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A Local and Global Well-Posedness Results for the General Stress-Assisted Diffusion Systems

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Abstract We prove the local and global in time existence of the classical solutions to two general classes of the stress-assisted diffusion systems. Our results are applicable in the context of the non-Euclidean elasticity and liquid crystal elastomers.

Keywords Stress-assisted diffusion systems · Residual stress · Evolutionary growth · Morphogenesis

Mathematics Subject Classification 35Q74 · 35Q35 · 74B20 · 7420

1 Introduction and the Main Results

There are a number of phenomena where inhomogeneous and incompatible pre-strain is observed in 3-dimensional bodies. Growing leaves, gels subjected to differential swelling, electrodes in electrochemical cells, edges of torn plastic sheets are but a few examples [22, 38, 39, 46]. It has also been recently suggested that such incompatible pre-strains may be exploited as means of actuation of micro-mechanical devices [40, 41]. The mathematical foundations for these theories has lagged behind but has recently been the focus of much attention. While the static theory involving thin structures such as pre-strained plates and shells is now reasonably well understood [8, 11, 24, 31, 35], leading to the variationally reduced models constrained to appropriate types of isometries [8, 12, 35], and requiring bringing together the differential geometry of surfaces with the theory of elasticity appropriately modified [24, 25, 30, 34], the parallel evolutionary PDE model seems to not have been considered in this context.

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1.1 The Model and the Main Results

In this paper, we are concerned with two systems of coupled PDEs in the description of stress-assisted diffusion. The first system:

$$\begin{cases} u_{tt} - \mathrm{div}(\partial_F W(\phi, \nabla u)) = 0 \\ \phi_t = \Delta(\partial_\phi W(\phi, \nabla u)) \end{cases} \tag{1.1}$$

consists of a balance of linear momentum in the deformation field $u : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3$, and the diffusion law of the scalar field $\phi : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$ representing the inhomogeneity factor in the elastic energy density W. The field ϕ may be interpreted as the local swelling/shrinkage rate in morphogenesis at polymerization, or the localized conformation in liquid crystal elastomers.

The second system is a quasi-static approximation of (1.1), in which we neglect the material inertia u_{tt}, consistent with the assumption that the diffusion time scale is much shorter than the time scale of elastic wave propagation:

$$\begin{cases} -\mathrm{div}(\partial_F W(\phi, \nabla u)) = 0 \\ \phi_s = \Delta(\partial_\phi W(\phi, \nabla u)). \end{cases} \tag{1.2}$$

In both systems, the deformation u induces the deformation gradient, and the velocity and velocity gradients, respectively denoted as:

$$F = \nabla u \in \mathbb{R}^{3\times 3}, \qquad v = \xi_t \in \mathbb{R}^3, \qquad Q = \nabla \xi_t = \nabla v = F_t \in \mathbb{R}^{3\times 3}.$$

We will be concerned with the local in time well-posedness of the classical solutions to (1.1), and the global well-posedness of (1.2), subject to the (subset of) initial data:

$$u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1 \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

$$\phi(0, \cdot) = \phi_0 \quad \text{in } \mathbb{R}^3, \tag{1.4}$$

and the non-interpenetration ansatz:

$$\det \nabla u > 0 \quad \text{in } \mathbb{R}^3. \tag{1.5}$$

The main results of this paper are the following:

Theorem 1.1 *Let* $u_0 - \mathrm{id} \in H^4(\mathbb{R}^3)$, $u_1 \in H^3(\mathbb{R}^3)$ *and* $\phi_0 \in H^3(\mathbb{R}^3)$. *Assume that* W *is as in Sect.* 1.2. *Fix* $T > 0$, *and assume that the following quantities*:

$$\|u_1, \nabla u_0 - \mathrm{Id}_3, \phi_0\|^2_{H^3} + \|u_0 - \mathrm{id}\|^2_{L^2} + \int_{\mathbb{R}^3} W(\phi_0, \nabla u_0)\,\mathrm{d}x \tag{1.6}$$

are sufficiently small in comparison with T, *and with the constant* γ *in* (1.7). *Then there exists a unique solution* (u, ϕ) *of the problem* (1.1) (1.3)–(1.5), *defined on the time interval* $[0, T]$, *and such that*:

$$u - \mathrm{id} \in L^\infty\big(0, T; H^4(\mathbb{R}^3)\big), \qquad u_{tt} \in L^\infty\big(0, T; H^2(\mathbb{R}^3)\big),$$
$$\phi \in L^\infty\big(0, T; H^3(\mathbb{R}^3)\big) \quad \textit{and} \quad \phi_t \in L^2\big(0, T; H^2(\mathbb{R}^3)\big).$$

Theorem 1.2 *Let $\phi_0 \in H^2(\mathbb{R}^3)$ and assume that W is as in Sect.* 1.2. *Assume that $\|\phi_0\|_{H^2}$ is sufficiently small. Then there exists a unique global in time solution (u, ϕ) to* (1.2) (1.4) (1.5) *such that*:

$$u - \mathrm{id} \in L^\infty\big(\mathbb{R}_+; L^6\big(\mathbb{R}^3\big)\big), \qquad \nabla^2 u \in L^2\big(\mathbb{R}_+; H^2\big(\mathbb{R}^3\big)\big),$$
$$\phi \in L^\infty\big(\mathbb{R}_+; H^2\big(\mathbb{R}^3\big)\big) \quad \textit{and} \quad \nabla\phi \in L^2\big(\mathbb{R}_+; H^2\big(\mathbb{R}^3\big)\big).$$

The proof of Theorem 1.1 relies on controlling the energy:

$$\int_{\mathbb{R}^3} \frac{1}{2}|u_t|^2 + W(\phi, \nabla u)\,\mathrm{d}x,$$

where the key observation is that the diffusion acts as a dissipative mechanism and drives the system towards an elastic equilibrium. The hyperbolic character of the first equation in (1.1) suggests to seek the a-priori bounds on higher norms of u and ϕ by the standard energy techniques. A detailed analysis reveals that the special structure of coupling in the stress-assisted diffusion system indeed allows for cancellation of those terms that otherwise prevent closing the bounds in each of the two equations in (1.1) alone. These terms are displayed in formulas (2.6) and (2.8) in the proof of Lemma 2.2. Existence of solutions in Theorem 1.1 is then shown via Galerkin's method, where we check that solutions to all appropriate ϵ-approximations of the original system (1.1) still enjoy the same a-priori bounds in Theorem 2.3. This is carried out in Sect. 3, while uniqueness of solutions is proved in Sect. 4.

The proof of Theorem 1.2, given in Sect. 5, is based on the L^2-approach as well. The system (1.2) is of elliptic-parabolic type, thus there is no loss of regularity with respect to the initial data (in contrast to (1.1)). The analysis here is simpler than for (1.1) and we are able to show the global in time existence of small solutions. The toolbox we use for the proofs of both results is universal for hyperbolic-parabolic and elliptic-parabolic systems. Similar methods have been applied in [7, 13, 43, 44, 51] to study models of elasticity and their couplings with flows of complex fluids. A key element in these methods is the basic conservation law of energy and entropy type.

1.2 The Energy Density *W*

We now introduce the assumptions on the inhomogeneous elastic energy density W in (1.1). Namely, the nonnegative scalar field $W : \mathbb{R}_+ \times \mathbb{R}^{3\times 3} \to \overline{\mathbb{R}}_+$ is assumed to be $\mathcal{C}^4$ in a neighborhood of $(0, \mathrm{Id}_3)$ and to satisfy, with some constant $\gamma > 0$:

$$\begin{aligned} &W(0, \mathrm{Id}) = 0, \qquad DW(0, \mathrm{Id}) = 0, \quad \text{and:} \\ &D^2W(0, \mathrm{Id}) : (\tilde{\phi}, \tilde{F})^{\otimes 2} \geq \gamma\big(|\tilde{\phi}|^2 + |\mathrm{sym}\tilde{F}|^2\big) \quad \text{for all } (\tilde{\phi}, \tilde{F}) \in \mathbb{R} \times \mathbb{R}^{3\times 3}. \end{aligned} \tag{1.7}$$

The two main examples of W that we have in mind, concern non-Euclidean elasticity and liquid crystal elastomers, where respectively:

$$\begin{aligned} W_1(\phi, F) &= W_0\big(FB(\phi)\big) + \frac{1}{2}|\phi|^2, \\ W_2(\phi, F) &= W_0\big(B(\phi)F\big) + \frac{1}{2}|\phi|^2 \end{aligned} \tag{1.8}$$

are given in terms of the homogeneous energy density $W_0 : \mathbb{R}^{3\times 3} \to \overline{\mathbb{R}}_+$ and the smooth tensor field $B : \mathbb{R} \to \mathbb{R}^{3\times 3}$. In both cases, we assume that $B(\phi)$ is symmetric and positive definite, and that $B(0) = \mathrm{Id}$. Further, the principles of material frame invariance, material consistency, normalisation, and non-degeneracy impose the following conditions on W_0, valid for all $F \in \mathbb{R}^{3\times 3}$ and all $R \in SO(3)$:

$$\begin{array}{rl} \text{(i)} & W_0(RF) = W_0(F). \\ \text{(ii)} & W_0(F) \to +\infty \quad \text{as } \det F \to 0. \\ \text{(iii)} & W_0(\mathrm{Id}) = 0. \\ \text{(iv)} & W_0(F) \geq c \,\mathrm{dist}^2(F, SO(3)). \end{array} \tag{1.9}$$

Examples of W_0 satisfying the above conditions are:

$$W_{0,1}(F) = \left|\left(F^T F\right)^{1/2} - \mathrm{Id}\right|^2 + |\log\det F|^q,$$
$$W_{0,2}(F) = \left|\left(F^T F\right)^{1/2} - \mathrm{Id}\right|^2 + \left|\frac{1}{\det F} - 1\right|^q \quad \text{for } \det F > 0,$$

where $q > 1$ and $W_{0,i}$ is intended to be $+\infty$ if $\det F \leq 0$ [42]. Another case-study example, satisfying (i), (iii) but not (iv) is: $W_0(F) = |F^T F - \mathrm{Id}|^2$.

We have the following observation, which we will prove in the Appendix:

Proposition 1.3 *For W_0 which is $\mathcal{C}^2$ in a neighborhood of $SO(3)$ and B which is $\mathcal{C}^2$ in a neighborhood of* 0, *assume* (1.9) *and assume that* $B(0) = \mathrm{Id}$. *Then W_1 and W_2 in* (1.8) *satisfy* (1.7).

1.3 Background and Relation to Some Previous Works

To put our results in a broader context, consider a general referential domain Ω which is an open, smooth and simply connected subset of $\mathbb{R}^3$. Let $G : \bar{\Omega} \to \mathbb{R}^{3\times 3}$ be a given smooth Riemann metric on Ω and denote its unique positive definite symmetric square root by $B = \sqrt{G}$. The "incompatible elastic energy" of a deformation u of Ω is then given by:

$$E(u, \Omega) = \int_\Omega W_0\big(\nabla u(x) B(x)^{-1}\big)\,\mathrm{d}x \quad \forall u \in W^{1,2}\big(\Omega, \mathbb{R}^3\big), \tag{1.10}$$

where the elastic energy density W_0 is as in (1.9). It has been proved in [35] that:

$$\inf_{u \in W^{1,2}(\Omega,\mathbb{R}^3)} E(u, \Omega) = 0 \quad \Longleftrightarrow \quad \mathit{Riem}(G) \equiv 0 \quad \text{in } \Omega,$$

i.e., when the Riemann curvature tensor of G vanishes identically in Ω and when (equivalently) the infimum above is achieved through a smooth isometric immersion u of G.

