CrossMark

The Exponentiated Hencky-Logarithmic Strain Energy. Part I: Constitutive Issues and Rank-One Convexity

Patrizio Neff[1] · Ionel-Dumitrel Ghiba[1,2,3] ·
Johannes Lankeit[4]

Abstract We investigate a family of isotropic volumetric-isochoric decoupled strain energies

$$F \mapsto W_{\mathrm{eH}}(F) := \widehat{W}_{\mathrm{eH}}(U) := \begin{cases} \dfrac{\mu}{k} e^{k\|\operatorname{dev}_n \log U\|^2} + \dfrac{\kappa}{2\widehat{k}} e^{\widehat{k}[\operatorname{tr}(\log U)]^2} & \text{if } \det F > 0, \\ +\infty & \text{if } \det F \leq 0, \end{cases}$$

based on the Hencky-logarithmic (true, natural) strain tensor $\log U$, where $\mu > 0$ is the infinitesimal shear modulus, $\kappa = \frac{2\mu+3\lambda}{3} > 0$ is the infinitesimal bulk modulus with λ the first Lamé constant, $k, \widehat{k}$ are additional dimensionless material parameters, $F = \nabla\varphi$ is the gradient of deformation, $U = \sqrt{F^T F}$ is the right stretch tensor and $\operatorname{dev}_n \log U = \log U - \frac{1}{n}\operatorname{tr}(\log U) \cdot \mathbb{1}$ is the n-dimensional deviatoric part of the strain tensor $\log U$. For small elastic strains, W_{eH} approximates the classical quadratic Hencky strain energy

$$F \mapsto W_{\mathrm{H}}(F) := \widehat{W}_{\mathrm{H}}(U) := \mu\|\operatorname{dev}_n \log U\|^2 + \frac{\kappa}{2}\big[\operatorname{tr}(\log U)\big]^2,$$

which is not everywhere rank-one convex. In plane elastostatics, i.e., $n = 2$, we prove the everywhere rank-one convexity of the proposed family W_{eH}, for $k \geq \frac{1}{4}$ and $\widehat{k} \geq \frac{1}{8}$. Moreover,

In memory of Albert Tarantola (⋆ 1949–† 2009), lifelong advocate of logarithmic measures.

✉ P. Neff
patrizio.neff@uni-due.de

I.-D. Ghiba
dumitrel.ghiba@uni-due.de; dumitrel.ghiba@uaic.ro

J. Lankeit
johannes.lankeit@math.uni-paderborn.de

1   Lehrstuhl für Nichtlineare Analysis und Modellierung, Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann Str. 9, 45127 Essen, Germany

2   Department of Mathematics, Alexandru Ioan Cuza University of Iaşi, Blvd. Carol I, no. 11, 700506 Iaşi, Romania

3   Octav Mayer Institute of Mathematics, Romanian Academy, Iaşi Branch, 700505 Iaşi, Romania

4   Institut für Mathematik, Universität Paderborn, Warburger Str. 100, 33098 Paderborn, Germany

we show that the corresponding Cauchy (true)-stress-true-strain relation is invertible for $n = 2, 3$ and we show the monotonicity of the Cauchy (true) stress tensor as a function of the true strain tensor in a domain of bounded distortions. We also prove that the rank-one convexity of the energies belonging to the family W_{eH} is not preserved in dimension $n = 3$ and that the energies

$$F \mapsto \frac{\mu}{k} e^{k \| \log U \|^2}, \qquad F \mapsto \frac{\mu}{k} e^{\frac{k}{\mu} \left(\mu \| \mathrm{dev}_n \log U \|^2 + \frac{\kappa}{2} [\mathrm{tr}(\log U)]^2 \right)}, \qquad F \in \mathrm{GL}^+(n), n \in \mathbb{N}, n \geq 2$$

are also not rank-one convex.

Keywords Idealized finite isotropic elasticity · Legendre-Hadamard ellipticity condition · Hyperelasticity · Constitutive inequalities · Stability · Hencky strain · Logarithmic strain · Natural strain · True strain · Hencky energy · Convexity · Rank-one convexity · Volumetric-isochoric split · Plane elastostatics · Monotonicity and invertibility of the constitutive law · Homogeneous symmetric bifurcations · Baker-Ericksen inequality · Bounded distortions · Elastic domain · Nonlinear Poisson's ratio

Mathematics Subject Classification 74B20 · 74G64 · 35Q74 · 35E10

1 Introduction

1.1 Logarithmic Strain and Geodesically Motivated Invariants

We introduce a modification of the well-known isotropic quadratic Hencky strain energy

$$W_{\mathrm{H}}(F) = \widehat{W}_{\mathrm{H}}(U) = \mu \| \mathrm{dev}_n \log U \|^2 + \frac{\kappa}{2} \big[\mathrm{tr}(\log U) \big]^2,$$

where $\mu > 0$ is the infinitesimal shear (distortional) modulus, $\kappa = \frac{2\mu + 3\lambda}{3} > 0$ is the bulk modulus with λ the first Lamé constant, $F = \nabla \varphi$ is the deformation gradient, $U = \sqrt{F^T F}$ is the right Biot stretch tensor, $\log U$ is the referential (Lagrangian) logarithmic strain tensor, $\| . \|$ is the Frobenius tensor norm, and $\mathrm{dev}_n X = X - \frac{1}{n} \mathrm{tr}(X) \cdot \mathbb{1}$ is the n-dimensional deviatoric part of a second order tensor $X \in \mathbb{R}^{n \times n}$ (see Sect. 2 for other notations).

It was recently discovered [170, 171] (see also [32, 132]) that the Hencky strain energy enjoys a surprising property, which singles it out among all other isotropic strain energy functions. Indeed, the Hencky energy measures the geodesic distance of the deformation gradient $F \in \mathrm{GL}^+(n)$ to the special orthogonal group $\mathrm{SO}(n)$, i.e.,

$$\mathrm{dist}_{\mathrm{geod}}^2 \big(F, \mathrm{SO}(n) \big) = \mu \| \mathrm{dev}_n \log U \|^2 + \frac{\kappa}{2} \big[\mathrm{tr}(\log U) \big]^2 = W_{\mathrm{H}}(F),$$

$$\mathrm{dist}_{\mathrm{geod}}^2 \big(F, \mathrm{SO}(n) \big) = 0 \quad \text{if and only if} \quad \varphi(x) = \widehat{Q} x + \widehat{b} \tag{1.1}$$

$$\text{for some fixed } \widehat{Q} \in \mathrm{SO}(3), \widehat{b} \in \mathbb{R}^3,$$

where the Lie-group $\mathrm{GL}^+(n)$ is viewed as a Riemannian manifold endowed with a certain left-invariant metric which is also right $\mathrm{O}(n)$-invariant[1] (isotropic). The use of the quadratic

---

[1]Although every such Riemannian metric is uniquely characterized by three coefficients, the geodesic distance to $\mathrm{SO}(n)$ in fact depends on only two of them, corresponding to the two material parameters μ and κ.

Hencky strain energy in nonlinear elasticity theory can therefore be motivated by purely geometric reasoning.

In contrast, for the case of the simple Euclidean distance on $\mathbb{R}^{n \times n}$ we note that

$$\text{dist}^2_{\text{euclid}}\big(F, \text{SO}(n)\big) \overset{\text{(def)}}{:=} \inf_{\overline{R} \in \text{SO}(n)} \|F - \overline{R}\|^2 = \inf_{\overline{R} \in \text{SO}(n)} \left\|\overline{R}^T F - \mathbb{1}\right\|^2 = \|U - \mathbb{1}\|^2, \quad (1.2)$$

which yields the Biot-stretch measure $U - \mathbb{1}$ without any possibility of weighting the deviatoric and volumetric contributions independently [174]. On the other hand, the additive *volumetric-isochoric split*

$$\widehat{W}_{\text{H}}(U) = \mu \|\text{dev}_n \log U\|^2 + \frac{\kappa}{2}\big[\text{tr}(\log U)\big]^2 = \underbrace{\mu \left\|\log \frac{U}{\det U^{1/n}}\right\|^2}_{\widehat{W}_{\text{H}}^{\text{iso}}\left(\frac{U}{\det U^{1/n}}\right)} + \underbrace{\frac{\kappa}{2}[\log \det U]^2}_{\widehat{W}_{\text{H}}^{\text{vol}}(\det U)} \quad (1.3)$$

of $\widehat{W}_H$ into an isochoric term $\widetilde{W}_{\text{H}}^{\text{iso}}$ depending only on $\frac{U}{\det U^{1/n}}$, i.e., on the isochoric part of U, and a volumetric term $\widetilde{W}_{\text{H}}^{\text{vol}}$ depending only on $\det U$ is characterized by means of the same geodesic distance as well: it can be shown that [170, 171, 177]

$$K_1^2 := \text{dist}^2_{\text{geod}}\big((\det F)^{1/n} \cdot \mathbb{1}, \text{SO}(n)\big)$$

$$= \text{dist}^2_{\text{geod},\mathbb{R}_+ \cdot \mathbb{1}}\big((\det F)^{1/n} \cdot \mathbb{1}, \mathbb{1}\big) = \left|\text{tr}(\log U)\right|^2 = \widetilde{W}_{\text{H}}^{\text{vol}}(\det U),$$

$$K_2^2 := \text{dist}^2_{\text{geod}}\left(\frac{F}{(\det F)^{1/n}}, \text{SO}(n)\right) = \text{dist}^2_{\text{geod},\text{SL}(n)}\left(\frac{F}{(\det F)^{1/n}}, \text{SO}(n)\right)$$

$$= \|\text{dev}_n \log U\|^2 = \widetilde{W}_{\text{H}}^{\text{iso}}\left(\frac{U}{\det U^{1/n}}\right),$$

where $\text{dist}^2_{\text{geod},\mathbb{R}_+ \cdot \mathbb{1}}$ and $\text{dist}^2_{\text{geod},\text{SL}(n)}$ are the canonical left invariant geodesic distances on the Lie-group $\text{SL}(n)$ and on the multiplicative group $\mathbb{R}_+ \cdot \mathbb{1}$, respectively. This result strongly suggests that the two quantities $K_1^2 = \|\text{dev}_n \log U\|^2$ and $K_2^2 = [\text{tr}(\log U)]^2$ should be considered separately as fundamental measures of elastic deformations, which motivates a family of elastic energy functions stated in terms of these two quantities alone [151]. It is clear, however, that it is not the strain measure $\log U$ itself which has any importance in this regard,[2] but the fundamentally motivated scalar geodesic invariants K_1^2, K_2^2. They restrict the form of the constitutive law. The mechanical and physical significance of the logarithmic strain tensor is further discussed in [169].

Moreover, in 2D, the purely isochoric term $\text{dist}^2_{\text{geod}}\big(\frac{F}{\det F^{1/2}}, \text{SL}(2)\big)$ penalyzes the departure from conformal (shape preserving) mappings, i.e., the absolute minimizer in dimension $n = 2$ is a deformation ϕ with $\nabla\phi$ satisfying

$$\log \nabla\phi^T \nabla\phi = \alpha(x, y) \cdot \mathbb{1}_2, \quad \alpha(x, y) \in \mathbb{R} \quad \Leftrightarrow \quad \nabla\phi^T \nabla\phi = e^{\alpha(x,y) \cdot \mathbb{1}_2}, \quad \alpha(x, y) \in \mathbb{R}$$
$$\nabla\phi \in \underbrace{\mathbb{R}_+ \cdot \text{SO}(2)}_{\text{the special conformal group CSO}(2)} \quad \Leftrightarrow \quad \phi : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{is holomorphic.}$$

---

[2]Truesdell writes [251]: "It is important to realize that since each of the several material tensors [the strain tensors like $U - \mathbb{1}$, $\mathbb{1} - U^{-1}$, $\log U$, $U - U^{-1}$] is an isotropic function of any one of the others, an exact description of strain in terms of any one is equivalent to a description in terms of any other; only when an approximation is to be made may the choice of a particular measure become important."

Since K_1^2, K_2^2 have this inherently fundamental differential geometric motivation, we propose to investigate a new constitutive framework for ideal isotropic elasticity. Then it is natural to consider the most primitive possible strain energy form satisfying:

(i) The elastic energy W can be written as a function of the *geodesic invariants*

$$W = \widetilde{W}(K_1^2, K_2^2), \quad \text{where} \quad K_1^2 := \| \mathrm{dev}_n \log U \|^2 \text{ and } K_2^2 := [\mathrm{tr}(\log U)]^2;$$

(ii) The energy is strictly increasing as a function of K_1^2, K_2^2;
(iii) The energy is strictly convex as a function of $\log U$ (Hill's inequality);
(iv) Preferably, the energy should be a rank-one convex (polyconvex, quasiconvex) function;
(v) The energy should satisfy a coercivity condition.

We observe that (iv) necessitates that W should grow at least exponentially (see [222]).

1.2 Scope of Investigation

Many elastic materials show a completely different response regarding shape changing deformations and purely volumetric deformations. Therefore, in concordance with our just stated requirements, we investigate in this paper a family of isotropic exponentiated Hencky-logarithmic strain type energies in which both contributions coming from dilatations and distortions are a priori additively separated[3] [78]

$$
W_{\mathrm{eH}}(F) := \widehat{W}_{\mathrm{eH}}(U) :=
\begin{cases}
\dfrac{\mu}{k} e^{kK_2^2} + \dfrac{\kappa}{2\widehat{k}} e^{\widehat{k}K_1^2} & \text{if } \det F > 0, \\
+\infty & \text{if } \det F \leq 0,
\end{cases}
$$

$$
=
\begin{cases}
\underbrace{\dfrac{\mu}{k} e^{k\| \mathrm{dev}_n \log U \|^2} + \dfrac{\kappa}{2\widehat{k}} e^{\widehat{k}(\mathrm{tr}(\log U))^2}}_{\text{volumetric-isochoric split}} & \text{if } \det F > 0, \\
+\infty & \text{if } \det F \leq 0,
\end{cases}
\tag{1.4}
$$

$$
=
\begin{cases}
\dfrac{\mu}{k} e^{k\| \log \frac{U}{\det U^{1/n}} \|^2} + \dfrac{\kappa}{2\widehat{k}} e^{\widehat{k}(\log \det U)^2} & \text{if } \det F > 0, \\
+\infty & \text{if } \det F \leq 0,
\end{cases}
$$

where $U = \sum_{i=1}^{n} \lambda_i N_i \otimes N_i$, $\log U = \sum_{i=1}^{n} \log \lambda_i (N_i \otimes N_i) = \lim_{r \to 0} \frac{1}{r}(U^r - \mathbb{1})$, λ_i and N_i are the eigenvalues and eigenvectors of U, respectively. The immediate importance of the family (1.4) of free-energy functions is seen by looking at small (but not infinitesimally small) elastic strains. Then the exponentiated Hencky energy $W_{\mathrm{eH}}(\cdot)$ reduces to first order to the quadratic Hencky energy based on the logarithmic strain tensor $\log U$

$$
W_{\mathrm{H}}(F) := \widehat{W}_{\mathrm{H}}(U) := \mu \| \mathrm{dev}_n \log U \|^2 + \frac{\kappa}{2}[\mathrm{tr}(\log U)]^2 + \underbrace{\left(\frac{\mu}{k} + \frac{\kappa}{2\widehat{k}} \right)}_{\text{const.}},
\tag{1.5}
$$

$$\tau_{\mathrm{H}} := D_{\log V} \widetilde{W}_{\mathrm{H}}(V) = 2\mu \, \mathrm{dev}_3 \log V + \kappa \, \mathrm{tr}(\log V) \cdot \mathbb{1},$$

$$\tau_{\mathrm{H}} = \det V \cdot \sigma_{\mathrm{H}},$$

---

[3]Such an assumption is especially suitable for only slightly compressible materials or under small elastic strains [98].

where $V = \sqrt{FF^T}$ is the left stretch tensor, $\widetilde{W}_{\mathrm H}(V) = \widehat{W}_{\mathrm H}(U)$, $\sigma_{\mathrm H}$ is the Cauchy stress tensor in the current configuration and $\tau_{\mathrm H}$ is the Kirchhoff stress tensor. The Hencky energy $W_{\mathrm H}$ has been introduced by Heinrich Hencky [243] starting from 1928 [24, 100–103, 186, 222, 238, 256] (see [169] for a recent English translation of Hencky's German original papers) and has since then acquired a unique status in finite strain elastostatics [1, 5, 6] and especially in finite strain elasto-plasticity.[4] Hencky himself used this constitutive law to study finite elastic deformations of rubber in some simple cases [100–104]. The modern applications seem to begin with the study of finite elastic and elasto-plastic bending of a long plate-strip (plane strain) in the cases of incompressible and compressible deformations [36–38, 60, 61]. The formulation based on the Hencky strain energy provides the greatest possible extent of elastic determinacy [169, page 19]: the Kirchhoff-stress response does not depend on a specific reference state or previously applied coaxial deformations. A similar property was postulated for an idealized law of elasticity by Murnaghan [161, 162], who argued that the dependence of the stress response on a specific position of zero strain was tantamount to an action at a distance and should therefore be avoided.

The first axiomatic study on the nonlinear stress-strain function involving a logarithmic strain tensor is, however, due to the famous geologist George Ferdinand Becker [27, 145] in 1893. Using a principle of superposition for the principal forces in the reference configuration he concludes with a stress-strain law in the form

$$T_{\mathrm{Biot}}(U) = 2\mu \, \mathrm{dev}_3 \log U + \kappa \, \mathrm{tr}(\log U) \cdot \mathbb{1}, \tag{1.6}$$

where $T_{\mathrm{Biot}}(U) = R^T \cdot S_1(F) = U \cdot S_2(C)$ is the (symmetric in case of isotropy) Biot-stress tensor, $F = RU$ is the right polar decomposition, S_2 is the symmetric second Piola-Kirchhoff stress tensor and $C = F^T F$, see [176] for detailed explanations. Even earlier, in 1880, Imbert [117] in his doctoral thesis considered the uniaxial tension of vulcanized rubber bands and obtained as a best fit a constitutive law[5] [117, page 53], which in modern notations reads

$$\langle T_{\mathrm{Biot}}.e_1, e_1 \rangle = E \langle \log U.e_1, e_1 \rangle, \tag{1.7}$$

for recoverable (fully elastic extensional) stretches $\lambda \in [1, e)$. Three years later, in 1893, Hartig [95] (see also [28]) used the same constitutive law for tension and compression data of rubber.

In [164] Nadai introduced the name "natural strain" tensor for the logarithmic strain tensor $\log U$ and motivated application of this concept in metal forming processes[6] in metallurgy. The strain measure (natural strain) then has been extensively used over the years to

---

[4]In Hencky's first paper [99], the constitutive law $\sigma_{\mathrm H} = 2\mu \, \mathrm{dev}_3 \log V + \kappa \, \mathrm{tr}(\log V) \cdot \mathbb{1}$ is proposed, which is Cauchy-elastic, tensorially correct, but not hyperelastic. This has been corrected by Hencky in later papers. Incidentally, Becker's law (1.6) is also Cauchy-elastic, tensorially correct, but hyperelastic only for $\nu = 0$ [33, 46] (see also [176, 265]).

[5]Note that (1.7) is the uniaxial specification of (1.6), and (1.6) is closely resembling $(1.5)_2$. A small calculation [176] shows $\tau_{\mathrm{Becker}} = V \cdot \tau_{\mathrm H}$, where τ is the corresponding Kirchhoff stress $\tau = (\det F) \cdot \sigma = D_{\log V} W(\log V)$ and V is the left stretch tensor. Moreover $\|\tau_{\mathrm{Becker}} - \tau_{\mathrm H}\| \le \|V - \mathbb{1}\| \cdot \|\tau_{\mathrm H}\|$. Hence, for small elastic strains $\|V - \mathbb{1}\| \ll 1$, Becker's law coincides with Hencky's model to first order in the nonlinear strain measure $V - \mathbb{1}$.

[6]In the German metal forming literature the logarithmic strain is also called "Umformgrad". In [138, page 17] Ludwik uses the "effective specific elongation" $\alpha = \int_{\ell_0}^{\ell} \frac{d\ell}{\ell} = \ln \frac{\ell}{\ell_0}$. It can be motivated by considering the summation over the infinitesimal increase in length as referred to the current length, i.e., $\ln \frac{\ell}{\ell_0} = \lim_{N \to \infty} \sum_{i=0}^{N-1} \frac{\ell_{i+1} - \ell_i}{\ell_i}$ [94, 261]. The scalar Hencky-type measure $\|\mathrm{dev}_3 \log U\|$ is sometimes

report experimental true-stress-true-strain data. More recently, in [98] a modified Hencky energy is proposed which is motivated by in depth molecular dynamics simulations for a metallic glass.[7] Hill [107, 108] (see also [53, 149, 150]) has discussed the advantage of the logarithmic strain measure[8] in setting up a class of constitutive inequalities, based on a family of measures of finite strain and their corresponding conjugate stresses, for both elastic and elasto-plastic solids. Hill showed that only one member of this class admits incompressibility, namely that corresponding to logarithmic strain. The special Hill's-inequality (which we will call KSTS-M$^+$ for reasons which become clear later) asserts in the hyperelastic case that the strain energy should be a convex function of logarithmic strain [179] and Hill argued that this inequality is the most suitable for compressible solids. Šilhavý [226] remarks that Hill's inequality is, up to date, not found to be in conflict with experimental facts.

The Hencky strain tensor appears also in much more diverse fields, such as image registration [188, 189] and relativistic elastomechanics [126]. Extensions to the anisotropic hyperelastic response based on the Hencky-logarithmic strain were investigated, e.g., in [66, 67, 90].

Let us now summarize some well-known unique features of the quadratic Hencky strain energy $W_{\rm H}$ based exclusively on the natural strain tensor $\log U$:

$\oplus_1$ The two isotropic Lamé constants $(\mu, \lambda) \rightsquigarrow (\mu, \kappa)$ (or (E, ν)), the shear modulus, the bulk modulus and the second Lamé constant, are determined in the infinitesimal strain regime, but the model based on the energy $W_{\rm H}(F)$ can well describe the nonlinear deformation response for moderate principal stretches $\lambda_i \in (0.7, 1.4)$ (see [5, 6, 34]). Of course, for a particular material, one may always get agreement with (a finite number of) experiments to any desired accuracy for another constitutive law with more adjustable parameters, e.g., Ogden's strain energy [182].

$\oplus_2$ The Hencky model outperforms other well known nonlinear elasticity models with equally few constitutive parameters, like Neo-Hooke or Mooney-Rivlin type elastic materials [48, 158, 182, 200] in the above-mentioned strain range.

$\oplus_3$ The geometrically nonlinear Poynting effect (a cylindrical bar of steel, copper, rubber or brass lengthens in torsion proportional to the square of the twist) is correctly described [6, 23, 28, 41, 62, 63, 187].

$\oplus_4$ $W_{\rm H}$ has the correct behavior for extreme strains in the sense that $W(F_e) \to \infty$ as $\det F_e \to 0$ and, likewise, $W(F_e) \to \infty$ as $\det F_e \to \infty$.

$\oplus_5$ The Hencky strain tensor $\log U$ puts extension and contraction on the same footing, its principal values vary from $-\infty$ to ∞, whereas those of $C = F^T F$ or $B = F F^T$ vary from 0 to ∞ and those of $C - \mathbb{1}$ vary from -1 to ∞.

---

used as "equivalent strain" in order to represent the degree of plastic deformation [184, 185]. Its use for severe shearing has been questioned in [224]. In our opinion the problematic issue is not the logarithmic measure itself, but its degenerate (sublinear) growth behavior for large strains. The opposing views may be reconciled by using $e^{\| \operatorname{dev}_3 \log U \|}$ as "exponentiated equivalent strain" measure.

[7]I.e., an amorphous metal which is very nearly isotropic with superior elastic deformability up to 1–2 % distortional strain, but which shows no ductility, in contrast to polycrystalline metals which typically show elastic strains up only to 0.1–0.2 %. Recently, Murphy [163] (see also [266]) has postulated a linear Cauchy stress-strain relation for some strain measure and gets as well $W_{\rm H}$ as a preferred solution. His corresponding "strain measure" E is then $E := \frac{1}{\det V} \cdot \log V$, so that $\sigma = 2\mu E + \lambda \operatorname{tr}(E) \cdot \mathbb{1}$, which is Hencky's relation in disguise. However, $V \mapsto E(V)$ is not invertible, thus E does not really qualify as a strain measure.

[8]Tarantola noted [245, page 15] that "Cauchy originally defined the strain as $E = \frac{1}{2}(C - \mathbb{1})$, but many lines of thought suggest that this was just a guess, that, in reality is just the first order approximation to the more proper definition $E = \log \sqrt{C} = \frac{1}{2}(C - \mathbb{1}) - \frac{1}{4}(C - \mathbb{1})^2 + \dots$, i.e., in reality, $E = \log U = (U - \mathbb{1}) - \frac{1}{2}(U - \mathbb{1})^2 + \dots$".

Fig. 1 The generic infinitesimal strain nonlinearity and second-order behavior in agreement with Bell's observation [28, 123]: decreasing elasticity modulus in tension, increasing modulus in compression
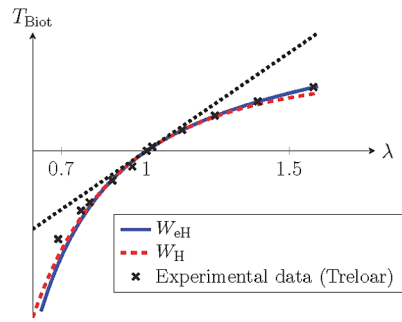
$\oplus_6$ The Hencky strain defines a strictly monotone primary matrix function [121, 176, 178], i.e.,

$$\langle \log U_1 - \log U_2, U_1 - U_2 \rangle > 0 \quad \forall U_1, U_2 \in \mathrm{PSym}(3), \ U_1 \neq U_2, \tag{1.8}$$

even for non-coaxial arguments U_1, U_2.

$\oplus_7$ Tension and compression are treated equivalently: $W_{\mathrm{H}}(F) = W_{\mathrm{H}}(F^{-1})$, i.e., invariance w.r.t. the Lagrangian or Eulerian description. Both the incompressible and compressible versions of J_2-finite strain deformation theory [116] usually assume identical true-stress-true-strain relations in tension and compression.

$\oplus_8$ The linear and second-order behavior of W_{H} is in agreement with Bell's experimental observations [28], i.e., in general, under small strain conditions the instantaneous elastic modulus E decreases for tension and increases in the case of compression (cf. Fig. 1).

$\oplus_9$ True strain for equivalent amounts of deformation in tension and compression is equal except for the sign: $\log V = -\log V^{-1}$.

$\oplus_{10}$ The Eulerian strain tensor $\log V$ (and the Lagrangian strain tensor $\log U$) is additive for coaxial stretches, i.e., $\log(V_1 V_2) = \log V_1 + \log V_2$ for $V_1 V_2 = V_2 V_1$. This implies the superposition principle for the Kirchhoff stress τ_{H} for coaxial strains [27, 176].

$\oplus_{11}$ For incompressibility (rubber for instance [104, 114, 115]), only one parameter, the shear (distortional) modulus $\mu = \frac{E}{3}$, suffices, where $E = \frac{\mu(2\mu+3\lambda)}{\lambda+\mu}$ is Young's modulus.

$\oplus_{12}$ The Hencky strain tensor $\log U$ has the advantage that it additively separates dilatation from pure distortion [52, 196–199]; there is an exact volumetric-isochoric decoupling by the properties of the logarithmic strain tensor:

$$\log \frac{U}{\det U^{1/n}} = \log\big[U \cdot (\det U)^{-1/n}\big] = \log U + \log\big[(\det U)^{-1/n} \cdot \mathbb{1}\big]$$

$$= \log U - \frac{1}{n}(\log \det U) \cdot \mathbb{1} = \log U - \frac{1}{n} \mathrm{tr}(\log U) \cdot \mathbb{1} = \mathrm{dev}_n \log U.$$

Among all finite strain measures from the Seth-Hill family [109, 223], only the spherical and deviatoric parts of the Hencky strain quadratic energy can additively separate the volumetric and the isochoric deformation [7, 114, 209].

$\oplus_{13}$ The volumetric expression $[\mathrm{tr}(\log U)]^2 = (\log \det U)^2$ has been motivated independently in [161, 192, 244] and found to be superior in describing the pressure-volume equation of state (EOS) for geomaterials under extreme pressure (see Sect. 3.5).

$\oplus_{14}$ The incompressibility condition $\det F = 1$ is the simple statement $\mathrm{tr}(\log U) = 0$.

Fig. 2 Nominal stress obtained from the exponentiated Hencky energy W_{eH} and the classical Hencky energy W_H for uniaxial deformation. Loss of monotonicity beyond e for W_H
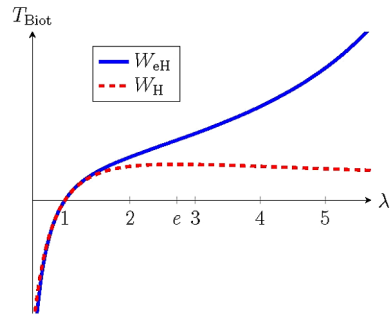
$\oplus_{15}$ For the Hencky energy W_H, uniaxial tension leads to uniaxial lateral contraction and a planar pure Cauchy shear stress produces biaxial pure shear strain [256], similar as in linear elasticity (see Fig. 2 and also Sect. 4), i.e., planar pure shear stress $\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathrm{tr}(\sigma) = 0$ corresponds to isochoric planar stretch $V = \begin{pmatrix} V_{11} & V_{12} & 0 \\ V_{12} & V_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\det V = 1$. For Poisson's number $\nu = 1/2$ exact incompressibility follows and for $\nu = 0$ there is no lateral contraction in uniaxial tension, exactly as in linear elasticity [256].

$\oplus_{16}$ The Hencky energy W_H has constant nonlinear Poisson's ratio $\widehat{\nu} = -\frac{(\log V)_{22}}{(\log V)_{11}} = \nu$ as in linear elasticity and $\lambda_2 = \lambda_1^{-\nu}$ in uniaxial extension [256].

$\oplus_{17}$ If $\Psi(\exp(S)) := W(S)$, then for isotropic response $D_S W(S) = D\Psi(\exp(S)) \cdot \exp(S)$ (see [130, 210, 256, 257]). Thus, $2S_2(C) = D_C \Psi(C) = D_{\log C} W(\log C) \cdot C^{-1}$, while $D_C[\log C].H = C^{-1} \cdot H$ is not true in general. Therefore $\tau = D_{\log V} W(\log V)$, $\sigma = \frac{1}{\det V} \cdot \tau = \frac{1}{\det V} \cdot D_{\log V} W(\log V)$, see Appendix A.2. Using this formula, the algorithmic tangent $D_F^2[W(F)].(H, H)$ for the isotropic Hencky energy in finite element simulations can be analyzed with knowledge of only the first Fréchet-derivative $D_C[\log C].H$ (see [119]).

$\oplus_{18}$ The Kirchhoff stress τ is conjugate to the strain measure $\log V$ [112, 113, 133, 178, 208, 214, 215, 257], where $V = \sqrt{FF^T}$ is the left stretch tensor, i.e., $\langle \tau, \frac{d}{dt} \log V \rangle = \det V \cdot \langle \sigma, D \rangle$ is equal to the power per unit volume element in the reference configuration. Here, D is the strain rate tensor $D = \mathrm{sym} L = \mathrm{sym}(\dot{F} F^{-1})$.

$\oplus_{19}$ Contrary to the arbitrary number of possible strain tensors in the Lagrangian setting, there is only one strain rate tensor D in the Eulerian setting. In the one dimensional case,[9] the logarithmic strain tensor $\log V$ is equal to the integrated strain rate. More generally,[10] $\frac{d}{dt}[\log V(t)] = D(t)$ for any coaxial stretch family $V(t)$.

$\oplus_{20}$ The logarithmic strain possesses certain intrinsic, far-reaching properties that also suggest its favored position among all possible strain measures: the Eulerian logarithmic strain $\log V$ is the unique strain measure whose corotational rate (associated with the so-called logarithmic spin) is the strain rate tensor D. In other words, the strain rate tensor is the co-rotational rate of the Hencky strain tensor associated with the logarithmic spin tensor. Such a result has been introduced by Reinhardt and Dubey [195] as D-rate and by Xiao et al. [262, 265] as log-rate (see also [179, 261, 264]). This is consistent with Truesdell's rate type concept of hypoelasticity based on a unique loga-

---

[9]In the one dimensional case $\varphi(x_1, t) = (\varphi_1(x_1, t), x_2, x_3)^T \Rightarrow F = \nabla \varphi = \mathrm{diag}(\varphi_{1,x_1}, 1, 1) \Rightarrow D = \mathrm{sym}(\dot{F} F^{-1}) = \mathrm{diag}(\frac{\dot{\varphi}_{1,x_1}}{\varphi_{1,x_1}}, 0, 0)$ and $\int_0^t \frac{\dot{\varphi}_{1,x_1}}{\varphi_{1,x_1}} ds = \log |\varphi_{1,x_1}| + C \cong \log U$.

[10]Computing the rates $\frac{d}{dt} \log U$ is more complicated because, in addition to the principal strains being a function of time, the principal directions also change in time [69, 93, 112, 120].

rithmic strain rate [93, 147, 263]. We need to emphasize that, contrary to hyperelastic models, hypo-elastic models [146, 160] ignore the potential character of the energy. Otherwise they are simply the hyperelastic models rewritten in a suitable incremental form. In case of the logarithmic rate, the hypo-elastic model integrates exactly to the hyperelastic quadratic Hencky model.

$\oplus_{21}$ The quadratic Hencky energy W_H satisfies the Baker-Ericksen (BE) inequalities everywhere, see Sect. 2.1 later in this paper.

$\oplus_{22}$ The Cauchy stress $\sigma = \sigma(\log V)$ induces an invertible true-stress-true-strain relation up to $\det F \le e$ [256, 257].

$\oplus_{23}$ The Kirchhoff stress $\tau = \tau(\log V)$ is invertible.

$\oplus_{24}$ The quadratic Hencky energy W_H satisfies Hill's inequality (KSTS-M$^+$) everywhere, i.e., the corresponding Kirchhoff stress $\tau_H = (\det F) \cdot \sigma = D_{\log V} W_H(\log V)$ is a monotone function of the logarithmic strain tensor $\log V$ and W_H is a convex function of $\log V$.

$\oplus_{25}$ The Kirchhoff stress τ_H has the symmetry property $\tau_H(V^{-1}) = -\tau_H(V)$. In fact, this relation is true whenever the energy satisfies the tension-compression symmetry.

$\oplus_{26}$ Since $\log B = \log V^2 = 2 \log V$, there is no need to compute the polar decomposition [119, 175] in order to evaluate $\log V$.

$\oplus_{27}$ There is a representation of $\| \operatorname{dev}_3 \log V \|^2$ and $[\operatorname{tr}(\log V)]^2$ in terms of principal invariants of V available [69, 77]: $\log V = \alpha_0 \mathbb{1} + \alpha_1 V + \alpha_2 V^2 = \beta_0 \mathbb{1} + \beta_1 V + \beta_{-1} V^{-1}$, $\alpha_h = \alpha_h(i_1, i_2, i_3)$, $\beta_r = \beta_r(i_1, i_2, i_3)$, $i_h = i_h(V)$, $h = 1, 2, 3$, $r = -1, 0, 1$. Moreover, it is always possible to express the strain energy terms via its representation in principal stretches from which we may infer, via Cardano's formula, a representation in terms of the principal invariants of B, i.e., $W_H = \widetilde{W}_H(i_1(B), i_2(B), i_3(B))$ [262]. Otherwise, calculation of $\log V$ needs diagonalization and determination of the principal axes. Then

$$\log V = Q^T \begin{pmatrix} \log \lambda_1 & 0 & 0 \\ 0 & \log \lambda_2 & 0 \\ 0 & 0 & \log \lambda_3 \end{pmatrix} Q \quad \text{for} \quad V = Q^T \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} Q$$

and $Q \in \mathrm{SO}(3)$.

$\oplus_{28}$ There are efficient methods for the explicit evaluation of the derivatives of the logarithm of an arbitrary tensor [2, 120].

