

Extension or Compression Induced Twisting in Fiber-Reinforced Nonlinearly Elastic Circular Cylinders

Cornelius O. Horgan¹ · Jeremiah G. Murphy²

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Abstract The problem of pure extension or compression of a solid circular fiber-reinforced cylinder is considered in the context of the theory of nonlinear elasticity. Here we are interested in a particular feature of this problem which only occurs for certain classes of incompressible anisotropic hyperelastic materials and is completely absent in the *isotropic* case. We show that pure extension or compression of a transversely isotropic fiber-reinforced cylinder can induce *twisting* of the cylinder. The transverse isotropy is due to a single family of helically wound extensible fibers distributed throughout the cylinder. For general pitch angles, it is shown that in the absence of an *applied torsional moment*, the cylinder subjected to an applied tensile or compressive force exhibits an induced *twisting* due to mechanical anisotropy. Such an induced twist does *not* occur when the fibers are oriented longitudinally or azimuthally. It is also shown that at a special pitch angle (a “magic” angle), the initial slope of the axial force and induced moment with respect to stretch is independent of the Young’s modulus of the fibers. The results on induced twist are described in detail for three specific strain-energy densities. These models are quadratic in the anisotropic invariants, linear in the isotropic strain invariants and are consistent with the linearized theory of elasticity for fiber-reinforced materials. The results are illustrated on using elastic moduli for muscles based on experimental findings of other authors.

Keywords Incompressible fiber-reinforced transversely isotropic nonlinearly elastic materials · Helically wound fibers · Pure extension or compression of solid circular cylinders · Induced twisting · Magic angle

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✉ C.O. Horgan
coh8p@virginia.edu

¹ School of Engineering and Applied Science, University of Virginia, Charlottesville, VA 22904, USA

² Department of Mechanical Engineering, Dublin City University, Glasnevin, Dublin 9, Ireland, UK

1 Introduction

In the context of the theory of nonlinear elasticity for rubber-like materials, the problem of extension and torsion of a circular bar or tube has been widely investigated. In a recent paper [11], the authors have considered this problem for a solid circular cylinder composed of an incompressible fiber-reinforced anisotropic hyperelastic material. The transverse isotropy considered is generated by a single family of extensible fibers helically wound throughout the cylinder. One of the results obtained in [11] that is of further concern here is that a twist is *induced* in the cylinder for the special case of pure extension or compression. This induced twist arises from the helically wound nature of the fibers and is absent for the case of longitudinally oriented fibers (see Sect. 3 of [11]). This result was obtained in [11] by formally setting the applied twist parameter in the extension plus torsion problem equal to zero and was derived for a subclass of transversely isotropic materials.

One of our purposes here is to demonstrate this induced twist effect *directly* by considering the basic problem of *pure* extension or compression of the cylinder. In this transparent simple setting, we treat the most general class of hyperelastic strain-energy densities that depends on *all* four of the relevant strain invariants. It should be noted that the presence of an induced twist has been observed previously by other authors in more elaborate contexts. Demirkoparan and Pence [5] have treated the problem of extension plus radial deformation of a hollow tube composed of a *compressible* hyperelastic material subject to finite deformation swelling. The tube is reinforced by a single family of helically wound fibers. The induced twist due to swelling found in [5] has been subsequently implemented in the design of a polymer torsional actuator [7]. As remarked in [5, 6, 10], the induced twist can be eliminated by introducing a counterbalanced family of fibers of opposite pitch angle. This might be expected intuitively. Several interesting results for general doubly reinforced *incompressible* tubes under pressure are considered in [6] and [10].

We note that the nonlinear elasticity theories for *anisotropic* materials considered here and in [5, 6, 10, 11] involving *extensible* fibers is different from the well known classical approach considered, for example, in Adkins and Rivlin [1] where *inextensible* fibers in an *isotropic* matrix are considered. That approach was motivated largely by applications to rubber reinforced by inextensible cords whereas one impetus for current anisotropic models has arisen from application to soft biological tissues where extensible collagen fibers are imbedded in an elastin matrix. It should be noted that hollow tubes reinforced by families of helically wound inextensible fibers are treated in [1] and an induced torsion of the type considered in the present paper was also suggested there.

In the next section, we discuss some preliminaries from the theory of nonlinear hyperelasticity for transversely isotropic incompressible solids. In Sect. 3, we consider the problem of pure extension or compression of a solid circular cylinder composed of an incompressible transversely isotropic hyperelastic material. The transverse isotropy is due to a single family of helically wound extensible fibers distributed throughout the cylinder. The main result established is that this deformation can induce twisting of the cylinder due to the fiber orientation. Such an induced twisting does *not* occur for longitudinally or azimuthally oriented fibers. In Sect. 4, we describe how at a special pitch angle (a “magic” angle), the initial slope of the axial force and induced moment with respect to stretch is independent of the Young’s modulus of the fibers. The same magic angle has been discussed in recent papers [6, 10, 15] in a variety of contexts for deformations of hyperelastic fiber-reinforced materials. The results of Sect. 3 are described in detail in Sect. 5 for three specific strain-energy densities. These models are quadratic in the anisotropic invariants, linear in the isotropic strain

invariants and are consistent with the linearized theory of elasticity for fiber-reinforced materials. The results are illustrated on using elastic moduli for muscles based on experimental findings of other authors.