Note that, at the formal level, the Euler-Lagrange equations of (1.10) are precisely the first equation in the system (1.2). The dynamical viscoelasticity has been the subject of many studies in the last decades (see for example [3–6, 9, 10, 45] and references therein), where results on existence, asymptotics and stability have been obtained for a large class of models. For the coupled systems of stress-assisted diffusion of the type (1.1) or (1.2), we found a substantial body of literature in the Applied Mechanics community [17–19, 49, 50], deriving these equations from basic principles of continuum mechanics and irreversible thermodynamics. For example, the system derived in [19] is quite close to (1.2) from the view point of theory of PDEs; in as much as the structures of nonlinearities in both systems are almost the same. Derivation from the first principles aside, it seems that the analytical study of the Cauchy problem, particularly in long temporal ranges, has not been yet carried out. The closest investigation in this direction has been proposed in [20] concerning existence of solutions for models of nonlinear thermoelasticity, and in [51] where the authors examine well-posedness of further models of thermoviscoelasticity. We refer here to [33, 47] as well.

Systems of the type (1.1), (1.2) are also similar to some PDEs appearing naturally in the Biomechanics literature on growth. In [16, 48] (see also the references therein) the authors consider morphoelasticity as a continuous dynamical process and include the effect of growth and remodelling within the framework of nonlinear elasticity under the assumption that the elastic tensor $(\nabla u)B^{-1}$ is diagonal. They also perform a numerical analysis of the stability of a fixed point for growth, assuming its constancy. The review article [21] features the interplay between growth patterns and mechanical stresses from different points of view, in particular assuming the multiplicative decomposition of the deformation gradient as in the argument of the density W_0 in (1.10), or the resulting additive decomposition in the displacement gradient at the linearised level. The paper [27] reviews the thermodynamics developed for multicomponent multiphase stressed crystalline solids, at the equilibria in which the solid is inhomogeneous both in stress and composition.

It is worth mentioning that in the context of thin films when $\Omega = \Omega^h = U \times (-\frac{h}{2}, \frac{h}{2})$ with some $U \subset \mathbb{R}^2$, there is a large body of literature relating the magnitude of curvatures of G to the scaling of $\inf E(\cdot, \Omega^h)$ in terms of the film's thickness h, and subsequently deriving the residual 2-dimensional energies using the variational techniques. In the Euclidean case of $G = \mathrm{Id}_3$, where the residual energies are driven by presence of applied forces $f^h \sim h^\alpha$, three distinct limiting theories have been obtained [14] for $\frac{1}{h}E(\cdot, \Omega^h) \sim h^\beta$ with $\beta > 2$ (equivalently $\alpha > 2$). Namely: $\beta \in (2, 4)$ corresponded to the linearized Kirchhoff model (nonlinear bending energy), $\beta = 4$ to the classical von-Kármán model, and $\beta > 4$ to the linear elasticity. For $\beta = 0$ the membrane energy has been derived in the seminal papers [28, 29], while the case $\beta = 2$ was considered in [15]. In the paper [36] a higher order (infinite) hierarchy of scalings and of the resulting elastic theories of shells, where the reference configuration is a thin curved film, has been derived by an asymptotic calculus.

In the context of the non-Euclidean energy (1.10), it has been shown in [8] that the scaling: $\inf \frac{1}{h}E(\cdot, \Omega^h) \sim h^2$ only occurs when the metric $G_{2\times 2}$ on the mid-plate U can be isometrically immersed in $\mathbb{R}^3$ with the regularity $W^{2,2}$ and when, at the same time, the three appropriate Riemann curvatures of G do not vanish identically; the relevant residual theory, obtained through Γ-convergence, yielded then a Kirchhoff-like residual energy. Further, in [37] the authors proved that the only outstanding nontrivial residual theory is a von Kàrmàn-like energy, valid when: $\inf \frac{1}{h}E(\cdot, \Omega^h) \sim h^4$. This scale separation, contrary to [14, 36], is due to the fact that while the magnitude of external forces is adjustable at will, it seems not to be the case for the interior mechanism of a given metric G which does not depend on h. In fact, it is the curvature tensor of G which induces the nontrivial stresses in the thin film and it has only six independent components, namely the six sectional curvatures created out of the three principal directions, which further fall into two categories: including or excluding the thin direction variable. The simultaneous vanishing of curvatures in each of these categories correspond to the two scenarios at hand in terms of the scaling of the residual energy.

Other types of residual energies pertaining to different contexts and scalings, have been studied and derived by the authors in [11, 12, 22, 24, 25, 30–32, 34, 38]. The analysis in the present paper can be seen as a prerequisite to the future considerations of dimension reduction for dynamical problems in nonlinear elasticity with prestrain, in the spirit of [1, 2] where dynamics of thin plates without incompatibility was treated.

1.4 Notation

Throughout the paper we use the following notation. In (1.1) the operator div stands for the spacial divergence of an appropriate field. We use the convention that the divergence of a matrix field is taken row-wise. We use the matrix norm $|F| = (\mathrm{tr}(F^T F))^{1/2}$, which is induced by the inner product: $\langle F_1 : F_2\rangle = \mathrm{tr}(F_1^T F_2)$.

The derivatives of W are denoted by DW, D^2W etc., while their action on the appropriate variations $(\tilde\phi, \tilde F) \in \mathbb{R} \times \mathbb{R}^{3\times 3}$ is denoted by: $DW(\phi, \nabla u) : (\tilde\phi, \tilde F)$, $D^2W(\phi, \nabla u) : (\tilde\phi, \tilde F)^{\otimes 2}$ etc., often abbreviating to $DW : (\tilde\phi, \tilde F)$ and $(D^2W) : (\tilde\phi, \tilde F)^{\otimes 2}$ when no confusion arises.

The partial derivative of W with respect to its second argument is denoted by $\partial_F W \in \mathbb{R}^{3\times 3}$. The derivative in the direction of the variation $\tilde F \in \mathbb{R}^{3\times 3}$ is then $\langle(\partial_F W) : \tilde F\rangle \in \mathbb{R}$. By $(\partial_F^k W) : (\tilde F_1 \otimes \tilde F_2 \ldots \otimes \tilde F_{k-1}) \in \mathbb{R}^{3\times 3}$ we denote the linear map acting on $F \in \mathbb{R}^{3\times 3}$ as the kth derivative of W in the direction of $\tilde F_1, \tilde F_2 \ldots \tilde F_{k-1}, F$. Hence, differentiating in F gives:

$$\big(\partial_F^k W\big) : (\tilde F_1 \otimes \ldots \tilde F_k) = \big\langle\big(\big(\partial_F^k W\big) : (\tilde F_1 \otimes \ldots \tilde F_{k-1})\big) : \tilde F_k\big\rangle \in \mathbb{R}.$$

Finally, $C, c > 0$ stand for universal constants, independent of the variable quantities at hand.

2 The Crucial A Priori Estimate

Lemma 2.1 *Every solution* (u, ϕ) *to* (1.1), *with regularity prescribed in Theorem* 1.1, *satisfies*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \frac{1}{2}|u_t|^2 + W(\phi, \nabla u)\,\mathrm{d}x + \int \big|\nabla(-\Delta)^{-1}\phi_t\big|^2\,\mathrm{d}x = 0. \tag{2.1}$$

Proof Testing the first equation in (1.1) by u_t and integrating by parts gives:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \frac{1}{2}|u_t|^2 + W(\phi, \nabla u)\,\mathrm{d}x \\
&\quad= \int \langle u_t, u_{tt}\rangle + \partial_\phi W(\phi, \nabla u)\phi_t + \big\langle \partial_F W(\phi, \nabla u) : \nabla u_t\big\rangle \mathrm{d}x \\
&\quad= \int \big\langle u_t, \mathrm{div}\big(\partial_F W(\phi, \nabla u)\big)\big\rangle + \big\langle \partial_F W(\phi, \nabla u) : \nabla u_t\big\rangle + \partial_\phi W(\phi, \nabla u)\phi_t\,\mathrm{d}x \\
&\quad= \int_{\mathbb{R}^3} \partial_\phi W(\phi, \nabla u)\phi_t\,\mathrm{d}x.
\end{aligned}$$

Define $\psi = (-\Delta)^{-1}\phi$ and integrate the second equation in (1.1) against ψ_t:

$$\int_{\mathbb{R}^3} |\nabla\psi_t|^2\,\mathrm{d}x = \int \phi_t\psi_t\,\mathrm{d}x = \int \psi_t\Delta\big(\partial_\phi W(\phi, \nabla u)\big)\,\mathrm{d}x = -\int_{\mathbb{R}^3} \partial_\phi W(\phi, \nabla u)\phi_t\,\mathrm{d}x.$$

Summing the above two equalities yields (2.1) and achieves the proof. □

For every $i, j, k \in \{1, 2, 3\}$ we now define the correction terms:

$$\mathcal{R}_{ijk} = \big(\partial_\phi \partial_F^2 W\big) : (\nabla u_{x_i,x_j} \otimes \nabla u_{x_k} + \nabla u_{x_i,x_k} \otimes \nabla u_{x_j} + \nabla u_{x_j,x_k} \otimes \nabla u_{x_i})$$

$$+ \big(\partial_\phi^2 \partial_F \partial_\phi W\big) : (\nabla u_{x_i,x_j}\phi_{x_k} + \nabla u_{x_i,x_k}\phi_{x_j} + \nabla u_{x_j,x_k}\phi_{x_i}$$
$$+ \nabla u_{x_i}\phi_{x_j,x_k} + \nabla u_{x_j}\phi_{x_i,x_k} + \nabla u_{x_k}\phi_{x_i,x_j})$$
$$+ \big(\partial_\phi \partial_F^3 W\big) : \nabla u_{x_i} \otimes \nabla u_{x_j} \otimes \nabla u_{x_k}$$
$$+ \big(\partial_\phi^2 \partial_F^2 W\big) : (\nabla u_{x_i} \otimes \nabla u_{x_j}\phi_{x_k} + \nabla u_{x_i} \otimes \nabla u_{x_k}\phi_{x_j} + \nabla u_{x_j} \otimes \nabla u_{x_k}\phi_{x_1})$$
$$+ \big(\partial_\phi^3 \partial_F W\big) : (\nabla u_{x_i}\phi_{x_j}\phi_{x_k} + \nabla u_{x_j}\phi_{x_i}\phi_{x_k} + \nabla u_{x_k}\phi_{x_i}\phi_{x_j})$$
$$+ \big(\partial_\phi^4 W\big)\phi_{x_i}\phi_{x_j}\phi_{x_k}$$
$$+ \big(\partial_\phi^3 W\big)(\phi_{x_i,x_j}\phi_{x_k} + \phi_{x_j,x_k}\phi_{x_i} + \phi_{x_i,x_k}\phi_{x_j}). \tag{2.2}$$