$\oplus_{29}$ The use of the logarithmic strain tensor $\log U$ leads to simple additive structures in algorithmic computational elasto-plasticity theory [3, 190, 203, 210, 235, 259].

For these reasons the quadratic Hencky model is used in theoretical investigations and in physical applications [8, 39, 40, 77, 83, 86–88, 148, 191]. We observe also a renewed interest in this class of isotropic slightly compressible hyperelastic solids originally proposed by Hencky [98, 100–103]. The strain energy W_H is also often used in commercial FEM-codes.

However, the quadratic Hencky energy has some serious shortcomings:

$\ominus_1$ Beyond $\det F \le e$, the Hencky energy W_H leads to no globally invertible Cauchy stress-logarithmic strain relation and the possibility for multiple symmetric homogeneous bifurcations may arise [121, page 48], see also [252, page 185], [157].

$\ominus_2$ The Cauchy stress tensor is degenerate in the sense that $\sigma_H \nrightarrow +\infty$ for $V \to +\infty$ and there are Cauchy stress distributions which cannot be reached by the constitutive law, i.e., $V \mapsto \sigma(V)$ is not surjective.

$\ominus_3$ The energy W_H does not satisfy the pressure-compression (PC) inequality (this is related to the non-convexity of $\det F \mapsto (\log \det F)^2$ for $\det F > e$).

$\ominus_4$ One may not guarantee real wave speeds over the entire deformation range [39, 116, 165]. Therefore, W_H is not quasiconvex (weakly lower semicontinuous) and not Legendre-Hadamard (LH)-elliptic (rank-one convex).

$\ominus_5$ The tension-extension (TE) inequalities (separate convexity) are not satisfied (see Proposition 5.9).

$\ominus_6$ The quadratic Hencky energy W_H is not coercive, i.e., an estimate of the type

$$W_H(F) \geq C_1 \|F\|^q - C_2, \quad q \geq 1, \ C_1, C_2 > 0$$

is not possible, since W_H growths only sublinearly.

$\ominus_7$ The true-stress-stretch invertibility (TSS-I) does not hold true everywhere.

These points being more or less well-known, it is clear that there cannot exist a general mathematical well-posedness result for the quadratic Hencky model W_H. Of course, in the vicinity of the stress free reference configuration, an existence proof for small loads based on the implicit function theorem will always be possible [48]. All in all, however, the status of Hencky's quadratic energy, despite its many attractive features, is thus put into doubt.

For sufficiently regular energies, Legendre-Hadamard ellipticity on $\mathrm{GL}^+(3)$ (LH-ellipticity, also known as rank-one-convexity)[11] [167, 182, 229–231, 233] is tantamount to

$$\left\langle D_F S_1(F).(\xi \otimes \eta), \xi \otimes \eta \right\rangle = D_F^2 W(F)(\xi \otimes \eta, \xi \otimes \eta) > 0,$$
$$\forall \xi, \eta \in \mathbb{R}^3 \setminus \{0\}, \quad \forall F \in \mathrm{GL}^+(3). \tag{1.10}$$

This condition stems from the study of wave propagation[12] or hyperbolicity of the dynamic problem and it is just what is needed for a good existence and uniqueness theory for linear elastostatics and elastodynamics (see [71, 79, 182, 237]). The failure of ellipticity [144, 234] may be related to the emergence of discontinuous deformation gradients [72, 127, 258]. Strict rank-one convexity in the solution of the boundary value problem is also necessary for the smoothness of weak solutions. While strong ellipticity apparently holds over wide ranges, including buckling, and is physically rather compelling, it is not necessarily universal [140, page 20] (see also [225]). However, from a numerical point of view in finite element simulations, loss of ellipticity manifests itself by a pathological dependence of the computed results on the size and distortion of the finite elements and should therefore be avoided.

Concerning our new formulation, it is clear that, up to moderate strains, for principal stretches $\lambda_i \in (0.7, 1.4)$, our exponentiated Hencky formulation (1.4) is de facto as good

---

[11]Since $\mathrm{GL}^+(3)$ is an open subset of $\mathbb{R}^{3\times3}$, in accordance with [15, page 352] we say that W is rank-one convex on $\mathrm{GL}^+(3)$ if it is convex on all closed line segments in $\mathrm{GL}^+(3)$ with end points differing by a matrix of rank one, i.e.,

$$W\big(F + (1-\theta)\xi \otimes \eta\big) \leq \theta W(F) + (1-\theta)W(F + \xi \otimes \eta), \tag{1.9}$$

for all $F \in \mathrm{GL}^+(3)$, $\theta \in [0, 1]$, and for all $\xi, \eta \in \mathbb{R}^3$, with $F + t\xi \otimes \eta \in \mathrm{GL}^+(3)$ for all $t \in [0, 1]$. In other words, the energy function W is rank-one convex on $\mathrm{GL}^+(3)$ if and only if the function $t \mapsto W(F + t\xi \otimes \eta)$ is convex $\forall \xi, \eta \in \mathbb{R}^3$, on all closed line segments in the set $\{t : F + t\xi \otimes \eta \in \mathrm{GL}^+(3)\}$.

[12]The condition $D_F^2 W(F)(\xi \otimes \xi, \xi \otimes \xi) > 0 \forall \xi \in \mathbb{R}^3 \setminus \{0\}$, i.e., the convexity of $t \mapsto W(F + t\xi \otimes \xi)$ for all $\xi \in \mathbb{R}^3$ with $F + t\xi \otimes \xi \in \mathrm{GL}^+(3)$ for all $t \in [0, 1]$, is a necessary condition for the existence of at least one longitudinal acceleration wave [4, 213, 270].

Fig. 3 $W_{\mathrm{H}}(F)$ is not rank-one convex
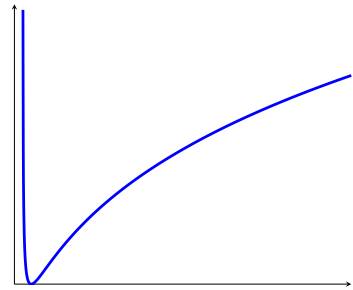
Fig. 4 $W_{\mathrm{eH}}(F)$ is rank-one convex
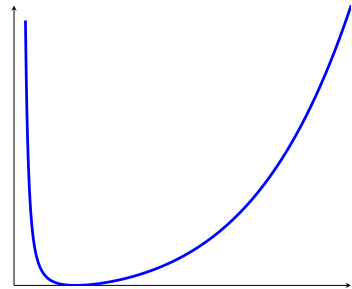
as the quadratic Hencky model W_{H} and in the large strain region it will improve several important features from a mathematical point of view.[13]

Having identified $K_2^2 = \|\operatorname{dev}_n \log U\|^2$ and $K_1^2 = [\operatorname{tr}(\log U)]^2$ as the basic input variables for a nonlinear elasticity formulation, this investigation started by numerically checking the ellipticity conditions for $e^{\|\log U\|^2}$ in the two-dimensional case.[14] In one space dimension it is readily observed that $t \mapsto (\log t)^2$ is not convex, but $t \mapsto e^{(\log t)^2}$ is convex (see Figs. 3 and 4). A similar effect appears for the Hencky energy (1.5): W_{H} is not LH-elliptic [30, 39, 232] (it is of the type given by Fig. 3), but we show that our energy $W_{\mathrm{eH}}(U)$ is LH-elliptic in the two dimensional case (it is of the type given by Fig. 4).

In this paper, then, we prove that the functions $W_{\mathrm{eH}}(F) := \frac{\mu}{k} e^{k\|\operatorname{dev}_n \log U\|^2} + \frac{\kappa}{2k} e^{\widehat{k}(\operatorname{tr}(\log U))^2}$ from the family of energies defined in (1.4) have the following attractive properties beyond those of W_{H}:[15]

---

[13] The domain where the Hencky energy W_{H} is rank-one convex is included in the domain for which the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of U satisfy $\lambda_1^2 \leq e^2 \lambda_2 \lambda_3$, $\lambda_2^2 \leq e^2 \lambda_3 \lambda_1$, $\lambda_3^2 \leq e^2 \lambda_1 \lambda_2$ (see Corollary 5.10). Moreover, this domain is included in the domain defined by $\|\operatorname{dev}_3 \log U\|^2 \leq \frac{4}{3}$. Numerical computations reveal that the exponentiated Hencky energy is rank-one convex in a domain for which $\|\operatorname{dev}_3 \log U\|^2 \leq a$ with $a > \frac{4}{3}$ (see Sect. 6.3).

[14] In this paper we also show that for planar elastostatics $F \mapsto e^{\|\log U\|^2}$ is *not rank-one convex*, a surprising observation which is difficult to obtain, since ellipticity is lost for extremely large principal stretches only.

[15] The idea of considering the exponential function in modelling of nonlinear elasticity is not entirely new. In fact $W(F) = \frac{\mu}{2k}[e^{k(I_1-3)} - 1]$, where $I_1 = \operatorname{tr}(F F^T)$, is a Fung-type model which is often used in the biomechanics literature to describe the nonlinearly elastic response of biological tissues [25, 85]. In the limit $\lim_{k \to 0} \frac{\mu}{2k}[e^{k(I_1-3)} - 1] = \frac{\mu}{2}(I_1 - 3)$, we recover the Neo-Hookean energy for elastic incompressible materials. Another Fung-type energy [25, 85] is $W(F) = \frac{\mu}{2k}[e^{k\|C-\mathbb{1}\|^2} - 1]$.
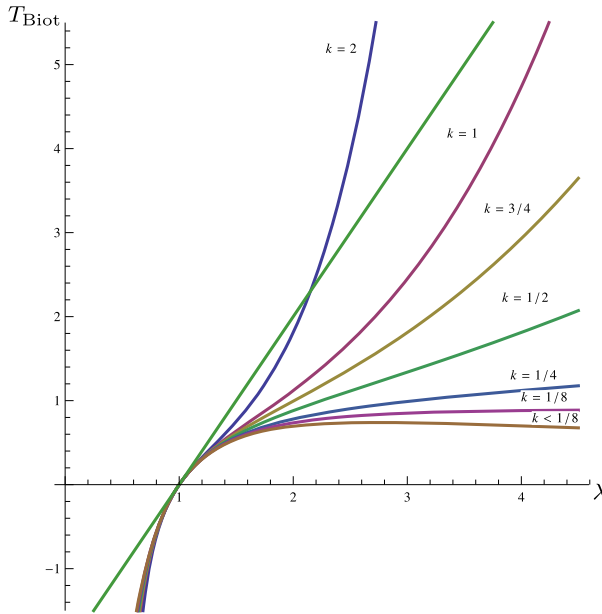
Fig. 5 Qualitative picture of nominal stress response of W_{eH} for uniaxial elongation, for different values of k, typical S-shape entropic elasticity response. We remark that for $k < \frac{1}{8}$ the function $\frac{\mathrm{d}}{\mathrm{d}\lambda}[\frac{\mu}{k}e^{k\log^2\lambda}]$ is not everywhere monotone increasing. The value of k determines the shape of the strain-hardening and strain-softening response, with larger k implying strong strain-hardening. Specific values of k only change the response for large elastic strains $|\log\lambda| > 0.1$. Classical Hencky's response is retrieved for $k \to 0$. Invertibility of the Cauchy stress-stretch response needs $k > \frac{1}{8}$. In the uniaxial case it is intuitively plausible that there should be a bijective constitutive relationship between stress and strain that defines the mechanical properties of an idealized elastic body. For large stretch values λ the function $\frac{\mathrm{d}}{\mathrm{d}\lambda}[\frac{\mu}{k}e^{k\log^2\lambda}]$ is monotone increasing for all $k > 0$. The influence of k on the response is easily understood and we have negligible effect of k under pressure

$\oplus_1$ For nonlinear incompressible material (like rubber) the new energy $\frac{\mu}{k}e^{k\|\mathrm{dev}_3\log U\|^2}$ has only two independent constants, which furthermore have a clear physical meaning, the infinitesimal shear modulus $\mu > 0$ and the distortional strain-stiffening parameter $k > 0$ (see Figs. 5 and 6). For nonlinear (slightly) compressible material, in addition, there is the infinitesimal bulk modulus $\kappa > 0$ and also the volumetric stiffening parameter $\widehat{k} > 0$.

$\oplus_2$ The Cauchy stress tensor satisfies $\sigma_{\mathrm{eH}} \to +\infty$ for $V \to +\infty$.

$\oplus_3$ We have: $\lim_{k,\widehat{k}\to 0}\sigma_{\mathrm{eH}} = \sigma_{\mathrm{H}},\ \lim_{k,\widehat{k}\to 0}\tau_{\mathrm{eH}} = \tau_{\mathrm{H}}$.

$\oplus_4$ At very large stretch ratios the model exhibits the strain stiffening behavior common to many elastomers.

$\oplus_5$ They satisfy the BE-inequalities.

$\oplus_6$ They satisfy the PC-inequalities.

$\oplus_7$ They satisfy the TE-inequalities in the planar case if $k \geq \frac{1}{4}$.

$\oplus_8$ They satisfy the TE-inequalities in the three dimensional case if $k \geq \frac{3}{16}$.

$\oplus_9$ They are rank one convex (LH-elliptic) in the planar case if $k \geq \frac{1}{4}$, in the entire deformation range.

$\oplus_{10}$ The corresponding Kirchhoff stress τ_{eH} has the property: $\tau_{\mathrm{eH}}(V^{-1}) = -\tau_{\mathrm{eH}}(V)$.

$\oplus_{11}$ The true-stress-true-strain invertibility (TSTS-I) holds true everywhere.

Fig. 6 Uniaxial Kirchhoff stress tensor as a function of logarithmic strain for the classical Hencky model W_H and our exponentiated family W_{eH}
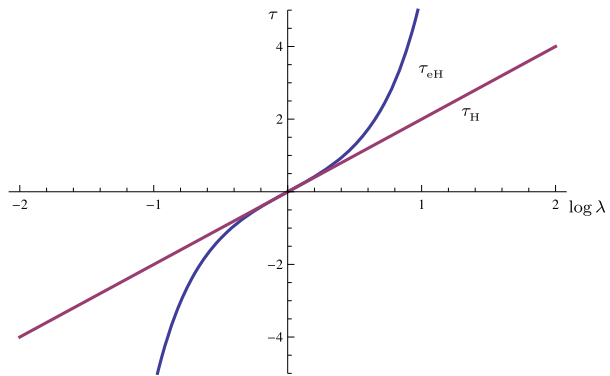
$\oplus_{12}$ The true-stress-stretch invertibility (TSS-I) holds true everywhere.

$\oplus_{13}$ The true-stress-true-strain monotonicity (TSTS-M) is satisfied for bounded distortions.

$\oplus_{14}$ Hill's inequality (KSTS-M$^+$) is satisfied, in the entire deformation range and τ_{eH} is invertible.

$\oplus_{15}$ Planar pure Cauchy shear stress produces biaxial pure shear strain and $\nu = \frac{1}{2}$ corresponds to exact incompressibility.

$\oplus_{16}$ For $n = 3$ among the family W_{eH} there exists a special ($k = \frac{2}{3}\widehat{k}$) three parameter subset such that uniaxial tension leads to no lateral contraction if and only if the Poisson's ratio $\nu = 0$, as in linear elasticity.

$\oplus_{17}$ There is no number $k > 0$ such that W_{eH} is rank one convex everywhere in the three dimensional case, but there is a built-in failure criterion active on extreme distortional strains: our conjecture is that the energy is rank-one convex in the cone-like elastic domain $\mathcal{E}^+(W_{eH}^{iso}, \text{LH}, U, 27) = \{U \in \text{PSym}(3) | \| \text{dev}_3 \log U \|^2 \leq 27\}$.

These results completely settle the status of the quadratic Hencky energy as a useful approximation in plane elasto-statics and lead to new perspectives for the three-dimensional idealized isotropic setting.

The contents of this paper in the order of their appearance are: (i) a further short discussion of the existing literature; (ii) notation; (iii) introduction of general constitutive requirements in idealized nonlinear elasticity; (iv) the invertible true-stress-true-strain relation; (v) rank-one convexity in the two-dimensional case; (vi) domains of rank-one convexity in the three-dimensional case; (vii) summary; (viii) extensive list of references; (ix) appendix.

1.3 Previous Work in the Spirit of Our Investigation

Rougée [207, pages 131, 302] (see also [82, 206] and later extensions by Fiala [73–76]) identifies Hencky's logarithmic strain measure $2 \log U = \log C$ as having (as its Frobenius tensor norm) the length of a geodesic joining two metric states: he endows the set of positive definite matrices PSym(3) (which is not a Lie-group w.r.t. matrix multiplication) with a Riemannian structure (see also [31, 155, 156]). In this case, geodesics joining the identity $\mathbb{1}$ with any metric tensor $C = F^T F$ are simply one-parameter groups $t \mapsto \exp(t \log C)$. This interpretation is fundamentally different from ours given in [170–172] and hinted at in the introduction.

Criscione et al. [52] proposed a new invariant basis for the natural strain $\log U$, which leads to a representation for the Cauchy stress σ as the sum of three response terms that are mutually orthogonal (see also [183]). In fact, Criscione et al. [52, 260] (see also

[65]) consider energies $W_{\text{Crisc}}(K_1, K_2, K_3)$ based on the Hencky-logarithmic strain, where (K_1, K_2, K_3) is a set of invariants for the isotropic case:[16]

$$\begin{cases} \text{"the amount-of-dilatation":} & K_1 = \text{tr}(\log U) = \log \det U = \log \det V = \log \det F, \\ \text{"the magnitude-of-distortion":} & K_2 = \|\operatorname{dev}_3 \log U\| = \|\operatorname{dev}_3 \log V\|, \\ \text{"the mode-of-distortion":} & K_3 = 3\sqrt{6} \det\left(\dfrac{\operatorname{dev}_3 \log U}{\|\operatorname{dev}_3 \log U\|}\right). \end{cases} \tag{1.11}$$

As it turns out, any isotropic energy can also be represented as a function $W_{\text{Crisc}} = W_{\text{Crisc}}(K_1, K_2, K_3)$ of Criscione's invariants (see [51, 52, 111]). In this paper, we use exclusively $|K_1|^2$ (which we may call accordingly the "magnitude-of-dilatation") and the magnitude-of-distortion K_2^2, but with our different geometric motivation.

In [222] some necessary conditions for the LH-ellipticity versus exponential-growth are discussed for energies depending on the Hencky strain $\log U$. In fact, Sendova and Walton [222] have considered the energy W to be a function of $K_2 = \|\operatorname{dev}_3 \log U\|$ and proved that W has to grow at least exponentially as a function of K_2. They note, however, that "constructing conditions that are both necessary and sufficient for strong ellipticity to hold for all deformations still seem[s to be] a daunting task". In [209] Sansour has discussed the multiplicative decomposition of the deformation gradient into its volumetric and isochoric parts and its implications in the case of anisotropy. Sansour's statement for isotropy is already contained in the paper by Richter [196, page 209]. This decomposition problem was studied later in the papers [7, 114]. Gearing and Anand [89] (see also [68, 98]) recently proposed an energy of the form[17]

$$W_{\text{Anand}}(\log U) = \mu(\log \det U) \cdot \|\operatorname{dev}_3 \log U\|^2 + h(\log \det U), \tag{1.12}$$

where $\det U \mapsto h(\log(\det U))$ is highly non-convex. The energy $W_{\text{Anand}}(\log U)$ couples volumetric and distortional response and is based on molecular dynamics simulations. The molecular dynamics simulation[18] is not in contradiction with an increasing generalized shear modulus μ as $\det U \to 0$, see [97, 98].

The Baker-Ericksen (BE) inequalities express the requirement that the greater principal Cauchy stress should occur in the direction of the greater principle stretch, while the tension-extension (TE) inequalities demand that each principal stress is a strictly increasing function of the corresponding principal stretch. The BE-inequalities and TE-inequalities arise in connection with propagation of waves in principal direction of strain [253]. The strong ellipticity condition for hyperelastic materials [267] was studied in [10, 110, 127, 213, 258], but the complete study seems to be presented first in Ogden's Ph.D.-thesis [180]. For an incompressible hyperelastic material corresponding conditions were given in [212]. A family of universal solutions in plane elastostatics for the quadratic Hencky model is obtained in [9].

A stronger constitutive requirement than rank-one convexity is Ball's-polyconvexity condition [14, 15]. A free energy function $W(F)$ is called polyconvex if and only if it is expressible in the form $W(F) = P(F, \operatorname{Cof} F, \det F)$, $P : \mathbb{R}^{19} \to \mathbb{R}$, where $P(\cdot, \cdot, \cdot)$ is convex.

---

[16]Richter in 1949 [197] already considers the following complete set of isotropic invariants: $K_1 = \text{tr}(\log U)$, $K_2^2 = \text{tr}((\operatorname{dev}_3 \log U)^2)$ and $\text{tr}((\operatorname{dev}_3 \log U)^3)$, see also [139]. A similar list of invariants was used by Lurie [139, page 189]: K_1, K_2 and $\widetilde{K}_3 = \arcsin(K_3)$.

[17]The energy (1.12) does not satisfy the tension-compression symmetry.

[18]The numerical results given by Hennan and Anand [98] correspond to the large volumetric strain range $0.75 \le \det F \le 1.16$ ($-0.3 \le \log \det F \le 0.15$) but small shear strain range $\|\operatorname{dev}_3 \log V\| \le 0.035$.

Polyconvexity implies weak lower semicontinuity, quasiconvexity and rank-one convexity. Quasiconvexity of the energy function W at $\overline{F}$ means that

$$\int_\Omega W(\overline{F} + \nabla\vartheta)dx \geq \int_\Omega W(\overline{F})dx = W(\overline{F}) \cdot |\Omega|,$$

for every bounded open set $\Omega \subset \mathbb{R}^3$ \hspace{2cm} (1.13)

holds for all $\vartheta \in C_0^\infty(\Omega)$ such that $\det(\overline{F} + \nabla\vartheta) > 0$. It implies that the homogeneous solution $\varphi(x) = \overline{F} \cdot x$, $x \in \mathbb{R}^3$ is always a global energy minimizer subject to its own Dirichlet boundary conditions.

In fact, polyconvexity is the cornerstone notion for a proof of the existence of minimizers by the direct methods of the calculus of variations for energy functions satisfying no polynomial growth conditions. This is typically the case in nonlinear elasticity since one has the natural requirement $W(F) \to \infty$ as $\det F \to 0$. Polyconvexity is best understood for isotropic energy functions, but it is not restricted to isotropic response. It was a long standing open question how to extend the notion of polyconvexity in a meaningful way to anisotropic materials [17]. The answer has been provided in a series of papers [18, 19, 70, 96, 144, 166, 216–220]. For isotropic strain energies, the polyconvexity condition in the case of space dimension 2 was conclusively discussed by Rosakis [205] and Šilhavý [227], while the case of arbitrary space dimension was studied by Mielke [152], by Dacorogna and Marcellini [57], Dacorogna and Koshigoe [56] and Dacorogna and Marechal [58].

1.4 Notation

For $a, b \in \mathbb{R}^n$ we let $\langle a, b \rangle_{\mathbb{R}^n}$ denote the scalar product on $\mathbb{R}^n$ with associated vector norm $\|a\|_{\mathbb{R}^n}^2 = \langle a, a \rangle_{\mathbb{R}^n}$. We denote by $\mathbb{R}^{n \times n}$ the set of real $n \times n$ second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{R}^{n \times n}$ is given by $\langle X, Y \rangle_{\mathbb{R}^{n \times n}} = \operatorname{tr}(XY^T)$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X \rangle_{\mathbb{R}^{n \times n}}$. In the following we do not adopt any summation convention and we omit the subscript $\mathbb{R}^{n \times n}$ in writing the Frobenius tensor norm. The identity tensor on $\mathbb{R}^{n \times n}$ will be denoted by $\mathbb{1}$, so that $\operatorname{tr}(X) = \langle X, \mathbb{1} \rangle$. We let $\operatorname{Sym}(n)$ and $\operatorname{PSym}(n)$ denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-group theory, i.e., $\operatorname{GL}(n) := \{X \in \mathbb{R}^{n \times n} | \det X \neq 0\}$ denotes the general linear group, $\operatorname{SL}(n) := \{X \in \operatorname{GL}(n) | \det X = 1\}$, $\operatorname{O}(n) := \{X \in \operatorname{GL}(n) | X^T X = \mathbb{1}\}$, $\operatorname{SO}(n) := \{X \in \operatorname{GL}(n, \mathbb{R}) | X^T X = \mathbb{1}, \det X = 1\}$, $\operatorname{GL}^+(n) := \{X \in \mathbb{R}^{n \times n} | \det X > 0\}$ is the group of invertible matrices with positive determinant, $\mathfrak{so}(3) := \{X \in \mathbb{R}^{3 \times 3} | X^T = -X\}$ is the Lie-algebra of skew symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{R}^{3 \times 3} | \operatorname{tr}(X) = 0\}$ is the Lie-algebra of traceless tensors. Here and in the following the superscript ^{T} is used to denote transposition, and $\operatorname{Cof} A = (\det A)A^{-T}$ is the cofactor of $A \in \operatorname{GL}(n)$. The set of positive real numbers is denoted by $\mathbb{R}_+ := (0, \infty)$, while $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$. For all vectors $\xi, \eta \in \mathbb{R}^3$ we have the (dyadic) tensor product $(\xi \otimes \eta)_{ij} = \xi_i \eta_j$.

Let us consider $W(F)$ to be the strain energy function of an elastic material in which F is the gradient of a deformation from a reference configuration to a configuration in the Euclidean 3-space; $W(F)$ is measured per unit volume of the reference configuration. The domain of $W(\cdot)$ is $\operatorname{GL}^+(n)$. We denote by $C = F^T F$ the right Cauchy-Green strain tensor, by $B = FF^T$ the left Cauchy-Green (or Finger) strain tensor, by U the right stretch tensor, i.e., the unique element of $\operatorname{PSym}(n)$ for which $U^2 = C$ and by V the left stretch tensor, i.e., the unique element of $\operatorname{PSym}(n)$ for which $V^2 = B$. Here, we are only concerned with rotationally symmetric functions (objective and isotropic), i.e., $W(F) = W(Q_1^T F Q_2)$

$\forall F = RU = VR \in \mathrm{GL}^+(n)$, $Q_1, Q_2, R \in \mathrm{SO}(n)$. We define $J = \det F$ and we denote by $S_1 = D_F[W(F)]$ the first Piola-Kirchhoff stress tensor, by $S_2 = F^{-1}S_1 = 2D_C[W(C)]$ the second Piola-Kirchhoff stress tensor, by $\sigma = \frac{1}{J}S_1 F^T$ the Cauchy stress tensor, and by $\tau = J \cdot \sigma$ the Kirchhoff stress tensor. In this paper, the scalar product will be always denoted by $\langle , \rangle$, while "·" will denote the multiplication with scalars or the multiplication of matrices. We will also use "." to denote by $D_F[W(F)].H$ the Fréchet derivative of the function $W(F)$ of a tensor F applied to the tensor-valued increment H and to denote by $D_F^2[W(F)].(H, H)$ the bilinear form induced by the second Fréchet derivative of the function $W(F)$ of a tensor F applied to (H, H).

2 Constitutive Requirements in Idealized Nonlinear Elasticity

2.1 The Baker-Ericksen Inequalities

An "ellipticity criterion" much weaker than the LH-ellipticity criterion (1.10) are the so called Baker-Ericksen (BE) inequalities. The Baker-Ericksen inequalities are arguably an absolutely necessary requirement for reasonable material behavior. Baker and Ericksen [13] considered a unit cube of isotropic elastic material to undergo a pure homogeneous deformation with principal directions parallel to the edges of the cube. They showed that the BE-inequalities are necessary and sufficient for the greater principal Cauchy stress to occur in the direction of the greater principal stretch. For an isotropic material, Rivlin [201] supposed that the unit cube considered by Baker and Ericksen is further subjected to a superposed infinitesimal simple shear with direction of shear parallel to an edge of the deformed cube and plane parallel to one of its faces. Rivlin [201] proved that the BE-inequalities are necessary and sufficient conditions for the incremental shear modulus to be positive. The order relation for Cauchy stresses requested by the cube problem considered by Baker and Ericksen then follows.

Let $\widehat{W} : \mathrm{GL}^+(3) \to \mathbb{R}$ be a function that can be written as a function of the singular values of U via $\widehat{W}(U) = g(\lambda_1, \lambda_2, \lambda_3)$. Then the BE-inequalities express the requirement that [13, 65, 80, 140, 260]:

$$(\sigma_i - \sigma_j)(\lambda_i - \lambda_j) \geq 0, \tag{2.1}$$

where $\sigma_i = \frac{1}{\lambda_1 \lambda_2 \lambda_3} \lambda_i \frac{\partial g}{\partial \lambda_i} = \frac{1}{\lambda_j \lambda_k} \frac{\partial g}{\partial \lambda_i}$, $i \neq j \neq k \neq i$, are the principal Cauchy stresses. Usually, in the literature, the BE-inequalities mean that the above inequalities are strict. In this paper, we prefer to denote these strict inequalities as BE$^+$-inequalities. The BE-inequalities are equivalent (see [140, page 17]) to

$$\frac{\lambda_i \frac{\partial g}{\partial \lambda_i} - \lambda_j \frac{\partial g}{\partial \lambda_j}}{\lambda_i - \lambda_j} \geq 0, \quad \text{for all } \lambda_i, \lambda_j \in \mathbb{R}^+, \ \lambda_i \neq \lambda_j. \tag{2.2}$$

We may also view the BE-inequalities as Cauchy **t**rue-**s**tress-**o**rder-**c**ondition (TS-OC).

2.2 Relation of Baker-Ericksen Inequalities to Other Constitutive Requirements

Marzano has shown [143] that the BE-inequalities are necessary and sufficient conditions for a simple extension (a deformation in which two, but not three, principal stretches are equal) to correspond to simple tension. In [153] it was proved that for a homogeneous isotropic

hyperelastic material subject to a pure Cauchy shear stress (a state of pure shear: $\text{tr}(\sigma) = 0$ [178]) of the form

$$\sigma = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.3}$$

the BE-inequalities are satisfied if and only if the corresponding left Cauchy-Green strain tensor $B = FF^T$ has the representation[19]

$$B = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{33} \end{pmatrix}, \tag{2.4}$$

where $B_{11} + B_{12} > B_{33} > B_{11} - B_{12} > 0$.

In general, there are many different possible ways of expressing the physically plausible requirement (the Drucker postulate) that stresses should increase with increasing stretch or strain[20] [250, 255]:

- TE-inequalities (**t**ension-**e**xtension-inequalities): each principal Cauchy stress is a strictly increasing function of the corresponding principal stretch, i.e., $\frac{\partial \sigma_i}{\partial \lambda_i} > 0$, $i = 1, 2, 3$. Since $\sigma_i = \frac{1}{\lambda_j \lambda_k} \frac{\partial g}{\partial \lambda_i}$, $i \neq j \neq k \neq i$, we obtain $\frac{\partial \sigma_i}{\partial \lambda_i} = \frac{1}{\lambda_j \lambda_k} \frac{\partial^2 g}{\partial \lambda_i^2}$ and the TE-inequalities are equivalent to the **s**eparate **c**onvexity (SC) of the function g, namely $\frac{\partial^2 g}{\partial \lambda_i^2} > 0$, $i = 1, 2, 3$.

- OF-inequalities[21] (**o**rdered-**f**orce-inequalities [253]): the greater principal force $T_i = \sigma_i \lambda_j \lambda_k = \frac{\partial g}{\partial \lambda_i}$, $i \neq j \neq k \neq i$, which is associated with the greater principal stretch is such that

$$(T_i - T_j)(\lambda_i - \lambda_j) \geq 0. \tag{2.5}$$

---

[19] Since $\text{tr}(\sigma) = 0$ one might rather expect the stronger statement $B = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, i.e., $B_{33} = 1$, as well as $\det B = 1$. However, this is not true in general for isotropic energies, e.g., it is not satisfied for Neo-Hooke or Mooney-Rivlin type materials.

[20] In the literature, all these concepts are defined using strict inequalities for $\lambda_i \neq \lambda_j$, $i \neq j$. In this paper these common cases will be denoted by TE^+, OF^+, E^+ and PC^+, respectively.

[21] These inequalities appear also, but not as strict inequalities, in the following theorem:

Theorem 2.1 ([16, Theorem 6.5]) *Let $W : \text{GL}^+(n) \to \mathbb{R}$ be an objective-isotropic function of class C^2 with the representation in terms of the singular values of U via $W(F) = \widehat{W}(U) = g(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Let $F \in \text{GL}^+(n)$ be given with the n-tuple of singular values $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then $D^2 W(F)[H, H] \geq 0$ for every $H \in \mathbb{R}^{n \times n}$ if and only if the following conditions hold simultaneously:*

(i) $\sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial \lambda_i \partial \lambda_j} a_i a_j \geq 0$ *for every* $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ *(convexity of g);*

(ii) *for every $i \neq j$,* $\underbrace{\frac{\frac{\partial g}{\partial \lambda_i} - \frac{\partial g}{\partial \lambda_j}}{\lambda_i - \lambda_j}}_{\text{"OF-inequality"}} \geq 0$ *if $\lambda_i \neq \lambda_j$,* $\frac{\partial^2 g}{\partial \lambda_i^2} - \frac{\partial^2 g}{\partial \lambda_i \partial \lambda_j} \geq 0$ *if $\lambda_i = \lambda_j$.*

(iii) $\frac{\partial g}{\partial \lambda_i} + \frac{\partial g}{\partial \lambda_j} \geq 0$ *for every $i \neq j$.*

Hence, if the function $F \mapsto W(F)$ is convex in $F \in \text{GL}^+(n)$, then the OF-inequalities hold true. However, the convexity of $F \mapsto W(F)$ is physically not acceptable, since it precludes buckling.

The physical meaning of the OF-inequalities is the following: if a block of isotropic material is supposed to be in equilibrium subject to pairs of equal and oppositely directed normal forces acting upon its faces, then the greater stretch will occur in the direction of the greater force. The OF-inequalities are therefore similar to the BE-inequalities, only that principal forces instead of principal stresses are concerned [253, page 158]. We observe that

$$RT_{\text{Biot}}(U)R^T = \tau(V)V^{-1} = J\sigma(V)V^{-1}, \tag{2.6}$$

where $F = RU = VR$, $FR^T = V$, $V^T = RF^T = V$. Hence, the Biot stress tensor T_{Biot} is symmetric [253, page 144] and represents "the principal forces acting in the reference system". Therefore, we may also denote the OF-inequalities as **B**iot **s**tress-**o**rder-**c**ondition (BS-OC). Using nearly incompressible materials like rubber, Ball [14] has described a reasonable situation for which the BE-inequalities are valid, while the OF-condition is violated. This fact was previously proved by Sidoroff [225, page 380].

- Convexity type conditions. The convexity of W as function of F means $D_F^2 W(F)$. $(H, H) > 0$, for all $H \neq 0$, and implies the monotonicity of the Piola-Kirchhoff stress

$$\langle S_1(F+H) - S_1(H), H \rangle > 0, \quad \forall H \neq 0. \tag{2.7}$$

This condition yields unqualified uniqueness of boundary value problems, it excludes therefore buckling and is unphysical [107]. For diagonal deformation gradients, the above convexity condition implies the monotonicity of the Biot stress tensor as a function of stretch (BSS-M$^+$).

- GCN-inequality (**G**eneralized-**C**oleman-**N**oll-inequality) [140, page 18]: The GCN-inequality requires that if $\Lambda \neq \mathbb{1}$ is a positive-definite symmetric matrix (a pure stretch), then the first Piola-Kirchhoff stress $S_1(F)$ satisfies

$$\langle S_1(\Lambda F) - S_1(F), \Lambda F - F \rangle > 0. \tag{2.8}$$

For homogeneous isotropic hyperelastic materials, the GCN-inequality implies strict convexity of $g(\lambda_1, \lambda_2, \lambda_3)$ in all variables, implying convexity of the strain energy in U [14, 140], which is known to be unreasonable [107, 108]. Moreover, the GCN-inequality implies the OF-condition which according to Sidoroff [225, page 380] is inadmissible for compressible materials. In order to circumvent the problems of the GCN-inequality, Sidoroff [225] proposed the condition

$$W(F) = \widehat{g}(\log \lambda_1, \log \lambda_2, \log \lambda_3), \quad \text{where } \widehat{g} \text{ should be strictly convex.} \tag{2.9}$$

This is nothing else than Hill's condition, i.e., our KSTS-M$^+$.