2 Preliminaries

The constitutive law for the Cauchy stress \mathbf{T} for incompressible transversely isotropic hyperelastic materials with strain-energy density $W = W(I_1, I_2, I_4, I_5)$ is given by

$$\mathbf{T} = -p\mathbf{I} + 2W_1\mathbf{B} - 2W_2\mathbf{B}^{-1} + 2W_4\mathbf{a} \otimes \mathbf{a} + 2W_5(\mathbf{a} \otimes \mathbf{B}\mathbf{a} + \mathbf{B}\mathbf{a} \otimes \mathbf{a}), \tag{2.1}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, where \mathbf{F} is the gradient of the deformation, and p is a hydrostatic pressure term associated with the incompressibility constraint $\det \mathbf{B} = 1$. In (2.1), the derivatives of the strain-energy are denoted by $W_i = \partial W / \partial I_i$ ($i = 1, 2, 4, 5$) and \otimes denotes the tensor product with Cartesian components $a_i a_j$. The three principal invariants of \mathbf{B} are defined as

$$I_1 = \text{tr} \mathbf{B}, \quad I_2 = \frac{1}{2}[(\text{tr} \mathbf{B})^2 - \text{tr} \mathbf{B}^2], \quad I_3 = \det \mathbf{B}.$$

On denoting the unit vector defining the direction of anisotropy in the undeformed configuration by \mathbf{A} , let $\mathbf{a} = \mathbf{F}\mathbf{A}$ and we have $I_4 = \mathbf{F}\mathbf{A} \cdot \mathbf{F}\mathbf{A}$, $I_5 = \mathbf{F}^T \mathbf{a} \cdot \mathbf{F}^T \mathbf{a}$. In order to simplify the mathematical complexity of the relation (2.1), specializations of the general form $W = W(I_1, I_2, I_4, I_5)$ have been used by several investigators. In particular, the forms $W = W(I_1, I_2, I_4)$, $W(I_1, I_2, I_5)$ and further specializations have been employed to focus attention on the influence of the individual anisotropic pseudo-invariants. The omission of the second isotropic invariant leads to some physical limitations (see, e.g., Horgan and Smayda [14] for a discussion). Moreover, as was pointed out by Murphy [17], simplifications where one of the anisotropic pseudo-invariants is omitted may also result in a lack of physical generality in certain deformations. See also [4] for numerical considerations of this issue. In Feng et al. [8], a similar conclusion is reached in the context of modeling brain white matter. As in [17], we assume here that the strain-energy and the stress are zero in the undeformed configuration and that, on linearization, the nonlinear theory should be compatible with the theory of incompressible transversely isotropic elasticity for infinitesimal strains. In this way, one can deduce that $W = W(I_1, I_2, I_4, I_5)$ should satisfy the conditions (see [17])

$$\hat{W} = 0, \quad 2\hat{W}_1 - 2\hat{W}_2 = \hat{p}, \quad \hat{W}_4 + 2\hat{W}_5 = 0, \tag{2.2}$$

and

$$2\hat{W}_1 + 2\hat{W}_2 = \mu_T, \quad 2\hat{W}_5 = \mu_L - \mu_T, \quad 4\hat{W}_{44} + 16\hat{W}_{45} + 16\hat{W}_{55} = E_L + \mu_T - 4\mu_L, \tag{2.3}$$

where the hat notation denotes evaluation at $I_1 = I_2 = 3$, $I_4 = I_5 = 1$, and \hat{p} is the value of the arbitrary pressure field in the undeformed configuration. Here μ_T, μ_L are the infinitesimal shear moduli for shearing in planes normal to the fibers and along the fibers respectively and E_L is the Young's modulus in the fiber direction. A fourth elastic constant E_T , the Young's modulus in the plane normal to the fibers, is related to these by the relation (see [17])

$$E_T = \frac{4E_L\mu_T}{E_L + \mu_T}. \tag{2.4}$$

There is significant experimental evidence, particularly for muscles, (see, e.g., [9, 18, 19] to suggest that

$$\mu_L > \mu_T \quad (2.5)$$

with an order of magnitude difference recorded in some instances. There is a lack of extensive experimental data for torsional testing of soft tissue. The work of Morrow et al. [16] on tensile testing showed that

$$E_L \gg \mu_L \quad (2.6)$$

for the extensor digitorum longus muscles of rabbits. These inequalities between the material constants will be assumed to hold in what follows.