Lemma 2.2 *Let* (u, ϕ) *be a solution to* (1.1), *with regularity prescribed in Theorem* 1.1. *For* $t > 0$, *define the two quantities*:

$$\mathcal{E}(t) = \int_{\mathbb{R}^3} |u_t|^2 + \left|\nabla^3 u_t\right|^2 + 2W(\phi, \nabla u)$$
$$+ \sum_{i,j,k=1..3} D^2 W(\phi, \nabla u) : (\phi_{x_i,x_j,x_k}, \nabla u_{x_i,x_j,x_k})^{\otimes 2} + 2 \sum_{i,j,k=1..3} \mathcal{R}_{ijk}\phi_{x_i,x_j,x_k}\,\mathrm{d}x,$$
$$\mathcal{Z}(t) = \|u_t\|^2_{H^3(\mathbb{R}^3)} + \|\nabla u - \mathrm{Id}\|^2_{H^3(\mathbb{R}^3)} + \|\phi\|^2_{H^3(\mathbb{R}^3)}.$$

Then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \le C\big(\mathcal{Z}^4 + \mathcal{Z}^{3/2}\big). \tag{2.3}$$

Proof **1.** We differentiate the first equation in (1.1) in a spacial direction $x_i \in \{x_1, x_2, x_3\}$:

$$u_{x_i,tt} - \mathrm{div}\big(\partial_F^2 W(\phi, \nabla u) : \nabla u_{x_i}\big) = \mathrm{div}\big(\partial_F \partial_\phi W(\phi, \nabla u)\phi_{x_i}\big).$$

We now differentiate the above twice more in the directions $x_i, x_j \in \{x_1, x_2, x_3\}$:

$$u_{x_i,x_j,x_k,tt} - \mathrm{div}\big(\partial_F^2 W(\phi, \nabla u) : \nabla u_{x_i,x_j,x_k}\big) = \mathrm{div}\big(\partial_F \partial_\phi W(\phi, \nabla u)\phi_{x_i,x_j,x_k}\big) + \mathcal{R}_1. \tag{2.4}$$

The error term $\mathcal{R}_1$ above has the following form, where we suppress the distinction between different x_i, x_j, x_k, retaining hence only the structure of different terms:

$$\mathcal{R}_1 = \mathrm{div}\Big(3\big(\partial_F^3 W\big) : \nabla u_x \otimes \nabla u_{xx} + 3\big(\partial_F^2 \partial_\phi W\big) : (\nabla u_{xx}\phi_x + \nabla u_x \phi_{xx})$$
$$+ 3\big(\partial_F \partial_\phi^2 W\big)\phi_x\phi_{xx} + \big(\partial_F^4 W\big) : (\nabla u_x)^{\otimes 3} + 3\big(\partial_F^3 \partial_\phi W\big) : (\nabla u_x)^{\otimes 2}\phi_x$$
$$+ 3\big(\partial_F^2 \partial_\phi^2 W\big) : \nabla u_x (\phi_x)^2 + \big(\partial_F \partial_\phi^3 W\big)(\phi_x)^3\Big). \tag{2.5}$$

Above and in what follows, we also write $(\partial_F^3 W)$ instead of $\partial_F^3 W(\phi, \nabla u)$, and $(\partial_F^2 \partial_\phi W)$ instead of $\partial_F^2 \partial_\phi W(\phi, \nabla u)$, etc. Integrating (2.4) by parts against $u_{x_i,x_j,x_k,t}$ we get:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \frac{1}{2}(u_{x_i,x_j,x_k,t})^2 + \phi_{x_i,x_j,x_k}\big\langle(\partial_F \partial_\phi W) : \nabla u_{x_i,x_j,x_k}\big\rangle$$
$$+ \frac{1}{2}\big(\partial_F^2 W\big) : \nabla u_{x_i,x_j,x_k} \otimes \nabla u_{x_i,x_j,x_k}\,\mathrm{d}x$$

$$
\begin{aligned}
&= \boxed{\int_{\mathbb{R}^3} \phi_{x_i,x_j,x_k,t} \big\langle (\partial_F \partial_\phi W) : \nabla u_{x_i,x_j,x_k} \big\rangle \,\mathrm{d}x} \\
&\quad + \int_{\mathbb{R}^3} \phi_{x_i,x_j,x_k} \big\langle (\partial_t \partial_F \partial_\phi W) : \nabla u_{x_i,x_j,x_k} \big\rangle \,\mathrm{d}x \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^3} \big(\partial_t \partial_F^2 \partial_\phi W\big) : \nabla u_{x_i,x_j,x_k} \otimes \nabla u_{x_i,x_j,x_k} \,\mathrm{d}x + \int_{\mathbb{R}^3} \mathcal{R}_1 u_{x_i,x_j,x_k,t} \,\mathrm{d}x. \qquad (2.6)
\end{aligned}
$$

2. Differentiate now the second equation in (1.1) in $x_i \in \{x_1, x_2, x_3\}$:

$$
\phi_{x_i,t} = \Delta\big(\big\langle \partial_\phi \partial_F W(\phi, \nabla u) : \nabla u_{x_i} \big\rangle + \partial_\phi^2 W(\phi, \nabla u) \phi_{x_i} \big).
$$

As before, differentiate twice more in $x_i, x_j \in \{x_1, x_2, x_3\}$, to obtain:

$$
\phi_{x_i,x_j,x_k,t} = \Delta\big((\partial_\phi \partial_F W) : \nabla u_{x_i,x_j,x_k} + \big(\partial_\phi^2 W\big) \phi_{x_i,x_j,x_k} + \mathcal{R}_{ijk} \big), \qquad (2.7)
$$

where $\mathcal{R}$ is given in (2.2). Testing (2.7) against $(-\Delta)^{-1} \phi_{x_i,x_j,x_k,t} = \psi_{x_i,x_j,x_k,t}$, we get:

$$
\begin{aligned}
&- \int_{\mathbb{R}^3} |\nabla \psi_{x_i,x_j,x_k,t}|^2 \,\mathrm{d}x \\
&= \boxed{\int_{\mathbb{R}^3} \phi_{x_i,x_j,x_k,t} \big\langle (\partial_F \partial_\phi W) : \nabla u_{x_i,x_j,x_k} \big\rangle \,\mathrm{d}x} \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \frac{1}{2} \big(\partial_\phi^2 W\big) (\phi_{x_i,x_j,x_k})^2 \,\mathrm{d}x \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^3} \big(\partial_t \big(\partial_\phi^2 W\big)\big) (\phi_{x_i,x_j,x_k})^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} \mathcal{R}_{ijk} \phi_{x_i,x_j,x_k,t} \,\mathrm{d}x. \qquad (2.8)
\end{aligned}
$$

Note now that the first terms in the right hand side of both (2.6) and (2.8), namely the terms displayed in boxes, are the same. Consequently, subtracting (2.8) from (2.6), we get:

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla \psi_{x_i,x_j,x_k,t}|^2 \,\mathrm{d}x \\
&\quad + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} (u_{x_i,x_j,x_k,t})^2 + D^2 W(\phi, \nabla u) : (\phi_{x_i,x_j,x_k}, \nabla u_{x_i,x_j,x_k})^{\otimes 2} \,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} \mathcal{R}_{ijk} \phi_{x_i,x_j,x_k,t} \,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \phi_{x_i,x_j,x_k} \big\langle (\partial_t \partial_F \partial_\phi W) : \nabla u_{x_i,x_j,x_k} \big\rangle \,\mathrm{d}x \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^3} \big(\partial_t \partial_F^2 \partial_\phi W\big) : \nabla u_{x_i,x_j,x_k} \otimes \nabla u_{x_i,x_j,x_k} \,\mathrm{d}x \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^3} \big(\partial_t \big(\partial_\phi^2 W\big)\big) (\phi_{x_i,x_j,x_k})^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} \mathcal{R}_1 u_{x_i,x_j,x_k,t} \,\mathrm{d}x. \qquad (2.9)
\end{aligned}
$$

3. We will now estimate terms in the right hand side of (2.9) and prove that:

$$
\int_{\mathbb{R}^3} |\phi_{x_i,x_j,x_k}| |\partial_t \partial_F \partial_\phi W| |\nabla u_{x_i,x_j,x_k}| \,\mathrm{d}x + \int_{\mathbb{R}^3} \big|\partial_t \partial_F^2 \partial_\phi W\big| |\nabla u_{x_i,x_j,x_k}|^2 \,\mathrm{d}x
$$

$$+\int_{\mathbb{R}^3}\left|\partial_t\left(\partial_\phi^2 W\right)\right||\phi_{x_i,x_j,x_k}|^2\,\mathrm{d}x \le C\left(\mathcal{Z}^{3/2}+\|\nabla\psi_t\|_{H^3(\mathbb{R}^3)}\mathcal{Z}\right) \tag{2.10}$$

and:

$$\left|\int_{\mathbb{R}^3}\mathcal{R}_1 u_{x_i,x_j,x_k,t}\,\mathrm{d}x\right| \le C\left(\mathcal{Z}^{3/2}+\mathcal{Z}^2+\mathcal{Z}^{5/2}\right)+C\|\nabla\psi_t\|_{H^3(\mathbb{R}^3)}\left(\mathcal{Z}^{3/2}+\mathcal{Z}^2\right). \tag{2.11}$$

For the first term in (2.10), we note that by the Sobolev embedding $\mathcal{C}^{0,1/2}(\mathbb{R}^3)\hookrightarrow H^2(\mathbb{R}^3)$ one easily gets:

$$\begin{aligned}
&\int_{\mathbb{R}^3}|\phi_{x_i,x_j,x_k}||\partial_t\partial_F\partial_\phi W||\nabla u_{x_i,x_j,x_k}|\\
&\quad\le \left\|\left(\partial_F^2\partial_\phi W\right):\nabla u_t+\left(\partial_F\partial_\phi^2 W\right)\phi_t\right\|_{L^\infty}\left\|\nabla^3\phi\right\|_{L^2}\left\|\nabla^4 u\right\|_{L^2}\\
&\quad\le C\left(\|\nabla u_t\|_{L^\infty}+\|\phi_t\|_{L^\infty}\right)\left\|\nabla^3\phi\right\|_{L^2}\left\|\nabla^4 u\right\|_{L^2}\\
&\quad\le C\left(\|\nabla u_t\|_{H^2}+\|\Delta\psi_t\|_{H^2}\right)\left\|\nabla^3\phi\right\|_{L^2}\left\|\nabla^4 u\right\|_{L^2}\\
&\quad\le C\left(\mathcal{Z}^{3/2}+\|\nabla\psi_t\|_{H^3(\mathbb{R}^3)}\mathcal{Z}\right).
\end{aligned}$$

Similarly, the other two terms in (2.10) are bounded by:

$$C\left(\|\nabla u_t\|_{L^\infty}+\|\phi_t\|_{L^\infty}\right)\left(\left\|\nabla^4 u\right\|_{L^2}^2+\left\|\nabla^3\phi\right\|_{L^2}^2\right),$$

which implies the same estimate as before.