- E-TSS-inequalities ("**e**mpirical"-inequalities): using the general representation of the Cauchy stress tensor for isotropic materials

$$\sigma = \sigma(B) = \beta_0 \mathbb{1} + \beta_1 B + \beta_{-1} B^{-1}, \tag{2.10}$$

where $\beta_0, \beta_1, \beta_{-1}$ are functions depending on the principal invariants of B, $I_1(B) = \text{tr}(B)$, $I_2(B) = \text{tr}(\text{Cof}B)$, $I_3(B) = \det B$, the E-TSS-inequalities require

$$\text{E-TSS:} \quad \beta_0 \leq 0, \quad \beta_1 > 0, \quad \beta_{-1} \leq 0, \tag{2.11}$$

while the strengthened E^+-TSS-inequalities require

$$E^+\text{-TSS:} \quad \beta_0 \le 0, \qquad \beta_1 > 0, \qquad \beta_{-1} < 0. \tag{2.12}$$

Some experimental data seem to support these inequalities in certain bounded deformation ranges [21, 22]. However, no theoretical motivation has been found for the empirical inequalities [26]. The connection of the E-TSS-inequalities with the Poynting effect is discussed in [154]. Batra [22] (see also [21]) proved that the E-TSS-inequalities are sufficient conditions for the simple extension to correspond to simple tension. Batra's result has been improved later by Marzano [143], who proved that the BE-inequalities are necessary and sufficient to have the equivalence between simple extension and simple tension. The BE-inequalities are weaker than the E-TSS-inequalities, because for $\beta_0 \le 0$ the BE-inequalities imply [143] only that

$$\beta_1 > 0. \tag{2.13}$$

Assuming that $\beta_{-1} < 0$, Johnson and Hoger [122] have shown that one may uniquely write

$$B = \psi_0 \mathbb{1} + \psi_1 \sigma + \psi_2 \sigma^2, \tag{2.14}$$

where $\quad \psi_i \;=\; \psi_i(\beta_0(I_1(B), I_2(B), I_3(B)), \beta_1(I_1(B), I_2(B), I_3(B)), \beta_{-1}(I_1(B), I_2(B), I_3(B)), I_3(B)), \; i = 0, 1, 2$. This means that E^+-TSS implies invertibility of the Cauchy stress-stretch relation if $\beta_0, \beta_1, \beta_{-1}$ do not depend on B.

Nothing can be said about the validity of the third inequality from (2.12), beyond their logical relation to the BE and OF inequalities, which may be abbreviated as follows [255]:

$$\text{E-TSS} \Rightarrow \text{BE} \quad \text{and} \quad \text{OF.}$$

While the OF and BE inequalities are equivalent in the linearized theory, in general [254, 255]

$$\text{OF} \nRightarrow \text{BE} \quad \text{and} \quad \text{BE} \nRightarrow \text{OF.}$$

Hence, the empirical inequalities imply the OF-inequality. Since the OF-condition is not a valid assumption in general (see above), the E-TSS-inequalities in general cannot be a valid assumption either. Rivlin [201] pointed out that the OF-conditions do not, in general, provide an appropriate restriction on the strain-energy function for an isotropic elastic material. Hence, OF is in general an independent statement and Rivlin [201] proved that it is unacceptable.

- E-BSS-inequalities: in view of the general representation of the Biot stress tensor for isotropic materials

$$T_{\text{Biot}} = \beta_0 \mathbb{1} + \beta_1 U + \beta_{-1} U^{-1}, \tag{2.15}$$

the E-BSS-inequalities require

$$\beta_0 \le 0, \qquad \beta_1 > 0, \qquad \beta_{-1} \le 0, \tag{2.16}$$

while the E^+-BSS-inequalities require

$$\beta_0 \le 0, \qquad \beta_1 > 0, \qquad \beta_{-1} < 0. \tag{2.17}$$

- IFS (**i**nvertible-**f**orce-**s**tretch relation): the invertibility of the map $(\lambda_1, \lambda_2, \lambda_3) \mapsto T_i(\lambda_1, \lambda_2, \lambda_3)$, where T_i are the principal forces [131, 254, 255]. We remark that if the GCN-inequality holds, then $g(\lambda_1, \lambda_2, \lambda_2)$ is strictly convex and IFS follows. IFS expresses the invertibility of the Biot stress tensor $T_{\text{Biot}}(U)$ [202], since, up to a superposed rotation, the Biot stress tensor defines the principal forces acting in the reference system [253, page 144] (see also [254]), see (2.6). Rivlin contested this condition since for Neo-Hookean incompressible materials different triples of $(\lambda_1, \lambda_2, \lambda_3)$ may correspond to the same T_{Biot} stress tensor [202].
- PC-inequality (**p**ressure-**c**ompression-inequality): the condition that the volume of a compressible isotropic material should be decreased by uniform pressure but increased by uniform tension is expressed by requiring the hydrostatic tension $\sigma = \sigma_1 = \sigma_2 = \sigma_3$ to be a strictly increasing function of the stretch $\lambda = \lambda_1 = \lambda_2 = \lambda_3$, i.e., $\frac{\partial \sigma}{\partial \lambda} \geq 0$.
- TSTS-M$^+$ (Jog and Patil's **t**rue-**s**tress-**t**rue-**s**train **m**onotonicity [121]): the monotonicity of the Cauchy stress tensor as a function of $\log B$ or $\log V$ (see Remark 4.1), i.e.,

$$\big\langle \sigma(\log B_1) - \sigma(\log B_2), \log B_1 - \log B_2 \big\rangle > 0, \quad \forall B_1, B_2 \in \text{PSym}^+(3), \, B_1 \neq B_2. \quad (2.18)$$

- TSTS-I (**t**rue-**s**tress-**t**rue-**s**train-**i**nvertibility): the map $\log B \mapsto \sigma(\log B)$ is invertible (see Sects. 3 and 4).
- TSS-M$^+$ (**t**rue-**s**tress-**s**tretch-**m**onotonicity): the monotonicity of the Cauchy stress tensor as a function of B or V, i.e.,

$$\big\langle \sigma(B_1) - \sigma(B_2), B_1 - B_2 \big\rangle > 0, \quad \forall B_1, B_2 \in \text{PSym}^+(3), \, B_1 \neq B_2. \quad (2.19)$$

- TSS-I (**t**rue-**s**tress-**s**tretch-**i**nvertibility): the map $B \mapsto \sigma(B)$ is invertible [45]. Since $\log : \text{PSym}(n) \to \text{Sym}(n)$ is invertible, TSTS-I and TSS-I are clearly equivalent. However, in order to be more precise we keep both definitions. Truesdell and Moon relate TSS-I with "semi-invertibility" [252]. There, they also implicitly show that the E-TSS-inequalities are not in general sufficient for TSS-I. Johnson and Hoger [122] have shown that E$^+$-TSS-inequalities together with constant coefficients are sufficient for TSS-I (see also [64]). Taking a compressible Neo-Hooke model in the form

$$W_{NH}^{\text{iso}}(F) = \frac{\mu}{2} \Big\langle \frac{B}{\det B^{1/3}} - \mathbb{1}, \mathbb{1} \Big\rangle + \kappa h(\det F), \quad (2.20)$$

which additively separates the isochoric and volumetric contributions it can be shown [91] that $B \mapsto \sigma(B)$ is invertible. Here $h : \mathbb{R}_+ \to \mathbb{R}$ must be a strictly convex function satisfying $\lim_{J \to 0} h'(J) = -\infty$ and $\lim_{J \to \infty} h'(J) = \infty$. For instance, suitable convex functions are $h : \mathbb{R}_+ \to \mathbb{R}$, $h(t) = e^{\log^2 t}$ and $h(t) = t^2 - 2\log t$. Therefore, TSS-I merits further investigation (see the discussion of IFS).
- KSTS-M$^+$ (Hill's **K**irchhoff-**s**tress-**t**rue-**s**train-**m**onotonicity [107]): the monotonicity of the Kirchhoff stress tensor as a function of $\log V$, i.e.,

$$\big\langle \tau(\log V_1) - \tau(\log V_2), \log V_1 - \log V_2 \big\rangle > 0, \quad \forall V_1, V_2 \in \text{PSym}^+(3), \, V_1 \neq V_2. \quad (2.21)$$

In [179] Ogden has proved that the later called Odgen's energy does not satisfy the KSTS-M$^+$ inequality, but it may satisfy KSTS-M$^+$ under some restrictions on deformations confirmed by experiments.
- KSTS-I (**K**irchhoff **s**tress-**t**rue-**s**train-**i**nvertibility): the map $\log V \mapsto \tau(\log V)$ is invertible.

- KSS-M$^+$ (**K**irchhoff **s**tress-**s**tretch **m**onotonicity): the monotonicity of the Kirchhoff stress tensor as a function of V, i.e.,

$$\langle \tau(V_1) - \tau(V_2), V_1 - V_2 \rangle > 0, \quad \forall V_1, V_2 \in \mathrm{PSym}^+(3), V_1 \neq V_2. \tag{2.22}$$

- KSS-I (**K**irchhoff **s**tress-**s**tretch-**i**nvertibility): the map $V \mapsto \tau(V)$ is invertible.
- BSTS-M$^+$ (**B**iot **s**tress-**t**rue **s**train **m**onotonicity): the monotonicity of the Biot stress tensor $T_{\mathrm{Biot}}(U) = R^T S_1(F)$ as a function of $\log B$, i.e.,

$$\langle T_{\mathrm{Biot}}(\log U_1) - T_{\mathrm{Biot}}(\log U_2), \log U_1 - \log U_2 \rangle > 0, \quad \forall U_1, U_2 \in \mathrm{PSym}^+(3), U_1 \neq U_2. \tag{2.23}$$

- BSTS-I (**B**iot **s**tress-**t**rue **s**train-**i**nvertibility): the map $\log U \mapsto T_{\mathrm{Biot}}(\log U)$ is invertible.
- BSS-M$^+$ (**B**iot **s**tress-**s**tretch-**m**onotonicity): the monotonicity of the Biot stress tensor [131] $T_{\mathrm{Biot}}(U) = R^T S_1(U)$ as a function of U, i.e.,

$$\langle T_{\mathrm{Biot}}(U_1) - T_{\mathrm{Biot}}(U_2), U_1 - U_2 \rangle > 0, \quad \forall U_1, U_2 \in \mathrm{PSym}^+(3), U_1 \neq U_2. \tag{2.24}$$

Krawietz [131] has shown that BSS-M$^+$ implies the generalized Colleman-Noll (GCN) inequality. The GCN-inequality in turn is known to be not acceptable from physical grounds [15]. Therefore BSS-M$^+$ is not an admissible requirement in general. However, Ogden [182, page 361] remarks that "there is a good physical reason for supposing that the inequality [(2.24)] holds for real elastic materials, at least for some bounded domain which encloses the stress free origin $U = \mathbb{1}$".

- BSS-I (**B**iot **s**tress-**s**tretch-**i**nvertibility): the map $U \mapsto T_{\mathrm{Biot}}(U)$ is invertible. In [181], Ogden suggested that T_{Biot} should be invertible in the domain of elastic response. However, BSS-I is in fact equivalent to Truesdell's notion IFS and to BSTS-I. This seems to have been overlooked in the literature [181, 182, 201]. In a forthcoming paper we will show that BSS-I excludes bifurcations in Rivlin's cube problem which is not necessarily a problematic feature.

In the following XSTS-M$^+$, XSTS-I, XSS-M$^+$, XSS-I, E-XSS, E$^+$-XSS have the obvious meaning once the stress tensor X is defined. It is easy to see that BE and TE are necessary for rank-one convexity (see Theorem 5.1), i.e.,

$$\text{LH-ellipticity} \quad \Rightarrow \quad \text{BE and TE.}$$

Moreover, because the constitutive inequalities are indifferent to superposed rotations, we have

$$\text{BSTS-M}^+ \quad \Rightarrow \quad \text{BSTS-I} \quad \Leftrightarrow \quad \text{BSS-I} \quad \Leftrightarrow \quad \text{IFS.}$$

In Fig. 7, we give a diagram showing the relation between some of the introduced constitutive requirements.

The KSTS-M$^+$ condition does not exclude loss of rank-one convexity (consider, e.g., the quadratic Hencky energy) but it is also in principle not in conflict with rank-one convexity. In order to prove this fact we consider a special Ciarlet-Geymonat energy (linear Poisson's ratio $\nu = 0$)

$$W_{\mathrm{CG}}^{\nu=0}(F) = \frac{\mu}{2} \big[\|F\|^2 - 2\log(\det F) - 3 \big]. \tag{2.25}$$
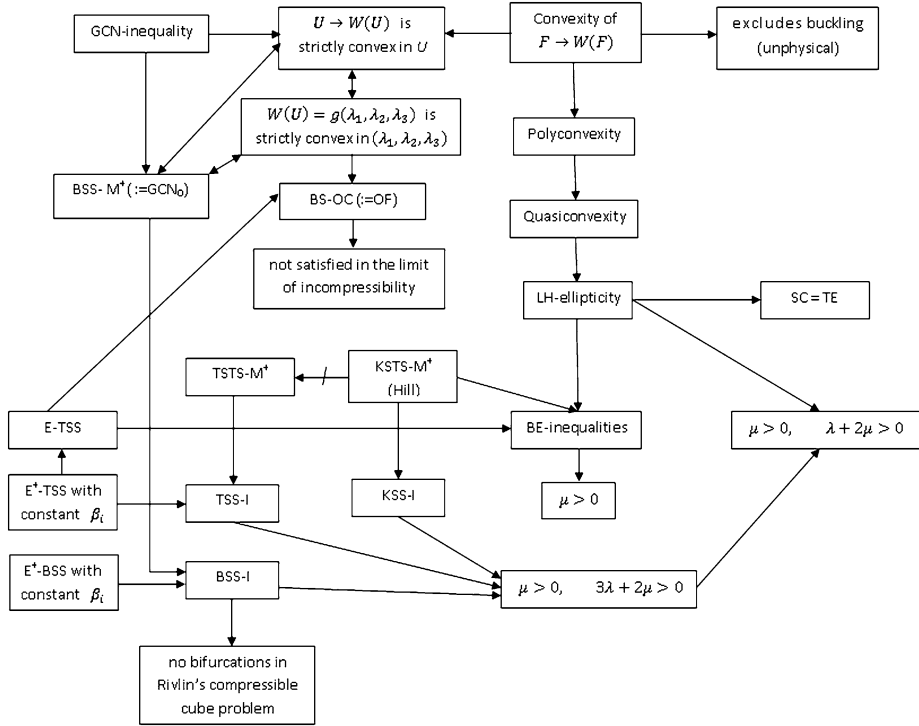
Fig. 7 Relations between some constitutive requirements in idealized isotropic compressible nonlinear elasticity. Whether TSTS-M$^+$ implies KSTS-M$^+$ is not clear

This uni-constant compressible Neo-Hooke energy was considered in [134] and it has been speculated that it has some advantageous properties [92, 134]. The energy $W_{CG}^{v=0}$ is LH-elliptic (it is even polyconvex). In the following we show that the energy $W_{CG}^{v=0}$ satisfies the KSTS-M$^+$ condition. First of all let us remark that

$$W_{CG}^{v=0}(F) = \frac{\mu}{2}\big[\|e^{\log U}\|^2 - 2\,\mathrm{tr}(\log U) - 3\big] = \frac{\mu}{2}\big[\|e^S\|^2 - 2\,\mathrm{tr}(S) - 3\big],$$

where $S = \log U \in \mathrm{Sym}(3)$,

and further

$$W_{CG}^{v=0}(F) = g_1(\mu_1, \mu_2, \mu_3) = \frac{\mu}{2}\big[e^{2\mu_1} + e^{2\mu_2} + e^{2\mu_3} - 2(\mu_1 + \mu_2 + \mu_3) - 3\big], \quad (2.26)$$

where μ_1, μ_2, μ_3 are the eigenvalues of $S = \log U$. The function g_1 being convex and non-decreasing in each variable μ_i, using the Davis-Lewis theorem [35, 59, 135–137] we have that $W_{CG}^{v=0}$ is convex in $S = \log U$. Thus, the energy $W_{CG}^{v=0}$ satisfies the KSTS-M$^+$ condition everywhere. Moreover, the BSS-M$^+$ condition is also satisfied, since

$$W_{CG}^{v=0}(F) = \frac{\mu}{2}\big[\|U\|^2 - 2\log(\det U) - 3\big] \quad (2.27)$$

is convex[22] in U [134]. On the other hand, the Mooney-Rivlin variant of the energy $W_{\mathrm{CG}}^{\nu=0}(F)$,

$$W_{\mathrm{CGMR}}(F) = \alpha_1 \|F\|^2 + \alpha_2 \|\mathrm{Cof}\,F\|^2 - \log(\det F) + e^{(\log \det F)^2} - 3\alpha_1 - 3\alpha_2 - 1 \quad (2.28)$$

is not convex considered as a function of $\log U$. We give the following conjecture:

Conjecture 2.2 *The energy W_{eH} does not satisfy the E^+-TSS-inequalities.*

However, we will show in this paper that:

Remark 2.3 The energy W_{eH} satisfies the TSS-I condition (see Sect. 3).

2.3 Baker-Ericksen Inequalities and Schur Convexity

The BE-inequalities related to the function g can be reformulated in terms of Schur-convexity. The connection between Schur-convexity and the Baker-Ericksen inequalities has been clearly pointed out by Šilhavý in [226, page 310] and in full explicitness in [231, pages 421, 429]. For our purpose here and in order to see the relation between Schur-convexity and BE-inequalities it is sufficient to know the following characterizations of Schur-convex functions (further information on Schur-convexity can be found in [141]):

Proposition 2.4 ([141, page 84]) *Let I be an open interval in $\mathbb{R}$ and let $\ell: I^n \to \mathbb{R}$ be continuously differentiable. Then ℓ is Schur convex if and only if ℓ is symmetric and $(x_i - x_j)(\frac{\partial \ell}{\partial x_i} - \frac{\partial \ell}{\partial x_j}) \geq 0$ for all $i \neq j$.*

Proposition 2.5 ([141, page 97]) *Let I be an open interval in $\mathbb{R}$ and let $\ell: I^n \to \mathbb{R}$. If the function ℓ is symmetric and convex in each pair of arguments, the other arguments being fixed, then ℓ is Schur-convex.*

This notion relates to the BE-inequalities as follows:

Proposition 2.6 ([42, Remark 5.1]) *Schur-convexity of the function*

$$\ell: \mathbb{R}_+^3 \to \mathbb{R}, \qquad \ell(x, y, z) = g(e^x, e^y, e^z) \quad (2.29)$$

is equivalent to the fulfillment of the Baker-Ericksen inequalities in terms of the function g.

This characterization makes the following theorem quickly conceivable.

Theorem 2.7 *Convex isotropic functions of $\log U$ always satisfy the BE-inequalities.*

Proof Convex (isotropic) functions of $\log U$ lead to

$$g(\lambda_1, \lambda_2, \lambda_3) = \ell(\log \lambda_1, \log \lambda_2, \log \lambda_3), \quad (2.30)$$

---

[22]Similarly, as shown in [134] the energy $C \mapsto \frac{\mu}{4}[\|C\|^2 - 2\log(\det C) - 3]$ is convex in C and indeed polyconvex. The convexity in C has been used by Fung [85] to invert the second Piola-Kirchhoff stress tensor $S_2 = 2D_C[W(C)]$.

where ℓ is a convex function. To see this, we apply Proposition 2.6: An energy function given by g satisfies BE if and only if the function $\ell : \mathbb{R}^3_+ \to \mathbb{R}$, $\ell(x, y, z) = g(e^x, e^y, e^z)$ is Schur-convex, hence it is sufficient to show that ℓ is convex in each pair of arguments and symmetric. Convexity follows from

$$g\big(e^x, e^y, e^z\big) = \ell\big(\log e^x, \log e^y, \log e^z\big) = \ell(x, y, z)$$

and convexity of ℓ, while the symmetry is obtained from the isotropy of W. From the Schur-convexity of ℓ it follows that the functions g satisfies the Baker-Ericksen-inequalities.   □

Remark 2.8 (Optimality of logarithmic strain and Baker-Ericksen inequalities) Theorem 2.7 shows that (Schur-)convex dependence on the logarithmic strain tensor somehow is the ideal form for BE. Isotropic functions W of $\log U$ satisfy the BE-inequalities if and only if ℓ from (2.29) is Schur-convex.

In the following remark we gather a few simple convexity properties, some of which can be derived with the results of this section:

Remark 2.9

(i) $e^{\|\operatorname{dev}_n \log U\|^2}$, $e^{\|\log U\|^2}$, $\|\operatorname{dev}_n \log U\|^2$, $\|\log U\|^2$ are all convex functions of $\log U$, i.e., satisfy Hill's inequality (KSTS-M).

(ii) $e^{\|\log U\|^2}$ satisfies BE, because $\|\cdot\|^2$ is convex and hence so is $e^{\|\cdot\|^2}$.

(iii) $e^{\|\operatorname{dev}_n \log U\|^2}$ satisfies the Baker-Ericksen inequalities in any dimension because $\|\operatorname{dev}_n \cdot\|^2$ is convex and $t \mapsto e^t$ is monotone increasing and convex.

(iv) $e^{\|\operatorname{dev}_n \log U\|^2}$, $e^{\|\log U\|^2}$ are SC (separately convex) in λ_i, $i = 1, 2, 3$ (direct calculations) but not convex in $(\lambda_1, \lambda_2, \lambda_3)$. Therefore, $e^{\|\operatorname{dev}_n \log U\|^2}$, $e^{\|\log U\|^2}$ is not convex in U and the energy terms do not satisfy BSS-M$^+$.

(v) $\|\operatorname{dev}_n \log U\|^2$, $\|\log U\|^2$ are not SC (separately convex) [226] in λ_i, $i = 1, 2, 3$ (they do not satisfy the TE-inequalities) and therefore are not rank-one convex [39, 165].

(vi) $W^{\nu=0}_{\text{Becker}}(U) = 2\mu\langle U, \log U - \mathbb{1}\rangle$ (the maximum entropy function) [176] does not satisfy the BE-inequalities but satisfies the TE-inequalities. The formulation of Becker [27] is hyperelastic for Poisson's ratio $\nu = 0$ (exclusively), which is the case for the modelling of cork. Moreover, since $T^{\text{Becker}}_{\text{Biot}}(U) = D_U W^{\nu=0}_{\text{Becker}}(U) = 2\mu \log U$ and since $\log$ is monotone, it follows $\langle T^{\text{Becker}}_{\text{Biot}}(U_1) - T^{\text{Becker}}_{\text{Biot}}(U_2), U_1 - U_2\rangle > 0$ which is BSS-M$^+$ for $\nu = 0$. Hence, it is clear that IFS (BSS-I) hold. Moreover, $T^{\text{Becker}}_{\text{Biot}}$ satisfies BSS-I for arbitrary $-1 < \nu \leq \frac{1}{2}$.

3 The Invertible True-Stress-True-Strain Relation

We consider the exponentiated Hencky energy

$$W_{\text{eH}}(\log V) := \frac{\mu}{k} e^{k\|\operatorname{dev}_3 \log V\|^2} + \frac{\kappa}{2k} e^{\widehat{k}[\operatorname{tr}(\log V)]^2}. \tag{3.1}$$

Here, we first show that the corresponding true-stress-true-strain relation

$$\sigma_{\text{eH}} : \operatorname{Sym}(3) \to \operatorname{Sym}(3), \qquad \sigma_{\text{eH}} = \sigma_{\text{eH}}(\log V)$$

is invertible for the exponentiated energy W_{eH}. Then we prove that a pure planar Cauchy shear stress σ produces a biaxial shear strain for general Hencky type energies. The invertibility of the true-stress-true-strain relation, i.e., of the map $\log V \mapsto \sigma(\log V)$, is denoted by TSTS-I as introduced previously. In the older literature, the requirement of an invertible stress-strain relation is tacitly assumed to always hold generally, even for nonlinear materials response [196].

The Kirchhoff stress tensor corresponding to (3.1) is given [182] by

$$D_{\log V} W_{\mathrm{eH}}(\log V) = \tau_{\mathrm{eH}} = (\det F) \cdot \sigma_{\mathrm{eH}} = e^{\log \det V} \cdot \sigma_{\mathrm{eH}} = e^{\mathrm{tr}(\log V)} \cdot \sigma_{\mathrm{eH}}, \quad (3.2)$$

where σ_{eH} is the Cauchy stress tensor. Hence, the Kirchhoff stress τ_{eH} has the expression

$$\tau_{\mathrm{eH}} = 2\mu e^{k\|\mathrm{dev}_3 \log V\|^2} \cdot \mathrm{dev}_3 \log V + \kappa e^{\widehat{k}[\mathrm{tr}(\log V)]^2} \mathrm{tr}(\log V) \cdot \mathbb{1}, \quad (3.3)$$

while the Cauchy stress tensor is

$$\sigma_{\mathrm{eH}} = e^{-\mathrm{tr}(\log V)} \cdot \tau_{\mathrm{eH}} = 2\mu e^{k\|\mathrm{dev}_3 \log V\|^2 - \mathrm{tr}(\log V)} \cdot \mathrm{dev}_3 \log V$$
$$+ \kappa e^{\widehat{k}[\mathrm{tr}(\log V)]^2 - \mathrm{tr}(\log V)} \mathrm{tr}(\log V) \cdot \mathbb{1}. \quad (3.4)$$

Moreover, by orthogonal projection onto the Lie-algebra $\mathfrak{sl}(3)$ and $\mathbb{R} \cdot \mathbb{1}$, respectively, we find

$$\mathrm{dev}_3 \, \sigma_{\mathrm{eH}} = 2\mu e^{k\|\mathrm{dev}_3 \log V\|^2 - \mathrm{tr}(\log V)} \cdot \mathrm{dev}_3 \log V,$$
$$\mathrm{tr}(\sigma_{\mathrm{eH}}) = 3\kappa e^{\widehat{k}[\mathrm{tr}(\log V)]^2 - \mathrm{tr}(\log V)} \mathrm{tr}(\log V). \quad (3.5)$$

Let us use the notation $x := \mathrm{tr}(\log V)$. In this notation, from (3.5), we have

$$\frac{\mathrm{tr}(\sigma_{\mathrm{eH}})}{3\kappa} = e^{\widehat{k}x^2 - x} x. \quad (3.6)$$

The function $x \mapsto e^{\widehat{k}x^2 - x} x$, $x \in \mathbb{R}$, is strictly monotone if $\widehat{k} > \frac{1}{8}$. Thus, in this case, Eq. (3.6) has a unique solution $x = \mathrm{tr}(\log V)$ as a function of $\mathrm{tr}(\sigma_{\mathrm{eH}})$. We substitute the solution x of Eq. (3.6) in Eq. (3.5)$_1$, to obtain

$$\frac{e^x \cdot \mathrm{dev}_3 \, \sigma_{\mathrm{eH}}}{\mu} = 2e^{k\|\mathrm{dev}_3 \log V\|^2} \cdot \mathrm{dev}_3 \log V, \quad (3.7)$$

and further

$$k \frac{e^x \cdot \mathrm{dev}_3 \, \sigma_{\mathrm{eH}}}{\mu} = D_{\mathrm{dev}_3 \log V} e^{k\|\mathrm{dev}_3 \log V\|^2}. \quad (3.8)$$

Using the substitution $Y = \mathrm{dev}_3 \log V$ we have

$$k \frac{e^x \cdot \mathrm{dev}_3 \, \sigma_{\mathrm{eH}}}{\mu} = D_Y e^{k\|Y\|^2}. \quad (3.9)$$

Because $Y \mapsto e^{k\|Y\|^2}$, $Y \in \mathrm{Sym}(3)$ is uniformly convex with respect to Y, it follows that $D_Y^2 e^{k\|Y\|^2}(H, H) > 0$, for all $Y \in \mathrm{Sym}(3)$, and for all $H \in \mathrm{Sym}(3)$. Hence, the function $Y \mapsto D_Y e^{\|Y\|^2}$ is a strictly monotone tensor function. Therefore, Eq. (3.9) has a unique solution

$Y = \text{dev}_3 \log V$ as a function of $\text{dev}_3 \sigma_{\text{eH}}$ and $x = \text{tr}(\log V)$. Hence, given the Cauchy stress σ_{eH}, we can always uniquely find $\text{tr}(\log V)$ and $\text{dev}_3 \log V$, i.e., $\log V$, such that (3.4) is satisfied. Therefore TSTS-I is true in the three-dimensional case. Simple changes of the computations show that TSTS-I is also true in the two-dimensional case.

Whether well known elastic strain energies like compressible Neo-Hooke, Mooney-Rivlin or Ogden type energies [48] give rise to an overall invertible Cauchy-stress-stretch relation $\sigma = \sigma(B)$ is not clear. This is connected to possible homogeneous bifurcations, e.g., in a hydrostatic loading problem [47, 125].

Let us consider, in the following, three particular cases for our energy W_{eH}: pure Cauchy shear stress, uniaxial tension and simple shear.

3.1 Pure Cauchy Shear Stress

In this subsection we consider the case of pure Cauchy shear stress, i.e.,

$$\sigma_{\text{eH}} = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad 0 = \text{tr}(\sigma_{\text{eH}}) = \text{tr}(\tau_{\text{eH}}). \tag{3.10}$$

We aim to find the corresponding form of the stretch tensor V. From (3.3), by considering the trace on both sides, it follows that in the case of pure shear stress, we must have $\text{tr}(\log V) = 0 \Leftrightarrow \det V = 1$. We need to remark that the conclusion that pure Cauchy shear stresses lead to an incompressible response is not verified, e.g., for Neo-Hooke, Mooney-Rivlin or Ogden-type materials. In our case, however, it remains to solve

$$2\mu e^{k\|\text{dev}_3 \log V\|^2} \text{dev}_3 \log V = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Leftrightarrow \quad 2\mu e^{k\|\log V\|^2} \log V = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.11}$$

Inspired by Vallée's result in [256], a solution of Eq. (3.11) can be found in the form of pure biaxial stretch[23]

$$V = \begin{pmatrix} \cosh \frac{\gamma}{2} & \sinh \frac{\gamma}{2} & 0 \\ \sinh \frac{\gamma}{2} & \cosh \frac{\gamma}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.12}$$

Corresponding to this ansatz for V, we have

$$\log V = \begin{pmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \det V = 1, \tag{3.13}$$

and Eq. (3.11) becomes

$$\sigma_{\text{eH}} = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2\mu e^{k\frac{\gamma^2}{2}} \begin{pmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.14}$$

---

[23]This is suggested by the formula presented in [29, page 736]: $e^{\alpha \cdot \widehat{A}} = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix}$ for $\widehat{A} = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$.

For all $s \in \mathbb{R}$ we always have a solution $\gamma = \gamma(s)$ of the above equation, because $\gamma \mapsto e^{k\frac{\gamma^2}{2}}\frac{\gamma}{2}$ is monotone increasing. Thus, we recover completely the classical statement that in linear elasticity, pure shear stresses (3.10) produces pure biaxial shear strains (3.18), i.e.,

$$\sigma = 2\mu\varepsilon = 2\mu \underbrace{\begin{pmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{``pure infinitesimal shear stress''}}$$

$$\Leftrightarrow \quad \varepsilon = \underbrace{\begin{pmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{``pure infinitesimal shear strain''}}, \qquad \underbrace{\operatorname{tr}(\varepsilon) = 0}_{\text{``linearized incompressibility''}}, \qquad (3.15)$$

where $\varepsilon = \operatorname{sym}\nabla u$. For the finite strain case, this equivalence seems to be true only for Hencky type energies [256].

3.2 Uniaxial Cauchy Tension

Next we consider the case of uniaxial tension

$$\sigma_{\text{eH}} = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (3.16)$$

From (3.4), by projection on the Lie-algebras $\mathfrak{sl}(n)$ and $\mathbb{R} \cdot \mathbb{1}$, we have

$$2\mu e^{k\|\operatorname{dev}_3 \log V\|^2 - \operatorname{tr}(\log V)} \operatorname{dev}_3 \log V = \operatorname{dev}_3 \sigma_{\text{eH}} = \begin{pmatrix} \frac{2}{3}s & 0 & 0 \\ 0 & -\frac{1}{3}s & 0 \\ 0 & 0 & -\frac{1}{3}s \end{pmatrix}, \qquad (3.17)$$

$$3\kappa e^{\widehat{k}[\operatorname{tr}(\log V)]^2 - \operatorname{tr}(\log V)} \operatorname{tr}(\log V) = \operatorname{tr}(\sigma_{\text{eH}}) = s.$$

This means that a suitable ansatz for V is similar to that considered by Vallée [256]

$$V = \begin{pmatrix} e^{a+\frac{1}{3}x} & 0 & 0 \\ 0 & e^{-\frac{1}{2}a+\frac{1}{3}x} & 0 \\ 0 & 0 & e^{-\frac{1}{2}a+\frac{1}{3}x} \end{pmatrix} = e^{\frac{1}{3}x} \begin{pmatrix} e^a & 0 & 0 \\ 0 & e^{-\frac{1}{2}a} & 0 \\ 0 & 0 & e^{-\frac{1}{2}a} \end{pmatrix}. \qquad (3.18)$$

It is easy to compute that, corresponding to this ansatz for V, we have

$$\det V = e^x, \qquad \log V = \begin{pmatrix} a+\frac{1}{3}x & 0 & 0 \\ 0 & -\frac{1}{2}a+\frac{1}{3}x & 0 \\ 0 & 0 & -\frac{1}{2}a+\frac{1}{3}x \end{pmatrix},$$

$$\operatorname{tr}(\log V) = x, \qquad \operatorname{dev}_3 \log V = \begin{pmatrix} a & 0 & 0 \\ 0 & -\frac{1}{2}a & 0 \\ 0 & 0 & -\frac{1}{2}a \end{pmatrix}, \qquad (3.19)$$

$$\|\operatorname{dev}_3 \log V\|^2 = \frac{3}{2}a^2$$

and Eq. (3.17) becomes

$$3\mu e^{k\frac{3}{2}a^2 - x} a = s, \qquad 3\kappa e^{\widehat{k}x^2 - x} x = s. \tag{3.20}$$

In terms of Poisson's ratio $\nu \in (-1, \frac{1}{2})$ and Young's modulus $E > 0$, we have

$$e^{k\frac{3}{2}a^2 - x}\frac{3}{2}a = \frac{1+\nu}{E}s, \qquad e^{\widehat{k}x^2 - x} x = \frac{1 - 2\nu}{E}s. \tag{3.21}$$

For all $s \in \mathbb{R}$ we always have a solution $x = x(s)$ of the second equation and the function $s \mapsto x(s)$ is monotone strictly increasing if $\kappa > 0$ and $\mathrm{sgn}[x(s)] = \mathrm{sgn}[s]$. Having $x(s)$ from $(3.21)_2$, we then find the unique solution $a(s)$ of $(3.21)_1$. Moreover, for $\mu > 0$ the function $s \mapsto a(s)$ is also monotone strictly increasing and $\mathrm{sgn}[a(s)] = \mathrm{sgn}[s]$.

Therefore, the ansatz

$$\log V = \begin{pmatrix} a + \frac{1}{3}x & 0 & 0 \\ 0 & -\frac{1}{2}a + \frac{1}{3}x & 0 \\ 0 & 0 & -\frac{1}{2}a + \frac{1}{3}x \end{pmatrix} \tag{3.22}$$

corresponds to

$$\sigma_{\mathrm{eH}} = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{3}{2}e^{k\frac{3}{2}a^2 - x}\frac{E}{1+\nu} \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.23}$$

In the limit case $\nu = \frac{1}{2}$ (linear incompressibility), we observe that $(3.21)_2$ implies $x = 0$. Therefore

$$\log V|_{\nu=\frac{1}{2}} = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & -\frac{1}{2}\gamma & 0 \\ 0 & 0 & -\frac{1}{2}\gamma \end{pmatrix}, \qquad \underbrace{\det V|_{\nu=\frac{1}{2}} = 1}_{\nu=\frac{1}{2}: \text{ exact incompressibility}}, \qquad e^{k\frac{3}{2}\gamma^2}\gamma = \frac{1}{E}s \tag{3.24}$$

and this corresponds to

$$\sigma_{\mathrm{eH}}|_{\nu=\frac{1}{2}} = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = E e^{k\frac{3}{2}\gamma^2} \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.25}$$

On the other hand, $\sigma_{\mathrm{eH}} = 0$ ($s = 0$) is equivalent with $\log V = 0$ ($x = 0, a = 0$).