The signs of the partial derivatives of the strain-energy function will also play a role in the following analysis. For *isotropic* materials, the so-called empirical inequalities given by

$$W_1 > 0, \quad W_2 \geq 0 \quad (2.7)$$

are often enforced and these are classically employed for rubber-like materials to ensure that specific choices for the strain-energy function are physically realistic. See Truesdell and Noll [20] and Beatty [2] for a discussion. In the absence of experimental data to suggest otherwise, they will be assumed to hold also for transversely isotropic materials.

3 Pure Extension or Compression of a Solid Circular Cylinder

Consider a long solid circular cylinder of radius A composed of an incompressible transversely-isotropic hyperelastic material subjected to a stretch in the axial direction. On using cylindrical coordinates (R, Θ, Z) in the undeformed configuration and (r, θ, z) in the current configuration, we may thus write

$$r = \gamma^{-1/2} R, \quad \theta = \Theta, \quad z = \gamma Z, \quad (3.1)$$

where γ denotes the axial stretch. The coefficient $\gamma^{-1/2}$ appears in (3.1)₁ in order to maintain incompressibility, so that for extension ($\gamma > 1$), the cylinder necessarily contracts laterally while for compression ($\gamma < 1$), one has a lateral expansion. The deformed radius a is given by $a = \gamma^{-1/2} A$. Corresponding to (3.1), one has the left Cauchy-Green tensor \mathbf{B} and its inverse

$$\mathbf{B} = \begin{bmatrix} \gamma^{-1} & 0 & 0 \\ 0 & \gamma^{-1} & 0 \\ 0 & 0 & \gamma^2 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma^{-2} \end{bmatrix} \quad (3.2)$$

and so the relevant invariants are

$$I_1 = \gamma^2 + 2\gamma^{-1}, \quad I_2 = \gamma^{-2} + 2\gamma, \quad I_3 = 1. \quad (3.3)$$

To describe the anisotropy of concern here, we write the unit vector \mathbf{A} in the undeformed configuration as $\mathbf{A} = (0, \sin \psi, \cos \psi)$ where ψ denotes a pitch angle measured relative to the horizontal axis with $0 \leq \psi \leq \pi/2$. Thus the fibers are considered to be helically wound throughout the cylinder. The special case where $\psi = 0$ and $\mathbf{A} = (0, 0, 1)$ so that the fibers

are orientated in the longitudinal direction will also be of particular interest in the sequel. For general angles ψ , on using the notation $c = \cos \psi$, $s = \sin \psi$, we find that

$$I_4 = \gamma^{-1}s^2 + \gamma^2c^2, \quad I_5 = \gamma^{-2}s^2 + \gamma^4c^2, \tag{3.4}$$

which reduce to

$$I_4 = \gamma^2, \quad I_5 = \gamma^4 \tag{3.5}$$

in the special case of transverse-isotropy about the axis.

On using (2.1), one finds that the only non-zero stresses are given by

$$\begin{aligned} T_{rr} &= -p + 2W_1\gamma^{-1} - 2W_2\gamma, \\ T_{\theta\theta} &= -p + 2W_1\gamma^{-1} - 2W_2\gamma + 2W_4\gamma^{-1}s^2 + 4W_5\gamma^{-2}s^2, \\ T_{zz} &= -p + 2W_1\gamma^2 - 2W_2\gamma^{-2} + 2W_4\gamma^2c^2 + 4W_5\gamma^4c^2, \\ T_{z\theta} &= 2\gamma^{1/2}[W_4 + W_5(\gamma^{-1} + \gamma^2)]sc, \end{aligned} \tag{3.6}$$

where p is the as yet unknown arbitrary pressure. For the homogeneous deformation at hand, the equilibrium equations in the absence of body forces are satisfied if p is constant. The lateral surface of the cylinder is assumed traction free so that the only boundary condition to be satisfied is $T_{rr} = 0$ on $a = \gamma^{-1/2}A$. This determines p as

$$p = 2W_1\gamma^{-1} - 2W_2\gamma, \tag{3.7}$$

and so the stresses (3.6) can be written as

$$\begin{aligned} T_{rr} &= 0, \\ T_{\theta\theta} &= 2W_4\gamma^{-1}s^2 + 4W_5\gamma^{-2}s^2, \\ T_{zz} &= 2W_1(\gamma^2 - \gamma^{-1}) + 2W_2(\gamma - \gamma^{-2}) + 2W_4\gamma^2c^2 + 4W_5\gamma^4c^2, \\ T_{z\theta} &= 2\gamma^{1/2}[W_4 + W_5(\gamma^{-1} + \gamma^2)]sc. \end{aligned} \tag{3.8}$$

