}$$

We have that,

$$\begin{aligned}&\int_\Omega A^{ijkl}(\varepsilon)\big(e_{k\|l}(\varepsilon) - e_{k\|l}\big)\big(e_{i\|j}(\varepsilon) - e_{i\|j}\big)\sqrt{g(\varepsilon)}dx \\ &\quad + \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega B^{ijkl}(\varepsilon)\big(e_{k\|l}(\varepsilon) - e_{k\|l}\big)\big(e_{i\|j}(\varepsilon) - e_{i\|j}\big)\sqrt{g(\varepsilon)}dx = \Psi(\varepsilon), \quad \text{a.e. } t \in (0,T).\end{aligned}$$

Integrating over the interval $[0, T]$, using (3.9) and (5.6) we find that

$$\int_0^T \left(\int_\Omega A^{ijkl}(\varepsilon)\big(e_{k\|l}(\varepsilon) - e_{k\|l}\big)\big(e_{i\|j}(\varepsilon) - e_{i\|j}\big)\sqrt{g(\varepsilon)}dx\right)dt \le \int_0^T \Psi(\varepsilon)dt,$$

Now, by (3.8) and (4.1)

$$C_e^{-1}g_0^{1/2}\sum_{i,j}|e_{i\|j}(\varepsilon) - e_{i\|j}|_{0,\Omega}^2 \le \int_\Omega A^{ijkl}(\varepsilon)\big(e_{k\|l}(\varepsilon) - e_{k\|l}\big)\big(e_{i\|j}(\varepsilon) - e_{i\|j}\big)\sqrt{g(\varepsilon)}dx$$

Therefore, together with the previous inequality leads to

$$C_e^{-1}g_0^{1/2}\int_0^T\left(\sum_{i,j}|e_{i\|j}(\varepsilon)(t) - e_{i\|j}(t)|_{0,\Omega}^2\right)dt \le \int_0^T \Psi(\varepsilon)dt. \tag{5.21}$$

Let $\varepsilon \to 0$. Taking into account the weak convergences studied in (i) and the asymptotic behaviour of the functions $A^{ijkl}(\varepsilon)$, $B^{ijkl}(\varepsilon)$ (see Theorem 3 and (4.2)–(4.3)) and $g(\varepsilon)$ (see Theorem 5), we find that

$$\Psi := \lim_{\varepsilon\to 0}\Psi(\varepsilon) = \int_\Omega f^i u_i\sqrt{a}dx - \int_\Omega A^{ijkl}(0)e_{k\|l}e_{i\|j}\sqrt{a}dx - \int_\Omega B^{ijkl}(0)\dot{e}_{k\|l}e_{i\|j}\sqrt{a}dx,$$

a.e. $t \in (0, T)$. By the expressions of $A^{ijkl}(0)$ and $B^{ijkl}(0)$ (see Theorem 3) we have that

$$\begin{aligned}&\int_\Omega A^{ijkl}(0)e_{k\|l}e_{i\|j}\sqrt{a}dx + \int_\Omega B^{ijkl}(0)\dot{e}_{k\|l}e_{i\|j}\sqrt{a}dx \\ &\quad = \int_\Omega \big(\lambda a^{\alpha\beta}a^{\sigma\tau} + \mu\big(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\big)\big)e_{\sigma\|\tau}e_{\alpha\|\beta}\sqrt{a}dx + \int_\Omega \lambda a^{\alpha\beta}e_{3\|3}e_{\alpha\|\beta}\sqrt{a}dx \\ &\qquad + \int_\Omega \big(\lambda a^{\sigma\tau}e_{\sigma\|\tau} + (\lambda + 2\mu)e_{3\|3}\big)e_{3\|3}\sqrt{a}dx\end{aligned}$$

$$+\int_{\Omega}\left(\theta a^{\alpha\beta}a^{\sigma\tau}+\frac{\rho}{2}\left(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}\right)\right)\dot{e}_{\sigma\|\tau}e_{\alpha\|\beta}\sqrt{a}dx+\int_{\Omega}\theta a^{\alpha\beta}\dot{e}_{3\|3}e_{\alpha\|\beta}\sqrt{a}dx$$
$$+\int_{\Omega}\left(\theta a^{\sigma\tau}\dot{e}_{\sigma\|\tau}+(\theta+\rho)\dot{e}_{3\|3}\right)e_{3\|3}\sqrt{a}dx,\quad \text{a.e. } t\in(0,T),$$