Regarding (2.11), the first term in $\int_{\mathbb{R}^3}\mathcal{R}_1 u_{x_i,x_j,x_k,t}\,\mathrm{d}x$, is bounded by:

$$\begin{aligned}
\int_{\mathbb{R}^3}\left|\mathrm{div}\left(\left(\partial_F^3 W\right):\nabla u_x\otimes\nabla u_{xx}\right)\right|\left|\nabla^3 u_t\right|\mathrm{d}x &\le C\Big(\left(\|\nabla u_t\|_{L^\infty}+\|\phi_t\|_{L^\infty}\right)\|\nabla^2 u\|_{L^\infty}\left\|\nabla^3 u\right\|_{L^2}\\
&\quad+\left\|\nabla^3 u\right\|_{L^4}^2+\left\|\nabla^2 u\right\|_{L^\infty}\left\|\nabla^4 u\right\|_{L^\infty}\Big)\left\|\nabla^3 u_t\right\|_{L^2}\\
&\le C\left(\mathcal{Z}^{3/2}+\mathcal{Z}^2+\|\nabla\psi_t\|_{H^3(\mathbb{R}^3)}\mathcal{Z}^{3/2}\right),
\end{aligned}$$

because of the Sobolev embedding $W^{1,2}(\mathbb{R}^3)\hookrightarrow L^p(\mathbb{R}^3)$ valid for any $p\in[2,6]$. Also:

$$\begin{aligned}
\int_{\mathbb{R}^3}\left|\mathrm{div}\left(\left(\partial_F^4 W\right):(\nabla u_x)^{\otimes 3}\right)\right|\left|\nabla^3 u_t\right|\mathrm{d}x &\le C\Big(\left(\|\nabla u_t\|_{L^\infty}+\|\phi_t\|_{L^\infty}\right)\left\|\nabla^2 u\right\|_{L^6}^3\\
&\quad+\left\|\nabla^2 u\right\|_{L^\infty}^2\left\|\nabla^3 u\right\|_{L^2}\Big)\left\|\nabla^3 u_t\right\|_{L^2}\\
&\le C\left(\mathcal{Z}^{5/2}+\mathcal{Z}^2+\|\nabla\psi_t\|_{H^3(\mathbb{R}^3)}\mathcal{Z}^2\right).
\end{aligned}$$

Other terms in $\mathcal{R}_1$ induce the same estimate as above. This establishes (2.11).

4. We now consider the last term in the right hand side of (2.9):

$$\int_{\mathbb{R}^3}\mathcal{R}_{ijk}\phi_{x_i,x_j,x_k,t}\,\mathrm{d}x = \left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\mathcal{R}_{ijk}\phi_{x_i,x_j,x_k}\,\mathrm{d}x\right)-\int_{\mathbb{R}^3}(\mathcal{R}_{ijk})_t\phi_{x_i,x_j,x_k}\,\mathrm{d}x. \tag{2.12}$$

We now prove that:

$$\left|\int_{\mathbb{R}^3}(\mathcal{R}_{ijk})_t\phi_{x_i,x_j,x_k}\,\mathrm{d}x\right| \le C\left(\mathcal{Z}^{3/2}+\mathcal{Z}^{5/2}\right)+C\|\nabla\psi_t\|_{H^3(\mathbb{R}^3)}\left(\mathcal{Z}^{3/2}+\mathcal{Z}^2\right). \tag{2.13}$$

First, using the notational convention as in (2.5), $\mathcal{R}$ can be replaced by:

$$\begin{aligned}\mathcal{R}_2 &= 3\big(\partial_\phi \partial_F^2 W\big) : \nabla u_x \otimes \nabla u_{xx} + 3\big(\partial_\phi^2 \partial_F \partial_\phi W\big) : (\nabla u_{xx}\phi_x + \nabla u_x \phi_{xx}) \\ &\quad + \big(\partial_\phi \partial_F^3 W\big) : (\nabla u_x)^{\otimes 3} + 3\big(\partial_\phi^2 \partial_F^2 W\big) : (\nabla u_x)^{\otimes 2}\phi_x \\ &\quad + 3\big(\partial_F \partial_\phi^3 W\big) : \nabla u_x (\phi_x)^2 + \big(\partial_\phi^4 W\big)(\phi_x)^3 + 3\big(\partial_\phi^3 W\big)\phi_x \phi_{xx}. \end{aligned} \tag{2.14}$$

The first term in (2.14) can be estimated as before, using embedding and interpolation theorems:

$$\begin{aligned}&\int_{\mathbb{R}^3} \big|\big(\big(\partial_\phi \partial_F^2 W\big) : \nabla u_x \otimes \nabla u_{xx}\big)_t\big| \big|\nabla^3 \phi\big| \,\mathrm{d}x \\ &\quad \le C\Big(\big(\|\nabla u_t\|_{L^\infty} + \|\phi_t\|_{L^\infty}\big)\big\|\nabla^2 u\big\|_{L^\infty}\big\|\nabla^3 u\big\|_{L^2} \\ &\qquad + \big\|\nabla^2 u_t\big\|_{L^4}\big\|\nabla^3 u\big\|_{L^4} + \big\|\nabla^2 u\big\|_{L^\infty}\big\|\nabla^3 u_t\big\|_{L^2}\Big)\big\|\nabla^3 \phi\big\|_{L^2} \\ &\quad \le C\big(\mathcal{Z}^{3/2} + \mathcal{Z}^2 + \|\nabla \psi_t\|_{H^3(\mathbb{R}^3)}\mathcal{Z}^{3/2}\big),\end{aligned}$$

while the third term in $\mathcal{R}_2$ is estimated by:

$$\begin{aligned}&\int_{\mathbb{R}^3} \big|\big(\big(\partial_\phi \partial_F^3 W\big) : (\nabla u_x)^{\otimes 3}\big)_t\big| \big|\nabla^3 \phi\big| \,\mathrm{d}x \\ &\quad \le C\Big(\big(\|\nabla u_t\|_{L^\infty} + \|\phi_t\|_{L^\infty}\big)\big\|\nabla^2 u\big\|_{L^6}^3 + \big\|\nabla^2 u\big\|_{L^\infty}^2\big\|\nabla^3 u_t\big\|_{L^2}\Big)\big\|\nabla^3 \phi\big\|_{L^2} \\ &\quad \le C\big(\mathcal{Z}^{5/2} + \mathcal{Z}^2 + \|\nabla \psi_t\|_{H^3(\mathbb{R}^3)}\mathcal{Z}^2\big).\end{aligned}$$

Other terms in $\mathcal{R}_2$ induce the same estimate as above. This establishes (2.13).

5. Summing now (2.9) over all triples x_i, x_j, x_k, adding (2.1), and taking into account (2.12), (2.10), (2.11) and (2.13), we obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \big(2\|\nabla \psi_t\|_{L^2}^2 + \big\|\nabla^4 \psi_t\big\|_{L^2}^2\big) &\le C\big(\mathcal{Z}^{5/2} + \mathcal{Z}^{3/2}\big) + C\|\nabla \psi_t\|_{W_2^3(\mathbb{R}^3)}\big(\mathcal{Z}^2 + \mathcal{Z}\big) \\ &\le \epsilon\|\nabla \psi_t\|_{H^3(\mathbb{R}^3)}^2 + C\big(\mathcal{Z}^4 + \mathcal{Z}^{3/2}\big)\end{aligned}$$

in view of Young's inequality. Consequently, (2.3) follows and the proof is complete. □

We now deduce the main a-priori estimate of this section:

Theorem 2.3 *Under the assumptions of Theorem 1.1, any solution on the time interval* $[0, T]$ *to* (1.1) (1.3)–(1.4) *satisfies*:

$$\sup_{t\le T} \mathcal{Z}(t) \le C\big(\mathcal{E}(0) + T^2\mathcal{E}_0(0) + \|u_0 - \mathrm{id}\|_{L^2}^2\big), \tag{2.15}$$

where: $\mathcal{E}_0(0) = \int_{\mathbb{R}^3} |u_1|^2 + 2W(\phi_0, \nabla u_0)\,\mathrm{d}x$, *and C is a universal constant.*

Proof Assume that the quantities in (1.6) are sufficiently small. In particular, we require that $\mathcal{Z}(0) \ll 1$ and that $\mathcal{Z}$ is sufficiently small on an interval $[0, t_0]$, where we also choose an appropriate $t_0 \ll T$. Lemma 2.2 implies: $\mathcal{E}'(t) \le C\mathcal{Z}^{3/2}(t)$, which is equivalent to:

$$\mathcal{E}(t) \le C\int_0^t \mathcal{Z}^{3/2}(s)\,\mathrm{d}s + \mathcal{E}(0). \tag{2.16}$$

Further, by (2.1) it follows that:

$$\sup_t \|u_t\|^2_{L^2} \leq \mathcal{E}_0(0). \tag{2.17}$$

Since:

$$\begin{aligned}\forall t \leq t_0 \quad \big\|u(t) - \mathrm{id}\big\|^2_{L^2} &= 2\int_0^t \int_{\mathbb{R}^3} \langle u - \mathrm{id}, u_t\rangle \,\mathrm{d}x + \|u_0 - \mathrm{id}\|^2_{L^2} \\ &\leq 2T\Big(\sup_{s\leq t} \|u_t\|_{L^2}\Big)\Big(\sup_{s\leq t} \|u - \mathrm{id}|_{L^2}\Big) + \|u_0 - \mathrm{id}\|^2_{L^2}, \end{aligned} \tag{2.18}$$

we easily obtain in view of (2.17):

$$\begin{aligned}\sup_{t\leq t_0}\big\|u(t) - \mathrm{id}\big\|^2_{L^2} &\leq 4t_0^2 \sup_{t\leq t_0}\big\|u_t(t)\big\|^2_{L^2} + 2\|u_0 - \mathrm{id}\|^2_{L^2} \\ &\leq 4t_0^2\mathcal{E}_0(0) + 2\|u_0 - \mathrm{id}\|^2_{L^2}. \end{aligned} \tag{2.19}$$

Further, we observe that thanks to (1.7), to Korn's inequality and to Poincaré's inequality, there exist constants $c, C > 0$ so that:

$$\begin{aligned} c\mathcal{Z}(t) \leq \int_{\mathbb{R}^3} |u_t|^2 + \big|\nabla^3 u_t\big|^2 + 2W(\phi, \nabla u) \\ + \sum_{i,j,k=1..3} D^2W(\phi, \nabla u) : (\phi_{x_i,x_j,x_k}, \nabla u_{x_i,x_j,x_k})^{\otimes 2}\,\mathrm{d}x + \int_{\mathbb{R}^3} |u - \mathrm{id}|^2\,\mathrm{d}x \leq C\mathcal{Z}(t), \end{aligned}$$

as well as:

$$\left|\int_{\mathbb{R}^3} \sum_{i,j,k=1..3} \mathcal{R}_{ijk}\phi_{x_i,x_j,x_k}\,\mathrm{d}x\right| \leq C\|\mathcal{R}\|_{L^2}\big\|\nabla^3\phi\big\|_{L^2} \leq C\mathcal{Z}^{3/2}(t)\mathcal{Z}^{1/2}(t) = C\mathcal{Z}^2(t),$$

where we estimated each term in (2.2) by the Cauchy-Schwarz inequality and noted the appropriate Sobolev embedding. Consequently, we arrive at:

$$\forall t \leq t_0 \quad \mathcal{E}(t) + \|u - \mathrm{id}\|^2_{L^2}(t) \geq c\mathcal{Z}(t) - C\mathcal{Z}^2(t) \geq c\mathcal{Z}(t), \tag{2.20}$$

provided that $\mathcal{Z} \ll 1$ is sufficiently small on the time interval we consider. In view of (2.16), (2.20) and (2.19), we now get:

$$\forall t \leq t_0 \quad \mathcal{Z}(t) \leq C\left(\int_0^t \mathcal{Z}^{3/2}(s)\,\mathrm{d}s + \mathcal{E}(0) + t_0^2\mathcal{E}_0(0) + \|u_0 - \mathrm{id}\|^2_{L^2}\right). \tag{2.21}$$

Calling $\bar{\mathcal{Z}} = \sup_{t\in[0,t_0]} \mathcal{Z}(t)$, we have:

$$\bar{\mathcal{Z}} \leq \big(Ct_0\bar{\mathcal{Z}}^{1/2}\big)\bar{\mathcal{Z}} + C\big(\mathcal{E}(0) + t_0^2\mathcal{E}_0(0) + \|u_0 - \mathrm{id}\|^2_{L^2}\big), \tag{2.22}$$

which combined with the requirement: $CT\bar{\mathcal{Z}}^{1/2} \leq \frac{1}{2}$ yields:

$$\mathcal{Z}(t_0) \leq \bar{\mathcal{Z}} \leq C\big(\mathcal{E}(0) + t_0^2\mathcal{E}_0(0) + \|u_0 - \mathrm{id}\|^2_{L^2}\big). \tag{2.23}$$

The above clearly implies the Theorem in view of the smallness of initial data in (1.6). □

3 Proof of Theorem 1.1: Existence of Solutions to (1.1)

In this section we construct approximate solutions to the Cauchy problem (1.1) (1.3)–(1.4), which satisfy the same a-priori bounds as in Sect. 2. Given $\epsilon > 0$, consider the regularized problem:

$$\begin{cases} u_{tt} - \operatorname{div}(\partial_F W(\phi, \nabla u)) - \epsilon \Delta u = 0 \\ \phi_t = \Delta(\partial_\phi W(\phi, \nabla u)) \end{cases} \tag{3.1}$$

with the same initial data as in (1.3)–(1.4).