The presence of a nonzero shear stress $T_{z\theta}$ in (3.8) is of primary interest here and such a stress might be expected intuitively as a consequence of the helically wound fiber reinforcement. This shear stress gives rise to an *induced* resultant twisting moment which is

$$M \equiv \int_0^{2\pi} \int_0^a T_{z\theta}r^2drd\theta = \frac{4\pi A^3}{3}[\gamma^{-1}W_4 + (\gamma^{-2} + \gamma)W_5]sc. \tag{3.9}$$

The corresponding resultant axial force is

$$N \equiv \int_0^{2\pi} \int_0^a T_{zz}rdrd\theta = 2\pi A^2[(\gamma - \gamma^{-2})W_1 + (1 - \gamma^{-3})W_2 + \gamma c^2W_4 + 2\gamma^3c^2W_5], \tag{3.10}$$

where the derivatives W_i in (3.9), (3.10) are evaluated at (3.3), (3.4).

Thus, we see from (3.9) that a resultant twisting moment is induced in the cylinder in pure extension or compression due to the angle of orientation of the fibers. For the special case

where $W = W(I_1, I_4)$, this result was obtained in [11]. For the special case of longitudinally oriented fibers, on setting $\psi = 0$ in (3.9), (3.10), we get

$$M = 0, \quad N = 2\pi A^2[(\gamma - \gamma^{-2})W_1 + (1 - \gamma^{-3})W_2 + \gamma W_4 + 2\gamma^3 W_5] \tag{3.11}$$

where the derivatives W_i are evaluated at (3.3), (3.5). Thus, there is no induced twisting moment in this case as might be anticipated due to axial symmetry. Another special (degenerate) case is apparent from (3.9) namely that when $\psi = \pi/2$ where now we have

$$M = 0, \quad N = 2\pi A^2[(\gamma - \gamma^{-2})W_1 + (1 - \gamma^{-3})W_2], \tag{3.12}$$

so that again there is no induced twisting moment for $\mathbf{A} = (0, 1, 0)$, i.e., for fibers oriented in the purely azimuthal direction. In this case, the resultant axial force coincides formally with that for an isotropic material.

The dimensionless ratio M/NA gives an indication of the relative size of the induced moment and resultant axial force. If one neglects the contribution of the isotropic component relative to the anisotropic in N , one obtains

$$\frac{M}{NA} = \frac{2}{3\gamma^2} \left[\frac{W_4 + (1 + \gamma^{-3})W_5}{W_4 + 2\gamma^2 W_5} \right] \tan \psi. \tag{3.13}$$

For materials independent of I_5 , this reduces to

$$\frac{M}{NA} = \frac{2}{3\gamma^2} \tan \psi, \tag{3.14}$$

while for materials independent of I_4 , we have

$$\frac{M}{NA} = \frac{1 + \gamma^{-3}}{3\gamma^4} \tan \psi. \tag{3.15}$$

The relations (3.14), (3.15) are *universal*, that is they are independent of the particular form of W considered. Each of these results show that for extension ($\gamma > 1$) the ratio M/NA decreases with axial stretch while for compression ($\gamma < 1$) this ratio increases with increasing compression.

The last expression in (3.11) gives the resultant axial force required to produce a stretch γ in *pure extension* of the cylinder with longitudinally oriented fibers. It can be easily shown that

$$\frac{1}{\pi A^2} \frac{dN}{d\gamma} \Big|_{\gamma=1} = 6(\hat{W}_1 + \hat{W}_2) + 2\hat{W}_4 + 12\hat{W}_5 + 4\hat{W}_{44} + 16\hat{W}_{45} + 16\hat{W}_{55}, \tag{3.16}$$

where we recall the notation used in Sect. 2 that the hat notation indicates evaluation of the corresponding quantity at (3, 3, 1, 1), i.e., in the undeformed state. It will be required that $N = 0$ when $\gamma = 1$ and hence from (3.11) we obtain

$$\hat{W}_4 + 2\hat{W}_5 = 0. \tag{3.17}$$

This is just the last assumption already made in (2.2). On using (3.17), Eq. (3.14) thus simplifies to

$$\frac{1}{\pi A^2} \frac{dN}{d\gamma} \Big|_{\gamma=1} = 6(\hat{W}_1 + \hat{W}_2) + 8\hat{W}_5 + 4\hat{W}_{44} + 16\hat{W}_{45} + 16\hat{W}_{55}. \tag{3.18}$$

On physical grounds, one expects the axial force required to produce a simple extension to be a monotone increasing function of γ and so in particular we require that the right hand side of (3.18) be positive. On using (2.3), this condition in terms of the infinitesimal longitudinal Young’s modulus just reads

$$E_L > 0, \tag{3.19}$$

which is of course a natural assumption to make.