which using the expressions of the limits $e_{i\|j}$ studied in (iii) can be written as

$$\int_{\Omega}\left(\left(\lambda-\frac{\theta}{\theta+\rho}(\theta\Lambda+\lambda)\right)a^{\alpha\beta}a^{\sigma\tau}+\mu\left(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}\right)\right)e_{\sigma\|\tau}e_{\alpha\|\beta}\sqrt{a}dx$$
$$+\int_{\Omega}\left(\frac{\theta\rho}{\theta+\rho}a^{\alpha\beta}a^{\sigma\tau}+\frac{\rho}{2}\left(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}\right)\right)\dot{e}_{\sigma\|\tau}e_{\alpha\|\beta}\sqrt{a}dx$$
$$-\int_{\Omega}\frac{(\theta\Lambda)^2}{\theta+\rho}\int_0^t e^{-k(t-s)}a^{\sigma\tau}e_{\sigma\|\tau}(s)ds\,a^{\alpha\beta}e_{\alpha\|\beta}(t)\sqrt{a}dx,$$

hence,

$$\Psi=\int_{\Omega}f^iu_i\sqrt{a}dx-\frac{1}{2}\int_{\Omega}a^{\alpha\beta\sigma\tau}e_{\sigma\|\tau}e_{\alpha\|\beta}\sqrt{a}dx-\frac{1}{2}\int_{\Omega}b^{\alpha\beta\sigma\tau}e_{\sigma\|\tau}\dot{e}_{\alpha\|\beta}\sqrt{a}dx$$
$$+\frac{1}{2}\int_0^t e^{-k(t-s)}\int_{\Omega}c^{\alpha\beta\sigma\tau}e_{\sigma\|\tau}(s)e_{\alpha\|\beta}(t)\sqrt{a}dxds,\quad \text{a.e. } t\in(0,T),$$

where $a^{\alpha\beta\sigma\tau}$, $b^{\alpha\beta\sigma\tau}$ and $c^{\alpha\beta\sigma\tau}$ denote the contravariant components of the fourth order two-dimensional tensors, defined in (5.1)–(5.3). Then, taking $\bar{\boldsymbol{v}}=\bar{\boldsymbol{u}}$ in (5.20) and using that $e_{\alpha\|\beta}=\gamma_{\alpha\beta}(\bar{\boldsymbol{u}})$ (see (iii)), we conclude that $\Psi=0$. As a consequence, using the Lebesgue dominated convergence theorem in (5.21), the strong convergences $e_{i\|j}(\varepsilon)\to e_{i\|j}$ in $L^2(0,T,L^2(\Omega))$ are satisfied. Analogously, if we define

$$\tilde{\Psi}(\varepsilon):=\int_{\Omega}A^{ijkl}(\varepsilon)\left(e_{k\|l}(\varepsilon)-e_{k\|l}\right)\left(\dot{e}_{i\|j}(\varepsilon)-\dot{e}_{i\|j}\right)\sqrt{g(\varepsilon)}dx$$
$$+\int_{\Omega}B^{ijkl}(\varepsilon)\left(\dot{e}_{k\|l}(\varepsilon)-\dot{e}_{k\|l}\right)\left(\dot{e}_{i\|j}(\varepsilon)-\dot{e}_{i\|j}\right)\sqrt{g(\varepsilon)}dx$$
$$=\int_{\Omega}f^i\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}dx+\int_{\Omega}A^{ijkl}(\varepsilon)\left(e_{k\|l}\dot{e}_{i\|j}-\frac{\partial}{\partial t}\left(e_{k\|l}(\varepsilon)e_{i\|j}\right)\right)\sqrt{g(\varepsilon)}dx$$
$$-\int_{\Omega}B^{ijkl}(\varepsilon)\left(2\dot{e}_{k\|l}(\varepsilon)-\dot{e}_{k\|l}\right)\dot{e}_{i\|j}\sqrt{g(\varepsilon)}dx.$$