In the case $\nu = 0$, the nonlinear system (3.21) becomes

$$e^{k\frac{3}{2}a^2 - x}\frac{3}{2}a = \frac{1}{E}s, \qquad e^{\widehat{k}x^2 - x} x = \frac{1}{E}s, \tag{3.26}$$

which implies $e^{k\frac{3}{2}a^2}a = e^{\widehat{k}x^2}\frac{2}{3}x$. Using the substitution $x = \frac{3}{2}y$, we have $e^{k\frac{3}{2}a^2}a = e^{\widehat{k}\frac{9}{4}y^2}y$. We choose the dimensionless material parameters $k, \widehat{k}$ such that $3\widehat{k} = 2k$ and we further deduce that $x = \frac{3}{2}a$. Thus, with the substitution $\gamma = \frac{3}{2}a$, we deduce

$$\log V|_{\nu=0} = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e^{k\frac{2}{3}\gamma^2 - \gamma}\gamma = \frac{1}{E}s, \tag{3.27}$$

which corresponds to

$$\sigma_{\mathrm{eH}}|_{\nu=0} = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = E e^{k\frac{2}{3}\gamma^2 - \gamma} \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.28}$$

Moreover, if there is no lateral contraction in uniaxial tension in the case $\nu = 0$, then from (3.18) we deduce that we must have $x = \frac{3}{2}a$. On the other hand, for $\nu = 0$, if $x = \frac{3}{2}a$, then using (3.26) we obtain that $3\widehat{k} = 2k$ must hold necessarily.

Thus, we have shown that uniaxial tension produces extension/contraction, as in linear elasticity, since for linear elasticity, using the inverted law $\varepsilon = \frac{1+\nu}{E}\sigma - \frac{\nu}{E}\mathrm{tr}(\sigma) \cdot \mathbb{1}$, we have

$$\sigma = 2\mu\varepsilon + \lambda \mathrm{tr}(\varepsilon) \cdot \mathbb{1} = E \underbrace{\begin{pmatrix} \gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{``uniaxial tension''}} \quad \Leftrightarrow \quad \varepsilon = \underbrace{\begin{pmatrix} \gamma & 0 & 0 \\ 0 & -\nu\gamma & 0 \\ 0 & 0 & -\nu\gamma \end{pmatrix}}_{\text{``extension/lateral contraction''}}, \tag{3.29}$$

where $\varepsilon = \mathrm{sym}\nabla u$. In the limit case $\nu = \frac{1}{2}$, we have $\mathrm{tr}(\varepsilon) = 0$, while for $\nu = 0$ there is no lateral contraction in uniaxial tension as in (3.28). In linear elasticity, the Poisson's ratio is defined by $\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}}$ [193], where the transverse strain ε_{22} and the longitudinal strain ε_{11} are computed in uniaxial extension.

Remark 3.1 (W_{eH} with no lateral contraction for $\nu = 0$) The above formula (3.28) is true if and only if the distortional stiffening parameter k and the volumetric strain stiffening parameter $\widehat{k}$ are such that $3\widehat{k} = 2k$. In this case $\nu = 0$ implies no lateral contraction for the exponentiated Hencky energy (3 parameter energy: ν, E, k)

$$W_{\mathrm{eH}}^{\sharp}(\log V) := \frac{\mu}{k} e^{k\|\mathrm{dev}_3 \log V\|^2} + \frac{3\kappa}{4k} e^{\frac{2}{3}k(\mathrm{tr}(\log V))^2}$$

$$= \frac{1}{2k}\left\{ \frac{E}{1+\nu} e^{k\|\mathrm{dev}_3 \log V\|^2} + \frac{E}{2(1-2\nu)} e^{\frac{2}{3}k[\mathrm{tr}(\log V)]^2} \right\}.$$

3.3 On the Nonlinear Poisson's Ratio

We define the nonlinear Poisson's ratio as negative ratio of the lateral contraction and axial extension measured in the logarithmic strain, i.e., according to (3.19)

$$\widehat{\nu}(s) = -\frac{(\log V)_{22}}{(\log V)_{11}} = \frac{\frac{1}{2}a - \frac{1}{3}x}{a + \frac{1}{3}x}. \tag{3.30}$$

The nonlinear Poisson's ratio [193] is a purely kinematical quantity which can be measured in the simple tension test. In [84, page 75] it is defined as $\widehat{\nu}(s) = -\frac{\lambda_2 - 1}{\lambda_1 - 1}$. The (linear) Poisson's ratio[24] $\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}}$ [193] for many materials is positive and not strain sensitive until

---

[24]In terms of the Young's modulus and the shear modulus ν is given by $\nu = \frac{E}{2\mu} - 1$, while in terms of the Young's modulus and the bulk modulus κ it is given by $\nu = \frac{1}{2} - \frac{E}{6\kappa}$.

nonelastic effects intervene [124, 239]. In view of our definition, we have

$$a\left(\frac{1}{2} - \widehat{v}\right) = \frac{x}{3}(1 + \widehat{v}).$$

(3.31)

Since $\mathrm{sgn}[a(s)] = \mathrm{sgn}[s] = \mathrm{sgn}[x(s)]$ we deduce that $\widehat{v} \in (-\frac{1}{2}, 1)$, which is in concordance with the definition from linear elasticity. From (3.31) we have $a = \frac{2}{3}\frac{1+\widehat{v}}{1-2\widehat{v}}x$. Moreover, the system (3.21) becomes

$$e^{k\frac{2}{3}(\frac{1+\widehat{v}}{1-2\widehat{v}})^2 x^2 - x} x = \frac{1-2\widehat{v}}{1+\widehat{v}}\frac{1+v}{E}s, \qquad e^{\widehat{k}x^2-x}x = \frac{1-2v}{E}s.$$

(3.32)

This system is also equivalent to

$$\left[k\frac{2}{3}\left(\frac{1+\widehat{v}}{1-2\widehat{v}}\right)^2 - \widehat{k}\right]x^2 = \log\left(\frac{1-2\widehat{v}}{1+\widehat{v}}\frac{1+v}{1-2v}\right), \qquad e^{\widehat{k}x^2-x}x = \frac{1-2v}{E}s.$$

(3.33)

In the following we consider the case of the three parameter energy $W_{\mathrm{eH}}^{\sharp}$, i.e., the case $k\frac{2}{3} = \widehat{k}$. In this case we obtain the system

$$\frac{\widehat{v}(2-\widehat{v})}{(1-2\widehat{v})^2}x^2 = \frac{1}{2k}\log\left(\frac{1-2\widehat{v}}{1+\widehat{v}}\frac{1+v}{1-2v}\right), \qquad e^{\widehat{k}x^2-x}x = \frac{1-2v}{E}s.$$

(3.34)

From the above equations we deduce that

$$\widehat{v} > 0 \quad \Leftrightarrow \quad \frac{1+v}{1-2v} > \frac{1+\widehat{v}}{1-2\widehat{v}} \quad \Leftrightarrow \quad v > \widehat{v}.$$

Hence, $\widehat{v} > 0$ implies $v > 0$. On the other hand, if we assume that there is $v > 0$ such that $\widehat{v} < 0$, then we obtain

$$\frac{1+v}{1-2v} < \frac{1+\widehat{v}}{1-2\widehat{v}}.$$

(3.35)

But $v > 0$ implies $\frac{1+v}{1-2v} > 1$, while $\widehat{v} < 0$ implies $\frac{1+\widehat{v}}{1-2\widehat{v}} < 1$. This is in clear contradiction with (3.35). Therefore $\widehat{v} > 0 \Leftrightarrow v > 0$. If $\widehat{v} = 0$, then from (3.34) it results that we have to have $v = 0$ and x is determined only by $e^{\widehat{k}x^2-x}x = \frac{s}{E}$ (see the discussion from Sect. 3.2 about the particular case $v = 0$).

If $\widehat{v} \neq 0$, then, since $\widehat{v} \in (-\frac{1}{2}, 1)$, we deduce that $\widehat{v}$ is given as solution of the following equation if $s > 0$:

$$\sqrt{\frac{\log\left(\frac{1-2\widehat{v}}{1+\widehat{v}}\frac{1+v}{1-2v}\right)}{\widehat{k}\left(\frac{1+\widehat{v}}{1-2\widehat{v}}\right)^2 - \widehat{k}}} \, e^{\frac{\log\left(\frac{1-2\widehat{v}}{1+\widehat{v}}\frac{1+v}{1-2v}\right)}{\left(\frac{1+\widehat{v}}{1-2\widehat{v}}\right)^2 - 1} - \sqrt{\frac{\log\left(\frac{1-2\widehat{v}}{1+\widehat{v}}\frac{1+v}{1-2v}\right)}{\widehat{k}\left(\frac{1+\widehat{v}}{1-2\widehat{v}}\right)^2 - \widehat{k}}}} = (1-2v)\frac{s}{E},$$

(3.36)

while for $s < 0$, $\widehat{v}$ is solution of the equation:

$$\sqrt{\frac{\log\left(\frac{1-2\widehat{v}}{1+\widehat{v}}\frac{1+v}{1-2v}\right)}{\widehat{k}\left(\frac{1+\widehat{v}}{1-2\widehat{v}}\right)^2 - \widehat{k}}} \, e^{\frac{\log\left(\frac{1-2\widehat{v}}{1+\widehat{v}}\frac{1+v}{1-2v}\right)}{\left(\frac{1+\widehat{v}}{1-2\widehat{v}}\right)^2 - 1} + \sqrt{\frac{\log\left(\frac{1-2\widehat{v}}{1+\widehat{v}}\frac{1+v}{1-2v}\right)}{\widehat{k}\left(\frac{1+\widehat{v}}{1-2\widehat{v}}\right)^2 - \widehat{k}}}} = -(1-2v)\frac{s}{E},$$

(3.37)
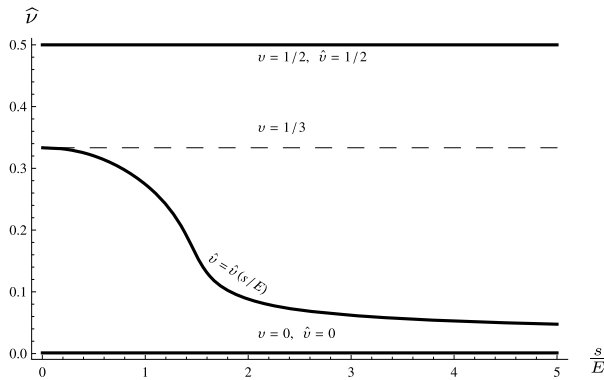
Fig. 8 The nonlinear Poisson's ratio $\widehat{\nu}$ for $\widehat{k} = \frac{1}{6}$ and for the following values of the (linear) Poisson's ratio: $\nu = 0$, $\nu = \frac{1}{3}$ and $\nu = \frac{1}{2}$. For $\nu = 0$ and $\nu = \frac{1}{2}$ the nonlinear Poisson's ratio is equal to the (linear) Poisson's ratio, while for $\nu \in (0, \frac{1}{2})$ the nonlinear Poisson ratio $\widehat{\nu}(\frac{s}{E}) = -\frac{(\log V)_{22}}{(\log V)_{11}}$ approximates the (linear) Poisson's ratio only in a small neighborhood of $\frac{s}{E} = 0$. The graphic of the map $\frac{s}{E} \mapsto \widehat{\nu}(\frac{s}{E})$ is tangent to the line $\widehat{\nu}(0) = \nu$, decreases and it is smaller than ν for non-infinitesimal values of the load parameter s. Moreover, the nonlinear Poisson's ratio $\widehat{\nu}$ remains positive whenever $\nu = \widehat{\nu}(0)$ is positive and $\widehat{\nu} \in (-1, \frac{1}{2})$
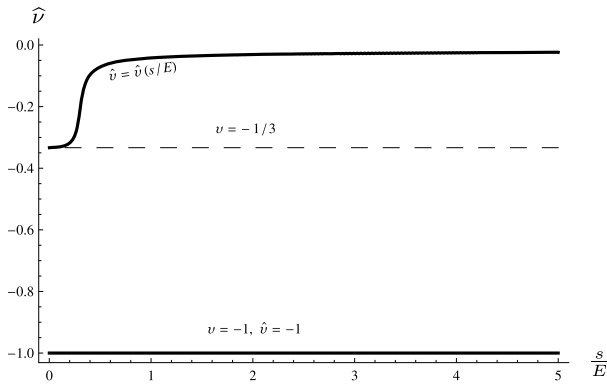

Fig. 9 The variation of the nonlinear Poisson's ratio $\widehat{\nu}$ for $\widehat{k} = \frac{1}{6}$ and negative (linear) Poisson ratio (e.g., auxetic materials). For $\nu = -1$ the nonlinear Poisson's ratio is equal to the (linear) Poisson's ratio, while for $\nu \in (-1, 0)$ the nonlinear Poisson's ratio approximates the (linear) Poisson's ratio only in a small neighborhood of $\frac{s}{E} = 0$. For negative (linear) Poisson's ratio the map $\frac{s}{E} \mapsto \widehat{\nu}(\frac{s}{E})$ is tangent to the line $\widehat{\nu}(0) = \nu$, increases and it is bigger than ν for non-infinitesimal values of the load parameter s. Moreover, the nonlinear Poisson's ratio $\widehat{\nu}$ remains negative whenever $\nu = \widehat{\nu}(0)$ is negative

with x given by the independent equation $e^{\widehat{k}x^2 - x} x = (1 - 2\nu)\frac{s}{E}$. In Figs. 8 and 9 we give the representation of the nonlinear Poisson's ratio $\widehat{\nu}$ as function of $\frac{s}{E}$, corresponding to different values of the (linear) Poisson's ratio. We also represent (see Fig. 10) the influence of the parameter $\widehat{k}$ on the nonlinear Poisson's ratio $\widehat{\nu}$.

We notice three particular cases. If $\nu = -1$, then it follows from $(3.21)_1$ that $a = 0$ and further from (3.30) that $\widehat{\nu} = -1$. If $\nu = \frac{1}{2}$, then $(3.21)_2$ leads to $x = 0$, while (3.30) implies $\widehat{\nu} = \frac{1}{2}$. Moreover, if $\nu = 0$, then (3.21) shows $x = \frac{3}{2}a$. Therefore, from (3.30) we obtain $\widehat{\nu} = 0$.

Fig. 10 Influence of the parameter $\widehat{k}$ on the nonlinear Poisson's ratio $\widehat{\nu}$. For $\widehat{k} < \frac{1}{8}$ the map $\frac{s}{E} \mapsto \widehat{\nu}(\frac{s}{E})$ is not well-defined. For $k \geq \frac{1}{8}$ the map $\frac{s}{E} \mapsto \widehat{\nu}(\frac{s}{E})$ is bijective
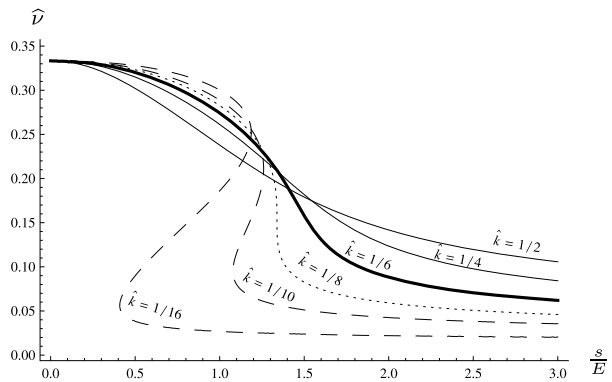
3.4 Cauchy Stress in Simple Shear for W_{H} and W_{eH}

Consider a simple glide deformation of the form

$$F = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.38}$$

with $\gamma > 0$. Then the polar decomposition of $F = R \cdot U = V \cdot R$ into the right Biot stretch tensor $U = \sqrt{F^T F}$ of the deformation and the orthogonal polar factor R is given by

$$U = \frac{1}{\sqrt{\gamma^2 + 4}} \begin{pmatrix} 2 & \gamma & 0 \\ \gamma & \gamma^2 + 2 & 0 \\ 0 & 0 & \sqrt{\gamma^2 + 4} \end{pmatrix}, \quad R = \frac{1}{\sqrt{\gamma^2 + 4}} \begin{pmatrix} 2 & \gamma & 0 \\ -\gamma & 2 & 0 \\ 0 & 0 & \sqrt{\gamma^2 + 4} \end{pmatrix}. \tag{3.39}$$

Further, U can be orthogonally diagonalized to

$$U = Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(\sqrt{\gamma^2 + 4} + \gamma) & 0 \\ 0 & 0 & \frac{1}{2}(\sqrt{\gamma^2 + 4} - \gamma) \end{pmatrix} Q^T = Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \frac{1}{\lambda_1} \end{pmatrix} Q^T, \tag{3.40}$$

where

$$Q = \begin{pmatrix} \dfrac{2}{\sqrt{\gamma^2 + 4} + \gamma} & \dfrac{-2}{\sqrt{\gamma^2 + 4} - \gamma} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.41}$$

and $\lambda_1 = \frac{1}{2}(\sqrt{\gamma^2 + 4} + \gamma)$ denotes the first eigenvalue of U. Hence, the principal logarithm of U is

$$\log U = Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & \log \lambda_1 & 0 \\ 0 & 0 & -\log \lambda_1 \end{pmatrix} Q^T = \frac{1}{\sqrt{\gamma^2 + 4}} \begin{pmatrix} -\gamma \log \lambda_1 & 2 \log \lambda_1 & 0 \\ 2 \log \lambda_1 & \gamma \log \lambda_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.42}$$

while the principal logarithm of V is given by

$$
\begin{aligned}
\log V = R \cdot \log U \cdot R^{-1} &= \frac{1}{\sqrt{\gamma^2+4}} R \cdot \begin{pmatrix} -\gamma \log \lambda_1 & 2 \log \lambda_1 & 0 \\ 2 \log \lambda_1 & \gamma \log \lambda_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot R^{-1} \\
&= \frac{1}{(\gamma^2+4)\sqrt{\gamma^2+4}} \begin{pmatrix} 2 & \gamma & 0 \\ -\gamma & 2 & 0 \\ 0 & 0 & \sqrt{\gamma^2+4} \end{pmatrix} \cdot \begin{pmatrix} -\gamma \log \lambda_1 & 2 \log \lambda_1 & 0 \\ 2 \log \lambda_1 & \gamma \log \lambda_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} 2 & -\gamma & 0 \\ \gamma & 2 & 0 \\ 0 & 0 & \sqrt{\gamma^2+4} \end{pmatrix} \\
&= \frac{1}{\sqrt{\gamma^2+4}} \begin{pmatrix} 0 & \log \lambda_1 & 0 \\ \log \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & -\gamma & 0 \\ \gamma & 2 & 0 \\ 0 & 0 & \sqrt{\gamma^2+4} \end{pmatrix} \\
&= \frac{\log \lambda_1}{\sqrt{\gamma^2+4}} \begin{pmatrix} \gamma & 2 & 0 \\ 2 & -\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{3.43}
$$

3.4.1 Cauchy Stress in Simple Shear for W_{H}

The Kirchhoff stress tensor τ_{H} corresponding to the Hencky energy W_{H} is given by

$$
\tau_{\mathrm{H}}(\log V) = 2\mu \, \mathrm{dev}_3 \log V + \kappa \, \mathrm{tr}(\log V) \cdot \mathbb{1}. \tag{3.44}
$$

Hence, in the case of simple shear, we have

$$
\tau_{\mathrm{H}} = 2\mu \frac{\log \lambda_1}{\sqrt{\gamma^2+4}} \begin{pmatrix} \gamma & 2 & 0 \\ 2 & -\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.45}
$$

Moreover, since $\det F = 1$ and $\sigma = \frac{1}{\det F} \tau$, we obtain

$$
\sigma_{\mathrm{H}} = 2\mu \frac{\log \lambda_1}{\sqrt{\gamma^2+4}} \begin{pmatrix} \gamma & 2 & 0 \\ 2 & -\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2\mu \frac{\log[\frac{1}{2}(\sqrt{\gamma^2+4}+\gamma)]}{\sqrt{\gamma^2+4}} \begin{pmatrix} \gamma & 2 & 0 \\ 2 & -\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.46}
$$

In particular, the simple shear stress $[\sigma_{\mathrm{H}}]_{12}$ corresponding to the amount of shear is given by

$$
[\sigma_{\mathrm{H}}]_{12} = 4\mu \frac{\log[\frac{1}{2}(\sqrt{\gamma^2+4}+\gamma)]}{\sqrt{\gamma^2+4}} = 2\frac{E}{1+\nu} \frac{\log[\frac{1}{2}(\sqrt{\gamma^2+4}+\gamma)]}{\sqrt{\gamma^2+4}}. \tag{3.47}
$$

The quadratic Hencky energy looses ellipticity in simple shear, see Sect. 5.3.

3.4.2 Cauchy Stress in Simple Shear for W_{eH}

In view of (3.3), the Kirchhoff stress tensor τ_{eH} is given by

$$
\tau_{\mathrm{eH}}(\log V) = 2\mu e^{k\|\mathrm{dev}_3 \log V\|^2} \cdot \mathrm{dev}_3 \log V + \kappa e^{\hat{k}[\mathrm{tr}(\log V)]^2} \mathrm{tr}(\log V) \cdot \mathbb{1}. \tag{3.48}
$$
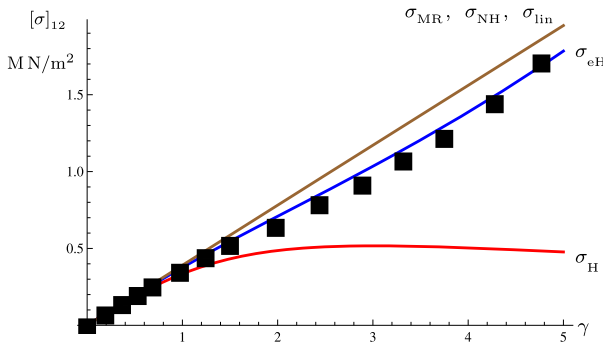
Fig. 11 The shear stress σ_{12} corresponding to the amount of shear γ for the energies W_{eH}, W_{H}, the Neo-Hooke energy W_{NH}, the Mooney-Rivlin energy W_{MR} and the infinitesimal case corresponding to rubber: $\mu = \frac{E}{2(1+\nu)} = 0.39\,\mathrm{M\,N/m^2}$ (according to Treloar's data [248]). For the exponentiated energy W_{eH} we have chosen $\frac{3}{16} < k = 0.243$. The *squares* (■) represent the experimental data for the simple shear deformation of vulcanized rubber, measured in 1944 by L.R.G. Treloar [247] and in 1975 by L.R.G. Treloar and D.F. Jones [123] (see also [248, 249]) and provided by courtesy of R. Ogden, in the form of tab-separated ASCII-files (see [246])

Since for simple shear $\det F = 1$ and $\mathrm{tr}(\log V) = 0$, we deduce

$$\sigma_{\mathrm{eH}}(\log V) = 2\mu e^{2k\log^2\lambda_1} \cdot \frac{\log\lambda_1}{\sqrt{\gamma^2+4}}\begin{pmatrix} \gamma & 2 & 0 \\ 2 & -\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.49}$$

$$= 2\mu e^{2k\log^2\left[\frac{1}{2}\left(\sqrt{\gamma^2+4}+\gamma\right)\right]} \cdot \frac{\log\left[\frac{1}{2}(\sqrt{\gamma^2+4}+\gamma)\right]}{\sqrt{\gamma^2+4}}\begin{pmatrix} \gamma & 2 & 0 \\ 2 & -\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.50}$$

For the exponentiated energy W_{eH} the simple shear stress $[\sigma_{\mathrm{eH}}]_{12}$ corresponding to the amount of shear γ is given by

$$[\sigma_{\mathrm{eH}}]_{12} = 2\frac{E}{1+\nu}e^{2k\log^2\left[\frac{1}{2}\left(\sqrt{\gamma^2+4}+\gamma\right)\right]} \cdot \frac{\log\left[\frac{1}{2}(\sqrt{\gamma^2+4}+\gamma)\right]}{\sqrt{\gamma^2+4}}. \tag{3.51}$$

The response of some rubbers is (more or less) linear under simple shear loading conditions (this is the raison d'être of the Mooney-Rivlin model [158], where $[\sigma_{\mathrm{MR}}]_{12} = 2(C_1 + C_2)\gamma = \frac{E}{2(1+\nu)}\gamma$). Let us therefore compare (Fig. 11) the simple shear stress σ_{12} corresponding to the amount of shear for the energies W_{eH}, W_{H}, for the Mooney-Rivlin energy (MR) and for the Neo-Hooke energy (NH).

Later in this paper we will implicitly show that W_{eH} remains rank-one convex in simple shear. Rubber becomes harder to deform at large strains, probably because of limited chain extendability. Many rubber materials are normally subjected to fairly small deformation, rarely exceeding 25 %, in tension/compression or 75 % in simple shear.

3.4.3 Cauchy Stress in Simple Shear in the Infinitesimal Case

It is well known that in the infinitesimal case the Cauchy stress tensor is given by

$$\sigma_{\mathrm{lin}} = 2\mu\,\mathrm{dev}_3\,\varepsilon + \kappa\,\mathrm{tr}(\varepsilon)\cdot\mathbb{1}, \tag{3.52}$$

where $\varepsilon = \text{sym}\nabla u$ is the linearized strain tensor of the deformation $\varphi(x) = x + u(x)$ with the displacement $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$. In the infinitesimal case, simple shear corresponds to the pure shear strain

$$\varepsilon = \begin{pmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.53}$$

The Cauchy stress tensor in simple shear is given by

$$\sigma_{\text{lin}} = 2\mu \begin{pmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad [\sigma_{\text{lin}}]_{12} = \mu\gamma = \frac{E}{2(1+\nu)}\gamma. \tag{3.54}$$

3.5 Response of Rubber Under Large Pressure. Equation of State

Rubber, if considered as a linear, isotropic solid very nearly satisfies $\nu = 0.5$ (i.e., for small loads, rubber responds practically incompressible). However, rubber under large pressure allows for an appreciable volume change [28]. This can be seen by experimentally determined equations of states (EOS), relating the mean stress (the pressure) $\frac{1}{3}\text{tr}(\sigma)$ to the relative volume change $\det F$. For the exponentiated Hencky energy this relation is given by

$$\frac{1}{3}\text{tr}(\sigma_{\text{eH}}) = \frac{d}{dt}\left[\frac{\kappa}{2\widehat{k}}e^{\widehat{k}(\log t)^2}\right]\Bigg|_{t=\det F} = \left(\kappa e^{\widehat{k}(\log \det F)^2}\frac{\log \det F}{\det F}\right), \tag{3.55}$$

while for the quadratic Hencky energy we have

$$\frac{1}{3}\text{tr}(\sigma_{\text{H}}) = \frac{d}{dt}\left[\frac{\kappa}{2}(\log t)^2\right]\Bigg|_{t=\det F} = \left(\kappa\frac{\log \det F}{\det F}\right). \tag{3.56}$$

We have found that the analytical expression of the pressure $\frac{1}{3}\text{tr}(\sigma)$ is in concordance with the classical Bridgman's compression data for natural rubber as reported in [28, page 497, Fig. 4.47] with $\kappa = 2.5 \cdot 10^9$ Pa $= 2.5 \cdot 10^9$ GPa (see Figs. 12, 13). Tabor [242] showed that the bulk modulus of rubber is of the order 1 GPa and found the value of the bulk modulus κ to be about 2 GPa. Recently, Zimmermann and Stommel [268] determined experimentally that κ is of the order $\kappa = 2.5$ GPa, which can be found in the literature as well (see, e.g., [115]).

From Fig. 13, certain threshold values seem unreachable by compression, unless an infinite amount of energy is spent. However, this impression is misleading: stresses and energy remain finite for any stretch $V \in \text{PSym}(3)$. Therefore, in our model the assumption of limited chain extensibility is not needed.

In Fig. 14 we represent the pressure $\frac{1}{3}\text{tr}(\sigma)$ as function of $\det F$ in the neighborhood of the identity $F = \mathbb{1}$ and we compare the analytical results obtained for the exponentiated volumetric Hencky energy $\frac{\kappa}{2\widehat{k}}e^{\widehat{k}(\log \det F)^2}$ with $\widehat{k} = 22$ with the analytical form corresponding to the volumetric quadratic Hencky energy $\frac{\kappa}{2}(\log \det F)^2$, as well with Bell's experimental data [28]. In the neighborhood of the identity $F = \mathbb{1}$, the quadratic Hencky energy gives also good results, while in large compression the values obtained using the quadratic Hencky energy are not in agreement with the experimental data (see Figs. 12, 13). Moreover, the EOS relation corresponding to the quadratic Hencky is not invertible for $\det F > e$ and it is not able to predict the response for $\frac{1}{3}\text{tr}(\sigma) > \frac{1}{e}$ [256].

Fig. 12 The pressure $\frac{1}{3}\operatorname{tr}(\sigma)$ as function of det F: Bridgman's experimental data [28] in compression (■), analytical form corresponding to the exponentiated volumetric Hencky energy $\frac{\kappa}{2\hat{k}}e^{\hat{k}(\log \det F)^2}$ with $\hat{k}=22$ (*continuous line*) and the analytical form corresponding to the volumetric quadratic Hencky energy $\frac{\kappa}{2}(\log \det F)^2$ (*dashed line*). The *dotted line* represents the tangent to these curves. The value of the bulk modulus of rubber is chosen to be $\kappa = 2.5$ GPa. We point out that in the experimental data reported in [28, page 487] the magnitude of the pressure $\frac{1}{3}\operatorname{tr}(\sigma)$ is expressed in $\frac{\text{kg}}{\text{cm}^2}$ (see Fig. 4.47 from [28, page 487]) which means in fact $9.81 \cdot 10^4 \frac{\text{kg}}{\text{m s}^2} = 9.81 \cdot 10^4$ Pa
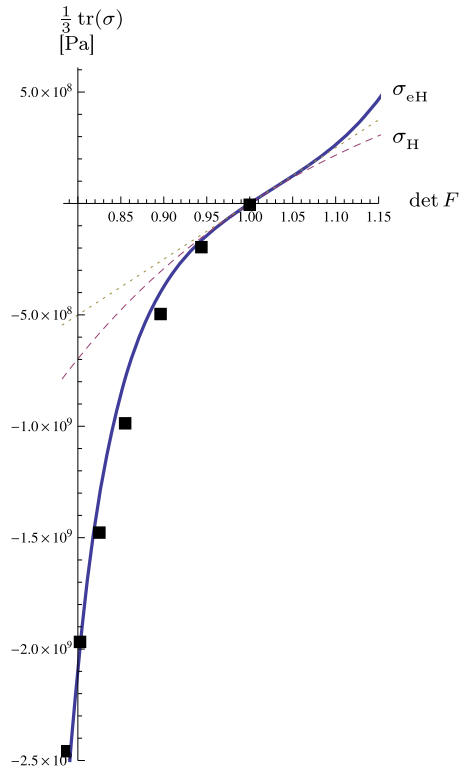
Fig. 13 The pressure $\frac{1}{3}\operatorname{tr}(\sigma)$ as function of det F. The pressure does not have a singularity in $(0, \infty)$, even if it seems that there is a singularity at det $F = 0.67$, meaning that this model would preclude compression beyond det $F = 0.67$. Moreover the mean stress (the pressure) corresponding to W_{eH} is invertible as function of the volume change. The considered values and the legend are the same as in Fig. 12
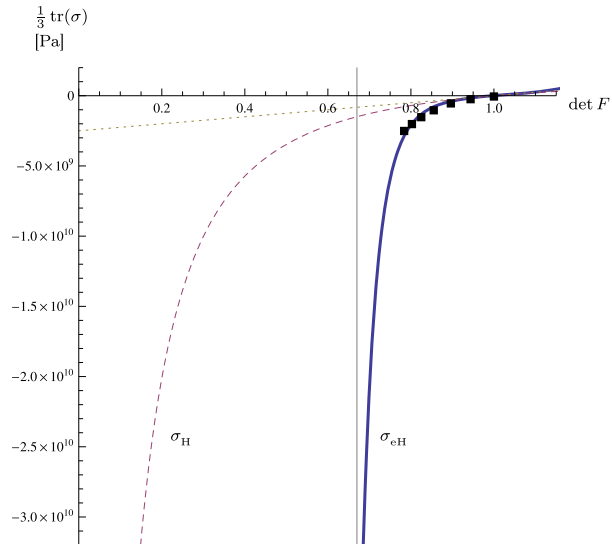
Fig. 14 The pressure $\frac{1}{3}\operatorname{tr}(\sigma)$ as function of $\det F$ in the neighborhood of identity $F = \mathbb{1}$. The considered values and the legend are the same as in Fig. 12
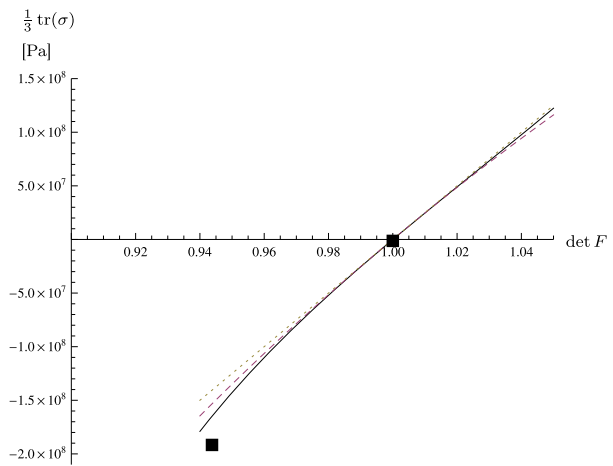
4 Monotonicity of the Cauchy Stress Tensor σ as a Function of $\log B$

Motivated by [121] we consider a novel constitutive requirement for an isotropic material, namely that the Cauchy stress tensor σ should be a monotone tensor function of $\log B$, $B = V^2$, i.e.,

$$\text{TSTS-M:} \quad \left\langle \sigma(\log B_1) - \sigma(\log B_2), \log B_1 - \log B_2 \right\rangle \geq 0, \quad \forall B_1, B_2 \in \mathrm{PSym}^+(3). \quad (4.1)$$

We will refer to (4.1) as true-stress-true-strain monotonicity (TSTS-M), and to

$$\begin{aligned}\text{TSTS-M}^+: \quad & \left\langle \sigma(\log B_1) - \sigma(\log B_2), \log B_1 - \log B_2 \right\rangle > 0, \\ & \forall B_1, B_2 \in \mathrm{PSym}^+(3), B_1 \neq B_2,\end{aligned} \quad (4.2)$$

as strict true-stress-true-strain monotonicity (TSTS-M$^+$). In a forthcoming paper [142] (see also [178]), it is shown that

$$\langle \log B_1 - \log B_2, B_1 - B_2 \rangle > 0, \quad \forall B_1, B_2 \in \mathrm{PSym}^+(3), B_1 \neq B_2. \quad (4.3)$$

Recall that Hill's monotonicity condition (KSTS-M) is monotonicity of the Kirchhoff stress tensor in terms of the logarithmic strain tensor, i.e.,

$$\text{KSTS-M:} \quad \left\langle \tau(\log B_1) - \tau(\log B_2), \log B_1 - \log B_2 \right\rangle \geq 0, \quad \forall B_1, B_2 \in \mathrm{PSym}^+(3), \quad (4.4)$$

where τ is the Kirchhoff stress tensor. The strict Hill's monotonicity condition is denoted by KSTS-M$^+$. Also, Hill has shown that convexity of the quadratic Hencky energy W_H in terms of $\log B$ implies the BE-inequalities.