4 Magic Angle

We now consider the quantity

$$\tilde{N} = \frac{N(\gamma)}{2\pi A^2} = (\gamma - \gamma^{-2})W_1 + (1 - \gamma^{-3})W_2 + \gamma c^2 W_4 + 2\gamma^3 c^2 W_5 \tag{4.1}$$

given in (3.10) for general W . On using the chain rule and the expressions (3.3), (3.4) for the invariants, we find that

$$\left(\frac{d\tilde{N}}{d\gamma}\right)_{\gamma=1} = 3(\hat{W}_1 + \hat{W}_2) + 4c^2\hat{W}_5 + c^2(2c^2 - s^2)(\hat{W}_{44} + 4\hat{W}_{54} + 4\hat{W}_{55}) \tag{4.2}$$

where we recall that the superimposed hat notation introduced in Sect. 2 indicates evaluation in the undeformed state. The last of (2.2) has also been used. On using (2.3), we simplify (4.2) to read

$$\frac{1}{\mu_T} \left(\frac{d\tilde{N}}{d\gamma}\right)_{\gamma=1} = \frac{3}{2} + 2c^2 \left(\frac{\mu_L}{\mu_T} - 1\right) + \frac{1}{4} \left(\frac{E_L}{\mu_T} - 4\frac{\mu_L}{\mu_T} + 1\right) c^2 (2c^2 - s^2). \tag{4.3}$$

We emphasize that (4.3) is valid for a general strain-energy $W = W(I_1, I_2, I_4, I_5)$.

The special angle for which

$$2c^2 - s^2 = 0 \tag{4.4}$$

is now seen to have the property that the initial slope of the axial force with respect to stretch is independent of the Young’s modulus of the fibers and is given by

$$\frac{1}{\mu_T} \left(\frac{d\tilde{N}}{d\gamma_0}\right)_{\gamma_0=1} = \frac{5}{6} + \frac{2\mu_L}{3\mu_T}. \tag{4.5}$$

The special pitch angle given by (4.4) may be equivalently written as

$$\arctan(\sqrt{2}) = \arcsin(\sqrt{2/3}) = \arccos(1/\sqrt{3}) \doteq 54.74^\circ \tag{4.6}$$

and is called a “magic” pitch angle. It arises naturally in a variety of contexts involving deformations of fiber-reinforced materials. See, e.g., Demirkoparan and Pence [6], Goriely and Tabor [10] and Kim and Segev [15] for recent discussions. In [6], this angle is shown to arise in the analysis of a (doubly) helically wound rubber hollow tube under pressure. A more general discussion is given in [10] where the “magic angle” terminology is used for the complement of this angle ($90^\circ - 54.74^\circ = 35.26^\circ$). As described in [10], the magic

angle concept was initially observed in biology in the context of locomotion and flattening of worms. See also [15] for an interesting application involving the motion of an octopus arm. It also arises in the field of soft robotics in connection with McKibben actuators (see [10] for pertinent references). The analysis carried out in [10] is given in the context of nonlinear elasticity modeling using doubly helically wound fiber-reinforced hollow cylinders under internal pressure.

Similarly, we consider the quantity

$$\tilde{M} = \frac{M(\gamma)}{\pi A^3} = \frac{4}{3}[\gamma^{-1}W_4 + (\gamma^{-2} + \gamma)W_5]_{sc} \tag{4.7}$$

given by (3.9) for general W . On using the chain rule, the expressions (3.3), (3.4) for the invariants and the last of (2.2) we find that

$$\frac{3}{4} \left(\frac{d\tilde{M}}{d\gamma} \right)_{\gamma=1} = [\hat{W}_5 + (2c^2 - s^2)(\hat{W}_{44} + 4\hat{W}_{54} + 4\hat{W}_{55})]_{sc}. \tag{4.8}$$

On using (2.3), we simplify (4.8) to read

$$\frac{3}{4\mu_T} \left(\frac{d\tilde{M}}{d\gamma} \right)_{\gamma=1} = \left[\frac{1}{2} \left(\frac{\mu_L}{\mu_T} - 1 \right) + \frac{1}{4} \left(\frac{E_L}{\mu_T} - 4 \frac{\mu_L}{\mu_T} + 1 \right) (2c^2 - s^2) \right]_{sc}. \tag{4.9}$$

For the special angle given by (4.4), i.e., the angle (4.6), we see again that the left hand side of (4.9) is independent of the fiber Young’s modulus and, on using (4.6), is simply

$$\frac{3}{4\mu_T} \left(\frac{d\tilde{M}}{d\gamma_0} \right)_{\gamma_0=1} = \frac{\sqrt{2}}{6} \left(\frac{\mu_L}{\mu_T} - 1 \right). \tag{4.10}$$

In Sect. 2, we have cited the papers [4, 8, 17] where the necessity of including *both* anisotropic invariants in the constitutive mix in order to capture the full range of the mechanical response of anisotropic hyperelastic materials is discussed. However, most transversely isotropic soft tissue models that have been used in the literature only include I_4 or less commonly I_5 . It follows from (2.2)₂, (2.3)₂ that for models including only one anisotropic invariant in the constitutive basis, one has $\mu_L = \mu_T$ and so (4.5), (4.10) show that such materials with fibers helically wound at the *magic angle* behave as if they were *isotropic* for infinitesimal deformations. See also [15] for related observations.