We have that,

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega}A^{ijkl}(\varepsilon)\left(e_{k\|l}(\varepsilon)-e_{k\|l}\right)\left(e_{i\|j}(\varepsilon)-e_{i\|j}\right)\sqrt{g(\varepsilon)}dx$$
$$+\int_{\Omega}B^{ijkl}(\varepsilon)\left(\dot{e}_{k\|l}(\varepsilon)-\dot{e}_{k\|l}\right)\left(\dot{e}_{i\|j}(\varepsilon)-\dot{e}_{i\|j}\right)\sqrt{g(\varepsilon)}dx=\tilde{\Psi}(\varepsilon),\quad \text{a.e. } t\in(0,T).$$

Integrating over the interval $[0,T]$, using (3.8) and (5.6) we find that

$$\int_0^T\left(\int_{\Omega}B^{ijkl}(\varepsilon)\left(\dot{e}_{k\|l}(\varepsilon)-\dot{e}_{k\|l}\right)\left(\dot{e}_{i\|j}(\varepsilon)-\dot{e}_{i\|j}\right)\sqrt{g(\varepsilon)}dx\right)dt\le\int_0^T\tilde{\Psi}(\varepsilon)dt,$$

Now, by (3.9) and (4.1)

$$C_v^{-1} g_0^{1/2} \sum_{i,j} \left|\dot{e}_{i\|j}(\varepsilon) - \dot{e}_{i\|j}\right|_{0,\Omega}^2 \leq \int_\Omega B^{ijkl}(\varepsilon)\big(\dot{e}_{k\|l}(\varepsilon) - \dot{e}_{k\|l}\big)\big(\dot{e}_{i\|j}(\varepsilon) - \dot{e}_{i\|j}\big)\sqrt{g(\varepsilon)}dx$$

Therefore, together with the previous inequality leads to

$$C_v^{-1} g_0^{1/2} \int_0^T \left(\sum_{i,j} \left|\dot{e}_{i\|j}(\varepsilon)(t) - \dot{e}_{i\|j}(t)\right|_{0,\Omega}^2\right) dt \leq \int_0^T \tilde{\Psi}(\varepsilon) dt, \tag{5.22}$$

which is similar with (5.21). Therefore, using analogous arguments as before, we find that

$$\tilde{\Psi} := \lim_{\varepsilon \to 0} \tilde{\Psi}(\varepsilon) = \int_\Omega f^i \dot{u}_i \sqrt{a} dx - \frac{1}{2}\int_\Omega a^{\alpha\beta\sigma\tau} e_{\sigma\|\tau} \dot{e}_{\alpha\|\beta} \sqrt{a} dx - \frac{1}{2}\int_\Omega b^{\alpha\beta\sigma\tau} \dot{e}_{\sigma\|\tau} \dot{e}_{\alpha\|\beta} \sqrt{a} dx$$
$$+ \frac{1}{2}\int_0^t e^{-k(t-s)} \int_\Omega c^{\alpha\beta\sigma\tau} e_{\sigma\|\tau}(s) \dot{e}_{\alpha\|\beta}(t) \sqrt{a} dx ds, \quad \text{a.e. } t \in (0,T),$$

Then, taking $\bar{\boldsymbol{v}} = \dot{\bar{\boldsymbol{u}}}$ in (5.20) and using that $e_{\alpha\|\beta} = \gamma_{\alpha\beta}(\bar{\boldsymbol{u}})$ (see (iii)), we conclude that $\tilde{\Psi} = 0$. As a consequence, using the Lebesgue dominated convergence theorem in (5.22), the strong convergences $\dot{e}_{i\|j}(\varepsilon) \to \dot{e}_{i\|j}$ in $L^2(0,T,L^2(\Omega))$ are satisfied. Therefore, we conclude that $e_{i\|j}(\varepsilon) \to e_{i\|j}$ in $W^{1,2}(0,T,L^2(\Omega))$.