In the linear theory of elasticity, $\sigma(\varepsilon) = 2\mu \operatorname{dev}_3 \varepsilon + \kappa \operatorname{tr}(\varepsilon) \cdot \mathbb{1}$, $\varepsilon = \operatorname{sym}\nabla u$, and the TSTS-M$^+$ condition implies, after linearization, $\langle \sigma(\varepsilon_1) - \sigma(\varepsilon_2), \varepsilon_1 - \varepsilon_2 \rangle > 0$ for all $\varepsilon_1, \varepsilon_2 \in \mathrm{Sym}(3)$, $\varepsilon_1 \neq \varepsilon_2$, and it is satisfied if and only if $\mu, \kappa > 0$. Therefore, in the linear setting, TSTS-M$^+$ is already stronger than rank-one convexity which only implies $\mu > 0$, $2\mu + \lambda > 0$.

The TSTS-M$^+$ condition caught our attention because of its possible relevance for the stability of nonlinear isotropic elastic bodies. Initially, its relation to loss of stability or loss of rank-one convexity was left unclear. Jog and Patil [121] have given a family of energies, including Neo-Hooke and Mooney-Rivlin energies, which does not satisfy TSTS-M$^+$. In this work we show (for the first time) that there exist free energies (namely W_{eH}) which do not satisfy TSTS-M$^+$ throughout but which are rank-one convex, while we also provide examples (namely $F \mapsto \frac{\mu}{k} e^{k\|\log V\|^2}, k \geq \frac{3}{8}$) which satisfy TSTS-M$^+$ but which are not rank-one convex. In [121, page 671] it is conjectured that the TSTS-M$^+$ condition is stronger than polyconvexity, which, however, is not true since TSTS-M$^+$ is not even stronger than rank-one convexity. The TSTS-M$^+$ condition implies that the Cauchy stress is an invertible function of the left stretch tensor (TSS-I); a property which could become important in FEM-computations based on the least squares finite element method [43, 44, 221, 240, 241].

For isotropic materials, TSTS-M$^+$ (and TSS-I) leads to a unique stress free reference (natural) configuration, up to a rigid deformation, i.e., $\sigma = 0$ implies $B = \mathbb{1}$ (or, equivalently, $\log B = 0$), since taking $B_2 = \mathbb{1}$ in (4.2) we deduce at once

$$\langle \sigma (\log B_1), \log B_1 \rangle > 0, \quad \forall B_1 \in \text{PSym}^+(3), B_1 \neq \mathbb{1} \quad \Rightarrow \quad \sigma (\log B_1) \neq 0. \tag{4.5}$$

We note the simple implications

$$\text{TSTS-M}^+ \quad \Rightarrow \quad \begin{cases} \text{TSTS-M} \\ \text{TSTS-I} \quad \Leftrightarrow \quad \text{TSS-I}. \end{cases}$$

The TSTS-M$^+$ and KSTS-M$^+$ condition are frame-indifferent in the following sense: superposing one time dependent rigid rotation field $Q(t) \in \text{SO}(3)$, we have

$$F_1 \mapsto F_1^* = Q(t)F_1, \qquad F_2 \mapsto F_2^* = Q(t)F_2, \tag{4.6}$$

$$B_1 = F_1 F_1^T \mapsto B_1^* = Q(t)B_1 Q^T(t), \qquad B_2 = F_2 F_2^T \mapsto B_2^* = Q(t)B_2 Q^T(t),$$

$$\log B_1 \mapsto \log B_1^* = Q(t)(\log B_1)Q^T(t), \qquad \log B_2 \mapsto \log B_2^* = Q(t)(\log B_2)Q^T(t),$$

and the identity

$$\langle \sigma (\log B_1^*) - \sigma (\log B_2^*), \log B_1^* - \log B_2^* \rangle = \langle \sigma (\log B_1) - \sigma (\log B_2), \log B_1 - \log B_2 \rangle,$$

holds, due to the isotropy of the formulation.

In Sect. 3 we have shown that

$$\tau = D_{\log V} W(\log V) = (\det V) \cdot \sigma = e^{\text{tr}(\log V)} \cdot \sigma, \tag{4.7}$$

where σ is the Cauchy stress and τ the Kirchhoff stress corresponding to the energy $F \mapsto W(\log V)$.

Remark 4.1 Sufficient for TSTS-M$^+$ is Jog and Patil's [121] constitutive requirement that

$$\mathbb{Z} := D_{\log V} \sigma (\log V) \tag{4.8}$$

is a positive definite fourth order tensor.

Proof Let us remark that for all $B_1, B_2 \in \mathrm{PSym}^+(3)$ and $0 \leq t \leq 1$, we have $2 \log V_1 = \log B_1, 2 \log V_2 = \log B_2$ and $t(\log V_1 - \log V_2) + \log V_2 \in \mathrm{Sym}(3)$, where $V_1^2 = B_1$, $V_2^2 = B_2$. Moreover, we have

$$\langle \sigma(\log B_1) - \sigma(\log B_2), \log B_1 - \log B_2 \rangle = 2\langle \sigma(2 \log V_1) - \sigma(2 \log V_2), \log V_1 - \log V_2 \rangle$$

$$= 2\left\langle \left[\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \sigma\left(2t(\log V_1 - \log V_2) + 2 \log V_2\right) \mathrm{d}t \right], \log V_1 - \log V_2 \right\rangle \tag{4.9}$$

$$= 4 \int_0^1 \langle \left[D_{\log V} \sigma\left(2t(\log V_1 - \log V_2) + 2 \log V_2\right).(\log V_1 - \log V_2) \right], \log V_1 - \log V_2 \rangle \mathrm{d}t.$$

Using that the integrand is non-negative, due to the assumption that $\mathbb{Z} = D_{\log V} \sigma(\log V)$ is positive definite, the TSTS-M$^+$ condition follows. $\qquad\square$

With the substitution $X = \log V$, the monotonicity of σ as a function of $X \in \mathrm{Sym}(3)$ means

$$\langle \sigma(X + H) - \sigma(X), H \rangle \geq 0 \quad \forall X, H \in \mathrm{Sym}(3), \tag{4.10}$$

and sufficient for monotonicity of σ is (proof as in Remark 4.1)

$$\langle D_X \sigma(X).H, H \rangle \geq 0 \quad \forall X, H \in \mathrm{Sym}(3). \tag{4.11}$$

Remark 4.2 Since $e^{\|\log U\|^2}$ is uniformly convex in $\log U$, KSTS-M$^+$ is satisfied everywhere.

4.1 TSTS-M$^+$ for the Energy $F \mapsto \frac{\mu}{k} e^{k\|\log V\|^2} + \frac{\lambda}{2k} e^{\widehat{k}[\mathrm{tr}(\log V)]^2}$

Proposition 4.3 *The Cauchy stress tensor σ corresponding to the energy $F \mapsto \frac{\mu}{k} e^{k\|\log V\|^2}$ satisfies TSTS-M for $k \geq \frac{3}{8}$ and TSTS-M$^+$ for $k > \frac{3}{8}$.*

Proof In order to show this, let us remark that for the energy $F \mapsto \frac{\mu}{k} e^{k\|\log V\|^2}$ we have

$$\widetilde{\tau}(\log V) = 2\mu e^{k\|\log V\|^2} \cdot \log V, \qquad \widetilde{\sigma}(\log V) = 2\mu e^{k\|\log V\|^2 - \mathrm{tr}(\log V)} \cdot \log V. \tag{4.12}$$

We compute

$$\langle D_X \widetilde{\sigma}(X).H, H \rangle = 2\mu e^{k\|X\|^2 - \mathrm{tr}(X)} \left[2k\langle X, H \rangle - \mathrm{tr}(H) \right] \langle X, H \rangle + 2\mu e^{k\|X\|^2 - \mathrm{tr}(X)} \|H\|^2$$

$$= 2\mu e^{k\|X\|^2 - \mathrm{tr}(X)} \{ 2k\langle X, H \rangle^2 - \mathrm{tr}(H)\langle X, H \rangle + \|H\|^2 \}. \tag{4.13}$$

If $\mathrm{tr}(H)\langle X, H \rangle < 0$, then obviously $\langle D_X \widetilde{\sigma}(X).H, H \rangle > 0$. Otherwise, for $k \geq \frac{3}{8}$ it follows

$$\langle D_X \widetilde{\sigma}(X).H, H \rangle \geq 2\mu e^{k\|X\|^2 - \mathrm{tr}(X)} \left\{ 2k\langle X, H \rangle^2 - 2\sqrt{\frac{2k}{3}} \mathrm{tr}(H)\langle X, H \rangle + \|H\|^2 \right\}$$

$$= 2\mu e^{k\|X\|^2 - \mathrm{tr}(X)} \left\langle H - \sqrt{\frac{2k}{3}}\langle X, H \rangle \cdot \mathbb{1}, H - \sqrt{\frac{2k}{3}}\langle X, H \rangle \cdot \mathbb{1} \right\rangle$$

$$= 2\mu e^{k\|X\|^2 - \operatorname{tr}(X)} \left\| H - \sqrt{\frac{2k}{3}} \langle X, H \rangle \cdot \mathbb{1} \right\|^2 \geq 0. \qquad (4.14)$$

Moreover, for $k > \frac{3}{8}$ we have $\langle D_X \widetilde{\sigma}(X).H, H \rangle > 0$ and the proof is complete. $\qquad \square$

Corollary 4.4 *The Cauchy stress tensor corresponding to the energy* $F \mapsto \frac{\mu}{k} e^{k\|\log V\|^2} + \frac{\lambda}{2k} e^{\widehat{k}[\operatorname{tr}(\log V)]^2}$ *satisfies TSTS-M for* $k \geq \frac{3}{8}, \widehat{k} \geq \frac{1}{8}$ *and* $\mu, \lambda > 0$ *and TSTS-M$^+$ for* $k > \frac{3}{8}, \widehat{k} \geq \frac{1}{8}$ *(or* $k \geq \frac{3}{8}, \widehat{k} > \frac{1}{8}$*) and* $\mu, \lambda > 0$.

Proof From direct calculations we have

$$\left\langle D_X e^{\widehat{k}[\operatorname{tr}(X)]^2 - \operatorname{tr}(X)} \operatorname{tr}(X) \cdot \mathbb{1}.H, H \right\rangle$$
$$= e^{\widehat{k}[\operatorname{tr}(X)]^2 - \operatorname{tr}(X)} \left\{ 2\widehat{k}[\operatorname{tr}(X)]^2 - \operatorname{tr}(X) + 1 \right\} [\operatorname{tr}(H)]^2. \qquad (4.15)$$

Thus, if $\widehat{k} \geq \frac{1}{8}$, then

$$\left\langle D_X e^{\widehat{k}[\operatorname{tr}(X)]^2 - \operatorname{tr}(X)} \operatorname{tr}(X) \cdot \mathbb{1}.H, H \right\rangle \geq e^{\widehat{k}[\operatorname{tr}(X)]^2 - \operatorname{tr}(X)} \left(\frac{1}{2} \operatorname{tr}(X) - 1 \right)^2 [\operatorname{tr}(H)]^2 \geq 0. \qquad (4.16)$$

The above inequality is strict for $\widehat{k} > \frac{1}{8}$. The rest of the proof follows from the previous theorem. $\qquad \square$

Since, however, we prove in Sect. 5.8 that $F \mapsto e^{\|\log V\|^2}$ is not LH-elliptic, we note that in general

$$\text{TSTS-M}^+ \quad \not\Rightarrow \quad \text{LH-ellipticity,}$$

answering a conjecture arising in [121]. It is also clear that

$$\text{LH-ellipticity} \quad \not\Rightarrow \quad \text{TSTS-M or TSTS-I,}$$

as already implied by some examples from the development in [121]. As a preliminary conclusion on the status of the TSTS-M-condition we can note that TSTS-M is an additional plausible criterion, basically independent of other conditions. It is compatible, in principle, with rank-one convexity, but does not imply it. It can be speculated that TSTS-M$^+$ should hold for some domain of bounded distortions.

The same remarks hold for the KSTS-M$^+$ condition, i.e., the notion is frame-indifferent and

$$\text{KSTS-M}^+ \quad \Rightarrow \quad \text{KSS-I}, \qquad \text{KSTS-M}^+ \quad \not\Rightarrow \quad \text{LH}, \qquad \text{LH} \quad \not\Rightarrow \quad \text{KSTS-M}^+.$$

4.2 TSTS-M$^+$ for the Family of Energies W_{eH}

Let us consider our exponentiated Hencky energy with volumetric-isochoric decoupled format

$$W_{\text{eH}}(\log V) := \frac{\mu}{k} e^{k\|\operatorname{dev}_3 \log V\|^2} + \frac{\kappa}{2\widehat{k}} e^{\widehat{k}[\operatorname{tr}(\log V)]^2}. \qquad (4.17)$$

Proposition 4.5 *The TSTS-M$^+$ condition* (4.11) *is not everywhere satisfied for the energy function* W_{eH} *defined by* (4.17) *for* $n = 2, 3$.

Proof In Sect. 3 we have shown that

$$\tau_{\mathrm{eH}}(\log V) = 2\mu e^{k\|\operatorname{dev}_3 \log V\|^2} \cdot \operatorname{dev}_3 \log V + \kappa e^{\widehat{k}[\operatorname{tr}(\log V)]^2} \operatorname{tr}(\log V) \cdot \mathbb{1}, \tag{4.18}$$

$$\sigma_{\mathrm{eH}}(\log V) = 2\mu e^{k\|\operatorname{dev}_3 \log V\|^2 - \operatorname{tr}(\log V)} \cdot \operatorname{dev}_3 \log V + \kappa e^{\widehat{k}[\operatorname{tr}(\log V)]^2 - \operatorname{tr}(\log V)} \operatorname{tr}(\log V) \cdot \mathbb{1}.$$

We compute

$$\begin{aligned}
&\langle D_X \sigma_{\mathrm{eH}}(X).H, H\rangle \\
&\quad = 2\mu e^{k\|\operatorname{dev}_3 X\|^2 - \operatorname{tr}(X)}\big[2k\langle \operatorname{dev}_3 X, H\rangle - \operatorname{tr}(H)\big]\langle \operatorname{dev}_3 X, H\rangle \\
&\qquad + 2\mu e^{k\|\operatorname{dev}_3 X\|^2 - \operatorname{tr}(X)}\|\operatorname{dev}_3 H\|^2 \\
&\qquad + \kappa e^{\widehat{k}[\operatorname{tr}(X)]^2 - \operatorname{tr}(X)}\big[2\widehat{k}\operatorname{tr}(X)\operatorname{tr}(H) - \operatorname{tr}(H)\big]\operatorname{tr}(X)\operatorname{tr}(H) + \kappa e^{\widehat{k}[\operatorname{tr}(X)]^2 - \operatorname{tr}(X)}\big[\operatorname{tr}(H)\big]^2 \\
&\quad = 2\mu e^{k\|\operatorname{dev}_3 X\|^2 - \operatorname{tr}(X)}\big\{2k\langle \operatorname{dev}_3 X, H\rangle^2 - \operatorname{tr}(H)\langle \operatorname{dev}_3 X, H\rangle + \|\operatorname{dev}_3 H\|^2\big\} \\
&\qquad + \kappa e^{\widehat{k}[\operatorname{tr}(X)]^2 - \operatorname{tr}(X)}\big\{2\widehat{k}\big[\operatorname{tr}(X)\big]^2 - \operatorname{tr}(X) + 1\big\} \cdot \big[\operatorname{tr}(H)\big]^2. \tag{4.19}
\end{aligned}$$

For $\widehat{k} > \frac{1}{8}$, it is easy to see that

$$\big\{2\widehat{k}\big[\operatorname{tr}(X)\big]^2 - \operatorname{tr}(X) + 1\big\} > 0, \quad \text{for all } X \in \operatorname{Sym}(3). \tag{4.20}$$

On the other hand, the first summand in (4.19)

$$\begin{aligned}
&\big\langle D_X\big[e^{k\|\operatorname{dev}_3 X\|^2 - \operatorname{tr}(X)} \cdot \operatorname{dev}_3 X\big].H, H\big\rangle \\
&\quad = 2k\langle \operatorname{dev}_3 X, H\rangle^2 - \operatorname{tr}(H)\langle \operatorname{dev}_3 X, H\rangle + \|\operatorname{dev}_3 H\|^2 \tag{4.21}
\end{aligned}$$

is not positive for all $H \in \operatorname{Sym}(3)$. For instance, we may choose

$$H_0 = \operatorname{dev}_3 X + a \cdot \mathbb{1}, \quad a \in \mathbb{R}_+, \tag{4.22}$$

and we obtain

$$\begin{aligned}
&2k\langle \operatorname{dev}_3 X, H_0\rangle^2 - \operatorname{tr}(H_0)\langle \operatorname{dev}_3 X, H_0\rangle + \|\operatorname{dev}_3 H_0\|^2 \\
&\quad = 2k\|\operatorname{dev}_3 X\|^4 - 3a\|\operatorname{dev}_3 X\|^2 + \|\operatorname{dev}_3 X\|^2, \tag{4.23}
\end{aligned}$$

which is negative for large values of a (in the two-dimensional case we may consider $H_0 = \operatorname{dev}_2 X + a \cdot \mathbb{1}, \quad a \in \mathbb{R}_+$). Hence, the TSTS-M condition is not satisfied for the energy $F \mapsto e^{k\|\operatorname{dev}_3 \log V\|^2}$ alone.

The next question is if one may control the negative part in (4.21) by adding the volumetric function $F \mapsto e^{\widehat{k}(\operatorname{tr}(\log V))^2}$. The answer is negative as we may see in the following. Let us consider the matrices

$$X_1 = \begin{pmatrix} 0 & t & 0 \\ t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \operatorname{Sym}(3), \qquad H_1 = \begin{pmatrix} q/3 & 1 & 0 \\ 1 & q/3 & 0 \\ 0 & 0 & q/3 \end{pmatrix} \in \operatorname{Sym}(3), \tag{4.24}$$

where, for large values of $t > 0$, q is chosen such that

$$\frac{8kt^2 + 4}{t} < q < \frac{2\mu}{\kappa} e^{2kt^2} t. \tag{4.25}$$

For the considered matrices, we deduce

$$\| \operatorname{dev}_3 X_1 \|^2 = 2t^2, \quad \operatorname{tr}(X_1) = 0, \quad \| \operatorname{dev}_3 H_1 \|^2 = 2,$$
$$\operatorname{tr}(H_1) = q, \quad \langle \operatorname{dev}_3 X_1, H_1 \rangle = 2t, \tag{4.26}$$

and

$$\begin{aligned}
\langle D_X \sigma_{\mathrm{eH}}(X_1).H_1, H_1 \rangle &= 2\mu e^{2kt^2} \left\{ 8kt^2 - 2qt + 4 \right\} + \kappa q^2 \\
&= 2\mu e^{2kt^2} \left\{ 8kt^2 - qt + 4 \right\} + q \left\{ -2\mu e^{2kt^2} + \kappa q \right\} < 0. \tag{4.27}
\end{aligned}$$

In the two-dimensional case, as counter-example we may consider the matrices

$$X_1 = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} \in \operatorname{Sym}(2), \qquad H_1 = \begin{pmatrix} q/2 & 1 \\ 1 & q/2 \end{pmatrix} \in \operatorname{Sym}(2), \tag{4.28}$$

where, for large values of $t > 0$, q satisfies (4.25). Therefore, the monotonicity condition is not satisfied and the proof is complete. $\qquad\square$

However, the energy W_{eH} satisfies the TSTS-M$^+$ condition by restricting it to some "elastic domain" in *stretch space* (a cone in PSym(3)) of bounded distortions

$$\mathcal{E}^+\left(W_{\mathrm{eH}}, \text{TSTS-M}^+, V, \frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2 \right) := \left\{ Y \in \operatorname{PSym}(3) \mid \| \operatorname{dev}_3 \log Y \|^2 \leq \frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2 \right\} \subset \operatorname{PSym}(3), \tag{4.29}$$

which is equivalent to restrict the energy $\overline{W}_{\mathrm{eH}}(\log V) = W_{\mathrm{eH}}(V)$ to the "elastic domain" in *strain space*

$$\mathcal{E}\left(W_{\mathrm{eH}}, \text{TSTS-M}^+, \log V, \frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2 \right) := \left\{ X \in \operatorname{Sym}(3) \mid \| \operatorname{dev}_3 X \|^2 \leq \frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2 \right\} \subset \operatorname{Sym}(3), \tag{4.30}$$

where $\widetilde{\sigma}_{\mathbf{y}}$ is a dimensionless quantity related to the so called yield stress $\sigma_{\mathbf{y}}$, whose dimension is [MPa], i.e., a critical value of shear stress, below which a plastic or viscoplastic material behaves like an elastic solid; above this value, a plastic material deforms and a viscoplastic material flows. This assumption is in complete concordance with the Huber-von-Mises-Hencky distortional strain energy hypothesis [105].

We also need to introduce the elastic domain in the *Kirchhoff-stress space*

$$\mathcal{E}\left(W_{\mathrm{eH}}, \text{TSTS-M}^+, \tau_{\mathrm{eH}}, \frac{2}{3}\sigma_{\mathbf{y}}^2 \right) := \left\{ \tau \in \operatorname{Sym}(3) \mid \| \operatorname{dev}_3 \tau \|^2 \leq \frac{2}{3}\sigma_{\mathbf{y}}^2 \right\} \subset \operatorname{Sym}(3). \tag{4.31}$$

Proposition 4.6 (TSTS-M$^+$ is satisfied for the energy function W_{eH} for bounded distortions) *If the material parameters $\mu, \kappa > 0, \widehat{k} > \frac{1}{8}$ and $\widetilde{\sigma}_{\mathbf{y}} \in \mathbb{R}$ are such that*

$$0 < \widetilde{\sigma}_{\mathbf{y}}^2 \leq \frac{6}{e} \frac{\kappa}{\mu} \frac{8\widehat{k} - 1}{8\widehat{k}}, \tag{4.32}$$

holds true, then there exists $k > 0$ such that

$$\forall X \in \mathcal{E}\begin{pmatrix} W_{\text{eH}}, \text{TSTS-M}^+, \\ \log V, \frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2 \end{pmatrix}, \quad \forall H \in \text{Sym}(3): \quad \langle D_X \sigma_{\text{eH}}(X).H, H \rangle > 0, \tag{4.33}$$

i.e., the TSTS-M$^+$ inequality is satisfied in $\mathcal{E}(W_{\text{eH}}, \text{TSTS-M}^+, \log V, \frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2)$ (or equivalently, the TSTS-M$^+$ inequality is satisfied in $\mathcal{E}^+(W_{\text{eH}}, \text{TSTS-M}^+, V, \frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2)$).

Proof Let us rewrite Eq. (4.19) as

$$\begin{aligned} \langle D_X \sigma_{\text{eH}}(X).H, H \rangle = e^{k\|\text{dev}_3 X\|^2 - \text{tr}(X)} &\Big\{ 4\mu k \langle \text{dev}_3 X, H \rangle^2 - 2\mu \, \text{tr}(H) \langle \text{dev}_3 X, H \rangle \\ &+ \kappa e^{\widehat{k}[\text{tr}(X)]^2 - k\|\text{dev}_3 X\|^2} \big\{ 2\widehat{k}[\text{tr}(X)]^2 - \text{tr}(X) + 1 \big\}[\text{tr}(H)]^2 \Big\} \\ &+ 2\mu e^{k\|\text{dev}_3 X\|^2 - \text{tr}(X)} \|\text{dev}_3 H\|^2. \end{aligned} \tag{4.34}$$

If $\widehat{k} > \frac{1}{8}$, then $2\widehat{k}[\text{tr}(X)]^2 - \text{tr}(X) + 1 > 0$ for all $X \in \text{Sym}(3)$. Hence, for

$$\begin{aligned} 4\mu k \langle \text{dev}_3 X, H \rangle^2 &- 2\mu \, \text{tr}(H) \langle \text{dev}_3 X, H \rangle \\ &+ \kappa e^{\widehat{k}[\text{tr}(X)]^2 - k\|\text{dev}_3 X\|^2} \big\{ 2\widehat{k}[\text{tr}(X)]^2 - \text{tr}(X) + 1 \big\}[\text{tr}(H)]^2 > 0 \end{aligned}$$

to hold for all $X, H \in \text{Sym}(3)$, it is sufficient to have

$$4\mu^2 - 16\mu k \kappa e^{\widehat{k}[\text{tr}(X)]^2 - k\|\text{dev}_3 X\|^2} \big\{ 2\widehat{k}[\text{tr}(X)]^2 - \text{tr}(X) + 1 \big\} < 0$$
$$\text{for all } X \in \text{Sym}(3). \tag{4.35}$$

Because $\mu, \kappa > 0$, for matrices $X \in \text{Sym}(3)$ which belong to the "elastic domain" $\mathcal{E}(W_{\text{eH}}, \text{TSTS-M}^+, \log V, \frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2)$ defined by (4.30) the above inequality is satisfied if

$$\frac{\mu}{4\kappa} < k e^{\widehat{k}[\text{tr}(X)]^2 - k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2} \big\{ 2\widehat{k}[\text{tr}(X)]^2 - \text{tr}(X) + 1 \big\}. \tag{4.36}$$

On the other hand, for $\widehat{k} > \frac{1}{8}$, we find

$$\inf_{X \in \text{Sym}(3)} \big\{ 2\widehat{k}[\text{tr}(X)]^2 - \text{tr}(X) + 1 \big\} = \frac{8\widehat{k} - 1}{8\widehat{k}} > 0. \tag{4.37}$$

Taking $\inf_{X \in \text{Sym}(3)}$ of the right hand side of (4.36), we obtain that if there exist $\widehat{k} > \frac{1}{8}$ and $k > 0$ such that

$$\frac{e^{k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2}}{k} \frac{\mu}{4\kappa} \leq \frac{8\widehat{k} - 1}{8\widehat{k}} < 1 \tag{4.38}$$

holds, then the inequality (4.36) follows. The question is whether there are always numbers $k > 0$ satisfying the above inequality. We have

$$\inf_{k > 0}\left\{ \frac{e^{k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2}}{k} \right\} = \inf_{k > 0}\left\{ \frac{e^{k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2}}{k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2} \frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2 \right\} = \inf_{r > 0}\left\{ \frac{e^r}{r} \right\} \frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2 = \frac{2}{3}e\widetilde{\sigma}_{\mathbf{y}}^2, \qquad \lim_{k \to \infty} \frac{e^{k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2}}{k} = \infty. \tag{4.39}$$

In view of (4.39) and using the continuity of the function $t \mapsto \frac{e^{t\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2}}{t}$, we conclude: if the material parameters $\mu, \kappa > 0, \widehat{k} > \frac{1}{8}$ and $\widetilde{\sigma}_{\mathbf{y}} \in \mathbb{R}$ are chosen such that

$$0 < \frac{2}{3}e\widetilde{\sigma}_{\mathbf{y}}^2 \leq \frac{4\kappa}{\mu}\frac{8\widehat{k}-1}{8\widehat{k}}, \tag{4.40}$$

then we may find a constant $k > 0$ which satisfies

$$\frac{2}{3}e\widetilde{\sigma}_{\mathbf{y}}^2\frac{\mu}{4\kappa} = \inf_{k>0}\left\{\frac{e^{k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2}}{k}\right\} \leq \frac{e^{k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2}}{k}\frac{\mu}{4\kappa} \leq \frac{8\widehat{k}-1}{8\widehat{k}}. \tag{4.41}$$

Using (4.37) we obtain that there is a constant $k > 0$ such that (4.38) is satisfied. Hence, there is a constant $k > 0$ such that (4.35) holds true, which in view of (4.34) implies (4.33) and the proof is complete. □

We remark that Proposition 4.6 is unspecific about the values for $k > 0$. Written in terms of Poisson's ratio[25] $-1 < \nu \leq \frac{1}{2}$, the extra constitutive assumption (4.32) becomes

$$0 < \widetilde{\sigma}_{\mathbf{y}}^2 \leq \frac{4}{e}\frac{1+\nu}{1-2\nu}\frac{8\widehat{k}-1}{8\widehat{k}} \quad \Leftrightarrow \quad \widehat{k} \geq \frac{1}{8 - \widetilde{\sigma}_{\mathbf{y}}^2\frac{e}{2}} > \frac{1}{8}. \tag{4.42}$$

Heinrich Hencky [99] offered a physical interpretation of the von Mises criterion suggesting that yielding begins when the elastic energy of distortion reaches a critical value [106] (see also [49, 50, 81]). For this, the von Mises criterion is also known as the maximum distortional strain energy criterion. This stems from the relation between the second deviatoric stress invariant J_2 and the elastic strain energy of distortion $W_D = \frac{J_2}{2\mu}$, with the elastic shear modulus $\mu = \frac{E}{2(1+\nu)}$, Young's modulus E and Poisson's ratio ν.

In the following we express the constitutive assumption (4.40) in terms of the yield stress $\sigma_{\mathbf{y}}$ and the Kirchhoff stress tensor τ_{eH}.

Proposition 4.7 (W_{eH} satisfies TSTS-M$^+$ for bounded distortions) *There exist $\widehat{k} > \frac{1}{8}$ and $k > 0$, such that for all $\sigma_{\mathbf{y}} \in \mathbb{R}$ for which*

$$0 < \sigma_{\mathbf{y}}^2 \leq \frac{3\mu\kappa}{e}\frac{8\widehat{k}-1}{\widehat{k}}e^{k\frac{\kappa}{e\mu}\frac{8\widehat{k}-1}{k}}, \tag{4.43}$$

holds true, the TSTS-M$^+$ inequality is satisfied for all $V \in \text{PSym}(3)$ (for all $\log V \in \text{Sym}(3)$) for which $\tau_{\text{eH}}(\log V) \in \mathcal{E}(W_{\text{eH}}, \text{TSTS-M}^+, \tau_{\text{eH}}, \frac{2}{3}\sigma_{\mathbf{y}}^2)$.

Proof Let us remark that any $X \in \text{Sym}(3)$ for which $\tau_{\text{eH}}(X)$ lies in the set $\mathcal{E}(W_{\text{eH}}, \text{TSTS-M}^+, \tau_{\text{eH}}, \frac{2}{3}\sigma_{\mathbf{y}}^2)$ satisfies

$$\left\|2\mu e^{k\|\text{dev}_3 X\|^2} \cdot \text{dev}_3 X\right\| \leq \sqrt{\frac{2}{3}}\sigma_{\mathbf{y}}.$$

Hence, $\|\text{dev}_3 X\|e^{k\|\text{dev}_3 X\|^2} \leq \sqrt{\frac{2}{3}}\frac{\sigma_{\mathbf{y}}}{2\mu}$. If the yield limit $\sigma_{\mathbf{y}}$ is chosen such that (4.43) is satisfied, then there is $\widetilde{\sigma}_{\mathbf{y}} > 0$ such that $0 < \sigma_{\mathbf{y}} \leq 2\mu\widetilde{\sigma}_{\mathbf{y}}e^{k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2}$, and $\widetilde{\sigma}_{\mathbf{y}}^2 \leq \frac{6\kappa}{e\mu}\frac{8\widehat{k}-1}{8\widehat{k}}$. Hence,

---

[25]We use that $\kappa = \frac{2\mu(1+\nu)}{3(1-2\nu)}, \nu = \frac{3\kappa-2\mu}{2(3\kappa+\mu)}$.

$0 < \sqrt{\frac{2}{3}}\frac{\sigma_{\mathbf{y}}}{2\mu} \le \sqrt{\frac{2}{3}}\widetilde{\sigma}_{\mathbf{y}}e^{k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2}$, which implies $\| \operatorname{dev}_3 X \|e^{k\| \operatorname{dev}_3 X\|^2} \le \sqrt{\frac{2}{3}}\widetilde{\sigma}_{\mathbf{y}}e^{k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2}$. In view of the monotonicity of $t \mapsto te^{kt^2}$, we deduce

$$\| \operatorname{dev}_3 X \| \le \sqrt{\frac{2}{3}}\widetilde{\sigma}_{\mathbf{y}}, \tag{4.44}$$

and $X \in \mathcal{E}(W_{\mathrm{eH}}, \text{TSTS-M}^+, \log V, \frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2)$. Since we have assumed that $\widetilde{\sigma}_{\mathbf{y}}^2$ and $\widehat{k} > \frac{1}{8}$ satisfy (4.32), then Proposition 4.6 ensures the existence of $k > 0$ such that the TSTS-M$^+$ inequality is satisfied and the proof is complete. $\qquad\square$

Remark 4.8

 (i) In terms of Young's modulus E and Poisson's ratio ν the condition imposed on the yield limit $\sigma_{\mathbf{y}}$ by Proposition 4.7 is

$$0 < \sigma_{\mathbf{y}}^2 \le \frac{1}{2e}\frac{E^2}{(1+\nu)(1-2\nu)}\frac{8\widehat{k}-1}{\widehat{k}}e^{k\frac{2}{3e}\frac{1+\nu}{1-2\nu}\frac{8\widehat{k}-1}{\widehat{k}}}. \tag{4.45}$$

 (ii) In the incompressible limit $\kappa \to \infty$, it follows that W_{eH} satisfies TSTS-M$^+$ everywhere since then $\mathcal{E}(W_{\mathrm{eH}}, \text{TSTS-M}^+, \tau_{\mathrm{eH}}, \frac{2}{3}\sigma_{\mathbf{y}}^2) = \mathrm{Sym}(3)$ and TSTS-M$^+ \Leftrightarrow$ KSTS-M$^+$.

4.3 TSTS-M$^+$ for Three-Parameter Energies $W_{\mathrm{eH}}^{\sharp}$

In this subsection we consider the set of energies of the family W_{eH} for which $\widehat{k} = \frac{2}{3}k$.

Proposition 4.9 (The exponentiated 3-parameter energy W_{eH} satisfies TSTS-M$^+$ for bounded distortions) *Let $\widetilde{\sigma}_{\mathbf{y}} > 0$ be such that $\widetilde{\sigma}_{\mathbf{y}}^2 e^{\frac{1}{8}\widetilde{\sigma}_{\mathbf{y}}^2+1} \le \frac{6\kappa}{\mu}$ holds true. Then there exists $k > \frac{3}{16}$ such that for all $\sigma_{\mathbf{y}}$ satisfying $0 < \sigma_{\mathbf{y}} \le 2\mu\widetilde{\sigma}_{\mathbf{y}}e^{k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2}$, the exponentiated 3-parameter energy*

$$W_{\mathrm{eH}}^{\sharp}(\log V) := \frac{\mu}{k}e^{k\| \operatorname{dev}_3 \log V\|^2} + \frac{3\kappa}{4k}e^{\frac{2}{3}k(\operatorname{tr}(\log V))^2}, \tag{4.46}$$

satisfies TSTS-M$^+$ for all $V \in \mathrm{PSym}(3)$ for which $\tau_{\mathrm{eH}}(\log V) \in \mathcal{E}(W_{\mathrm{eH}}, \text{TSTS-M}^+, \tau_{\mathrm{eH}}, \frac{2}{3}\sigma_{\mathbf{y}}^2)$.