5 Response for Specific Strain-Energies

As discussed in [17], the restrictions (2.2), (2.3) on the strain-energy function suggest simple polynomial models for transversely isotropic elastomers or soft tissue: the strain energies must be at least quadratic in I_4, I_5 and at least linear in I_1, I_2 . As remarked earlier, Murphy [17] and Feng et al. [8] have argued that it is essential that the strain-energy function be a function of *both* pseudo-invariants when modeling soft tissue and this will be assumed here, as well as a linear dependence on the two strain isotropic invariants. We will consider three specific models of this type. The first of these emphasizes a quadratic dependence on I_4 , the second a quadratic dependence on I_5 while the third involves a product of these invariants. The first model considered is

$$W = \frac{\mu_T}{2} [\alpha(I_1 - 3) + (1 - \alpha)(I_2 - 3)] + \frac{\mu_T - \mu_L}{2} (2I_4 - I_5 - 1) + \frac{E_L + \mu_T - 4\mu_L}{8} (I_4 - 1)^2 \tag{5.1}$$

where α is a dimensionless constant such that $0 < \alpha \leq 1$. When $\alpha = 1$, this reduces to the modified standard reinforcing model proposed in [17]. The form (5.1) was used in recent papers [3, 13] in investigations of reverse Poynting effects in simple shear and torsion respectively. See also [12]. The isotropic part of (5.1) is the usual Mooney-Rivlin strain-energy. It can be easily verified that the form (5.1) satisfies all the conditions (2.2), (2.3) provided that

$$\mu_T(2\alpha - 1) = \hat{p}. \tag{5.2}$$

For the model (5.1), it is easily verified that (3.9) reduces to

$$\frac{M}{\pi A^3 \mu_T} = \frac{1}{3} \left[\begin{aligned} &\left(\frac{E_L}{\mu_T} - 4 \frac{\mu_L}{\mu_T} + 1 \right) \gamma^{-1} (\gamma^{-1} s^2 + \gamma^2 c^2 - 1) \\ &+ 2 \left(1 - \frac{\mu_L}{\mu_T} \right) (2\gamma^{-1} - \gamma - \gamma^{-2}) \end{aligned} \right] sc. \tag{5.3}$$

This is the induced non-dimensional twisting moment generated in the cylinder due to a pure extension or compression of amount γ . Note that M is independent of the parameter α . The corresponding value of N is given by (3.10) as

$$\begin{aligned} \frac{N}{\pi A^2 \mu_T} &= \alpha(\gamma - \gamma^{-2}) + (1 - \alpha)(1 - \gamma^{-3}) \\ &+ \frac{1}{2} \left(\frac{E_L}{\mu_T} - 4 \frac{\mu_L}{\mu_T} + 1 \right) \gamma c^2 (\gamma^{-1} s^2 + \gamma^2 c^2 - 1) + 2(\gamma - \gamma^3) c^2 \left(1 - \frac{\mu_L}{\mu_T} \right). \end{aligned} \tag{5.4}$$

In order to make further progress, estimates of the values for the material constant ratios are needed. Our choice will be motivated by data from muscles. In particular, we note that $\mu_L/\mu_T = 5$, corresponding to a typical relationship between the shear moduli for the experimental data for muscles given in [9, 18, 19] and $E_L/\mu_T = 75$, motivated by the data of Morrow et al. [16] for rabbit muscles. For these values of the elastic constant ratios, the result (5.3) reduces to

$$\frac{M}{\pi A^3 \mu_T} = \frac{8}{3} [7(\gamma^{-2} s^2 + \gamma c^2) + \gamma + \gamma^{-2} - 9\gamma^{-1}] sc. \tag{5.5}$$

The corresponding value of N is

$$\begin{aligned} \frac{N}{\pi A^2 \mu_T} &= \alpha(\gamma - \gamma^{-2}) + (1 - \alpha)(1 - \gamma^{-3}) \\ &+ 28\gamma c^2 (\gamma^{-1} s^2 + \gamma^2 c^2 - 1) - 8c^2 (\gamma - \gamma^3). \end{aligned} \tag{5.6}$$