(vi) The family $(\bar{\boldsymbol{u}}(\varepsilon))_{\varepsilon>0}$ converges strongly to $\bar{\boldsymbol{u}}$ (when $\varepsilon \to 0$) in $W^{1,2}(0,T,V_M(\omega))$, that is,

$$\bar{u}_\alpha(\varepsilon) \to \bar{u}_\alpha \quad \text{in } W^{1,2}\big(0,T,H^1(\omega)\big), \qquad \bar{u}_3(\varepsilon) \to \bar{u}_3 \quad \text{in } W^{1,2}\big(0,T,L^2(\omega)\big).$$

This proof is a corollary of the step (vi) in Theorem 4.4-1 [6]. In order to do that, we follow the same arguments made there to prove that $\bar{u}_\alpha(\varepsilon) \to \bar{u}_\alpha$ in $L^2(0,T,H^1(\omega))$, $\bar{u}_3(\varepsilon) \to \bar{u}_3$ in $L^2(0,T,L^2(\omega))$ and the corresponding convergences of the time derivatives in the same spaces. Then the conclusion follows.

(vii) The convergence $u_3(\varepsilon) \rightharpoonup u_3$ in $W^{1,2}(0,T,L^2(\Omega))$ is, in fact, strong.

Indeed, by (3.7) and step (i), we have $\partial_3 u_3(\varepsilon) = \varepsilon e_{3\|3}(\varepsilon) \to 0$ in $W^{1,2}(0,T,L^2(\Omega))$. On the other hand, we have $\bar{u}_3(\varepsilon) \to \bar{u}_3$ in $W^{1,2}(0,T,L^2(\omega))$. Hence by Theorem 7 (c), the conclusion follows.

(viii) The convergences $u_\alpha(\varepsilon) \to u_\alpha$ are strong in $W^{1,2}(0,T,H^1(\Omega))$. This proof is a corollary of the step (viii) in Theorem 4.4-1 [6]. In order to do that, we follow the same arguments made there to prove that $u_\alpha(\varepsilon) \to u_\alpha$ in $L^2(0,T,H^1(\omega))$ and the corresponding convergences of the time derivatives in the same spaces. Then the conclusion follows.

(ix) Let $X(0,T;\Omega) := \{\boldsymbol{v} \in W^{1,2}(0,T,L^2(\Omega)); \partial_3 v \in W^{1,2}(0,T,L^2(\Omega))\}$. The trace $\boldsymbol{v}(\cdot,s)$ of whatever function $\boldsymbol{v} \in X(0,T;\Omega)$ is well defined by a function in $W^{1,2}(0,T,L^2(\omega))$ for every $s \in [-1,1]$ and the trace operator defined in this fashion is continuous. In particular, there exists a constant $c_1 > 0$ such that:

$$\|\boldsymbol{v}\|_{W^{1,2}(0,T,L^2(\Gamma_+\cup\Gamma_-))} \leq c_1\big(|\boldsymbol{v}|^2_{W^{1,2}(0,T,L^2(\Omega))} + |\partial_3 v|^2_{W^{1,2}(0,T,L^2(\Omega))}\big)^{1/2}$$

for all $\boldsymbol{v} \in X(\Omega)$. As consequence there exists a constant $c_2 > 0$ such that

$$\|v_3\|_{W^{1,2}(0,T,L^2(\Gamma_+\cup\Gamma_-))} \leq c_2\left(\sum_{i,j}\left|e_{i\|j}(\varepsilon;\boldsymbol{v})\right|^2_{W^{1,2}(0,T,L^2(\Omega))}\right)^{1/2}$$

for all $\boldsymbol{v} \in V(\Omega)$.

This proof is a corollary of the step (ix) in Theorem 4.4-1 [6]. In order to do that, we use the same arguments made there obtaining the analogous inequalities for $\|v\|_{L^2(0,T,L^2(\Gamma_+\cup\Gamma_-))}$ and then, for the norm of the corresponding time derivative $\|\dot{v}\|_{L^2(0,T,L^2(\Gamma_+\cup\Gamma_-))}$. Therefore, together we claim the conclusion.

(x) We add in this step the surface forces.