Proof Similar as in the proof of Proposition 4.7, we deduce that $\tau_{\mathrm{eH}}(X) \in \mathcal{E}(W_{\mathrm{eH}}, \text{TSTS-M}^+, \tau_{\mathrm{eH}}, \frac{2}{3}\sigma_{\mathbf{y}}^2)$ implies

$$\| \operatorname{dev}_3 X \|e^{k\| \operatorname{dev}_3 X\|^2} \le \sqrt{\frac{2}{3}}\frac{\sigma_{\mathbf{y}}}{2\mu}. \tag{4.47}$$

On the other hand, in view of (4.34)–(4.40), in order to have $\langle D_X\sigma_{\mathrm{eH}}(X).H, H\rangle > 0$ for $X \in \mathrm{Sym}(3)$ which belong also to the "elastic domain" $\mathcal{E}(W_{\mathrm{eH}}, \text{TSTS-M}^+, \log V, \frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2)$ defined by (4.30), we already know that it is sufficient to prove that there are $k > 0$ and $\widehat{k} > \frac{1}{8}$ which satisfy (4.38), that is

$$\frac{e^{k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2}}{k}\frac{\mu}{4\kappa} \le \frac{8\widehat{k}-1}{8\widehat{k}} < 1. \tag{4.48}$$

For the 3-parameter energy we have $2k = 3\widehat{k}$. Hence, in this case we have to prove that there is $k > \frac{3}{16}$ such that

$$\frac{e^{k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2}}{k}\frac{\mu}{4\kappa} \leq \frac{16k-3}{16k}. \tag{4.49}$$

Let us rewrite (4.49) in the form

$$\frac{e^{k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2}}{16k-3} \leq \frac{\kappa}{4\mu} \quad \Leftrightarrow \quad \frac{e^{(k\frac{2}{3}-\frac{1}{8})\widetilde{\sigma}_{\mathbf{y}}^2}}{(k\frac{2}{3}-\frac{1}{8})\widetilde{\sigma}_{\mathbf{y}}^2}\frac{1}{24}\widetilde{\sigma}_{\mathbf{y}}^2 e^{\frac{1}{8}\widetilde{\sigma}_{\mathbf{y}}^2} \leq \frac{\kappa}{4\mu}. \tag{4.50}$$

We have

$$\inf_{k>\frac{3}{16}}\left\{\frac{e^{(k\frac{2}{3}-\frac{1}{8})\widetilde{\sigma}_{\mathbf{y}}^2}}{(k\frac{2}{3}-\frac{1}{8})\widetilde{\sigma}_{\mathbf{y}}^2}\right\} = e, \qquad \lim_{k\to\infty}\frac{e^{(k\frac{2}{3}-\frac{1}{8})\widetilde{\sigma}_{\mathbf{y}}^2}}{(k\frac{2}{3}-\frac{1}{8})\widetilde{\sigma}_{\mathbf{y}}^2} = \infty. \tag{4.51}$$

In view of (4.51) and using the continuity of the function $t \mapsto \frac{e^{(t\frac{2}{3}-\frac{1}{8})\widetilde{\sigma}_{\mathbf{y}}^2}}{(t\frac{2}{3}-\frac{1}{8})\widetilde{\sigma}_{\mathbf{y}}^2}$, we conclude: if the material parameters $\mu, \kappa > 0$ and $\widetilde{\sigma}_{\mathbf{y}} \in \mathbb{R}$ are chosen such that

$$0 < \widetilde{\sigma}_{\mathbf{y}}^2 e^{\frac{1}{8}\widetilde{\sigma}_{\mathbf{y}}^2+1} \leq \frac{6\kappa}{\mu}, \tag{4.52}$$

then we may find a constant $k > \frac{3}{16}$ which satisfies (4.49). If the yield limit $\boldsymbol{\sigma}_{\mathbf{y}}$ is chosen such that

$$0 < \sqrt{\frac{2}{3}}\frac{\boldsymbol{\sigma}_{\mathbf{y}}}{2\mu} \leq \sqrt{\frac{2}{3}}\widetilde{\sigma}_{\mathbf{y}} e^{k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2}, \tag{4.53}$$

then in view of the monotonicity $t \mapsto te^{kt^2}$, by (4.47) and (4.53) we have that $\|\operatorname{dev}_3 X\| \leq \sqrt{\frac{2}{3}}\widetilde{\sigma}_{\mathbf{y}}$, which means that $X \in \mathcal{E}(W_{\mathrm{eH}}, \text{TSTS-M}^+, \log V, \frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2)$. Since $\widetilde{\sigma}_{\mathbf{y}}$ satisfies (4.52), it follows that there is $k > \frac{3}{16}$ satisfying (4.49). For the 3-parameter energy $(2k = 3\widehat{k})$, in view of (4.34)–(4.40), if (4.49) is satisfied, then it follows that the TSTS-M$^+$ inequality is satisfied and the proof is complete. $\qquad\square$

Remark 4.10

(i) In terms of Young's modulus and Poisson's ratio, the condition imposed on the yield limit $\boldsymbol{\sigma}_{\mathbf{y}}$ by Proposition 4.9 may be written in the form

$$0 < \boldsymbol{\sigma}_{\mathbf{y}} \leq \frac{E}{1+\nu}\widetilde{\sigma}_{\mathbf{y}} e^{k\frac{2}{3}\widetilde{\sigma}_{\mathbf{y}}^2}, \quad \text{where} \quad \widetilde{\sigma}_{\mathbf{y}}^2 e^{\frac{1}{8}\widetilde{\sigma}_{\mathbf{y}}^2+1} \leq \frac{4(1+\nu)}{1-2\nu}. \tag{4.54}$$

(ii) For illustrating purposes let us consider the case of $\nu = 1/3$ and an extremely large domain of roughly 10 % distortional strain, i.e., $\|\operatorname{dev}_3 \log U\| \leq 0.1$. To this specification corresponds $\widetilde{\sigma}_{\mathbf{y}} = \sqrt{\frac{3}{2}}0.1$, which is in concordance with the values considered for the yield stress $\widetilde{\sigma}_{\mathbf{y}} = \sqrt{\frac{3}{2}}0.1$ ($\|\operatorname{dev}_3 \log U\| \leq 0.1$), since $\frac{3}{2}0.01e^{\frac{1}{8}\frac{3}{2}0.01+1} \simeq 0.041$. Moreover, the required inequality (4.50), $\frac{e^{0.01\cdot k}}{16k-3} \leq \frac{1+\nu}{6(1-2\nu)}$ is satisfied if the parameter k belongs to the interval $[0.29, 919] \subset [\frac{3}{16}, 919]$.

(iii) We will encounter $k > \frac{3}{16}$ also later on with regard to rank-one convexity conditions for W_{eH}.

4.4 TSTS-M$^+$ for the Quadratic Hencky Energy

For comparison, we also consider the quadratic Hencky energy

$$\widehat{W}_{\mathrm{H}}(U) := \mu \|\operatorname{dev}_n \log U\|^2 + \frac{\kappa}{2}\big[\operatorname{tr}(\log U)\big]^2. \tag{4.55}$$

We recall that the corresponding Kirchhoff and the Cauchy stress tensors are given by

$$
\begin{aligned}
\tau_{\mathrm{H}} &= D_{\log V}\,\widetilde{W}_{\mathrm{H}}(V) = 2\mu\,\operatorname{dev}_3 \log V + \kappa\,\operatorname{tr}(\log V)\cdot\mathbb{1}, \\
\sigma_{\mathrm{H}} &= \big[2\mu\,\operatorname{dev}_3 \log V + \kappa\,\operatorname{tr}(\log V)\cdot\mathbb{1}\big]e^{-\operatorname{tr}(\log V)}.
\end{aligned}
\tag{4.56}
$$

The monotonicity inequality (4.11) becomes

$$
\begin{aligned}
\big\langle D_X\sigma_{\mathrm{H}}(X).H, H\big\rangle = \big\{ & 2\mu\|\operatorname{dev}_3 H\|^2 + \kappa\big[\operatorname{tr}(H)\big]^2 - 2\mu\,\operatorname{tr}(H)\langle\operatorname{dev}_3 X, \operatorname{dev}_3 H\rangle \\
& - \kappa\,\operatorname{tr}(X)\big[\operatorname{tr}(H)\big]^2\big\}e^{-\operatorname{tr}(X)} \ge 0.
\end{aligned}
$$

If $X = \mathbb{1}$ we have

$$\big\langle D_X\sigma_{\mathrm{H}}(\mathbb{1}).H, H\big\rangle = \big[2\mu\|\operatorname{dev}_3 H\|^2 - 2\kappa\big[\operatorname{tr}(H)\big]^2\big]e^{-3}, \tag{4.57}$$

which is negative for all H such that $\|\operatorname{dev}_3 H\|^2 < [\operatorname{tr}(H)]^2$. Jog and Patil [121, page 676] have proved that the quadratic Hencky energy satisfies the TSTS-M$^+$ conditions only for those deformations for which $\det V < e$. This bound coincides, incidentally, with the loss of ellipticity for W_{eH} in a uniaxial setting.

4.5 TSTS-M$^+$ for the Energy $F \mapsto \mu e^{a\|\operatorname{dev}_3 \log V\|^2 + \frac{\widehat{a}}{2}(\operatorname{tr}(\log V))^2}$

At the end of this subsection we consider the energy

$$W(\log V) := \mu e^{a\|\operatorname{dev}_3 \log V\|^2 + \frac{\widehat{a}}{2}(\operatorname{tr}(\log V))^2} \tag{4.58}$$

with the corresponding Kirchhoff and Cauchy stress, respectively

$$
\begin{aligned}
\widehat{\tau}(\log V) &= \mu e^{a\|\operatorname{dev}_3 \log V\|^2 + \frac{\widehat{a}}{2}\operatorname{tr}(\log V)^2}\big\{2a\,\operatorname{dev}_3 \log V + \widehat{a}\,\operatorname{tr}(\log V)\cdot\mathbb{1}\big\}, \\
\widehat{\sigma}(\log V) &= \mu e^{a\|\operatorname{dev}_3 \log V\|^2 + \frac{\widehat{a}}{2}\operatorname{tr}(\log V)^2 - \operatorname{tr}(\log V)}\big\{2a\,\operatorname{dev}_3 \log V + \widehat{a}\,\operatorname{tr}(\log V)\cdot\mathbb{1}\big\},
\end{aligned}
\tag{4.59}
$$

and we try to determine $a, \widehat{a}$ such that this energy satisfies the TSTS-M condition.

The monotonicity inequality (4.11) becomes

$$
\begin{aligned}
\big\langle & D_X\widehat{\sigma}(X).H, H\big\rangle \\
&= \mu e^{a\|\operatorname{dev}_3 X\|^2 + \frac{\widehat{a}}{2}\operatorname{tr}(X)^2 - \operatorname{tr}(X)}\big\{\big[2a\langle\operatorname{dev}_3 X, H\rangle + \widehat{a}\,\operatorname{tr}(X)\operatorname{tr}(H)\big]^2 \\
&\quad - \operatorname{tr}(H)\big[2a\langle\operatorname{dev}_3 X, H\rangle + \widehat{a}\,\operatorname{tr}(X)\operatorname{tr}(H)\big]\big\} \\
&\quad + \mu e^{a\|\operatorname{dev}_3 X\|^2 + \frac{\widehat{a}}{2}\operatorname{tr}(X)^2 - \operatorname{tr}(X)}\big\{2a\|\operatorname{dev}_3 H\|^2 + \widehat{a}\big[\operatorname{tr}(H)\big]^2\big\} \\
&= \mu e^{a\|\operatorname{dev}_3 X\|^2 + \frac{\widehat{a}}{2}\operatorname{tr}(X)^2 - \operatorname{tr}(X)}\big\{\big[2a\langle\operatorname{dev}_3 X, H\rangle + \widehat{a}\,\operatorname{tr}(X)\operatorname{tr}(H)\big]^2
\end{aligned}
$$

$$- \operatorname{tr}(H) \big[2a \langle \operatorname{dev}_3 X, H \rangle + \widehat{a} \operatorname{tr}(X) \operatorname{tr}(H) \big] + 2a \| \operatorname{dev}_3 H \|^2 + \widehat{a} \big[\operatorname{tr}(H) \big]^2 \big\}. \quad (4.60)$$

Using the inequality of means, $xy < \frac{\alpha}{2} x^2 + \frac{1}{2\alpha} y^2$, $\alpha > 0$, we deduce

$$\big[2a \langle \operatorname{dev}_3 X, H \rangle + \widehat{a} \operatorname{tr}(X) \operatorname{tr}(H) \big]^2 - \operatorname{tr}(H) \big[2a \langle \operatorname{dev}_3 X, H \rangle + \widehat{a} \operatorname{tr}(X) \operatorname{tr}(H) \big] + \widehat{a} \big[\operatorname{tr}(H) \big]^2$$

$$\geq \left(1 - \frac{\alpha}{2} \right) \big[2a \langle \operatorname{dev}_3 X, H \rangle + \widehat{a} \operatorname{tr}(X) \operatorname{tr}(H) \big]^2 + \left(\widehat{a} - \frac{1}{2\alpha} \right) \big[\operatorname{tr}(H) \big]^2 \quad \forall \alpha > 0. \quad (4.61)$$

Hence, choosing the dimensionless parameters $\widehat{a}, \alpha > 0$ such that

$$\frac{1}{4} < \widehat{a}, \qquad \frac{1}{2\widehat{a}} < \alpha < 2, \quad (4.62)$$

we have

$$\big\langle D_X \widehat{\sigma}(X).H, H \big\rangle \geq 0 \quad \forall X, H \in \operatorname{Sym}(3). \quad (4.63)$$

Therefore, if $\widehat{a} > \frac{1}{4}$, then the energy $F \mapsto \mu e^{a \| \operatorname{dev}_3 \log V \|^2 + \frac{\widehat{a}}{2} \operatorname{tr}(\log V)^2}$ satisfies the TSTS-M condition. For instance, if we choose $\widehat{a} = \frac{\kappa}{\mu}$, the condition $\widehat{a} > \frac{1}{4}$ is equivalent to

$$\frac{1}{4} < \frac{\kappa}{\mu} \quad \Leftrightarrow \quad 1 < \frac{8(1 + \nu)}{3(1 - 2\nu)} \quad \Leftrightarrow \quad -\frac{5}{14} < \nu. \quad (4.64)$$

If we choose $\widehat{a} = \frac{\kappa}{4\mu}$, the condition $\widehat{a} > \frac{1}{4}$ is equivalent to[26]

$$\mu < \kappa \quad \Leftrightarrow \quad 1 < \frac{2(1 + \nu)}{3(1 - 2\nu)} \quad \Leftrightarrow \quad \frac{1}{8} < \nu < \frac{1}{2}. \quad (4.65)$$

In conclusion, we observe that we do not need to consider a restricted domain for the energy (4.58) in order to enforce the TSTS-M$^+$ condition.

5 Rank-One Convexity

5.1 Criteria for Rank-One Convexity

In this subsection we recall some criteria for rank-one-convexity that we will use throughout the rest of this paper. Knowles and Sternberg [128, 129] (see also [11, 12, 127]) have given the following result:

Theorem 5.1 (Knowles and Sternberg [226, page 318]) *Let $W : \operatorname{GL}^+(n) \to \mathbb{R}$ be an objective-isotropic function of class C^2 with the representation in terms of the singular values of U via $W(F) = \widehat{W}(U) = g(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $g \in C^2(\mathbb{R}^n_+, \mathbb{R})$. Let $F \in \operatorname{GL}^+(n)$ be given with an n-tuple of singular values $\lambda_1, \lambda_2, \ldots, \lambda_n$. If $D^2 W(F)[a \otimes b, a \otimes b] \geq 0$ for every $a, b \in \mathbb{R}^n$ (i.e., $F \mapsto W(F)$ is rank-one convex), the following conditions hold:*

(i) *$\frac{\partial^2 g}{\partial \lambda_i^2} \geq 0$ for every $i = 1, 2, \ldots, n$, i.e., separate convexity (SC) and the TE-inequalities hold;*

---

[26]In [159] it is claimed that the classical elasticity formulation is applicable only for $\frac{1}{5} < \nu < \frac{1}{2}$.

(ii) *for every $i \neq j$,*

$$\underbrace{\frac{\lambda_i \frac{\partial g}{\partial \lambda_i} - \lambda_j \frac{\partial g}{\partial \lambda_j}}{\lambda_i - \lambda_j} \geq 0}_{\text{"BE-inequalities"}} \quad \text{if} \quad \lambda_i \neq \lambda_j, \quad \text{and} \quad \frac{\partial^2 g}{\partial \lambda_i^2} - \frac{\partial^2 g}{\partial \lambda_i \partial \lambda_j} + \frac{\partial g}{\partial \lambda_i} \frac{1}{\lambda_i} \geq 0 \quad \text{if} \ \lambda_i = \lambda_j,$$

$$\sqrt{\frac{\partial^2 g}{\partial \lambda_i^2} \frac{\partial^2 g}{\partial \lambda_j^2}} + \frac{\partial^2 g}{\partial \lambda_i \partial \lambda_j} + \frac{\frac{\partial g}{\partial \lambda_i} - \frac{\partial g}{\partial \lambda_j}}{\lambda_i - \lambda_j} \geq 0 \quad \text{if} \ \lambda_i \neq \lambda_j, \tag{5.1}$$

$$\sqrt{\frac{\partial^2 g}{\partial \lambda_i^2} \frac{\partial^2 g}{\partial \lambda_j^2}} - \frac{\partial^2 g}{\partial \lambda_i \partial \lambda_j} + \frac{\frac{\partial g}{\partial \lambda_i} + \frac{\partial g}{\partial \lambda_j}}{\lambda_i + \lambda_j} \geq 0.$$

If $n = 2$, then conditions (i) *and* (ii) *are also sufficient.*

From the above theorem we can easily see that LH-ellipticity implies the BE-inequalities and TE-inequalities. Necessary and sufficient conditions for LH-ellipticity in the three-dimensional case are given in [204, 236] and more recently by Dacorogna [54], also for compressible materials.

Theorem 5.2 (Dacorogna [54, page 5]) *Let $W : \mathrm{GL}^+(3) \to \mathbb{R}$ be an objective-isotropic function of class C^2 with the representation in terms of the singular values of U via $W(F) = \widehat{W}(U) = g(\lambda_1, \lambda_2, \lambda_3)$, where $g \in C^2(\mathbb{R}_+^3, \mathbb{R})$ and g is symmetric. Then $F \mapsto W(F)$ is rank one convex if and only if the following four sets of conditions hold for every $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_+$*

(i) $\frac{\partial^2 g}{\partial \lambda_i^2} \geq 0$ *for every $i = 1, 2, 3$, i.e., separate convexity* (SC) *and the TE-inequalities hold;*

(ii) *for every $i \neq j$,*

$$\underbrace{\frac{\lambda_i \frac{\partial g}{\partial \lambda_i} - \lambda_j \frac{\partial g}{\partial \lambda_j}}{\lambda_i - \lambda_j} \geq 0}_{\text{"BE-inequalities"}} \quad \text{if} \ \lambda_i \neq \lambda_j, \quad \text{and} \quad \sqrt{\frac{\partial^2 g}{\partial \lambda_i^2} \frac{\partial^2 g}{\partial \lambda_j^2}} + m_{ij}^\varepsilon \geq 0, \quad \text{and either}$$

$$m_{12}^\varepsilon \sqrt{\frac{\partial^2 g}{\partial \lambda_3^2}} + m_{13}^\varepsilon \sqrt{\frac{\partial^2 g}{\partial \lambda_2^2}} + m_{23}^\varepsilon \sqrt{\frac{\partial^2 g}{\partial \lambda_1^2}} + \sqrt{\frac{\partial^2 g}{\partial \lambda_1^2} \frac{\partial^2 g}{\partial \lambda_2^2} \frac{\partial^2 g}{\partial \lambda_3^2}} \geq 0 \quad \text{or} \tag{5.2}$$

$$\det M^\varepsilon \geq 0,$$

where $M^\varepsilon = (m_{ij}^\varepsilon)$ is symmetric and

$$m_{ij}^\varepsilon = \begin{cases} \frac{\partial^2 g}{\partial \lambda_i^2} & \text{if } i = j \text{ or if } i < j \text{ and } \lambda_i = \lambda_j, \\ \varepsilon_i \varepsilon_j \frac{\partial^2 g}{\partial \lambda_i \partial \lambda_j} + \frac{\frac{\partial g}{\partial \lambda_i} - \varepsilon_i \varepsilon_j \frac{\partial g}{\partial \lambda_j}}{\lambda_i - \varepsilon_i \varepsilon_j \lambda_j} & \text{if } i < j \text{ and } \lambda_i \neq \lambda_j \text{ or } \varepsilon_i \varepsilon_j \neq 1, \end{cases} \tag{5.3}$$

for any choice of $\varepsilon_i \in \{\pm 1\}$.

The last one is taken from Buliga [42]:

Theorem 5.3 (Buliga [42, page 1538]) *A twice continuously differentiable function $W : \mathrm{GL}^+(n) \to \mathbb{R}$ that can be written as a function of the singular values of U via $W(F) = \widehat{W}(U) = g(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is rank-one-convex if and only if*

(i) *the function* $(\lambda_1, \lambda_2, \ldots, \lambda_n) \mapsto g(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n})$ *is Schur-convex and*
(ii) *for all* $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}_+^n$

$$\sum_{i,j} H_{ij}(\lambda_1, \lambda_2, \ldots, \lambda_n)a_i a_j + G_{ij}(\lambda_1, \lambda_2, \ldots, \lambda_n)|a_i||a_j| \geq 0, \tag{5.4}$$

where

$$G_{ij}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{\lambda_i \frac{\partial g}{\partial \lambda_i}(\lambda_1, \lambda_2, \ldots, \lambda_n) - \lambda_j \frac{\partial g}{\partial \lambda_j}(\lambda_1, \lambda_2, \ldots, \lambda_n)}{\lambda_i^2 - \lambda_j^2}$$

for $i \neq j$, $\quad G_{ii}(\lambda_1, \lambda_2, \ldots, \lambda_n) = 0$,

$$H_{ij}(\lambda_1, \lambda_2, \ldots, \lambda_n)) = \overline{H}_{ij}(\lambda_1, \lambda_2, \ldots, \lambda_n) + \left(D^2 g(\lambda_1, \lambda_2, \ldots, \lambda_n)\right)_{ij}, \tag{5.5}$$

$$\overline{H}_{ij}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{\lambda_j \frac{\partial g}{\partial \lambda_i}(\lambda_1, \lambda_2, \ldots, \lambda_n) - \lambda_i \frac{\partial g}{\partial \lambda_j}(\lambda_1, \lambda_2, \ldots, \lambda_n)}{\lambda_i^2 - \lambda_j^2}$$

for $i \neq j$, $\quad H_{ii}(\lambda_1, \lambda_2, \ldots, \lambda_n) = 0$.

Šilhavý [228] has previously given a similar result in terms of the copositivity of some matrices (see also [54]).

5.2 The LH-Condition for Incompressible Media

In this subsection we consider the case of incompressible materials, i.e., we consider objective-isotropic energies $W : \mathrm{SL}(3) \to \mathbb{R}$. The restrictions imposed by rank-one convexity are less strict in this case. The rank-one convexity for such a function W means that W still has to satisfy

$$D^2 W(F)(\xi \otimes \eta, \xi \otimes \eta) > 0, \tag{5.6}$$

(similar to the LH-ellipticity condition), but now only for all vectors $\xi, \eta \neq 0$ with the additional property that

$$\det(F + \xi \otimes \eta) = 1.$$

Remark 5.4 For $F, H \in \mathbb{R}^{3 \times 3}$ we have

$$\det(F + H) = \det F + \langle \mathrm{Cof}\, F, H \rangle + \langle F, \mathrm{Cof}\, H \rangle + \det H. \tag{5.7}$$

Proof Let us first consider that $\det H \neq 0$ and $\det F \neq 0$. We recall the formal expansion of the determinant

$$\det(\mathbb{1} + H) = 1 + \mathrm{tr}\, H + \mathrm{tr}(\mathrm{Cof}\, H) + \det H. \tag{5.8}$$

We have

$$\det(F + H) = \det\left((\mathbb{1} + HF^{-1})F\right) = \det(\mathbb{1} + HF^{-1})\det(F)$$
$$= \det(F)\left(\det(\mathbb{1}) + \mathrm{tr}(HF^{-1}) + \mathrm{tr}(\mathrm{Cof}(HF^{-1})) + \det(HF^{-1})\right)$$

$$\begin{aligned}
&= \det(F) + \det(F)\langle HF^{-1}, \mathbb{1}\rangle + \det(F)\langle\det\!\left(HF^{-1}\right)\!\left(HF^{-1}\right)^{-T}, \mathbb{1}\rangle \\
&\quad + \det(F)\det(H)\det\!\left(F^{-1}\right) \\
&= \det(F) + \det(F)\langle HF^{-1}, \mathbb{1}\rangle + \det(F)\det(H)\det\!\left(F^{-1}\right)\langle\mathbb{1}, FH^{-1}\rangle + \det(H) \\
&= \det(F) + \det(F)\langle HF^{-1}, \mathbb{1}\rangle + \det(H)\langle F, H^{-T}\rangle + \det(H) \\
&= \det(F) + \langle H, \mathrm{Cof}\,F\rangle + \langle F, \mathrm{Cof}(H)\rangle + \det(H).
\end{aligned} \tag{5.9}$$

By continuity this result holds as well for non-invertible matrices $F, H \in \mathbb{R}^{3\times3}$. $\qquad\square$

Thus, for $F \in \mathbb{R}^{3\times3}$

$$\det(F + \xi\otimes\eta) = \det F + \langle\mathrm{Cof}\,F, \xi\otimes\eta\rangle + \langle F, \underbrace{\mathrm{Cof}[\xi\otimes\eta]}_{=0}\rangle + \underbrace{\det[\xi\otimes\eta]}_{=0}$$

$$= \det F\left[1 + \langle F^{-T}, \xi\otimes\eta\rangle\right] = \det F\left[1 + \mathrm{tr}\!\left(F^{-1}\xi\otimes\eta\right)\right], \tag{5.10}$$

since $\mathrm{rank}(\xi\otimes\eta) = 1$. Hence, it follows that $\xi, \eta \neq 0$ have to satisfy

$$\det F \cdot \mathrm{tr}\!\left(F^{-1}\xi\otimes\eta\right) = \det F \cdot \langle F^{-1}\xi, \eta\rangle = 0 \quad\Leftrightarrow\quad \langle F^{-1}\xi, \eta\rangle = 0. \tag{5.11}$$

Necessary conditions for LH-ellipticity of incompressible, isotropic hyperelastic solids were obtained by Sawyers and Rivlin [211, 213], while necessary and sufficient conditions were established by Zubov and Rudev [269, 270].

Theorem 5.5 (Zubov's LH-ellipticity criterion for incompressible materials [270, page 437]) *Let $W : \mathrm{SL}(3) \to \mathbb{R}$ be an objective-isotropic function of class C^2 with the representation in terms of the singular values of U via $W(F) = \widehat{W}(U) = g(\lambda_1, \lambda_2, \lambda_3)$, where $g \in C^2(\mathbb{R}_+^3, \mathbb{R})$ and g is symmetric. Then $F \mapsto W(F)$ is rank one convex on $\mathrm{SL}(3)$ if and only if the following nine inequalities hold for every $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_+, \ \lambda_1\lambda_2\lambda_3 = 1$:*

(i) *for every $i \neq j$ and for any arbitrary permutation (i, j, k) of the numbers $1, 2, 3$*

$$\underbrace{\alpha_k := \frac{\lambda_i\frac{\partial g}{\partial\lambda_i} - \lambda_j\frac{\partial g}{\partial\lambda_j}}{\lambda_i - \lambda_j} > 0}_{\text{"BE-inequalities"}} \quad \text{if } \lambda_i \neq \lambda_j; \tag{5.12}$$

(ii) $\delta_k := \beta_i\lambda_i^2 + \beta_j\lambda_j^2 + 2\gamma_k^-\lambda_i\lambda_j > 0$, *where (i, j, k) is any arbitrary permutation of the numbers $1, 2, 3$, and*

$$\beta_i := \frac{\partial^2 g}{\partial\lambda_i^2}, \qquad \gamma_k^\pm = \pm\frac{\partial^2 g}{\partial\lambda_i\partial\lambda_j} + \frac{\frac{\partial g}{\partial\lambda_i} \mp \frac{\partial g}{\partial\lambda_j}}{\lambda_i \mp \lambda_j}; \tag{5.13}$$

(iii) $\epsilon_k + \sqrt{\delta_i\delta_j} > 0$, *where $\epsilon_k := \beta_k\lambda_k^2 + \gamma_i^+\lambda_i\lambda_j + \gamma_i^-\lambda_k\lambda_j + \gamma_j^-\lambda_k\lambda_i$ and (i, j, k) is any arbitrary permutation of the numbers $1, 2, 3$.* $\qquad\square$

5.3 The Quadratic Hencky Energy W_{H} is not Rank-One Convex

In this subsection we re-examine a counter-example first considered by Neff [165] in order to prove that the quadratic Hencky energy function W_{H} defined by (1.5) is not rank-one

convex even when restricted to SL(3). A domain where W_H is LH-elliptic has been given in [39] under some strong conditions upon the constitutive coefficients, i.e., $\mu, \lambda > 0$. The first proof of the non-ellipticity of a related energy expression $\| \text{dev}_3 \log U \|^N$, $0 < N \leq 1$ seems to be due to Hutchinson et al. [116].

Proposition 5.6 *The function* $W : \text{SL}(3) \to \mathbb{R}$, $W(F) = \| \text{dev}_3 \log U \|^2$ *is not LH-elliptic.*

Proof The proof of this remark is adapted from [165]. We consider the function $h : \mathbb{R} \to \mathbb{R}$,

$$h(t) = W\big(\mathbb{1} + t(\eta \otimes \xi)\big). \tag{5.14}$$

We choose the vectors $\eta, \xi \in \mathbb{R}^3$ so that (i.e., the family of simple shears)

$$\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \eta \otimes \xi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.15}$$

Hence

$$\big(\mathbb{1} + t(\eta \otimes \xi)\big)^T \big(\mathbb{1} + t(\eta \otimes \xi)\big) = \begin{pmatrix} 1 & t & 0 \\ t & 1 + t^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.16}$$

and from

$$\det \begin{pmatrix} 1 - \lambda & t & 0 \\ t & 1 + t^2 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{pmatrix} = (1 - \lambda)\big[\lambda^2 - \lambda(2 + t^2) + 1\big]$$

the eigenvalues of the matrix $(\mathbb{1} + y(\eta \otimes \xi))^T (\mathbb{1} + y(\eta \otimes \xi))$ can be seen to be

$$\lambda_1 = 1, \qquad \lambda_2 = \frac{1}{2}\big(2 + t^2 + t\sqrt{4 + t^2}\big), \qquad \lambda_3 = \frac{1}{2}\big(2 + t^2 - t\sqrt{4 + t^2}\big). \tag{5.17}$$

The matrix U is positive definite and symmetric and therefore can be assumed diagonal, to obtain

$$\| \text{dev}_3 \log U \|^2 = \frac{1}{3}\left(\log^2 \frac{\lambda_1}{\lambda_2} + \log^2 \frac{\lambda_2}{\lambda_3} + \log^2 \frac{\lambda_3}{\lambda_1}\right). \tag{5.18}$$

An analogous expression for $\| \text{dev}_n \log U \|^2$ can be given in any dimension $n \in \mathbb{N}$, see Appendix A.1. In terms of the eigenvalues, the function h is given by

$$h(t) = \frac{1}{3}\left(\log^2 \frac{\lambda_1}{\lambda_2} + \log^2 \frac{\lambda_2}{\lambda_3} + \log^2 \frac{\lambda_3}{\lambda_1}\right). \tag{5.19}$$

Since $\lambda_1 \lambda_2 \lambda_3 = 1$, see (5.17), it follows $0 = \log(\lambda_1 \lambda_2 \lambda_3) = \log \lambda_2 + \log \lambda_3$, $\log \lambda_2 = -\log \lambda_3$. Thus, $h(t) = 2 \log^2 \lambda_2 = 2[\log(2 + t^2 + t\sqrt{4 + t^2}) - \log 2]^2$. This function is not convex in t (see Fig. 15), as can be easily deduced. Let us remark that $\mathbb{1} \in \text{SL}(3)$ and also $(\mathbb{1} + t(\eta \otimes \xi)) \in \text{SL}(3)$. Therefore, the function W is not rank-one convex in SL(3). Hence, W is not elliptic in SL(3). $\qquad\square$

A direct consequence of the previous proposition is

Fig. 15 The graphical representation of $h : \mathbb{R} \to \mathbb{R}$, $h(t) = 2\log^2 \lambda_2(t) = 2[\log(2 + t^2 + t\sqrt{4 + t^2}) - \log 2]^2$
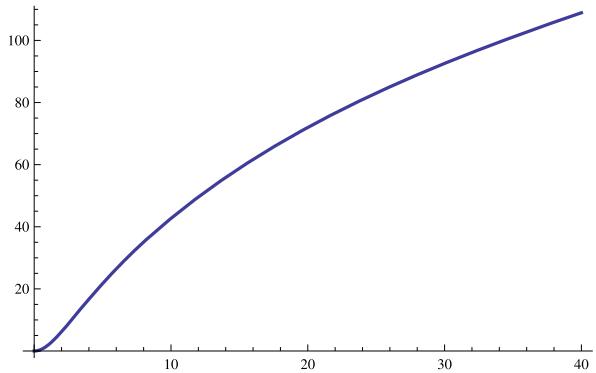
Remark 5.7 (Three-dimensional case) The function $W : \mathrm{SL}(3) \to \mathbb{R}$, $W(F) = \mu \| \mathrm{dev}_3 \log U \|^2 + \frac{\kappa}{2} (\mathrm{tr}(\log U))^2$, for any $\mu, \kappa > 0$, is not LH-elliptic.

Proof The counterexample is the one as in the proof of the previous remark because corresponding to this counterexample we have $\frac{\kappa}{2} (\mathrm{tr}(\log U))^2 = \log(\lambda_1 \lambda_2 \lambda_3) = \log 1 = 0$. □

Remark 5.8 (Two-dimensional case) The function $W : \mathrm{SL}(2) \to \mathbb{R}$, $W(F) = \mu \| \mathrm{dev}_2 \log U \|^2 + \frac{\kappa}{2} (\mathrm{tr}(\log U))^2$, for any $\mu, \kappa > 0$, is not LH-elliptic.

Proof The proof is similar to the proof in the 3D case. The vectors $\eta, \xi \in \mathbb{R}^2$ are now

$$\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

□

Proposition 5.9 *The energy* $W : \mathrm{SL}(3) \to \mathbb{R}$, $W(F) = \| \mathrm{dev}_3 \log U \|^2$ *satisfies the TE-inequalities (SC) only for those U such that the eigenvalues μ_1, μ_2, μ_3 of $\mathrm{dev}_3 \log U$ are smaller than $\frac{2}{3}$.*

Proof The corresponding function $g : \mathbb{R}_+^3 \to \mathbb{R}$ for the isotropic energy $W : \mathrm{SL}(3) \to \mathbb{R}$, $W(F) = \| \mathrm{dev}_3 \log U \|^2$ is

$$g(\lambda_1, \lambda_2, \lambda_3) := \frac{1}{3} \left[\log^2 \frac{\lambda_1}{\lambda_2} + \log^2 \frac{\lambda_2}{\lambda_3} + \log^2 \frac{\lambda_3}{\lambda_1} \right]. \tag{5.20}$$

Hence, we have to check where the function g is separately convex. We deduce

$$\frac{\partial^2 g}{\partial \lambda_1^2} = \frac{2}{\lambda_1^2} \left(2 - \log \frac{\lambda_1}{\lambda_2} + \log \frac{\lambda_3}{\lambda_1} \right) = \frac{6}{\lambda_1^2} \left(\frac{2}{3} - \frac{2}{3} \log \lambda_1 + \frac{1}{3} \log \lambda_2 + \frac{1}{3} \log \lambda_3 \right),$$

$$\frac{\partial^2 g}{\partial \lambda_2^2} = \frac{2}{\lambda_2^2} \left(2 - \log \frac{\lambda_2}{\lambda_3} + \log \frac{\lambda_1}{\lambda_2} \right) = \frac{6}{\lambda_2^2} \left(\frac{2}{3} - \frac{2}{3} \log \lambda_2 + \frac{1}{3} \log \lambda_3 + \frac{1}{3} \log \lambda_1 \right), \tag{5.21}$$

$$\frac{\partial^2 g}{\partial \lambda_3^2} = \frac{2}{\lambda_3^2} \left(2 - \log \frac{\lambda_3}{\lambda_1} + \log \frac{\lambda_2}{\lambda_3} \right) = \frac{6}{\lambda_3^2} \left(\frac{2}{3} - \frac{2}{3} \log \lambda_3 + \frac{1}{3} \log \lambda_1 + \frac{1}{3} \log \lambda_2 \right).$$

On the other hand, the eigenvalues μ_1, μ_2, μ_3 of $\text{dev}_3 \log U$ are

$$
\begin{aligned}
\mu_1 &= \frac{2}{3} \log \lambda_1 - \frac{1}{3} \log \lambda_2 - \frac{1}{3} \log \lambda_3, \\
\mu_2 &= -\frac{1}{3} \log \lambda_1 + \frac{2}{3} \log \lambda_2 - \frac{1}{3} \log \lambda_3, \\
\mu_3 &= -\frac{1}{3} \log \lambda_1 - \frac{1}{3} \log \lambda_2 + \frac{2}{3} \log \lambda_3,
\end{aligned}
\tag{5.22}
$$

and the proof is complete. $\qquad\square$

We can obtain a similar condition in terms of the eigenvalues of U instead of those of $\text{dev}_3 \log U$:

Corollary 5.10 *The energy* $W : \text{SL}(3) \to \mathbb{R}$, $W(F) = \| \text{dev}_3 \log U \|^2$ *satisfies the TE-inequalities only for those* U *such that the eigenvalues* $\lambda_1, \lambda_2, \lambda_3$ *of* U *satisfy*

$$
\lambda_1^2 \le e^2 \lambda_2 \lambda_3, \qquad \lambda_2^2 \le e^2 \lambda_3 \lambda_1, \qquad \lambda_3^2 \le e^2 \lambda_1 \lambda_2.
\tag{5.23}
$$

Proof From (5.21) we find that $g(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{3}[\log^2 \frac{\lambda_1}{\lambda_2} + \log^2 \frac{\lambda_2}{\lambda_3} + \log^2 \frac{\lambda_3}{\lambda_1}]$ is separately convex if and only if

$$
2 - \log \frac{\lambda_1^2}{\lambda_2 \lambda_3} \ge 0, \qquad 2 - \log \frac{\lambda_2^2}{\lambda_1 \lambda_3} \ge 0, \qquad 2 - \log \frac{\lambda_3^2}{\lambda_1 \lambda_2} \ge 0,
\tag{5.24}
$$

which are equivalent to the inequalities (5.23). $\qquad\square$

5.4 Convexity of the Volumetric Response $F \mapsto e^{\widehat{k}(\log \det F)^m}$

In the family of energies (1.4) which we consider, the volumetric response is modeled by a term of the form $F \mapsto e^{\widehat{k}(\log \det F)^m}$. In deriving convexity conditions, we first examine the conditions under which the more general form $\det F \mapsto h(\log \det F)$ is convex in $\det F$, which is clearly sufficient for LH-ellipticity (details can be found in Appendix A.3, see also [55, page 213] and [134]). Hence, we ask for the second derivative of $t \mapsto h(\log t)$ to be positive:

$$
\frac{d^2}{dt^2} h(\log t) = \frac{d}{dt} \left[h'(\log t) \frac{1}{t} \right] = h''(\log t) \frac{1}{t^2} - h'(\log t) \frac{1}{t^2} \ge 0.
\tag{5.25}
$$

Obviously, this is the case if and only if $h''(\log t) \ge h'(\log t)$ for all $t > 0$ and hence, if and only if for all $\xi \in \mathbb{R}$ $h''(\xi) \ge h'(\xi)$. Thus, $t \mapsto h(\log t)$ is convex if and only if h grows at least exponentially (see also Appendix A.3). This result is in concordance with the necessary conditions derived in the paper of Sendova and Walton [222].