Plots of (5.5), (5.6) are given below in Figs. 1, 2 for representative pitch angles. In Fig. 1, we take $\psi = \pi/6$ and $\alpha = 1$ while for Fig. 2, we consider the larger pitch angle $\psi = \pi/3$. In both figures, the force curve behaves as one might expect intuitively. However note that in Fig. 1, for extension the moment curve is monotonic increasing with stretch while for compression, the moment curve is non-monotonic and is negative in the initial stages of compression. This type of non-monotone behavior has also been observed in a wider context in [10] and has interesting applications in biology. See also [5]. A similar plot is given in Fig. 2 for the larger pitch angle $\psi = \pi/3$ and $\alpha = 1$. In this case, the moment curve is

Fig. 1 Non-dimensional moment and force curves plotted against axial stretch for the model (5.1) with $\psi = \pi/6$, $\alpha = 1$ using data for muscles. Note that for extension, the moment is monotonic increasing with stretch while for compression, the moment curve is non-monotonic

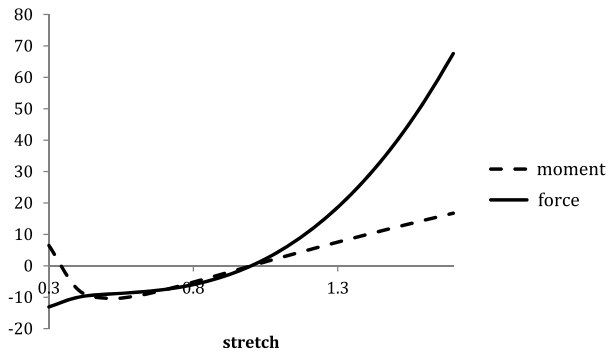
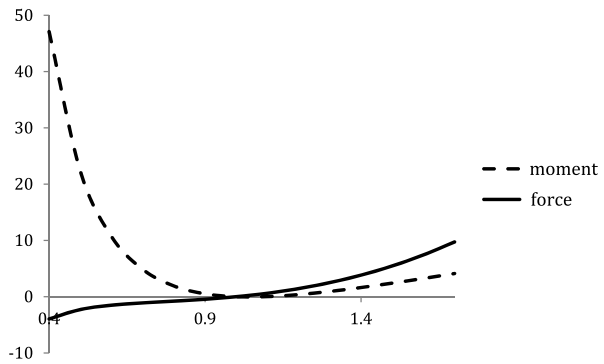


Fig. 2 Non-dimensional moment and force curves plotted against axial stretch for the model (5.1) with $\psi = \pi/3$ and $\alpha = 1$ using data for muscles. Note that for extension, the moment is monotonic increasing with stretch while for compression, the moment curve is monotonic decreasing but remains positive throughout



monotonically increasing for extension and monotonically decreasing for compression but remains positive.

The other two strain-energies considered in [3, 13] are

$$W = \frac{\mu_T}{2} [\alpha(I_1 - 3) + (1 - \alpha)(I_2 - 3)] + \frac{\mu_T - \mu_L}{2} (2I_4 - I_5 - 1) + \frac{E_L + \mu_T - 4\mu_L}{32} (I_5 - 1)^2 \tag{5.7}$$

and

$$W = \frac{\mu_T}{2} [\alpha(I_1 - 3) + (1 - \alpha)(I_2 - 3)] + \frac{\mu_T - \mu_L}{2} (2I_4 - I_5 - 1) + \frac{E_L + \mu_T - 4\mu_L}{16} (I_4 - 1)(I_5 - 1) \tag{5.8}$$

which, following the nomenclature of those papers, will be called materials *I, II* respectively. The first of these emphasizes the quadratic dependence on I_5 rather than I_4 while the second involves a product of these invariants. As remarked on by Murphy [17] in the context of simple shear, the focused dependence on I_5 in (5.7) is motivated by the fact that this model captures a more complete range of shear response than model (5.1). These models generalize those considered in [17] to include a dependence on the second strain invariant. The model (5.1) is called model *III*. It can be readily verified that these models satisfy the conditions (2.2), (2.3) provided that (5.2) holds.