Lemma 3.1 *Assume that all quantities in* (1.6) *are sufficiently small. Then, there exists* $T_\epsilon > 0$ *and a solution* $(u^\epsilon, \phi^\epsilon)$ *of* (3.1) (1.3)–(1.4) *on* $\mathbb{R}^3 \times [0, T_\epsilon)$, *such that*:

$$\begin{aligned} &u^\epsilon - \mathrm{id} \in L^\infty\big(0, T; H^4\big(\mathbb{R}^3\big)\big), \qquad u^\epsilon_{tt} \in L^\infty\big(0, T; H^2\big(\mathbb{R}^3\big)\big), \\ &\phi^\epsilon \in L^\infty\big(0, T; H^3\big(\mathbb{R}^3\big)\big) \quad \textit{and} \quad \phi^\epsilon_t \in L^2\big(0, T; H^2\big(\mathbb{R}^3\big)\big). \end{aligned} \tag{3.2}$$

Proof **1.** Since $\epsilon > 0$ is fixed, we drop the superscript $^\epsilon$ in order to lighten the notation in the next two steps. We proceed by the Galerkin method. Choose an orthonormal base $\{w^k\}_{k=1}^\infty$ in the space $H^4(\mathbb{R}^3, \mathbb{R}^3)$ equipped with the scalar product:

$$\langle w, \tilde{w}\rangle_{H^4} = \langle w, \tilde{w}\rangle_{L^2} + \big\langle \nabla^4 w : \nabla^4 \tilde{w}\big\rangle_{L^2}. \tag{3.3}$$

Similarly, let $\{v^k\}_{k=1}^\infty$ be an orthonormal basis in $H^3(\mathbb{R}^3)$ equipped with:

$$\langle v, \tilde{v}\rangle_{H^3} = \langle v, \tilde{v}\rangle_{L^2} + \big\langle \nabla^3 v : \nabla^3 \tilde{v}\big\rangle_{L^2}. \tag{3.4}$$

Denote: $W^N = \operatorname{span}\{w^1, \ldots, w^N\}$ and $V^N = \operatorname{span}\{v^1, \ldots, v^N\}$.

We now introduce the auxiliary scalar products:

$$\begin{aligned} &\langle w, \tilde{w}\rangle_W = \langle w, \tilde{w}\rangle_{L^2} + \big\langle \nabla^3 w : \nabla^3 \tilde{w}\big\rangle_{L^2} \quad \forall w, \tilde{w} \in H^4\big(\mathbb{R}^3, \mathbb{R}^3\big), \\ &\langle v, \tilde{v}\rangle_V = \big\langle v, (-\Delta)^{-1}\tilde{v}\big\rangle_{L^2} + \big\langle \nabla^3 v : (-\Delta)^{-1}\nabla^3 \tilde{v}\big\rangle_{L^2} \quad \forall v, \tilde{v} \in H^3\big(\mathbb{R}^3, \mathbb{R}\big). \end{aligned} \tag{3.5}$$

Clearly, these products are not equivalent to (3.3), (3.4), however their properties will allow for using the energy estimates of the proof of Lemma 2.2 to prove the regularity of approximate solutions (u^N, ϕ^N) which we define below.

Let $(u^N, \phi^N) \in W^N \times V^N$ be the solution to:

$$\begin{cases} \big\langle u^N_{tt} - \operatorname{div} \partial_F W\big(\phi^N, \nabla u^N\big) - \epsilon \Delta u^N, w^l\big\rangle_W = 0, \\ \big\langle \phi^N_t - \Delta \partial_\phi W\big(\phi^N, \nabla u^N\big), v^l\big\rangle_V = 0, \quad \forall l : 1 \ldots N \\ u^N(0, \cdot) = \mathbb{P}_{W^N}(u_0), \qquad \big(u^N\big)_t(0, \cdot) = \mathbb{P}_{W^N}(u_1), \qquad \phi^N(0, \cdot) = \mathbb{P}_{V^N}(\phi_0). \end{cases} \tag{3.6}$$

By $\mathbb{P}$ we denote here the orthogonal projections on appropriate subspaces. The classical theory of systems of ODEs guarantees existence of solutions to (3.6) on some time interval $[0, T_N)$. We now prove that these time intervals may be taken uniform for all sequences $\{u^N, \phi^N\}$.

2. Since $u^N_t \in W^N$ and $\phi^N_t \in V^N$, (3.6) implies:

$$\begin{aligned} &\big\langle u^N_{tt} - \operatorname{div} \partial_F W\big(\phi^N, \nabla u^N\big) - \epsilon \Delta u^N, u^N_t\big\rangle_W = 0, \\ &\big\langle \phi^N_t - \Delta \partial_\phi W\big(\phi^N, \nabla u^N\big), \phi^N_t\big\rangle_V = 0. \end{aligned} \tag{3.7}$$

Note that the first equation in (3.7) is equivalent to:

$$
\begin{aligned}
&\big\langle u^N_{tt} - \operatorname{div} \partial_F W\big(\phi^N, \nabla u^N\big) - \epsilon \Delta u^N, u^N_t \big\rangle_{L^2} \\
&\quad + \sum_{i,j,k=1..3} \big\langle u^N_{x_i,x_j,x_k,tt} - \operatorname{div}\big(\partial^2_F W\big(\phi^N, \nabla u^N\big) : \nabla u^N_{x_i,x_j,x_k}\big) - \epsilon \Delta u^N_{x_i,x_j,x_k}, u^N_{x_i,x_j,x_k,t}\big\rangle_{L^2} \\
&= \sum_{i,j,k=1..3} \big\langle \operatorname{div}\big(\partial_F \partial_\phi W\big(\phi^N, \nabla u^N\big)\phi_{x_i,x_j,x_k}\big), u^N_{x_i,x_j,x_k,t}\big\rangle_{L^2} + \big\langle \mathcal{R}^N_1 : \nabla^3 u^N_t\big\rangle,
\end{aligned}
$$

where by $\mathcal{R}^N_1$ we denote the error terms induced by the functions u^N, ϕ^N as in (2.4), (2.5). Likewise, the second equation in (3.7) becomes:

$$
\begin{aligned}
&\big\langle \phi^N_t - \Delta \partial_\phi W\big(\phi^N, \nabla u^N\big), (-\Delta)^{-1}\phi^N_t\big\rangle_{L^2} \\
&\quad + \sum_{i,j,k=1..3} \big\langle \phi^N_{x_i,x_j,x_k,t} - \Delta\big(\big(\partial_\phi \partial_F W^N\big) : \nabla u^N_{x_i,x_j,x_k} - \big(\partial^2_\phi W^N\big)\phi^N_{x_i,x_j,x_k}\big), \\
&\qquad\qquad (-\Delta)^{-1}\phi^N_{x_i,x_j,x_k,t}\big\rangle_{L^2} \\
&= \sum_{i,j,k=1..3} \big\langle \Delta \mathcal{R}^N_{ijk}, (-\Delta)^{-1}\phi^N_{x_i,x_j,x_k,t}\big\rangle_{L^2},
\end{aligned}
$$

where we used the identity (2.7) and the notation (2.2), with the superscript ^{N} indicating that they concern u^N and ϕ^N.

Let $\mathcal{E}[u^N, \phi^N](t)$ be as in Lemma 2.2 with (u, ϕ) replaced by (u^N, ϕ^N), and define:

$$
\mathcal{E}_\epsilon\big[u^N, \phi^N\big](t) = \mathcal{E}\big[u^N, \phi^N\big](t) + \epsilon\big\langle(-\Delta)u^N, u^N\big\rangle_V, \tag{3.8}
$$

so that:

$$
\begin{aligned}
\mathcal{E}_\epsilon\big[\phi^N, u^N\big](t) = \int_{\mathbb{R}^3} &\big|u^N_t\big|^2 + \big|\nabla^3 u^N_t\big|^2 + 2W\big(\phi^N, \nabla u^N\big) + \epsilon\big|\nabla u^N\big|^2 \\
&+ \sum_{i,j,k=1..3} D^2 W\big(\phi^N, \nabla u^n\big) : \big(\phi^N_{x_i,x_j,x_k}, \nabla u^N_{x_i,x_j,x_k}\big)^{\otimes 2} + \epsilon\big|\nabla u^N_{x_i,x_j,x_k}\big|^2 \\
&+ 2 \sum_{i,j,k=1..3} \mathcal{R}^N_{ijk}\phi^N_{x_i,x_j,x_k}\,\mathrm{d}x.
\end{aligned}
$$

Following the proof of Lemmas 2.1 and 2.2, we find the counterpart of the inequality (2.3):

$$
\mathcal{E}_\epsilon\big[u^N, \phi^N\big](t) \le C \int_0^t \mathcal{Z}^{3/2}\big[u^N, \phi^N\big](s)\,\mathrm{d}s + \mathcal{E}_\epsilon(0), \tag{3.9}
$$

where the constant C is independent from ϵ, and where:

$$
\mathcal{Z}\big[u^N, \phi^N\big](t) = \big\|u^N_t\big\|^2_{H^3(\mathbb{R}^3)} + \big\|\nabla u^N - \mathrm{Id}\big\|^2_{H^3(\mathbb{R}^3)} + \big\|\phi^N\big\|^2_{H^3(\mathbb{R}^3)}.
$$

Note that in order to obtain (3.9) we use only the equivalent formulations of (3.7) above, hence indeed all the steps from the proof of Lemma 2.2 are valid with universal constants. Since the initial data in (3.6) consists of projections of the original data, their norms are uniformly controlled as well.