Assume that the problem is only subjected to surface forces. That is, find $\boldsymbol{u}(\varepsilon)$ such that satisfies the following variational problem,

$$\begin{aligned}
&\boldsymbol{u}(\varepsilon)(t,\cdot)\in V(\Omega) \quad \forall t\in[0,T]\\
&\int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}(\varepsilon)e_{i\|j}(\varepsilon,\boldsymbol{v})\sqrt{g(\varepsilon)}dx+\int_\Omega B^{ijkl}(\varepsilon)\dot{e}_{k\|l}(\varepsilon)e_{i\|j}(\varepsilon,\boldsymbol{v})\sqrt{g(\varepsilon)}dx\\
&\quad=\int_{\Gamma_+\cup\Gamma_-} h^i v_i\sqrt{g(\varepsilon)}d\Gamma \quad \forall \boldsymbol{v}\in V(\Omega), \quad \text{a.e. } t\in(0,T)\\
&\boldsymbol{u}(\varepsilon)(0,\cdot)=\boldsymbol{u}_0(\varepsilon)(\cdot),
\end{aligned}$$

where we identified $h^i\equiv h^{i,1}$ for notational brevity. The proof follows the same arguments made in (x) of Theorem 4.4-1, [6], which will need step (ix). Hence, with minor changes, we can conclude for this problem the same results found in steps (i)–(viii).

Therefore, the proof of the theorem is complete. □

Remark 9 For each $\varepsilon>0$, let $\sigma^{ij,\varepsilon}=A^{ijkl,\varepsilon}e^\varepsilon_{i\|j}(\boldsymbol{u}^\varepsilon)+B^{ijkl,\varepsilon}e^\varepsilon_{i\|j}(\dot{\boldsymbol{u}}^\varepsilon)$ denote the contravariant components of the linearized stress tensor field for a family of linearly viscoelastic shells that satisfy the conditions of Theorem 11 and let us define the scaled stresses $\sigma^{ij}(\varepsilon):\bar{\Omega}\longleftrightarrow\mathbb{R}$ by letting $\sigma^{ij,\varepsilon}(\boldsymbol{x}^\varepsilon)=:\sigma^{ij}(\varepsilon)(\boldsymbol{x})$ for all $\boldsymbol{x}^\varepsilon=\pi^\varepsilon(\boldsymbol{x})\in\bar{\Omega}^\varepsilon$. Then, the scaled stresses satisfy

$$\sigma^{ij}(\varepsilon)=A^{ijkl}(\varepsilon)e_{i\|j}(\varepsilon)+B^{ijkl}(\varepsilon)\dot{e}_{i\|j}(\varepsilon).$$

Hence, using the asymptotic behaviour of $A^{ijkl}(\varepsilon)$, $B^{ijkl}(\varepsilon)$ (see Theorem 3) and the strong convergences of $e_{i\|j}(\varepsilon)(t,\cdot)$ in $W^{1,2}(0,T,L^2(\Omega))$ and their independence of the transversal variable x_3 found in Theorem 11, we can prove that $\sigma^{\alpha\beta}(\varepsilon)$ converge in $L^2(0,T,L^2(\Omega))$ and that $\frac{1}{\varepsilon}\sigma^{i3}(\varepsilon)$ converge in $L^2(0,T,H^1(-1,1,H^{-1}(\omega)))$. To obtain these results we follow similar arguments to those used in [32] for the elastic case (see Exercise 4.4 in [6], as well).

It remains to be proved an analogous result to the previous theorem but in terms of de-scaled unknowns. The convergences $u_\alpha(\varepsilon)\to u_\alpha$ in $W^{1,2}(0,T,H^1(\Omega))$ and $u_3(\varepsilon)\to u_3$ in $W^{1,2}(0,T,L^2(\Omega))$ from Theorem 11, the scaling proposed in Sect. 3, the de-scalings $\xi_i^\varepsilon:=\xi_i$ for each $\varepsilon>0$ and Theorem 7 together lead to the following convergences:

$$\begin{aligned}
&\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}u_\alpha^\varepsilon dx_3^\varepsilon\to\xi_\alpha \quad \text{in } W^{1,2}\big(0,T,H^1(\omega)\big),\\
&\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}u_3^\varepsilon dx_3^\varepsilon\to\xi_3 \quad \text{in } W^{1,2}\big(0,T,L^2(\omega)\big).
\end{aligned}$$

Furthermore, we can prove the following theorem regarding the convergences of the averages of the tangential and normal components of the three-dimensional displacement vector field:

Theorem 12 *Assume that $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{R}^3)$. Consider a family of viscoelastic elliptic membrane shells with thickness 2ε approaching zero and with each having the same elliptic middle surface $S = \boldsymbol{\theta}(\bar{\omega})$, and let the assumptions on the data be as in Theorem 10.*