For the model (5.7), one finds that (3.9) reads

$$\frac{M}{\pi A^3 \mu_T} = \frac{1}{3} \left[\begin{aligned} &\frac{1}{4} \left(\frac{E_L}{\mu_T} - 4 \frac{\mu_L}{\mu_T} + 1 \right) (\gamma^{-2} s^2 + \gamma^4 c^2 - 1) (\gamma + \gamma^{-2}) \\ &+ 2 \left(1 - \frac{\mu_L}{\mu_T} \right) (2\gamma^{-1} - \gamma - \gamma^{-2}) \end{aligned} \right]_{sc} \quad (5.9)$$

while for the model (5.8) one obtains

$$\frac{M}{\pi A^3 \mu_T} = \frac{1}{3} \left[\begin{aligned} &\frac{1}{4} \left(\frac{E_L}{\mu_T} - 4 \frac{\mu_L}{\mu_T} + 1 \right) [2(\gamma^{-3} s^2 + \gamma^3 c^2) + 1 - \gamma - \gamma^{-1} - \gamma^{-2}] \\ &+ 2 \left(1 - \frac{\mu_L}{\mu_T} \right) (2\gamma^{-1} - \gamma - \gamma^{-2}) \end{aligned} \right]_{sc}. \quad (5.10)$$

The corresponding values of N follow from (3.10) as

$$\begin{aligned} \frac{N}{\pi A^2 \mu_T} &= \alpha(\gamma - \gamma^{-2}) + (1 - \alpha)(1 - \gamma^{-3}) \\ &+ \frac{1}{4} \left(\frac{E_L}{\mu_T} - 4 \frac{\mu_L}{\mu_T} + 1 \right) \gamma^3 c^2 (\gamma^{-2} s^2 + \gamma^4 c^2 - 1) \\ &+ 2 \left(1 - \frac{\mu_L}{\mu_T} \right) (\gamma - \gamma^3) c^2 \end{aligned} \quad (5.11)$$

and

$$\begin{aligned} \frac{N}{\pi A^2 \mu_T} &= \alpha(\gamma - \gamma^{-2}) + (1 - \alpha)(1 - \gamma^{-3}) \\ &+ \frac{1}{8} \left(\frac{E_L}{\mu_T} - 4 \frac{\mu_L}{\mu_T} + 1 \right) \gamma c^2 \{ 2\gamma^2 (\gamma^{-1} s^2 + \gamma^2 c^2 - 1) + (\gamma^{-2} s^2 + \gamma^4 c^2 - 1) \} \\ &+ 2 \left(1 - \frac{\mu_L}{\mu_T} \right) (\gamma - \gamma^3) c^2, \end{aligned} \quad (5.12)$$

respectively. On using the values of the elastic constant ratios used previously, the results (5.9) and (5.10) reduce to

$$\frac{M}{\pi A^3 \mu_T} = \frac{2}{3} [7(\gamma^{-2} s^2 + \gamma^4 c^2 - 1)(\gamma + \gamma^{-2}) - 4(2\gamma^{-1} - \gamma - \gamma^{-2})]_{sc} \quad (5.13)$$

and

$$\frac{M}{\pi A^3 \mu_T} = \frac{2}{3} [7\{2(\gamma^{-3} s^2 + \gamma^3 c^2) + 1 - \gamma - \gamma^{-1} - \gamma^{-2}\} - 4(2\gamma^{-1} - \gamma - \gamma^{-2})]_{sc}, \quad (5.14)$$

respectively. The corresponding values of N are

$$\frac{N}{\pi A^2 \mu_T} = \alpha(\gamma - \gamma^{-2}) + (1 - \alpha)(1 - \gamma^{-3}) + 14\gamma c^2 (\gamma^{-2} s^2 + \gamma^4 c^2 - 1) - 8(\gamma - \gamma^3) c^2 \quad (5.15)$$

and

$$\frac{N}{\pi A^2 \mu_T} = \alpha(\gamma - \gamma^{-2}) + (1 - \alpha)(1 - \gamma^{-3}) + 7\gamma c^2 \{2\gamma^2(\gamma^{-1}s^2 + \gamma^2c^2 - 1) + (\gamma^{-2}s^2 + \gamma^4c^2 - 1)\} - 8(\gamma - \gamma^3)c^2, \quad (5.16)$$

respectively.

It turns out that the plots of the foregoing response for materials *I, II* are similar to those shown in Figs. 1, 2 and so will not be displayed.

6 Concluding Remarks

We have investigated an interesting induced twisting effect that arises in the simple problem of pure extension or compression of an anisotropic nonlinearly elastic solid circular cylinder. The cylinder is composed of a transversely isotropic incompressible hyperelastic material described by a strain-energy function that depends on the full set of relevant invariants. The transverse isotropy is due to a single family of helically wound extensible fibers distributed throughout the cylinder. It is shown that, for general pitch angles, a twisting moment is induced in the cylinder whereas when the fibers are oriented longitudinally or azimuthally no such induced twisting occurs. It is also shown that at a special pitch angle (a “magic” angle), the initial slope of the axial force and induced moment with respect to stretch is independent of the Young’s modulus of the fibers. The character of the induced twist has been described in detail by using three specific strain-energy density functions that model soft tissues. These models are quadratic in the anisotropic invariants, linear in the isotropic strain invariants and are consistent with the linear theory. The results have been illustrated on using elastic moduli for muscles based on experimental findings of other authors.

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