3. We now consider the equivalence of $\mathcal{E}_\epsilon$ with $\mathcal{Z}$. Since for small $\mathcal{Z}$ one has:

$$\int \mathcal{R}^N_{ijk}\phi^N_{x_i,x_j,x_k}\,\mathrm{d}x \le C\mathcal{Z}^{3/2}\big[u^N,\phi^N\big],$$

we easily see that:

$$\mathcal{E}_\epsilon\big[u^N,\phi^N\big] \le C\mathcal{Z}\big[u^N,\phi^N\big]. \tag{3.10}$$

On the other hand, in view of (3.8):

$$\mathcal{E}_\epsilon\big[u^N,\phi^N\big] \ge c_\epsilon\big(\mathcal{Z} - C_\epsilon\mathcal{Z}^{3/2}\big) \ge c_\epsilon\mathcal{Z}\big[u^N,\phi^N\big], \tag{3.11}$$

where by c_ϵ, C_ϵ we denote positive constants independent of N but depending on ϵ. By (3.9) we now arrive at:

$$\mathcal{Z}\big[u^N,\phi^N\big](t) \le C_\epsilon\int_0^t \mathcal{Z}^{3/2}\big[u^N,\phi^N\big](s)\,\mathrm{d}s + C_\epsilon\mathcal{E}_\epsilon(0).$$

Consequently, for $t_{0,\epsilon}$ sufficiently small, we have:

$$\sup_{t\le t_{0,\epsilon}} \mathcal{Z}\big[u^N,\phi^N\big](t) \le C_\epsilon\mathcal{E}_\epsilon(0).$$

The above estimates, in particular (3.10) and (3.11) imply the uniform in N boundedness of the following quantities, on their common interval of existence $[0, T_\epsilon]$:

$$\begin{aligned} &u^N - \mathrm{id} \in L^\infty\big(0,T_\epsilon; H^4\big(\mathbb{R}^3\big)\big), && u^N_t \in L^\infty\big(0,T_\epsilon; H^3\big(\mathbb{R}^3\big)\big),\\ &\phi^N \in L^\infty\big(0,T_\epsilon; H^3\big(\mathbb{R}^3\big)\big), && \phi^N_t \in L^2\big(0,T_\epsilon; H^2\big(\mathbb{R}^3\big)\big), \end{aligned} \tag{3.12}$$

yielding the weak-$*$ convergence in L^∞ as $N\to\infty$ (up to a subsequence), of the quantities: $u^N - \mathrm{id}$, u^N_t, ϕ^N, ϕ^N_t to the limiting quantities: $u^\epsilon - \mathrm{id}$, u^ϵ_t, ϕ^ϵ, ϕ^ϵ_t. Additionally, passing if necessary to a further subsequence and invoking a diagonal argument, we may also assure that:

$$\nabla u^N \to \nabla u^\epsilon \quad \text{and} \quad \phi^N \to \phi^\epsilon \quad \text{point-wise in } \mathbb{R}^3.$$

The Sobolev compact embedding: $H^1(0,T_\epsilon; H^2(B(R))) \hookrightarrow \mathcal{C}^\alpha((0,T_\epsilon)\times B(R))$, valid on any ball $B(R)\subset\mathbb{R}^3$, justifies now that $\phi^\epsilon\in\mathcal{C}^\alpha((0,T_\epsilon)\times B(R))$. Thus, in particular:

$$(\partial_\phi W,\ \partial_F W)\big(\phi^N,\nabla u^N\big) \to (\partial_\phi W,\ \partial_F W)\big(\phi^\epsilon,\nabla u^\epsilon\big) \quad \text{as } N\to\infty. \tag{3.13}$$

It follows that $(\phi^\epsilon, u^\epsilon)$ is a distributional solution to (3.1) (1.3)–(1.4). By (3.12) we obtain the desired regularity, completing the proof of Lemma 3.1. □

Proof of Theorem 1.1 (Existence part) Let ϕ^ϵ, u^ϵ be as in Lemma 3.1. We first observe that a common interval of existence of $(\phi^\epsilon, u^\epsilon)$ can be taken as $[0,T]$ with T prescribed by Theorem 1.1. This follows through repeating the estimates in Sect. 2, dealing with estimates of the first and the third order separately, and noting that the ϵ-term appears exclusively in $\mathcal{E}_\epsilon$ with a "good" sign. Consequently:

$$\sup_{t\le t_0}\mathcal{Z}\big[u^\epsilon,\phi^\epsilon\big] \le C(t_0, \text{initial data}), \tag{3.14}$$

and we see that indeed the solutions $\phi^\epsilon, u^\epsilon$ can be extended over appropriate $[0, T]$ with the quantities in (3.2) enjoying common bounds, independent of ϵ.

The same argument as in the last part of the proof of Lemma 3.1 implies now that the weak-$*$ limit (up to a subsequence) of $(\phi^\epsilon, u^\epsilon)$ yield the desired regular solution (ϕ, u) to the original problem (1.1) (1.3)–(1.4). Condition (1.5) is automatically satisfied because of the smallness of initial data. $\square$

4 Proof of Theorem 1.1: Uniqueness of solutions to (1.1)

Let (ϕ, u) and $(\bar\phi, \bar u)$ be two solutions to (1.1) with the same initial data. Define:

$$(\delta\phi) = \phi - \bar\phi, \qquad (\delta u) = u - \bar u,$$

and observe that:

$$\begin{aligned}
&\partial_F W(\phi, \nabla u) - \partial_F W(\bar\phi, \nabla\bar u)\\
&\quad = \partial_F^2 W(\bar\phi, \nabla\bar u) : \nabla(\delta u) + \partial_\phi\partial_F W(\bar\phi, \nabla\bar\phi)(\delta\phi) + D^2\partial_F W(\tilde\phi, \nabla\tilde u) : \big((\delta\phi), \nabla(\delta u)\big)^{\otimes 2},\\
&\partial_\phi W(\phi, \nabla u) - \partial_\phi W(\bar\phi, \nabla\bar u)\\
&\quad = \partial_\phi^2 W(\bar\phi, \nabla\bar u)(\delta\phi) + \partial_\phi\partial_F W(\bar\phi, \nabla\bar u) : \nabla(\delta u) + D^2\partial_\phi W(\tilde\phi, \nabla\tilde u) : \big((\delta\phi), \nabla(\delta u)\big)^{\otimes 2},
\end{aligned} \tag{4.1}$$

where $\tilde\phi$ and $\tilde u$ are suitable linear combinations of $\phi, \bar\phi$ and $u, \bar u$, given by the application of the Taylor formula. Subtracting equations (1.1) for (ϕ, u) and $(\bar\phi, \bar u)$ and using (4.1), it follows that:

$$\begin{aligned}
&(\delta u)_{tt} - \operatorname{div}\big(\partial_F^2 W(\bar\phi, \nabla\bar u) : \nabla(\delta u) + \partial_\phi\partial_F W(\bar\phi, \nabla\bar u)(\delta\phi)\big)\\
&\quad = \operatorname{div}\big(D^2\partial_F W(\tilde\phi, \nabla\tilde u) : \big((\delta\phi), \nabla(\delta u)\big)^{\otimes 2}\big),\\
&(\delta\phi)_t - \Delta\big(\partial_\phi^2 W(\bar\phi, \nabla\bar u)(\delta\phi) + \partial_\phi\partial_F W(\bar\phi, \nabla\bar u) : \nabla(\delta u)\big)\\
&\quad = \Delta\big(D^2\partial_\phi W(\tilde\phi, \nabla\tilde u) : \big((\delta\phi), \nabla(\delta u)\big)^{\otimes 2}\big).
\end{aligned}$$

We now test the first equation above by $(\delta u)_t$, while the second equation by $(-\Delta)^{-1}(\delta\phi)_t$, to obtain:

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3} |\delta u_t|^2 + \partial_F^2 W(\bar\phi, \nabla\bar u) : \big(\nabla(\delta u)\big)^{\otimes 2}\,\mathrm{d}x + \int_{\mathbb{R}^3} \partial_\phi\partial_F W(\bar\phi, \nabla\bar u) : (\delta\phi)\nabla(\delta u)_t\,\mathrm{d}x\\
&\quad = \int_{\mathbb{R}^3} D^2\partial_F W(\tilde\phi, \nabla\tilde u) : \Big(\big((\delta\phi), \nabla(\delta u)\big)^{\otimes 2} \otimes \nabla(\delta u)_t\Big)\,\mathrm{d}x,\\
&\int_{\mathbb{R}^3} \big|\nabla(\delta\phi)_t\big|^2\,\mathrm{d}x + \int_{\mathbb{R}^3} \big(\partial_\phi^2 W(\bar\phi, \nabla\bar u)(\delta\phi) + \partial_\phi\partial_F W(\bar\phi, \nabla\bar u) : \nabla(\delta u)\big)(\delta\phi)_t\,\mathrm{d}x\\
&\quad = \int_{\mathbb{R}^3} D^2\partial_\phi W(\tilde\phi, \nabla\tilde u) : \big((\delta\phi), \nabla(\delta u)\big)^{\otimes 2}(\delta\phi)_t\,\mathrm{d}x.
\end{aligned} \tag{4.2}$$

Consequently:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \partial_\phi^2 W(\bar\phi, \nabla\bar u)(\delta\phi)^2\,\mathrm{d}x + \int_{\mathbb{R}^3} \partial_\phi\partial_F W(\bar\phi, \nabla\bar u) : (\delta\phi_t)\nabla(\delta u)\,\mathrm{d}x$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^3}\left|\nabla(\delta\phi)_t\right|^2\mathrm{d}x\\
&=\int_{\mathbb{R}^3} D^2\partial_\phi W(\tilde\phi,\nabla\tilde u):\big((\delta\phi),\nabla(\delta u)\big)^{\otimes 2}(\delta\phi)_t\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\partial_t\big(\partial^2_\phi W(\bar\phi,\nabla\bar u)\big)(\delta\phi)^2\,\mathrm{d}x. \qquad (4.3)
\end{aligned}
$$

Adding (4.2) and (4.3), we arrive at:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{R}^3}\left|(\delta u)_t\right|^2+\partial_F^2 W(\bar\phi,\nabla\bar u):\big(\nabla(\delta u)\big)^{\otimes 2}+\partial^2_\phi W(\bar\phi,\nabla\bar u)(\delta\phi)^2\\
&\qquad+2\partial_\phi\partial_F W(\bar\phi,\nabla\bar u):(\delta\phi)\nabla(\delta u)\,\mathrm{d}x\\
&\quad\le C\left\|(\delta\phi)_t,\phi_t,\bar\phi_t\right\|_{L_\infty(\mathbb{R}^3)}(t)\cdot\sup_t\left\|(\delta\phi),\nabla(\delta u)\right\|^2_{L^2(\mathbb{R}^3)}(t),
\end{aligned}
$$

which implies that:

$$
\begin{aligned}
&\sup_t\int_{\mathbb{R}^3}|\delta u_t|^2+D^2W(\bar\phi,\nabla\bar u):\big((\delta\phi),\nabla(\delta u)\big)^{\otimes 2}\,\mathrm{d}x\\
&\quad\le C\sup_t\left\|(\delta\phi),\nabla(\delta u)\right\|^2_{L^2}\int_0^t\left\|(\delta\phi)_t,\phi_t,\bar\phi_t\right\|_{L_\infty}(s)\,\mathrm{d}s. \qquad (4.4)
\end{aligned}
$$

As before, assumptions on W guarantee that the left hand side in (4.4) bounds from above the quantity: $\sup_t\|(\delta\phi),\nabla(\delta u)\|^2_{L^2}$. Since the integral quantity above is small for $t\ll 1$, it follows by (4.4) that $(\delta\phi)$ and $\nabla(\delta u)$ are zero.

5 Proof of Theorem 1.2: The Elliptic-Parabolic Problem (1.2)

As in Sect. 3, we first derive an a priori estimate for solutions of (1.2), whose existence will follow then via Galerkin's method, in the same manner as for the system (1.1).