Let $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon) \in W^{1,2}(0, T, V(\Omega^\varepsilon))$ and $\boldsymbol{\xi}^\varepsilon = (\xi_i^\varepsilon) \in W^{1,2}(0, T, V_M(\omega))$ respectively denote for each $\varepsilon > 0$ the solutions to the three-dimensional and two-dimensional Problems 4 and 1. Moreover, let $\boldsymbol{\xi} = (\xi_i) \in W^{1,2}(0, T, V_M(\omega))$ denote the solution to Problem 6. Then we have that

$$\xi_\alpha^\varepsilon = \xi_\alpha \quad \text{and thus } \xi_\alpha^\varepsilon \boldsymbol{a}^\alpha = \xi_\alpha \boldsymbol{a}^\alpha \text{ in } W^{1,2}\big(0, T, H^1(\omega)\big), \quad \forall \varepsilon > 0,$$

$$\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon \boldsymbol{g}^{\alpha,\varepsilon} dx_3^\varepsilon \to \xi_\alpha \boldsymbol{a}^\alpha \text{ in } W^{1,2}\big(0, T, H^1(\omega)\big) \text{ as } \varepsilon \to 0,$$

and

$$\xi_3^\varepsilon = \xi_3 \quad \text{and thus } \xi_3^\varepsilon \boldsymbol{a}^3 = \xi_3 \boldsymbol{a}^3 \text{ in } W^{1,2}\big(0, T, L^2(\omega)\big), \quad \forall \varepsilon > 0,$$

$$\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon \boldsymbol{g}^{3,\varepsilon} dx_3^\varepsilon \to \xi_3 \boldsymbol{a}^3 \text{ in } W^{1,2}\big(0, T, L^2(\omega)\big) \text{ as } \varepsilon \to 0.$$

Proof Since $\boldsymbol{\theta} \in \mathcal{C}^3(\bar{\omega}; \mathbb{R}^3)$ the vector fields $\boldsymbol{g}^\alpha(\varepsilon) : \bar{\Omega} \to \mathbb{R}^3$ defined by $\boldsymbol{g}^\alpha(\varepsilon) := \boldsymbol{g}^{\alpha,\varepsilon}(\boldsymbol{x}^\varepsilon)$ for all $\boldsymbol{x}^\varepsilon = \pi(\boldsymbol{x}) \in \bar{\Omega}^\varepsilon$ are such that $\boldsymbol{g}^\alpha(\varepsilon) - \boldsymbol{a}^\alpha = O(\varepsilon)$, where the fields $\boldsymbol{a}^\alpha$ have been identified with vector fields defined over the whole set $\bar{\Omega}$. Now we have that,

$$\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon \boldsymbol{g}^{\alpha,\varepsilon} dx_3^\varepsilon - \xi_\alpha^\varepsilon \boldsymbol{a}^\alpha = \frac{1}{2} \int_{-1}^{1} u_\alpha(\varepsilon) \boldsymbol{g}^\alpha(\varepsilon) dx_3 - \xi_\alpha \boldsymbol{a}^\alpha$$

$$= \frac{1}{2} \int_{-1}^{1} u_\alpha(\varepsilon)\big(\boldsymbol{g}^\alpha(\varepsilon) - \boldsymbol{a}^\alpha\big) dx_3 - \big(\overline{u_\alpha(\varepsilon)} - \xi_\alpha\big)\boldsymbol{a}^\alpha.$$

On one hand, since $u_\alpha(\varepsilon) \to u_\alpha$ in $W^{1,2}(0, T; H^1(\Omega))$ and $\boldsymbol{g}^\alpha(\varepsilon) \to \boldsymbol{a}^\alpha$ in $\mathcal{C}^1(\bar{\Omega})$ imply that

$$u_\alpha(\varepsilon)\big(\boldsymbol{g}^\alpha(\varepsilon) - \boldsymbol{a}^\alpha\big) \to 0 \quad \text{in } W^{1,2}\big(0, T; H^1(\Omega)\big),$$

hence, applying Theorem 7 (b) we have that

$$\frac{1}{2} \int_{-1}^{1} u_\alpha(\varepsilon)\big(\boldsymbol{g}^\alpha(\varepsilon) - \boldsymbol{a}^\alpha\big) dx_3 \to 0 \quad \text{in } W^{1,2}\big(0, T; H^1(\omega)\big),$$