Fix $m \in \mathbb{N}$. We want to find $\widehat{k}$ such that $h(\xi) = e^{\widehat{k}\xi^m}$ matches this criterion, i.e.,

$$
\widehat{k}^2 m^2 \xi^{2m-2} e^{\widehat{k}\xi^m} + \widehat{k} m(m-1) \xi^{m-2} e^{\widehat{k}\xi^m} \ge \widehat{k} m \xi^{m-1} e^{\widehat{k}\xi^m},
\tag{5.26}
$$

which is equivalent to $\widehat{k} m \xi^m - \xi + (m-1) \ge 0$. We compute the minimum of this expression. To this aim we solve the equation $\widehat{k} m^2 \xi^{m-1} - 1 = 0$ and we obtain $\xi = \widehat{k}^{-\frac{1}{m-1}} m^{-\frac{2}{m-1}}$.

Therefore

$$\min_{\xi \in \mathbb{R}} \left\{ \widehat{k} m \xi^m - \xi + (m-1) \right\} = \widehat{k} m \widehat{k}^{-\frac{m}{m-1}} m^{-\frac{2m}{m-1}} - \widehat{k}^{-\frac{1}{m-1}} m^{-\frac{2}{m-1}} + (m-1)$$

$$= \widehat{k}^{-\frac{1}{m-1}} m^{-\frac{m+1}{m-1}} (1-m) + (m-1). \tag{5.27}$$

This minimum is nonnegative if and only if $-\widehat{k}^{-\frac{1}{m-1}} m^{-\frac{m+1}{m-1}} + 1 \geq 0$. Thus $\widehat{k}$ has to be chosen such that $\widehat{k} \geq m^{-(m+1)}$. In conclusion:

Lemma 5.11 *Let* $m \in \mathbb{N}$. *Then the function* $t \mapsto e^{\widehat{k}(\log(t))^m}$ *is convex if and only if* $\widehat{k} \geq \frac{1}{m^{(m+1)}}$.

This implies our next result:

Proposition 5.12 *The function*

$$\det F \mapsto e^{\widehat{k}(\log \det F)^m}, \quad F \in \mathrm{GL}^+(n)$$

is convex in $\det F$ *for* $\widehat{k} \geq \frac{1}{m^{(m+1)}}$. (*More explicitly, for* $m = 2$ *this means* $\widehat{k} \geq \frac{1}{8}$, *in case of* $m = 3$ *convexity holds for* $\widehat{k} \geq \frac{1}{81}$.)

In view of Proposition A.2 (see also [55, page 213]), we have

Corollary 5.13 *The function*

$$F \mapsto e^{\widehat{k}(\log \det F)^m}, \quad F \in \mathrm{GL}^+(n)$$

is rank-one convex in F *for* $\widehat{k} \geq \frac{1}{m^{(m+1)}}$. (*More explicitly, for* $m = 2$ *this means* $\widehat{k} \geq \frac{1}{8}$, *in case of* $m = 3$ *rank-one convexity holds for* $\widehat{k} \geq \frac{1}{81}$.)

5.5 Rank-One Convexity of the Isochoric Exponentiated Hencky Energy in Plane Elastostatics

In this subsection we consider a variant of the exponentiated Hencky energy in plane strain, with isochoric part

$$W_{\mathrm{eH}}^{\mathrm{iso}}(F) = e^{k\| \operatorname{dev}_2 \log U \|^2} = e^{k\| \log \frac{U}{\det U^{1/2}} \|^2}. \tag{5.28}$$

Let us first recall that for small strains the exponentiated Hencky energy turns into the well-known quadratic Hencky energy:

$$W_{\mathrm{eH}}(F) - \left(\frac{\mu}{k} + \frac{\kappa}{2\widehat{k}} \right) = \underbrace{\frac{\mu}{k} e^{k\| \operatorname{dev}_n \log U \|^2} + \frac{\kappa}{2\widehat{k}} e^{\widehat{k}[\operatorname{tr}(\log U)]^2}}_{\text{fully nonlinear elasticity}} - \left(\frac{\mu}{k} + \frac{\kappa}{2\widehat{k}} \right)$$

$$= \underbrace{\mu \| \operatorname{dev}_n \log U \|^2 + \frac{\kappa}{2} \left[(\log \det U) \right]^2}_{\text{materially linear, geometrically nonlinear elasticity}} + \text{h.o.t.}$$

$$= \underbrace{W_{\mathrm{H}}(F)}_{\text{quadratic Hencky energy}} + \text{h.o.t.}$$

$$= \underbrace{\mu \| \operatorname{dev}_n \operatorname{sym}\nabla u \|^2 + \frac{\kappa}{2}\left[\operatorname{tr}(\operatorname{sym}\nabla u)\right]^2}_{\text{linear elasticity}} + \text{h.o.t.,} \qquad (5.29)$$

where $u : \mathbb{R}^n \to \mathbb{R}^n$ is the displacement and $F = \nabla\varphi = \mathbb{1} + \nabla u$ is the gradient of the deformation $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ and h.o.t. denotes higher order terms of $\| \operatorname{dev}_n \log U \|^2$ and $\frac{\kappa}{2}[(\log \det U)]^2$.

Remark 5.14

(i) If $F \mapsto W(F)$ is rank-one convex in $\operatorname{GL}^+(n)$ and if $Z : \mathbb{R}_+ \to \mathbb{R}$ is a convex and monotone non-decreasing function, then the composition function $F \mapsto (Z \circ W)(F)$ is also rank-one convex in $\operatorname{GL}^+(n)$. This follows from the fact that if $t \mapsto h(t), t \in \mathbb{R}$, $h(t) = W(F + t(\eta \otimes \xi))$ is convex, then $t \mapsto Z(h(t)), t \in \mathbb{R}$, is also convex.

(ii) If $F \mapsto W(F)$ is quasi-convex in $\operatorname{GL}^+(n)$ and if $Z : \mathbb{R}_+ \to \mathbb{R}$ a convex and monotone non-decreasing function, then the function $F \mapsto (Z \circ W)(F)$ is also quasi-convex in $\operatorname{GL}^+(n)$. To prove this fact, let us recall that quasiconvexity of the energy function W at $\overline{F}$ means that $\frac{1}{|\Omega|} \int_\Omega W(\overline{F} + \nabla\vartheta) dx \geq W(\overline{F})$, holds, for every bounded open set $\Omega \subset \mathbb{R}^n$ and for all $\vartheta \in C_0^\infty(\Omega)$ such that $\det(\overline{F} + \nabla\vartheta) > 0$. Using the monotonicity of Z we deduce $Z(\frac{1}{|\Omega|} \int_\Omega W(\overline{F} + \nabla\vartheta) dx) \geq Z(W(\overline{F}))$. Hence, using the convexity and Jensen's inequality, we obtain $\frac{1}{|\Omega|} \int_\Omega Z(W(\overline{F} + \nabla\vartheta)) dx \geq Z(W(\overline{F}))$.

(iii) If $F \mapsto W(F)$ is polyconvex in $\operatorname{GL}^+(3)$ and if $Z : \mathbb{R}_+ \to \mathbb{R}$ is a convex and monotone non-decreasing function, then the function $F \mapsto (Z \circ W)(F)$ is also polyconvex in $\operatorname{GL}^+(3)$. A free energy function $W(F)$ is called polyconvex if and only if it is expressible in the form $W(F) = P(F, \operatorname{Cof} F, \det F)$, $P : \mathbb{R}^{19} \to \mathbb{R}$, where $P(\cdot, \cdot, \cdot)$ is convex. If P is convex, then $Z \circ P$ is also convex. In this case, we have $Z(W(F)) = (Z \circ P)(F, \operatorname{Cof} F, \det F)$, which means that $F \mapsto (Z \circ W)(F)$ is polyconvex in $\operatorname{GL}^+(3)$.

An example of a convex and monotone non-decreasing function $Z : \mathbb{R}_+ \to \mathbb{R}$ is the exponential function $Z(\xi) = e^\xi$.

We prove in this subsection that although $F \mapsto \| \operatorname{dev}_2 \log U \|^2$ is not rank-one convex, the function $F \mapsto e^{k\| \operatorname{dev}_2 \log U \|^2}$, $k > \frac{1}{4}$ is indeed rank-one convex.

Lemma 5.15 *Let $F \in GL^+(2)$ with singular values λ_1, λ_2. Then*

$$W(F) = e^{k\| \operatorname{dev}_2 \log U \|^2} = e^{k\| \log \frac{U}{\det U^{1/2}} \|^2} = g(\lambda_1, \lambda_2),$$

$$\text{where } g : \mathbb{R}_+^2 \to \mathbb{R}, g(\lambda_1, \lambda_2) := e^{\frac{k}{2}(\log \frac{\lambda_1}{\lambda_2})^2}. \qquad (5.30)$$

Proof The matrix U is positive definite and symmetric and therefore can be assumed diagonal, and we obtain

$$\| \operatorname{dev}_2 \log U \|^2 = \left\| \log U - \frac{1}{2}(\log \lambda_1 + \log \lambda_2)\mathbb{1} \right\|^2$$

$$= \left\| \begin{pmatrix} \frac{1}{2}\log \lambda_1 - \frac{1}{2}\log \lambda_2 & 0 \\ 0 & \frac{1}{2}\log \lambda_2 - \frac{1}{2}\log \lambda_1 \end{pmatrix} \right\|^2$$

$$= \frac{1}{4}\Big[2(\log\lambda_1 - \log\lambda_2)^2\Big] = \frac{1}{2}\bigg(\log\frac{\lambda_1}{\lambda_2}\bigg)^2.$$

With this, the proof is complete. □

In this subsection we apply Theorem 5.1 in order to prove that the function $F \mapsto e^{k\|\operatorname{dev}_2 \log U\|^2}$ is LH-elliptic. Thus, according to Lemma 5.15, we have to prove that the function

$$g : \mathbb{R}^2_+ \to \mathbb{R}, \qquad g(\lambda_1, \lambda_2) := e^{\frac{k}{2}(\log\frac{\lambda_1}{\lambda_2})^2}$$

satisfies all the necessary and sufficient conditions established by Knowles and Sternberg's Theorem 5.1. The first condition from Theorem 5.1 requests separate convexity in each variable λ_1, λ_2.

Lemma 5.16 *The function g is separately convex in each variable λ_1, λ_2, i.e., $\frac{\partial^2 g}{\partial\lambda_1^2} \geq 0$, $\frac{\partial^2 g}{\partial\lambda_2^2} \geq 0$, if and only if $k \geq \frac{1}{4}$.*

Proof We need to compute

$$\frac{\partial g}{\partial\lambda_1} = \frac{k\log\frac{\lambda_1}{\lambda_2}e^{\frac{k}{2}\log^2\frac{\lambda_1}{\lambda_2}}}{\lambda_1}, \qquad \frac{\partial g}{\partial\lambda_2} = -\frac{k\log\frac{\lambda_1}{\lambda_2}e^{\frac{k}{2}\log^2\frac{\lambda_1}{\lambda_2}}}{\lambda_2},$$

$$\frac{\partial^2 g}{\partial\lambda_1^2} = \frac{ke^{\frac{k}{2}\log^2\frac{\lambda_1}{\lambda_2}}}{\lambda_1^2}\bigg(k\log^2\frac{\lambda_1}{\lambda_2} - \log\frac{\lambda_1}{\lambda_2} + 1\bigg), \tag{5.31}$$

$$\frac{\partial^2 g}{\partial\lambda_2^2} = \frac{ke^{\frac{k}{2}\log^2\frac{\lambda_1}{\lambda_2}}}{\lambda_2^2}\bigg(k\log^2\frac{\lambda_1}{\lambda_2} + \log\frac{\lambda_1}{\lambda_2} + 1\bigg).$$

We introduce the function $r : \mathbb{R} \to \mathbb{R}$ given by $r(t) = kt^2 - t + 1$. It is clear that if $k \geq \frac{1}{4}$, then $r(t) = kt^2 - t + 1 \geq (\frac{1}{2}t - 1)^2 \geq 0$ for all $t \in \mathbb{R}$. Moreover, if $r(t) \geq 0$ for all $t \in \mathbb{R}$, then $k \geq \frac{1}{4} = \max_{t\in(0,\infty)}\{\frac{t-1}{t^2}\}$. Thus, $r(t) \geq 0$ for all $t \in \mathbb{R}$ if and only if $k \geq \frac{1}{4}$. In consequence, we deduce

$$\frac{\partial^2 g}{\partial\lambda_1^2}(\lambda_1, \lambda_2) = ke^{\frac{k}{2}\log^2\frac{\lambda_1}{\lambda_2}}\frac{1}{\lambda_1^2}r\bigg(\log\bigg(\frac{\lambda_1}{\lambda_2}\bigg)\bigg) \geq 0 \quad \text{if and only if } k \geq \frac{1}{4}. \tag{5.32}$$

Analogously, we have $\frac{\partial^2 g}{\partial\lambda_2^2}(\lambda_1, \lambda_2) \geq 0$ if and only if $k \geq \frac{1}{4}$. □

Lemma 5.17 *The function g satisfies the BE-inequalities.*

Proof For the function g defined by (5.30), the BE-inequalities become

$$\frac{\lambda_1\frac{\partial g}{\partial\lambda_1} - \lambda_2\frac{\partial g}{\partial\lambda_2}}{\lambda_1 - \lambda_2} = \frac{2k\log\frac{\lambda_1}{\lambda_2}e^{\frac{k}{2}\log^2\frac{\lambda_1}{\lambda_2}}}{\lambda_1 - \lambda_2} \geq 0 \quad \text{if } \lambda_1 \neq \lambda_2, \tag{5.33}$$

which is always true. Indeed, this fact also follows directly from Theorem 2.7 because g is convex as a function of $\log U$ (see Remark 2.9). □

Let us also compute

$$\frac{\partial^2 g}{\partial \lambda_1 \partial \lambda_2} = -\frac{k e^{\frac{k}{2} \log^2 \frac{\lambda_1}{\lambda_2}}}{\lambda_1 \lambda_2} \left(k \log^2 \frac{\lambda_1}{\lambda_2} + 1 \right). \tag{5.34}$$

The next set of inequalities from Knowles and Sternberg's criterion requires that the following quantities

$$\frac{\partial^2 g}{\partial \lambda_1^2} - \frac{\partial^2 g}{\partial \lambda_1 \partial \lambda_2} + \frac{1}{\lambda_1} \frac{\partial g}{\partial \lambda_1} = \frac{k(\lambda_1 + \lambda_2) e^{\frac{k}{2} \log^2 \frac{\lambda_1}{\lambda_2}} (k \log^2 \frac{\lambda_1}{\lambda_2} + 1)}{\lambda_1^2 \lambda_2},$$

$$\frac{\partial^2 g}{\partial \lambda_2^2} - \frac{\partial^2 g}{\partial \lambda_1 \partial \lambda_2} + \frac{1}{\lambda_2} \frac{\partial g}{\partial \lambda_2} = \frac{k(\lambda_1 + \lambda_2) e^{\frac{k}{2} \log^2 \frac{\lambda_1}{\lambda_2}} (k \log^2 \frac{\lambda_1}{\lambda_2} + 1)}{\lambda_1 \lambda_2^2}, \tag{5.35}$$

are positive for $\lambda_1 = \lambda_2$. This condition is always satisfied because $\lambda_1, \lambda_2, k > 0$.

In order to show that the last two inequalities from Knowles and Sternberg's Theorem 5.1 are satisfied, we compute

$$\sqrt{\frac{\partial^2 g}{\partial \lambda_1^2} \frac{\partial^2 g}{\partial \lambda_2^2}} + \frac{\partial^2 g}{\partial \lambda_1 \partial \lambda_2} + \frac{\frac{\partial g}{\partial \lambda_1} - \frac{\partial g}{\partial \lambda_2}}{\lambda_1 - \lambda_2} = \frac{k e^{\frac{k}{2} \log^2 \frac{\lambda_1}{\lambda_2}}}{\lambda_1 \lambda_2} \widetilde{g}(\lambda_1, \lambda_2), \quad \lambda_1 \neq \lambda_2,$$

$$\sqrt{\frac{\partial^2 g}{\partial \lambda_1^2} \frac{\partial^2 g}{\partial \lambda_2^2}} - \frac{\partial^2 g}{\partial \lambda_1 \partial \lambda_2} + \frac{\frac{\partial g}{\partial \lambda_1} + \frac{\partial g}{\partial \lambda_2}}{\lambda_1 + \lambda_2} = \frac{k e^{\frac{k}{2} \log^2 \frac{\lambda_1}{\lambda_2}}}{\lambda_1 \lambda_2} \widehat{g}(\lambda_1, \lambda_2), \tag{5.36}$$

where the functions $\widetilde{g} : \mathbb{R}_+^2 \setminus \{(x, x); x \in \mathbb{R}\} \to \mathbb{R}, \widehat{g} : \mathbb{R}_+^2 \to \mathbb{R}$ are defined by

$$\widetilde{g}(\lambda_1, \lambda_2) = \sqrt{\left(k \log^2 \left(\frac{\lambda_1}{\lambda_2} \right) + 1 \right)^2 - \log^2 \frac{\lambda_1}{\lambda_2}} - k \log^2 \frac{\lambda_1}{\lambda_2} - 1 + \frac{(\lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)} \log \frac{\lambda_1}{\lambda_2},$$

$$\widehat{g}(\lambda_1, \lambda_2) = \sqrt{\left(k \log^2 \left(\frac{\lambda_1}{\lambda_2} \right) + 1 \right)^2 - \log^2 \frac{\lambda_1}{\lambda_2}} + k \log^2 \frac{\lambda_1}{\lambda_2} + 1 - \frac{(\lambda_1 - \lambda_2)}{(\lambda_1 + \lambda_2)} \log \frac{\lambda_1}{\lambda_2}. \tag{5.37}$$

Let us remark that the functions $\widetilde{g}$ and $\widehat{g}$ can be written in terms of functions of a single variable only, i.e.,

$$\widetilde{g}(\lambda_1, \lambda_2) = \widetilde{r}\left(\frac{\lambda_1}{\lambda_2} \right), \qquad \widehat{g}(\lambda_1, \lambda_2) = \widehat{r}\left(\frac{\lambda_1}{\lambda_2} \right), \tag{5.38}$$

where $\widetilde{r} : \mathbb{R}_+ \setminus \{1\} \to \mathbb{R}, \widehat{r} : \mathbb{R}_+ \to \mathbb{R}$ are defined by

$$\widetilde{r}(t) = \sqrt{\left(k \log^2 t + 1 \right)^2 - \log^2 t} - \left(k \log^2 t + 1 \right) + \frac{t + 1}{t - 1} \log t,$$

$$\widehat{r}(t) = \sqrt{\left(k \log^2 t + 1 \right)^2 - \log^2 t} + \left(k \log^2 t + 1 \right) - \frac{t - 1}{t + 1} \log t. \tag{5.39}$$

Hence, Knowles and Sternberg's criterion is completely satisfied if and only if

$$\widetilde{r}(t) \geq 0 \quad \text{for all } t \in \mathbb{R}_+ \setminus \{1\} \quad \text{and} \quad \widehat{r}(t) \geq 0 \quad \text{for all } t \in \mathbb{R}_+. \tag{5.40}$$

We have to show the following inequality, which is the same as $(5.40)_1$:

$$\sqrt{\left(k\log^2 t + 1\right)^2 - \log^2 t} + 1 \geq \left(k\log^2 t + 1\right) - \frac{t+1}{t-1}\log t + 1. \tag{5.41}$$

In order to transform it equivalently by squaring both sides, first we prove the following lemma:

Lemma 5.18 *The inequality*

$$\left(k\log^2 t + 1\right) - \frac{t+1}{t-1}\log t + 1 \geq 0 \tag{5.42}$$

is satisfied for all $t \in \mathbb{R}_+ \setminus \{1\}$ if and only if $k \geq \frac{1}{6}$.

Proof Let us consider the function $\tilde{s} : \mathbb{R}_+ \setminus \{1\} \to \mathbb{R}$ by $\tilde{s}(t) := (\frac{1}{6}\log^2 t + 1) - \frac{t+1}{t-1}\log t$. For the function $\tilde{s}$ we compute

$$\tilde{s}'(t) = \frac{1}{3(-1+t)^2 t}\hat{s}(t), \tag{5.43}$$

where $\hat{s} : \mathbb{R}_+ \to \mathbb{R}$, $\hat{s}(t) = 3(1 - t^2) + (1 + 4t + t^2)\log t$.

On the other hand $\hat{s}'(t) = -5t + \frac{1}{t} + 2(t+2)\log t + 4$, $\hat{s}''(t) = \frac{4t-1}{t^2} + 2\log t - 3$, $\hat{s}'''(t) = \frac{2(t-1)^2}{t^3} \geq 0$ for all $t \in \mathbb{R}_+$, $\hat{s}(1) = 0$, $\hat{s}'(1) = 0$, $\hat{s}''(1) = 0$. Thus $\hat{s}''(t) \geq 0$ if $t \geq 1$ and $\hat{s}''(t) \leq 0$ if $0 < t \leq 1$, which implies further that $\hat{s}'$ is monotone decreasing on $(0, 1)$ and monotone increasing on $(1, \infty)$. We deduce $\hat{s}'(t) \geq \hat{s}'(1) = 0$ for all $0 < t \leq 1$ and $\hat{s}'(t) \geq \hat{s}'(1) = 0$ for all $t \geq 1$.

Hence, $\hat{s}$ is monotone increasing in $\mathbb{R}_+$, i.e., $\hat{s}(t) \leq \hat{s}(1) = 0$ for all $0 < t < 1$ and $\hat{s}(t) \geq \hat{s}(1) = 0$ for all $t > 1$. In view of (5.43), we have $\tilde{s}'(t) \leq 0$ for all $0 < t < 1$ and $\tilde{s}'(t) \geq 0$ for all $t > 1$. Because $\lim_{t\to 1}\tilde{s}(t) = -1$, the monotonicity of $\tilde{s}(t)$ implies $\tilde{s}(t) = (\frac{1}{6}\log^2 t + 1) - \frac{t+1}{t-1}\log t \geq \lim_{t_0\to 1}\tilde{s}(t_0) = -1$, for all $t \in \mathbb{R}_+ \setminus \{1\}$. For $k \geq \frac{1}{6}$, we have

$$\left(k\log^2 t + 1\right) - \frac{t+1}{t-1}\log t \geq \left(\frac{1}{6}\log^2 t + 1\right) - \frac{t+1}{t-1}\log t \geq -1, \tag{5.44}$$

for all $t \in \mathbb{R}_+ \setminus \{1\}$. On the other hand, if $(k\log^2 t + 1) - \frac{t+1}{t-1}\log t \geq -1$ for all $t \in \mathbb{R}_+ \setminus \{1\}$, then

$$k \geq \frac{1}{6} = \sup_{t \in \mathbb{R}_+}\left\{\frac{1}{\log^2 t}\left(-2 + \frac{t+1}{t-1}\log t\right)\right\}, \tag{5.45}$$

since the function $\tilde{s}_1 : \mathbb{R}_+ \to \mathbb{R}$, $\tilde{s}_1(t) = \log^2 t - 6\frac{t+1}{t-1}\log t + 12$ is monotone decreasing on $(0, 1]$, monotone increasing on $[1, \infty)$, $\tilde{s}_1(1) = 0$, and $\lim_{t \in \mathbb{R}_+}\{\frac{1}{\log^2 t}(-2 + \frac{t+1}{t-1}\log t)\} = \frac{1}{6}$. Thus, the inequality (5.42) holds for all $t \in \mathbb{R}_+ \setminus \{1\}$ if and only if $k \geq \frac{1}{6}$, $\qquad \square$

Lemma 5.19 *The inequality $\tilde{r}(t) \geq 0$ is satisfied for all $t \in \mathbb{R}_+ \setminus \{1\}$ if $k \geq \frac{1}{4}$.*

Proof Let us first remark that, in view of Lemma 5.18, we obtain $(k\log^2 t + 1) - \frac{t+1}{t-1}\log t + 1 \geq 0$ for all $k \geq \frac{1}{6}$. Hence, the inequality $\tilde{r}(t) \geq 0$ is equivalent to the inequality

$$\sqrt{\left(k\log^2 t + 1\right)^2 - \log^2 t} + 1 \geq \left(k\log^2 t + 1\right) - \frac{t+1}{t-1}\log t + 1 \geq 0, \tag{5.46}$$

for $t \in \mathbb{R}_+ \setminus \{1\}$, which can, by squaring and multiplication with $\frac{(t-1)^2}{2}$, equivalently be written in the following form:

$$k(t-1)\left[1 - t + (t+1)\log t\right]\log^2 t - \left\{\left[2(1-t^2) + (t^2+1)\log t\right]\log t + (t-1)^2\right\}$$

$$+ (t-1)^2\sqrt{\left(k\log^2 t + 1\right)^2 - \log^2 t} \geq 0. \tag{5.47}$$

Our next step is to prove that $s(t) \leq 0$ if $t < 1$ and $s(t) \geq 0$ if $t > 1$, where $s : \mathbb{R}_+ \to \mathbb{R}$ is defined by $s(t) = 1 - t + (t+1)\log t$. This follows from $s'(t) = \frac{1}{t} + \log t$, $s''(t) = \frac{t-1}{t^2}$, $s'(1) = 1$, $s(1) = 0$. Moreover, if $k \geq \frac{1}{4}$, we deduce

$$\sqrt{\left(k\log^2 t + 1\right)^2 - \log^2 t} \geq \sqrt{\left(\frac{1}{4}\log^2 t + 1\right)^2 - \log^2 t} = \sqrt{\left(\frac{1}{4}\log^2 t - 1\right)^2}$$

$$= \left|\frac{1}{4}\log^2 t - 1\right|, \tag{5.48}$$

and, due to the nonnegativity of $(t-1)s(t)$,

$$k(t-1)\left[1 - t + (t+1)\log t\right]\log^2 t - \left\{\left[2(1-t^2) + (t^2+1)\log t\right]\log t + (t-1)^2\right\}$$

$$\geq \frac{1}{4}(t-1)\left[1 - t + (t+1)\log t\right]\log^2 t - \left\{\left[2(1-t^2) + (t^2+1)\log t\right]\log t + (t-1)^2\right\}$$

$$= \frac{1}{4}\left\{8(t^2-1)\log t + (t^2-1)\log^3 t + (-5t^2 + 2t - 5)\log^2 t - 4(t-1)^2\right\}. \tag{5.49}$$

Hence, it is sufficient to prove that

$$(t-1)^2\left(\frac{1}{4}\log^2 t - 1\right) + \frac{1}{4}\left\{8(t^2-1)\log t + (t^2-1)\log^3 t\right.$$

$$\left. + (-5t^2 + 2t - 5)\log^2 t - 4(t-1)^2\right\}$$

$$= \frac{t^2-1}{4}\left(\log^3 t - 4\log^2 t + 8\log t - \frac{8(t-1)}{t+1}\right) \geq 0. \tag{5.50}$$

Employing the substitution $x = \log t$, we are going to show that

$$s_0(x) = x^3 - 4x^2 + 8x - 8\frac{e^x - 1}{e^x + 1} = x^3 - 4x^2 + 8x - 8 + \frac{16}{e^x + 1}$$

is negative for $x < 0$ and positive for $x > 0$.

Firstly, we observe that $s_0(0) = 0$ and $\lim_{x\to\infty} s_0(x) = \infty$. We then compute $s_0'(x) = s_1(x) - s_2(x)$, where we denote $s_1(x) = 3(x - \frac{4}{3})^2 + \frac{8}{3}$, $s_2(x) = 16\frac{e^x}{(e^x+1)^2}$. Due to the fact that $\frac{y}{(1+y)^2} \in (0, 4]$ for $y = e^x > 0$,

$$s_0'(x) \geq 3\left(\frac{4}{3}\right)^2 + \frac{8}{3} - 4 = 4 > 0 \quad \text{for } x < 0,$$

so that clearly $s_0(x) < 0$ for $x < 0$. To deduce $s_0(x) > 0$ for $x > 0$, we will prove that all local minima of s_0 are located in $(1, \infty)$ and that the value of s_0 is positive there. Because

$s_2''(x) = \frac{e^x}{(1+e^x)^4}(1 - 4e^x + e^{2x})$ is negative on $(-\infty, \log(\sqrt{3} + 2)) \supset (0, 1)$ and hence s_2 is concave and s_1 convex on $(0, 1)$, s_1 and s_2 can intersect in at most two points in $(0, 1)$. Thanks to the fact that $s_1(0) > s_2(0)$ and $s_1(1) < s_2(1)$, there is only one $x_m \in (0, 1)$, where $s_1(x_m) = s_2(x_m)$ and hence $s_0'(x_m) = 0$. In x_m, s_0 attains a maximum (s_0' is positive for smaller and negative for larger values of x), hence local minima of s must lie in $(1, \infty)$. In any such place x_0, from $s_0'(x_0) = 0$ we know $\frac{16}{e^{x_0}+1} = \frac{e^{x_0}+1}{e^{x_0}}(3x_0^2 - 8x_0 + 8)$ and hence

$$s_0(x_0) = x_0^3 - 4x_0^2 + 8x_0 - 8 + \left(1 + e^{-x_0}\right)\left(3x_0^2 - 8x_0 + 8\right)$$

$$= x_0^2(x_0 - 1) + 3e^{-x_0}\left(\left(x_0 - \frac{4}{3}\right)^2 + \frac{8}{3}\right) > 0,$$

because $x_0 \geq 1$. In conclusion, s_0 is positive on all of $(0, \infty)$, and negative in $(-\infty, 0)$.

Thus, the inequality (5.50) is satisfied. Therefore (5.46) is also satisfied and the proof is complete. $\qquad\square$

Lemma 5.20 *If $k \geq \frac{1}{4}$, then the inequality $\widehat{r}(t) \geq 0$ is satisfied for all $t \in \mathbb{R}_+$.*

Proof It is easy to see that for all $t \in \mathbb{R}_+ \setminus \{1\}$ and if $k \geq \frac{1}{4}$, we have

$$\left(k \log^2 t + 1\right) - \frac{t - 1}{t + 1} \log t \geq \frac{1}{4} \log^2 t - \frac{t - 1}{t + 1} \log t + 1.$$

Let us remark that $\frac{1}{4}\xi^2 - \frac{t-1}{t+1}\xi + 1 > 0$ for all $\xi \in \mathbb{R}$, since $(\frac{t-1}{t+1})^2 - 1 < 0$ and $\frac{1}{4} > 0$. Hence, taking $\xi = \log t \in \mathbb{R}$, we have $\frac{1}{4}\log^2 t - \frac{t-1}{t+1}\log t + 1 > 0$ for all $t \in \mathbb{R}_+$. Therefore,

$$\widehat{r}(t) = \sqrt{\left(k \log^2 t + 1\right)^2 - \log^2 t} + \left(k \log^2 t + 1\right) - \frac{t - 1}{t + 1} \log t > 0 \quad \text{for all } t \in \mathbb{R}_+,$$

which completes the proof. $\qquad\square$

Collecting Lemmas 5.16, 5.19, 5.20 and Eq. (5.33), we can finally conclude:

Proposition 5.21 *If $k \geq \frac{1}{4}$, then the function $F \mapsto e^{k\|\operatorname{dev}_2 \log U\|^2}$ is rank-one convex in $\mathrm{GL}^+(2)$.*

5.6 The Main Rank-One Convexity Statement

In view of the results established in Sects. 5.5 and 5.4 we conclude that:

Theorem 5.22 (Planar rank-one convexity) *The functions $W_{\mathrm{eH}} : \mathbb{R}^{n \times n} \to \overline{\mathbb{R}}_+$ from the family of exponentiated Hencky type energies*

$$W_{\mathrm{eH}}(F) = W_{\mathrm{eH}}^{\mathrm{iso}}\left(\frac{F}{\det F^{\frac{1}{n}}}\right) + W_{\mathrm{eH}}^{\mathrm{vol}}\left(\det F^{\frac{1}{n}} \cdot \mathbb{1}\right)$$

$$= \begin{cases} \dfrac{\mu}{k} e^{k\|\operatorname{dev}_n \log U\|^2} + \dfrac{\kappa}{2\widehat{k}} e^{\widehat{k}[(\log \det U)]^2} & \text{if } \det F > 0, \\ +\infty & \text{if } \det F \leq 0, \end{cases} \qquad (5.51)$$

are rank-one convex *for the two-dimensional situation $n = 2$, $\mu > 0$, $\kappa > 0$, $k \geq \frac{1}{4}$ and $\widehat{k} \geq \frac{1}{8}$.*

Conjecture 5.23 (Planar polyconvexity) *The functions* $W_{\mathrm{eH}} : \mathbb{R}^{n \times n} \to \overline{\mathbb{R}}_+$ *from the family of exponentiated Hencky type energies defined by* (5.51) *are* polyconvex[27] *for the two-dimensional situation* $n = 2$, $\mu > 0$, $\kappa > 0$, $k \geq \frac{1}{4}$ *and* $\widehat{k} \geq \frac{1}{8}$.

The rank-one convex energy $W_{\mathrm{eH}}(F)$ is applicable to the bending or shear of long strips and to all cases in which symmetry arguments can be applied to reduce the formulation to a planar deformation.