Lemma 5.1 *Assume that* (ϕ,u) *is a sufficiently smooth solution to* (1.2) *which remains in a vicinity of* $(0,\mathrm{id})$ *for all* $t\ge 0$, *in the sense that*:

$$
\varXi[\phi,\nabla u-\mathrm{Id}]:=\sup_{t\ge 0}\big(\|\phi\|^2_{H^2(\mathbb{R}^3)}+\|\nabla u-\mathrm{Id}\|_{H^2}\big)+c\int_0^\infty\left\|\nabla\phi,\nabla^2 u\right\|^2_{H^2(\mathbb{R}^3)}\mathrm{d}t\ll 1.
$$

Then:

$$
\sup_{t\ge 0}\Big(\left\|\nabla u(t)-\mathrm{Id}\right\|^2_{H^2}+\left\|u(t)-\mathrm{id}\right\|^2_{L^6}\Big)+\varXi[\phi,\nabla u-\mathrm{Id}]\le C\|\phi_0\|^2_{H^2(\mathrm{R}^3)}.
$$

Proof **1.** Observe first the following elementary fact:

$$
\left\|\nabla^2 u(t)\right\|_{H^1(\mathbb{R}^3)}\le C\left\|\nabla\phi(t)\right\|_{H^1(\mathbb{R}^3)}. \qquad (5.1)
$$

To see (5.1), consider the first equation in (1.2):

$$
\partial^2_F W(\phi,\nabla u):\nabla u_{x_i}=-\partial_\phi\partial_F W(\phi,\nabla u)\phi_{x_i},\quad i=1\dots 3. \qquad (5.2)
$$

Condition (1.7) and Korn's inequality imply that the system (5.2) is elliptic, hence its solutions (normalised so that $\nabla u - \mathrm{Id} \in L^6(\mathbb{R}^3)$) obey:

$$\left\|\nabla^2 u\right\|_{L^p(\mathbb{R}^3)} \le C\|\nabla\phi\|_{L^p(\mathbb{R}^3)}, \quad p = 2, 4.$$

Differentiating (5.2) with respect to x leads further to:

$$\begin{aligned}\left\|\nabla^3 u\right\|_{L^2} &\le C\big(\left\|\nabla^2\phi\right\|_{L^2} + \left\|\nabla\phi, \nabla^2 u\right\|_{L^4}^2\big)\\ &\le C\big(\left\|\nabla^2\phi\right\|_{L^2} + \|\nabla\phi\|_{L^4}^2\big) \le C\big(\left\|\nabla^2\phi\right\|_{L^2} + \|\nabla\phi\|_{H^1}^2\big),\end{aligned}$$

proving (5.1) in view of the assumption in the Lemma.

We also observe the resulting control of pointwise smallness of ϕ and $(\nabla u - \mathrm{Id})$, by the Sobolev embedding:

$$\sup_{t\ge 0}\|\phi, \nabla u - \mathrm{Id}\|_{L^\infty} \le C\sup_{t\ge 0}\|\phi, \nabla u - \mathrm{Id}\|_{H^2} \le C\varXi^{1/2} \ll 1. \tag{5.3}$$

2. Testing the first equation in (1.2) by u_t and the second one by $\psi_t = (-\Delta)^{-1}\phi_t$, we obtain the energy estimate, as in Lemma 2.1:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} W(\phi, \nabla u)\,\mathrm{d}x + \int_{\mathbb{R}^3}\left|\nabla(-\Delta)^{-1}\phi_t\right|^2\mathrm{d}x = 0.$$

To derive the second energy estimate we proceed slightly differently. Differentiating (1.2) in a spatial direction $x_i \in \{x_1, x_2, x_3\}$, we get:

$$\begin{aligned}&\mathrm{div}\left(\partial_F^2 W(\phi, \nabla u) : \nabla u_{x_i} + \partial_F\partial_\phi W(\phi, \nabla u)\phi_{x_i}\right) = 0,\\ &\phi_{x_i,t} = \Delta\big(\partial_\phi\partial_F W(\phi, \nabla u) : \nabla u_{x_i} + \partial_\phi^2 W(\phi, \nabla u)\phi_{x_i}\big).\end{aligned} \tag{5.4}$$

Now, testing the first equation above by u_{x_i}, testing the second one by $\psi_{x_i} = (-\Delta)^{-1}\phi_{x_i}$, and summing up the results, yields:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left|\nabla(-\Delta)^{-1}\phi_{x_i}\right|^2\mathrm{d}x + \int_{\mathbb{R}^3} D^2 W(\phi, \nabla u) : (\phi_{x_i}, \nabla u_{x_i})^{\otimes 2}\,\mathrm{d}x = 0. \tag{5.5}$$

Consequently, thanks to (1.7), the strict convexity of W implies:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left|\nabla^2\psi\right|^2 + \frac{\gamma}{2}\int_{\mathbb{R}^3}\left(\left|\nabla\phi\right|^2 + \sum_{i=1}^{3}\left|(\mathrm{sym}\nabla u_{x_i})\right|^2\right) \le 0.$$

Using Korn's inequality and integrating in time we see that:

$$\sup_{t>0}\int_{\mathbb{R}^3}\phi^2\,\mathrm{d}x + c\int_0^\infty\int_{\mathbb{R}^3}\left(|\nabla\phi|^2 + \left|\nabla^2 u\right|^2\right)\mathrm{d}x\,\mathrm{d}t \le C\|\phi_0\|_{L^2(\mathbb{R}^3)}^2. \tag{5.6}$$

3. We now differentiate (5.4) in a spatial direction $x_j \in \{x_1, x_2, x_3\}$, getting:

$$\begin{aligned}&\mathrm{div}\left(\partial_F^2 W(\phi, \nabla u) : \nabla u_{x_i,x_j} + \partial_\phi\partial_F W(\phi, \nabla u)\phi_{x_i,x_j}\right) = \mathrm{div}\,\mathcal{R}_1,\\ &\phi_{x_i,x_j,t} - \Delta\big(\partial_\phi^2 W(\phi, \nabla u)\phi_{x_i,x_j} + \partial_\phi\partial_F W(\phi, \nabla u) : \nabla u_{x_i,x_j}\big) = \Delta\mathcal{R}_2,\end{aligned} \tag{5.7}$$

where the error terms $\mathcal{R}_1$ and $\mathcal{R}_1$ have the following structure (we suppress the distinction between different x_i, x_j):

$$\mathcal{R}_1, \mathcal{R}_2 \sim D^3 W(\phi, \nabla u) : \big((\nabla u_x)^{\otimes 2} + (\nabla u_x)\phi_x + (\phi_x)^2\big). \tag{5.8}$$

Integrating (5.7) by parts against u_{x_i,x_j} and $(-\Delta)^{-1}\phi_{x_i,x_j}$, respectively, it follows that:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} |\nabla \psi_{x_i,x_j}|^2\,\mathrm{d}x \ &+ c\int_{\mathbb{R}^3} D^2 W(\phi, \nabla u) : (\phi_{x_i,x_j}, \nabla u_{x_i,x_j})^{\otimes 2}\,\mathrm{d}x \\ &\leq C\int_{\mathbb{R}^3} \left|\nabla^2 u\right|^4 + |\nabla\phi|^4\,\mathrm{d}x,\end{aligned} \tag{5.9}$$

because of (5.8) and:

$$\begin{aligned}&\left|\int_{\mathbb{R}^3} (\operatorname{div}\mathcal{R}_1) u_{x_i,x_j}\,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} (\Delta\mathcal{R}_2)(-\Delta)^{-1}\phi_{x_i,x_j}\,\mathrm{d}x\right| \\ &\quad\leq \epsilon\|\nabla u_{x_i,x_j}, \phi_{x_i,x_j}\|^2_{L^2(\mathbb{R}^3)} + C\|\mathcal{R}_1, \mathcal{R}_2\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Differentiating (5.7) further, we obtain:

$$\begin{aligned}&\operatorname{div}\big(\partial_F^2 W(\phi, \nabla u) : \nabla u_{x_i,x_j,x_k} + \partial_\phi\partial_F W(\phi, \nabla u)\phi_{x_i,x_j,x_k}\big) = \operatorname{div}\mathcal{R}_3, \\ &\phi_{x_i,x_j,x_k,t} - \Delta\big(\partial_\phi^2 W(\phi, \nabla u)\phi_{x_i,x_j,x_k} + \partial_\phi\partial_F W(\phi, \nabla u) : \nabla u_{x_i,x_j,x_k}\big) = \Delta\mathcal{R}_4,\end{aligned}$$

where, as before:

$$\begin{aligned}\mathcal{R}_3, \mathcal{R}_4 \sim\ & D^3 W(\phi, \nabla u) : (\nabla u_{xx}\otimes\nabla u_x + \nabla u_{xx}\phi_x + \nabla u_x\phi_{xx} + \phi_x\phi_{xx}) \\ &+ D^4 W(\phi, \nabla u) : \big((\nabla u_x)^{\otimes 3} + (\nabla u_x)^{\otimes 2}\phi_x + \nabla u_x(\phi_x)^2 + \nabla u_x(\phi_x)^2 + (\phi_x)^2\big).\end{aligned}$$

Testing by u_{x_i,x_j,x_k} and ψ_{x_i,x_j,x_k}, we find:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} |\nabla\psi_{x_i,x_j,x_k}|^2\,\mathrm{d}x &+ c\int_{\mathbb{R}^3} D^2 W(\phi, \nabla u) : (\phi_{x_i,x_j,x_k}, \nabla u_{x_i,x_j,x_k})^{\otimes 2}\,\mathrm{d}x \\ &\leq C\int_{\mathbb{R}^3} \left|\nabla^3 u\right|^4 + \left|\nabla^2 u\right|^4 + |\nabla\phi|^4 + \left|\nabla^2\phi\right|^4 + \left|\nabla^2 u\right|^6 + |\nabla\phi|^6\,\mathrm{d}x.\end{aligned} \tag{5.10}$$

Summing (5.5), (5.9), (5.10), integrating the result in time in the same manner as in (5.6), and recalling (5.1), we obtain:

$$\begin{aligned}\Xi[\phi, \nabla u - \mathrm{Id}] &\leq C\int_0^\infty\int_{\mathbb{R}^3} \left|\nabla^2 u\right|^4 + |\nabla\phi|^4 + \left|\nabla^2 u\right|^6 + |\nabla\phi|^6 \\ &\qquad + \left|\nabla^3 u\right|^4 + \left|\nabla^2 u\right|^4 + \left|\nabla^2\phi\right|^4\,\mathrm{d}x\,\mathrm{d}t + C\|\phi_0\|^2_{H^2(\mathbb{R}^3)} \\ &\leq C\int_0^\infty\int_{\mathbb{R}^3} |\nabla\phi|^4 + |\nabla\phi|^6 + \left|\nabla^2\phi\right|^4\,\mathrm{d}x\,\mathrm{d}t + C\|\phi_0\|^2_{H^2(\mathbb{R}^3)}.\end{aligned} \tag{5.11}$$

We further have:

$$\int_0^\infty\int_{\mathbb{R}^3} |\nabla\phi|^4 + |\nabla\phi|^6 + \left|\nabla^2\phi\right|^4\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq C \sup_{t\geq 0}\big(\|\nabla\phi\|_{L^\infty}^2 + \|\nabla\phi\|_{L^\infty}^4 + \big\|\nabla^2\phi\big\|_{L^\infty}^2\big)\int_0^\infty \|\nabla\phi\|_{H^1(\mathbb{R}^3)}^2\,\mathrm{d}t \leq C\big(\varXi^2+\varXi^3\big)$$