and by using the same argument we have that $(\overline{u_\alpha(\varepsilon)} - \xi_\alpha)\boldsymbol{a}^\alpha \to 0$ in $W^{1,2}(0, T; H^1(\omega))$. For the normal components we have that $\boldsymbol{g}^{3,\varepsilon} = \boldsymbol{a}^3$, then

$$\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon \boldsymbol{g}^{3,\varepsilon} dx_3^\varepsilon - \xi_3^\varepsilon \boldsymbol{a}^3 = \big(\overline{u_3(\varepsilon)} - \xi_3\big)\boldsymbol{a}^3,$$

hence applying Theorem 7 (a) we have that $(\overline{u_3(\varepsilon)} - \xi_3)\boldsymbol{a}^3 \to 0$ in $W^{1,2}(0, T; L^2(\omega))$. □

Remark 10 The fields $\tilde{\boldsymbol{\xi}}_T^\varepsilon, \tilde{\boldsymbol{\xi}}_N^\varepsilon : [0, T] \times \bar{\omega} \to \mathbb{R}^3$ defined by $\tilde{\boldsymbol{\xi}}_T^\varepsilon := \xi_\alpha^\varepsilon \boldsymbol{a}^\alpha$ and $\tilde{\boldsymbol{\xi}}_N^\varepsilon := \xi_3^\varepsilon \boldsymbol{a}^3$, are known as the limit tangential and normal displacement fields, respectively, of the middle surface S of the shell. If we denote the limit displacement field of S by $\tilde{\boldsymbol{\xi}}^\varepsilon := \xi_i \boldsymbol{a}^i$ then $\tilde{\boldsymbol{\xi}}^\varepsilon = \tilde{\boldsymbol{\xi}}_T^\varepsilon + \tilde{\boldsymbol{\xi}}_N^\varepsilon$.

6 Conclusions

We have found and mathematically justified a model for viscoelastic shells in the particular case of the so-called elliptic membranes. To this end we used the insight provided by the asymptotic expansion method (presented in our previous work [27]) and we have justified this approach by obtaining convergence theorems.

The main novelty that this model presented is a long-term memory, represented by an integral on the time variable, more specifically

$$M(t, \boldsymbol{\eta}) = \int_0^t e^{-k(t-s)} \int_\omega c^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}\big(\boldsymbol{\xi}(s)\big)\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}dyds,$$

for all $\boldsymbol{\eta} \in V_M(\omega)$. An analogous behaviour has been found in beam models for the bending-stretching of viscoelastic rods [28], obtained by using asymptotic methods as well. Also, this kind of viscoelasticity has been described in [10, 12], for instance.

As an example, this two-dimensional model could be useful in order to model the viscoelastic behaviour of a ventricle or atria wall of the human heart, considering that its surface is elliptic and taking the boundary condition of place accordingly.

As the viscoelastic case differs from the elastic case on time dependent constitutive law and external forces, we must consider the possibility that these models and the convergence result generalize the elastic case (studied in [6]). However, analogously to the asymptotic analysis made in [27], the reader can easily check that when the ordinary differential equation (5.18) was presented, we had to use assumptions that make it impossible to include the elastic case. Hence, the viscoelastic and elastic problems must be treated separately in order to reach reasonable and justified conclusions.

In this paper we have presented the convergence results concerning the models for the so-called viscoelastic elliptic membrane shells which implied that $V_0(\omega) = \{\mathbf{0}\}$ and also, it is easy to check that the space

$$\begin{aligned} V_F(\omega) := \big\{&\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \\ &\eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0, \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\big\} = \{\mathbf{0}\}. \end{aligned}$$

In [33] we shall consider the cases when the membrane is not elliptic or the shell is clamped only in a portion of its lateral face, but still $V_F(\omega) = \{\mathbf{0}\}$. For these cases, additional spaces must be considered in order to obtain well posed problems. They are the so-called viscoelastic generalized membranes, where we also distinguish the cases where $V_0(\omega)$ contains only the zero function (first kind) or not (second kind). Further, regarding the case where the space $V_F(\omega)$ contains non-zero functions, in [34] we shall study the problem of viscoelastic flexural shells.

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