Consequently, (5.11) becomes: $\varXi \leq C(\varXi^2+\varXi^3) + C\|\phi_0\|_{H^2}^2$. By the assumed smallness of $\varXi[\phi, \nabla u - \mathrm{Id}]$, we see that:

$$\varXi \leq 2C\|\phi_0\|_{H^2}^2. \tag{5.12}$$

4. We now conclude the proof of the a-priori bound. Test (5.2) by $(u - \mathrm{id})$ to get:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \partial_F^2 W(\phi,\nabla u) : (\nabla u - \mathrm{Id})^{\otimes 2}\,\mathrm{d}x \\
&\quad\leq C\int_{\mathbb{R}^3} |\phi||\nabla u - \mathrm{Id}| + |u-\mathrm{id}|\big(|\nabla u - \mathrm{Id}| + |\phi|\big)\big(|\nabla\phi| + \big|\nabla^2 u\big|\big)\,\mathrm{d}x \\
&\quad\leq C\|\phi\|_{L^2}^2 + \big(\epsilon + C\varXi^{1/2}\big)\|\nabla u - \mathrm{Id}\|_{L^2}^2
\end{aligned}$$

Indeed, by (5.3) and (5.1):

$$\begin{aligned}
&\int_{\mathbb{R}^3} |u-\mathrm{id}|\big(|\nabla u - \mathrm{Id}| + |\phi|\big)\big(|\nabla\phi| + \big|\nabla^2 u\big|\big)\,\mathrm{d}x \\
&\quad\leq C\|u-\mathrm{id}\|_{L^6}\|\nabla u - \mathrm{Id}\|_{L^2}\big\|\nabla\phi, \nabla^2\phi\big\|_{L^3} + C\|u-\mathrm{id}\|_{L^6}\|\phi\|_{L^2}\big\|\nabla\phi,\nabla^2\phi\big\|_{L^3} \\
&\quad\leq C\|\nabla u - \mathrm{Id}\|_{L^2}^2\varXi^{1/2} + C\|\phi\|_{L^2}^2 + C\varXi\|\nabla u - \mathrm{id}\|_{L^2}^2
\end{aligned}$$

Thus, we obtain the bound on $\|\nabla u - \mathrm{Id}\|_{L^2}$, and subsequently on $\|u-\mathrm{id}\|_{L^6}$. □

A proof of Theorem 1.2 Given $(\bar\phi, \bar u)$, consider the following problem which is the linearization of (1.2) at $(0, \mathrm{id})$:

$$\begin{aligned}
&\mathrm{div}\big(\partial_F^2 W(0,\mathrm{Id})(\nabla u - \mathrm{Id}) + \partial_\phi\partial_F W(0,\mathrm{Id})\phi\big) = \mathrm{div}\,A, \\
&\phi_t - \Delta\big(\partial_\phi^2 W(0,\mathrm{Id})\phi + \partial_\phi\partial_F W(0,\mathrm{Id})(\nabla u - \mathrm{Id})\big) = \Delta B,
\end{aligned} \tag{5.13}$$

where:

$$\begin{aligned}
A &= \partial_F^2 W(0,\mathrm{Id})(\nabla\bar u - \mathrm{Id}) + \partial_\phi\partial_F W(0,\mathrm{Id})\bar\phi - \partial_F W(\bar\phi, \nabla\bar u), \\
B &= \partial_\phi W(\bar\phi,\nabla\bar u) - \partial_\phi^2 W(0,\mathrm{Id})\bar\phi + \partial_\phi\partial_F W(0,\mathrm{Id})(\nabla\bar u - \mathrm{Id}).
\end{aligned} \tag{5.14}$$

Let $\mathcal{T}$ be its solution operator, so that $\mathcal{T}[\bar\phi,\bar u] = (\phi, u)$. We will prove that $\mathcal{T}$ has a fixed point in the space X, where:

$$\begin{aligned}
X = \Big\{(\phi,u);\ \ &\phi\in L^\infty\big(\mathbb{R}_+; H^2\big(\mathbb{R}^3\big)\big),\ \nabla(\nabla u - \mathrm{Id}) \in L^\infty\big(\mathbb{R}_+; H^1\big(\mathbb{R}^3\big)\big), \\
&\nabla\phi\in L^2\big(\mathbb{R}_+; H^2\big(\mathbb{R}^3\big)\big),\ \nabla(\nabla u - \mathrm{Id})\in L^2\big(\mathbb{R}_+; H^2\big(\mathbb{R}^3\big)\big)\Big\}.
\end{aligned}$$

Note first that the well-posedness of the system (5.13) follows by the Galerkin method in exactly the same manner as in Sect. 3, under the regularity of the right hand side:

$$A\in L^\infty\big(\mathbb{R}_+; H^2\big(\mathbb{R}^3\big)\big)\cap L^2\big(\mathbb{R}_+; H^3\big(\mathbb{R}^3\big)\big), \qquad B\in L^2\big(\mathbb{R}_+; H^3\big(\mathbb{R}^3\big)\big)$$

Approximative spaces are constructed for $\phi \in H^3(\mathbb{R}^3)$ and for $u - \mathrm{id}$ such that $\nabla u - \mathrm{Id} \in H^3(\mathbb{R}^3)$. We leave this construction to the reader and note that it is simpler than the one for the system (1.1). As in the proof of Lemma 5.1, solutions to (5.13) then satisfy:

$$\sup_{t\geq 0} \int_{\mathbb{R}^3} \phi^2 \,\mathrm{d}x + \sum_i \int_0^\infty \int_{\mathbb{R}^3} D^2W(0,\mathrm{Id}) : (\phi_{x_i}, \nabla u_{x_i})^{\otimes 2} \,\mathrm{d}x\,\mathrm{d}t$$
$$\leq C\|\nabla A, \nabla B\|^2_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^3))} + C\|\phi_0\|^2_{L^2},$$

which is obtained by testing with u_{x_i} and $(-\Delta)^{-1}\phi_{x_i}$. Similarly, the second and third derivatives bounds eventually yield:

$$\sup_{t\geq 0} \|\phi\|^2_{H^2(\mathbb{R}^3)} + \big\|\nabla\phi, \nabla(\nabla u - \mathrm{Id})\big\|^2_{L^2(\mathbb{R}_+;H^2(\mathbb{R}^3))}$$
$$\leq C\|\nabla A, \nabla B\|^2_{L^2(\mathbb{R}_+;H^2(\mathbb{R}^3))} + C\|\phi_0\|^2_{H^2(\mathbb{R}^3))}, \tag{5.15}$$

while:

$$\sup_{t\geq 0} \|\nabla u - \mathrm{Id}\|^2_{H^2(\mathbb{R}^3)} \leq C\|A\|^2_{L^\infty(\mathbb{R}_+;H^2(\mathbb{R}^3))}.$$

Directly from (5.14) we observe that:

$$\|A\|_{L^\infty(\mathbb{R}_+;H^2(\mathbb{R}^3))\cap L^2(\mathbb{R}_+;H^3(\mathbb{R}^3))} \leq C\ \Xi[\bar{\phi}, \nabla\bar{u} - \mathrm{Id}]$$
$$\|B\|_{L^2(\mathbb{R}_+;H^3(\mathbb{R}^3))} \leq C\ \Xi[\bar{\phi}, \nabla\bar{u} - \mathrm{Id}]$$

provided the quantity Ξ is small. Then, by (5.15):

$$\Xi[\phi, \nabla u - \mathrm{Id}] \leq C\,\Xi[\bar{\phi}, \nabla\bar{u} - \mathrm{Id}]^2 + C_0\|\phi_0\|^2_{H^2(\mathbb{R}^3)}.$$

Based on the considerations from the part about the a priori bound we observe that:

$$\Xi[\phi, \nabla u - \mathrm{Id}] \leq 2C_0\|\phi_0\|^2_{H^2(\mathbb{R}^3)},$$

provided that Ξ is sufficiently small. Hence the operator $\mathcal{T}$ maps a ball $\mathcal{B} \subset X$ with a sufficiently small radius, into itself. Observe further that $\mathcal{T}$ is a contraction over $\mathcal{B}$, whose fixed point yields the unique solution to the system (1.2). Theorem 1.2 is proved. □

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Appendix: Proof of Proposition 1.3

The first condition in (1.7) is obvious. A direct calculation shows that:

$$DW_1(\phi, F) : (\tilde{\phi}, \tilde{F}) = \phi\tilde{\phi} + \big\langle DW_0\big(FB(\phi)\big) : \tilde{\phi}FB'(\phi)\big\rangle + \big\langle DW_0\big(FB(\phi)\big) : \tilde{F}B(\phi)\big\rangle,$$

which implies the second condition in (1.7). Further:

$$D^2W_1(0,\mathrm{Id}) : (\tilde{\phi}, \tilde{F})^{\otimes 2} = |\tilde{\phi}|^2 + D^2W_0(\mathrm{Id}) : \big(\tilde{\phi}B'(0)\big)^{\otimes 2}$$

$$
\begin{aligned}
&+ 2D^2W_0(\mathrm{Id}) : \big(\tilde{B}'(0) \otimes \tilde{F}\big) + D^2W_0(\mathrm{Id}) : \tilde{F}^{\otimes 2} \\
&= |\tilde{\phi}|^2 + D^2W_0(\mathrm{Id}) : \big(\tilde{F} + \tilde{\phi}B'(0)\big)^{\otimes 2} \\
&\geq |\tilde{\phi}|^2 + c\big|\mathrm{sym}\ \tilde{F} + \tilde{\phi}B'(0)\big|^2,
\end{aligned}
$$

where we concluded from (1.9) that $DW_0(\mathrm{Id}) = 0$ and that $D^2W_0(\mathrm{Id})$ is positive definite on symmetric matrices. We also note that: $D^2W_2(0, \mathrm{Id}) = D^2W_1(0, \mathrm{Id})$. To conclude the proof, it is hence enough to show that:

$$
|\tilde{\phi}|^2 + \big|\mathrm{sym}\ \tilde{F} + \tilde{\phi}B'(0)|^2 \geq c\big(|\tilde{\phi}\big|^2 + |\mathrm{sym}\ \tilde{F}|^2\big), \tag{6.1}
$$

for all $\tilde{\phi}$ and $\tilde{F}$. Expanding the square in the left hand side, dividing by $|\tilde{\phi}|$ and collecting terms, this is equivalent to:

$$
\big(1 - c + \big|B'(0)\big|^2\big) + (1 - c)\left|\mathrm{sym}\left(\frac{1}{\tilde{\phi}}\tilde{F}\right)\right|^2 + 2\left\langle \mathrm{sym}\left(\frac{1}{\tilde{\phi}}\tilde{F}\right) : B'(0)\right\rangle \geq 0,
$$

which becomes:

$$
1 - c - \frac{c}{1-c}\big|B'(0)\big|^2 + \left|\sqrt{1-c}\ \mathrm{sym}\left(\frac{1}{\tilde{\phi}}\tilde{F}\right) + \frac{1}{\sqrt{1-c}}B'(0)\right|^2 \geq 0.
$$

The above inequality follows from: $1 - c - \frac{c}{1-c}|B'(0)|^2 > 0$, which is true whenever $c > 0$ is sufficiently small.

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