5.7 Formulation of the Dynamic Problem in the Planar Case

For the convenience of the reader we state the complete dynamic setting. The dynamic problem in the planar case consists in finding the solution $\varphi : \Omega \times (0, \infty) \to \mathbb{R}^2$, $\Omega \subset \mathbb{R}^2$ of the equation of motion

$$\varphi_{,tt} = \mathrm{Div} S_1(\nabla \varphi) \quad \text{in } \Omega \times (0, \infty), \tag{5.52}$$

where the first Piola-Kirchhoff stress tensor $S_1 = D_F[W(F)]$ corresponding to the energy $W_{\mathrm{eH}}(F)$ is given by the constitutive equation

$$S_1 = D_F\big[W(F)\big] = J\sigma F^{-T} = \tau F^{-T} \tag{5.53}$$

$$= \big[2\mu e^{k\|\operatorname{dev}_2 \log U\|^2} \cdot \operatorname{dev}_2 \log U + \kappa e^{\widehat{k}[\operatorname{tr}(\log U)]^2} \operatorname{tr}(\log U) \cdot \mathbb{1}\big]F^{-T} \quad \text{in } \overline{\Omega} \times [0, \infty),$$

with $F = \nabla \varphi$, $U = \sqrt{F^T F}$. The above equations are supplemented, in the case of the mixed problem, by the boundary conditions

$$\begin{aligned} \varphi(x, t) &= \widehat{\varphi}_i(x, t) \quad \text{on} \quad \Gamma_D \times [0, \infty), \\ S_1(x, t).n &= \widehat{s}_1(x, t) \quad \text{on} \quad \Gamma_N \times [0, \infty), \end{aligned} \tag{5.54}$$

and the initial conditions

$$\varphi(x, 0) = \varphi_0(x), \qquad \varphi_{,t}(x, 0) = \psi_0(x) \quad \text{in } \Omega, \tag{5.55}$$

where Γ_D, Γ_N are subsets of the boundary $\partial\Omega$, so that $\Gamma_D \cup \overline{\Gamma}_N = \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$, n is the unit outward normal to the boundary and $\widehat{\varphi}_i$, $\widehat{s}_1$, φ_0, ψ_0 are prescribed fields.

5.8 The Non-deviatoric Planar Case: $F \mapsto e^{\|\log U\|^2}$

We consider the function $W : \mathrm{GL}^+(2) \to \mathbb{R}$, defined by $W(F) := \widehat{W}(U) = e^{\|\log U\|^2}$. We have

$$e^{\|\log U\|^2} = g(\lambda_1, \lambda_2), \tag{5.56}$$

where λ_1, λ_2 are the singular values of U and $g : \mathbb{R}_+^2 \to \mathbb{R}$ is defined by

$$g(\lambda_1, \lambda_2) = e^{\log^2 \lambda_1 + \log^2 \lambda_2}. \tag{5.57}$$

---

[27]We use the definition of polyconvexity given by Ball [15] (see also [216, 220]). Polyconvexity implies LH-ellipticity and may lead to an existence theorem based on the direct methods of the calculus of variations, provided that proper growth conditions are satisfied [18, 20, 96, 167, 168].

In order to check the rank-one convexity of the function $F \mapsto e^{\|\log U\|^2}$, we will use Buliga's criterion given by Theorem 5.3. As we will need the derivatives of g, we compute:

$$\frac{\partial g}{\partial \lambda_1} = e^{\log^2 \lambda_1 + \log^2 \lambda_2} \frac{2\log \lambda_1}{\lambda_1}, \qquad \frac{\partial g}{\partial \lambda_2} = e^{\log^2 \lambda_1 + \log^2 \lambda_2} \frac{2\log \lambda_2}{\lambda_2},$$

$$\frac{\partial^2 g}{\partial \lambda_1^2} = e^{\log^2 \lambda_1 + \log^2 \lambda_2} \left(\frac{4\log^2 \lambda_1}{\lambda_1^2} + \frac{2 - 2\log \lambda_1}{\lambda_1^2} \right),$$

$$\frac{\partial^2 g}{\partial \lambda_1 \partial \lambda_2} = e^{\log^2 \lambda_1 + \log^2 \lambda_2} \frac{4\log \lambda_1 \log \lambda_2}{\lambda_1 \lambda_2},$$

$$\frac{\partial^2 g}{\partial \lambda_2^2} = e^{\log^2 \lambda_1 + \log^2 \lambda_2} \left(\frac{4\log^2 \lambda_2}{\lambda_2^2} + \frac{2 - 2\log \lambda_2}{\lambda_2^2} \right).$$

For our function, the matrices $G(\lambda_1, \lambda_2)$ and $H(\lambda_1, \lambda_2)$ from Theorem 5.3 are then

$$G(\lambda_1, \lambda_2) = 2e^{\log^2 \lambda_1 + \log^2 \lambda_2} \begin{pmatrix} 0 & \frac{\log \lambda_1 - \log \lambda_2}{\lambda_1^2 - \lambda_2^2} \\ \frac{\log \lambda_1 - \log \lambda_2}{\lambda_1^2 - \lambda_2^2} & 0 \end{pmatrix},$$

$$H(\lambda_1, \lambda_2) = 2e^{\log^2 \lambda_1 + \log^2 \lambda_2} \left[\begin{pmatrix} 0 & \frac{\frac{\lambda_2 \log \lambda_1}{\lambda_1} - \frac{\lambda_1 \log \lambda_2}{\lambda_2}}{\lambda_1^2 - \lambda_2^2} \\ \frac{\frac{\lambda_2 \log \lambda_1}{\lambda_1} - \frac{\lambda_1 \log \lambda_2}{\lambda_2}}{\lambda_1^2 - \lambda_2^2} & 0 \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} \frac{2\log^2 \lambda_1 - \log \lambda_1 + 1}{\lambda_1^2} & \frac{2\log \lambda_1 \log \lambda_2}{\lambda_1 \lambda_2} \\ \frac{2\log \lambda_1 \log \lambda_2}{\lambda_1 \lambda_2} & \frac{2\log^2 \lambda_2 - \log \lambda_2 + 1}{\lambda_2^2} \end{pmatrix} \right]$$

(5.58)

$$= 2e^{\log^2 \lambda_1 + \log^2 \lambda_2} \begin{pmatrix} \frac{2\log^2 \lambda_1 - \log \lambda_1 + 1}{\lambda_1^2} & \frac{2\log \lambda_1 \log \lambda_2}{\lambda_1 \lambda_2} + \frac{\frac{\lambda_2 \log \lambda_1}{\lambda_1} - \frac{\lambda_1 \log \lambda_2}{\lambda_2}}{\lambda_1^2 - \lambda_2^2} \\ \frac{2\log \lambda_1 \log \lambda_2}{\lambda_1 \lambda_2} + \frac{\frac{\lambda_2 \log \lambda_1}{\lambda_1} - \frac{\lambda_1 \log \lambda_2}{\lambda_2}}{\lambda_1^2 - \lambda_2^2} & \frac{2\log^2 \lambda_2 - \log \lambda_2 + 1}{\lambda_2^2} \end{pmatrix},$$

respectively. The first condition of Buliga's criterion is obviously satisfied because of the symmetry and convexity and hence Schur-convexity of the function $\ell : \mathbb{R}_+^2 \to \mathbb{R}$, $\ell(\lambda_1, \lambda_2) := g(e^{\lambda_1}, e^{\lambda_2}) = e^{\lambda_1^2 + \lambda_2^2}$ (see Theorem 2.5).

Hence, the energy is rank-one-convex if and only if the following inequality holds true for all $a_1, a_2 \in \mathbb{R}$ and for all $\lambda_1, \lambda_2 > 0$:

$$H_{11}(\lambda_1, \lambda_2)a_1^2 + 2H_{12}(\lambda_1, \lambda_2)a_1 a_2 + H_{22}(\lambda_1, \lambda_2)a_2^2 + 2G_{12}(\lambda_1, \lambda_2)|a_1 a_2| \geq 0.$$

Applied to our function (and upon division by $2e^{\log^2 \lambda_1 + \log^2 \lambda_2} > 0$) this corresponds to

$$\left(\frac{2\log^2 \lambda_1 - \log \lambda_1 + 1}{\lambda_1^2} \right) a_1^2 + \left(\frac{2\log^2 \lambda_2 - \log \lambda_2 + 1}{\lambda_2^2} \right) a_2^2$$

$$+ \left(\frac{2\log \lambda_1 \log \lambda_2}{\lambda_1 \lambda_2} + \frac{\frac{\lambda_2 \log \lambda_1}{\lambda_1} - \frac{\lambda_1 \log \lambda_2}{\lambda_2}}{\lambda_1^2 - \lambda_2^2} \right) 2a_1 a_2$$

$$+ \frac{\log \lambda_1 - \log \lambda_2}{\lambda_1^2 - \lambda_2^2} |2a_1 a_2| \geq 0, \quad \forall \lambda_1, \lambda_2 > 0, \forall a_1, a_2 \in \mathbb{R}. \tag{5.59}$$

To see that this does not hold true, we set

$$\lambda_1 = e^2, \quad \lambda_2 = e^{11}, \quad a_1 = -e^{15}, \quad a_2 = e^{22}.$$

Upon these choices, the inequality turns into

$$0 \leq \frac{2 \cdot 2^2 - 2 + 1}{e^4} e^{30} + \frac{2 \cdot 11^2 - 11 + 1}{e^{22}} e^{44} - \left(\frac{2 \cdot 2 \cdot 11}{e^{13}} + \frac{2e^9 - 11e^{-9}}{e^4 - e^{22}} \right) \cdot 2 \cdot e^{37}$$

$$+ \frac{2 - 11}{e^4 - e^{22}} 2e^{37}$$

$$= 7e^{26} + 111e^{22} - 88e^{24} + \frac{4e^9 - 22e^{-9}}{e^{18} - 1} e^{33} + \frac{18}{e^{18} - 1} e^{33} \leq 7e^{26} + 111e^{22} - 88e^{24}$$

$$+ 4e^{33+9-17} + 18e^{16}$$

$$\leq 7e^{26} + 4e^{25} + 112e^{22} - 88e^{24} = e^{22} \left(7e^4 + 4e^3 + 112 - 88e^2 \right) < -75e^{22},$$

and it is obviously not satisfied.

In view of Theorem 5.3, we conclude that $F \mapsto e^{\| \log U \|^2}$ is not rank-one convex in 2D. Of course, this shows that $F \mapsto e^{\| \log U \|^2}$ is also not rank-one convex in 3D.

Conjecture 5.24 *It seems that the function* $F \mapsto e^{\| \log U \|^2 - \frac{\alpha}{2} \operatorname{tr}(\log U)^2}$ *is*

(i) *not separately convex as a function of the principal stretches (which implies it is not rank-one convex) for* $\alpha > 1$;
(ii) *is not rank-one convex for* $\alpha < 1$.

If this conjecture is true, then the function $F \mapsto e^{\| \log U \|^2 - \frac{\alpha}{2} \operatorname{tr}(\log U)^2}$ *is rank-one convex in* 2D *if and only if* $\alpha = 1$, *i.e., only for the function* $F \mapsto e^{\| \operatorname{dev}_2 \log U \|^2}$.

Hence, the form of energies (1.4) may not just be an arbitrary choice, but additively splitting into isochoric and volumetric parts seems to be the only useful version of an additive split in plane elasto-statics. The reason to believe that Conjecture 5.24 is true consists in the fact that for $\lambda_1 = e^n$ and $\lambda_2 = e^{n-3}$, the last inequality from Knowles and Sternberg's Theorem 5.1 seems to be satisfied in the limit $n \to \infty$ only if $\alpha = 1$.

6 Outlook for Three Dimensions

The 3D-case is, as usual, much more involved. In this section we show that a similar calculus as in 2D can be applied in principle. However, while we consider the obvious generalization of the 2D result, the answer is in general negative: the necessary conditions from Knowles and Sternberg's Theorem 5.1 or Dacorogna's Theorem 5.2 are not satisfied for the energy

$$\widehat{W}(U) = e^{k \| \operatorname{dev}_3 \log U \|^2}. \tag{6.1}$$

This implies that this energy is *not rank-one convex* [165, 194]. We have already shown that $F \mapsto \| \operatorname{dev}_3 \log U \|^2$ is not rank-one convex even in the case of incompressible materials (see Proposition 5.6 or [165], page 197).

Lemma 6.1 *Let* $F \in \mathrm{GL}^+(3)$ *with singular values* $\lambda_1, \lambda_2, \lambda_3$. *Then*

$$W(F) = \widehat{W}(U) = g(\lambda_1, \lambda_2, \lambda_3),$$

$$\text{where } g : \mathbb{R}_+^3 \to \mathbb{R}, \, g(\lambda_1, \lambda_2, \lambda_3) := e^{\frac{k}{3}[\log^2 \frac{\lambda_1}{\lambda_2} + \log^2 \frac{\lambda_2}{\lambda_3} + \log^2 \frac{\lambda_3}{\lambda_1}]}. \tag{6.2}$$

Proof The proof follows from relation (5.18). □

This lemma remains true in all dimension $n \in \mathbb{N}$, see Appendix A.1.

6.1 $F \mapsto e^{k \| \operatorname{dev}_3 \log U \|^2}$ Is not Rank-One Convex

We begin our 3D investigation by proving that

Lemma 6.2 *For all* $k > 0$ *the function*

$$F \mapsto e^{k \| \operatorname{dev}_3 \log U \|^2}, \quad F \in \mathrm{GL}^+(3) \tag{6.3}$$

is not rank-one convex.

Proof In the following we prove that two necessary conditions given by Knowles and Sternberg's criterion are not satisfied for the function g defined by (6.2). Our goal is to prove that there does not exist a number $k > 0$ such that the inequalities

$$\frac{\partial^2 g}{\partial \lambda_1^2} \geq 0, \qquad \frac{\partial^2 g}{\partial \lambda_2^2} \geq 0, \qquad \sqrt{\frac{\partial^2 g}{\partial \lambda_1^2} \frac{\partial^2 g}{\partial \lambda_2^2}} - \frac{\partial^2 g}{\partial \lambda_2 \partial \lambda_1} + \frac{\frac{\partial g}{\partial \lambda_1} + \frac{\partial g}{\partial \lambda_2}}{\lambda_1 + \lambda_2} \geq 0 \tag{6.4}$$

are simultaneously satisfied. The inequalities (6.4) are equivalent to

$$\frac{2\frac{k}{3} e^{\frac{k}{3}(\log^2 \frac{\lambda_1}{\lambda_2} + \log^2 \frac{\lambda_3}{\lambda_1} + \log^2 \frac{\lambda_2}{\lambda_3})}}{\lambda_1^2} g_1(\lambda_1, \lambda_2, \lambda_3) \geq 0,$$

$$\frac{2\frac{k}{3} e^{\frac{k}{3}(\log^2 \frac{\lambda_1}{\lambda_2} + \log^2 \frac{\lambda_3}{\lambda_1} + \log^2 \frac{\lambda_2}{\lambda_3})}}{\lambda_2^2} g_2(\lambda_1, \lambda_2, \lambda_3) \geq 0, \tag{6.5}$$

$$\frac{2\frac{k}{3} e^{\frac{k}{3}(\log^2 \frac{\lambda_1}{\lambda_2} + \log^2 \frac{\lambda_3}{\lambda_1} + \log^2 \frac{\lambda_2}{\lambda_3})}}{\lambda_1 \lambda_2} g_3(\lambda_1, \lambda_2, \lambda_3) \geq 0,$$

where

$$g_1(\lambda_1, \lambda_2, \lambda_3) = 2\frac{k}{3} \left(\log \frac{\lambda_1}{\lambda_2} - \log \frac{\lambda_3}{\lambda_1} \right)^2 + \log \frac{\lambda_3}{\lambda_1} - \log \frac{\lambda_1}{\lambda_2} + 2,$$

$$g_2(\lambda_1, \lambda_2, \lambda_3) = 2\frac{k}{3} \left(\log \frac{\lambda_1}{\lambda_2} - \log \frac{\lambda_2}{\lambda_3} \right)^2 + \log \frac{\lambda_1}{\lambda_2} - \log \frac{\lambda_2}{\lambda_3} + 2,$$

$$g_3(\lambda_1, \lambda_2, \lambda_3) = 2\frac{k}{3}\left(\log\frac{\lambda_3}{\lambda_1}\log\frac{\lambda_2}{\lambda_3} + 2\log^2\frac{\lambda_1}{\lambda_2}\right) + 1 + \frac{\lambda_2(\log\frac{\lambda_1}{\lambda_2} - \log\frac{\lambda_3}{\lambda_1})}{\lambda_1 + \lambda_2} + \frac{\lambda_1(\log\frac{\lambda_2}{\lambda_3} - \log\frac{\lambda_1}{\lambda_2})}{\lambda_1 + \lambda_2}$$

$$+ 2\frac{k}{3}\sqrt{\left[\left(\log\frac{\lambda_1}{\lambda_2} - \log\frac{\lambda_2}{\lambda_3}\right)^2 + \log\frac{\lambda_1}{\lambda_2} - \log\frac{\lambda_2}{\lambda_3} + 2\right]\left[\left(\log\frac{\lambda_1}{\lambda_2} - \log\frac{\lambda_3}{\lambda_1}\right)^2 - \log\frac{\lambda_1}{\lambda_2} + \log\frac{\lambda_3}{\lambda_1} + 2\right]}.$$

We compute that, for extremely large principal stretches $(\lambda_1, \lambda_2, \lambda_2) = (e^{11}, e^7, e^{-1})$

$$g_1(e^{11}, e^7, e^{-1}) = 2\left(256\frac{k}{3} - 7\right), \qquad g_2(e^{11}, e^7, e^{-1}) = 2\left(16\frac{k}{3} - 1\right),$$

$$g_3(e^{11}, e^7, e^{-1}) = -128\frac{k}{3} + \frac{12}{1 + e^4} + 5 + 2\sqrt{\left(16\frac{k}{3} - 1\right)\left(256\frac{k}{3} - 7\right)}, \tag{6.6}$$

and we remark that

$$g_1(e^{11}, e^7, e^{-1}) > 0 \quad \Leftrightarrow \quad \frac{k}{3} > \frac{7}{256} \quad \text{and} \quad g_2(e^{11}, e^7, e^{-1}) > 0 \quad \Leftrightarrow \quad \frac{k}{3} > \frac{1}{16}. \tag{6.7}$$

For $\frac{k}{3} > \frac{1}{16}$ we have $-5 - \frac{12}{1+e^4} + 128\frac{k}{3} > 0$. Hence, $g_3(e^{11}, e^7, e^{-1}) \geq 0$ is equivalent to

$$4\left(16\frac{k}{3} - 1\right)\left(256\frac{k}{3} - 7\right) - \left(-5 - \frac{12}{1 + e^4} + 128\frac{k}{3}\right)^2 \geq 0$$

$$\Leftrightarrow \quad -64(e^4 - 15)(1 + e^4)\frac{k}{3} + e^8 - 38e^4 - 87 \geq 0,$$

which is not satisfied for $\frac{k}{3} > \frac{1}{16}$. Hence, for $0 < k \leq \frac{3}{16}$ the function is not separately convex, while for $\frac{k}{3} > \frac{1}{16}$ one of the condition (6.4)$_3$ given by Knowles and Sternberg's criterion is also not satisfied. Thus, the proof is complete. □

However, the function g defined by (6.2) satisfies the Baker-Ericksen (BE) inequalities

$$\frac{\lambda_i\frac{\partial g}{\partial \lambda_i} - \lambda_j\frac{\partial g}{\partial \lambda_j}}{\lambda_i - \lambda_j} = 2\frac{k}{3}e^{\frac{k}{3}(\log^2\frac{\lambda_i}{\lambda_j} + \log^2\frac{\lambda_r}{\lambda_i} + \log^2\frac{\lambda_j}{\lambda_r})}\frac{2\log\frac{\lambda_i}{\lambda_j} - \log\frac{\lambda_r}{\lambda_i} - \log\frac{\lambda_j}{\lambda_r}}{\lambda_i - \lambda_j}$$

$$= 2ke^{\frac{k}{3}(\log^2\frac{\lambda_i}{\lambda_j} + \log^2\frac{\lambda_r}{\lambda_i} + \log^2\frac{\lambda_j}{\lambda_r})}\frac{\log\frac{\lambda_i}{\lambda_j}}{\lambda_i - \lambda_j} > 0, \tag{6.8}$$

for any permutation of i, j, r. Moreover,

$$\frac{\partial^2 g}{\partial \lambda_i^2} = \frac{2\frac{k}{3}e^{\frac{k}{3}(\log^2\frac{\lambda_i}{\lambda_j} + \log^2\frac{\lambda_r}{\lambda_i} + \log^2\frac{\lambda_j}{\lambda_r})}}{\lambda_i^2}\left[2\frac{k}{3}\left(\log\frac{\lambda_r}{\lambda_i} - \log\frac{\lambda_i}{\lambda_j}\right)^2 + \log\frac{\lambda_r}{\lambda_i} - \log\frac{\lambda_i}{\lambda_j} + 2\right] \geq 0 \tag{6.9}$$

for any permutation of i, j, r and for all $\frac{k}{3} \geq \frac{1}{16}$. Thus, g is separately convex for $\frac{k}{3} \geq \frac{1}{16}$. This is not in contradiction to the 2D result where $k \geq \frac{1}{4}$ was needed for separate convexity since the function g in 2D is not obtained by choosing $\lambda_3 = 1$ in the 3D expression of the function g.

It is easy to see that the condition

$$\frac{\partial^2 g}{\partial \lambda_1^2} \geq 0, \qquad \frac{\partial^2 g}{\partial \lambda_2^2} \geq 0, \qquad \sqrt{\frac{\partial^2 g}{\partial \lambda_1^2} \frac{\partial^2 g}{\partial \lambda_2^2}} + m_{12}^\varepsilon \geq 0 \qquad (6.10)$$

from Dacorogna's criterion (Theorem 5.2) are also not simultaneously satisfied for the values considered in (6.6). Let us recall that for $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$

$$m_{12}^\varepsilon = \varepsilon_1 \varepsilon_2 \frac{\partial^2 g}{\partial \lambda_1 \partial \lambda_2} + \frac{\frac{\partial g}{\partial \lambda_1} - \varepsilon_1 \varepsilon_2 \frac{\partial g}{\partial \lambda_2}}{\lambda_1 - \varepsilon_1 \varepsilon_2 \lambda_2} \quad \text{if } \lambda_1 \neq \lambda_2 \text{ or } \varepsilon_1 \varepsilon_2 \neq 1. \qquad (6.11)$$

We choose $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$. For these values, the inequalities (6.11) become

$$\frac{\partial^2 g}{\partial \lambda_1^2} \geq 0, \qquad \frac{\partial^2 g}{\partial \lambda_2^2} \geq 0, \qquad \sqrt{\frac{\partial^2 g}{\partial \lambda_1^2} \frac{\partial^2 g}{\partial \lambda_2^2}} - \frac{\partial^2 g}{\partial \lambda_1 \partial \lambda_2} + \frac{\frac{\partial g}{\partial \lambda_1} + \frac{\partial g}{\partial \lambda_2}}{\lambda_1 + \lambda_2} \geq 0. \qquad (6.12)$$

We remark that the conditions (6.10) are in fact equivalent to the inequalities (6.4) from Knowles and Sternberg's criterion, and they cannot be simultaneously satisfied for the values defined in (6.6).

Moreover, direct and similar calculations as above give:

Remark 6.3

- The function

$$g : \mathbb{R}_+^3 \to \mathbb{R}, \qquad g(\lambda_1, \lambda_2, \lambda_3) := e^{\frac{k}{3}\left[\log^2 \frac{\lambda_1}{\lambda_2} + \log^2 \frac{\lambda_2}{\lambda_3} + \log^2 \frac{\lambda_3}{\lambda_1}\right]} + \frac{\kappa}{2} e^{\widehat{k} \log^2 (\lambda_1 \lambda_2 \lambda_3)} \qquad (6.13)$$

  does not satisfy the inequalities from Knowles and Sternberg's criterion because it does not even satisfy Zubov's criterion for incompressible elastic materials as we prove in the next subsection.
- While we have shown ellipticity of $F \mapsto e^{k\|\operatorname{dev}_2 \log U\|^2}$, we cannot infer (and it does not hold) that $F \mapsto e^{k\|\operatorname{dev}_3 \log U\|^2}$, evaluated and restricted to plane strain deformation $(\lambda_1, \lambda_2, 1)$ is elliptic.

Motivated by the preceding negative development, we were inclined to try other, similar Hencky type energies as candidates for an overall elliptic formulation. However:

- The function $g : \mathbb{R}_+^3 \to \mathbb{R}, g(\lambda_1, \lambda_2, \lambda_3) := e^{\frac{k}{3} \log^2 \frac{\lambda_1}{\lambda_2}} + e^{\frac{k}{3} \log^2 \frac{\lambda_2}{\lambda_3}} + e^{\frac{k}{3} \log^2 \frac{\lambda_3}{\lambda_1}}$ does not satisfy the inequalities from Knowles and Sternberg's criterion.
- The function $g : \mathbb{R}_+^3 \to \mathbb{R}, g(\lambda_1, \lambda_2, \lambda_3) := \mu(e^{\frac{k}{3} \log^2 \frac{\lambda_1}{\lambda_2}} + e^{\frac{k}{3} \log^2 \frac{\lambda_2}{\lambda_3}} + e^{\frac{k}{3} \log^2 \frac{\lambda_3}{\lambda_1}}) + \frac{\kappa}{2} e^{\widehat{k} \log^2 (\lambda_1 \lambda_2 \lambda_3)}$ does not satisfy the inequalities from Knowles and Sternberg's criterion because it does not satisfy Zubov's criterion for incompressible elastic materials.
- The function $g : \mathbb{R}_+^3 \to \mathbb{R}, g(\lambda_1, \lambda_2, \lambda_3) := \mu(e^{k \log^2 \lambda_1} + e^{k \log^2 \lambda_2} + e^{k \log^2 \lambda_3}) + \frac{\kappa}{2} e^{\widehat{k} \log^2 (\lambda_1 \lambda_2 \lambda_3)}$ does not satisfy the inequalities from Knowles and Sternberg's criterion.

6.2 The Ideal Nonlinear Incompressible Elasticity Model

Whereas $e^{k\|\operatorname{dev}_3 \log U\|^2}$ is not rank-one convex on $\operatorname{GL}^+(3)$, one might hope that perhaps its restriction to $\operatorname{SL}(3)$ might be rank-one convex. In the following, for simplicity, we consider

only the case $k = 1$. Thus, a first open problem is if the following energy $W : \mathrm{GL}^+(3) \to \mathbb{R}_+$,

$$W(F) = \begin{cases} e^{\|\operatorname{dev}_3 \log U\|^2} & \text{if } \det F = 1, \\ +\infty & \text{if } \det F \neq 1, \end{cases} \tag{6.14}$$

is rank-one convex. To this aim, we use Zubov's Theorem 5.5 to show that this energy is not even rank-one-convex on SL(3). According to (6.2), we check the conditions of this theorem for the function defined by (6.2). The answer is negative as can be seen by the counterexample

$$\lambda_1 = e^4, \qquad \lambda_2 = e^{-4}, \qquad \lambda_3 = 1, \qquad \lambda_1 \lambda_2 \lambda_3 = 1. \tag{6.15}$$

For these values we will prove that the condition

$$\sqrt{\delta_1 \delta_2} + \epsilon_3 > 0, \tag{6.16}$$

from Zubov's Theorem 5.5 is not satisfied. Let us recall that

$$\beta_1 = \frac{\partial^2 g}{\partial \lambda_1^2}, \qquad \beta_2 = \frac{\partial^2 g}{\partial \lambda_2^2}, \qquad \beta_3 = \frac{\partial^2 g}{\partial \lambda_3^2}, \qquad \gamma_1^- = -\frac{\partial^2 g}{\partial \lambda_2 \partial \lambda_3} + \frac{\frac{\partial g}{\partial \lambda_2} + \frac{\partial g}{\partial \lambda_3}}{\lambda_2 + \lambda_3},$$

$$\gamma_2^- = -\frac{\partial^2 g}{\partial \lambda_1 \partial \lambda_3} + \frac{\frac{\partial g}{\partial \lambda_1} + \frac{\partial g}{\partial \lambda_3}}{\lambda_1 + \lambda_3}, \qquad \gamma_3^+ = \frac{\partial^2 g}{\partial \lambda_1 \partial \lambda_2} + \frac{\frac{\partial g}{\partial \lambda_1} - \frac{\partial g}{\partial \lambda_2}}{\lambda_1 - \lambda_2},$$

$$\delta_1 = \beta_2 \lambda_2^2 + \beta_3 \lambda_3^2 + 2\gamma_1^- \lambda_2 \lambda_3, \qquad \delta_2 = \beta_3 \lambda_3^2 + \beta_1 \lambda_1^2 + 2\gamma_2^- \lambda_3 \lambda_1,$$

$$\epsilon_3 = \beta_3 \lambda_3^2 + \gamma_3^+ \lambda_1 \lambda_2 + \gamma_1^- \lambda_3 \lambda_2 + \gamma_2^- \lambda_3 \lambda_1.$$

In view of (6.2), we have

$$\beta_i = \frac{\frac{2}{3} e^{\frac{1}{3}\left(\log^2 \frac{\lambda_i}{\lambda_j} + \log^2 \frac{\lambda_r}{\lambda_i} + \log^2 \frac{\lambda_j}{\lambda_r}\right)}}{\lambda_i^2} \left[\frac{2}{3}\left(\log \frac{\lambda_r}{\lambda_i} - \log \frac{\lambda_i}{\lambda_j}\right)^2 + \log \frac{\lambda_r}{\lambda_i} - \log \frac{\lambda_i}{\lambda_j} + 2\right], \tag{6.17}$$

for any permutation of i, j, r. Moreover, we have

$$\gamma_1^- = \frac{2e^{\frac{1}{3}\left(\log^2 \frac{\lambda_1}{\lambda_2} + \log^2 \frac{\lambda_3}{\lambda_1} + \log^2 \frac{\lambda_2}{\lambda_3}\right)}}{9\lambda_2 \lambda_3 (\lambda_2 + \lambda_3)} \left[\log \frac{\lambda_3}{\lambda_1}\left(2(\lambda_2 + \lambda_3)\left(\log \frac{\lambda_1}{\lambda_2} - \log \frac{\lambda_2}{\lambda_3}\right) + 3\lambda_2\right)\right.$$

$$- \left(2(\lambda_2 + \lambda_3) \log \frac{\lambda_2}{\lambda_3} + 3\lambda_3\right)\left(\log \frac{\lambda_1}{\lambda_2} - \log \frac{\lambda_2}{\lambda_3}\right)$$

$$\left. + 3\left(\lambda_2\left(-\log \frac{\lambda_2}{\lambda_3}\right) + \lambda_2 + \lambda_3\right)\right],$$

$$\gamma_2^- = \frac{2e^{\frac{1}{3}\left(\log^2 \frac{\lambda_1}{\lambda_2} + \log^2 \frac{\lambda_3}{\lambda_1} + \log^2 \frac{\lambda_2}{\lambda_3}\right)}}{9\lambda_1 \lambda_3 (\lambda_1 + \lambda_3)} \left[\log \frac{\lambda_3}{\lambda_1}\left(3(\lambda_1 - \lambda_3) - 2(\lambda_1 + \lambda_3)\left(\log \frac{\lambda_2}{\lambda_3} - \log \frac{\lambda_3}{\lambda_1}\right)\right)\right.$$

$$+ \log \frac{\lambda_1}{\lambda_2}\left(2(\lambda_1 + \lambda_3)\left(\log \frac{\lambda_2}{\lambda_3} - \log \frac{\lambda_3}{\lambda_1}\right) + 3\lambda_3\right)$$

$$\left. + 3\left(\lambda_1\left(-\log \frac{\lambda_2}{\lambda_3}\right) + \lambda_1 + \lambda_3\right)\right], \tag{6.18}$$

$$\gamma_3^+ = -\frac{2e^{\frac{1}{3}\left(\log^2\frac{\lambda_1}{\lambda_2}+\log^2\frac{\lambda_3}{\lambda_1}+\log^2\frac{\lambda_2}{\lambda_3}\right)}}{9\lambda_1\lambda_2(\lambda_1-\lambda_2)}\left[-\log\frac{\lambda_1}{\lambda_2}\left(2(\lambda_1-\lambda_2)\left(\log\frac{\lambda_3}{\lambda_1}+\log\frac{\lambda_2}{\lambda_3}\right)+3(\lambda_1+\lambda_2)\right)\right.$$

$$+3\left(\lambda_2\log\frac{\lambda_3}{\lambda_1}+\lambda_1-\lambda_2\right)+\log\frac{\lambda_2}{\lambda_3}\left(2(\lambda_1-\lambda_2)\log\frac{\lambda_3}{\lambda_1}+3\lambda_1\right)$$

$$\left.+2(\lambda_1-\lambda_2)\log^2\frac{\lambda_1}{\lambda_2}\right].$$

By direct substitution we deduce

$$\beta_1\left(e^4,e^{-4},1\right)=\frac{172e^{24}}{3},\qquad \beta_2\left(e^4,e^{-4},1\right)=\frac{220e^{40}}{3},\qquad \beta_3\left(e^4,e^{-4},1\right)=\frac{4e^{32}}{3},$$

$$\gamma_1^-\left(e^4,e^{-4},1\right)=\frac{2}{3}\left(1-\frac{12}{1+\frac{1}{e^4}}\right)e^{36},\qquad \gamma_2^-\left(e^4,e^{-4},1\right)=\frac{2e^{28}(13+e^4)}{3(1+e^4)},$$

$$\gamma_3^+\left(e^4,e^{-4},1\right)=\frac{2e^{32}(109-85e^8)}{3(e^8-1)},\qquad \delta_1\left(e^4,e^{-4},1\right)=\frac{4e^{32}(19+15e^4)}{1+e^4},$$

$$\delta_2\left(e^4,e^{-4},1\right)=\frac{4e^{32}(19+15e^4)}{1+e^4},\qquad \epsilon_3\left(e^4,e^{-4},1\right)=\frac{2e^{32}(31+8e^4-31e^8)}{e^8-1},$$

and

$$\sqrt{\delta_1\left(e^4,e^{-4},1\right)\delta_2\left(e^4,e^{-4},1\right)}+\epsilon_3\left(e^4,e^{-4},1\right)=-\frac{2e^{32}(7-16e^4+e^8)}{e^8-1}<0. \qquad (6.19)$$

This means that the necessary and sufficient conditions from Zubov's Theorem 5.5 are not satisfied. Hence, we conclude

Proposition 6.4 *The function $F\mapsto e^{\|\operatorname{dev}_3\log U\|^2}$ is not rank-one convex on* SL(3).

6.3 Rank-One Convexity Domains for the Energy $F\mapsto e^{k\|\operatorname{dev}_3\log U\|^2}$

The understanding of loss of ellipticity may become important for severe strains and stresses at crack tips. The analysis in this subsection is motivated by the results established by Bruhns et al. [39, 40] (see also [92, 127] in order to compare the domains of ellipticity obtained in nonlinear elastostatics for a special material),[28] in which it is proved that the quadratic Hencky strain energy function W_H with non-negative Lamé constants, $\mu,\lambda>0$, fulfils the Legendre-Hadamard condition for all principal stretches with

$$\lambda_i\in\left[0.21162\ldots,\sqrt[3]{e}\right]=[0.21162\ldots,1.39561\ldots]. \qquad (6.20)$$

The LH-ellipticity of the quadratic Hencky strain energy function W_H for all principal stretches in this cube $[0.21162\ldots,1.39561\ldots]^3$ implies (see Remark 5.14) that the exponentiated energy e^{W_H} is also LH-elliptic for all principal stretches in this box and for non-negative Lamé constants $\mu,\lambda>0$.

---

[28]For this special material the energy is elliptic for $\rho<\frac{\lambda_1}{\lambda_2}<\frac{1}{\rho}$, $\rho=2-\sqrt{3}=0.268$.

Let us first remark that the function $g : \mathbb{R}_+^3 \to \mathbb{R}$, $g(\lambda_1, \lambda_2, \lambda_3) := e^{\frac{k}{3}[\log^2 \frac{\lambda_1}{\lambda_2} + \log^2 \frac{\lambda_2}{\lambda_3} + \log^2 \frac{\lambda_3}{\lambda_1}]}$ corresponding to our energy $F \mapsto e^{k\|\operatorname{dev}_3 \log U\|^2}$ is invariant under scaling:[29]

$$g(a\lambda_1, a\lambda_2, a\lambda_3) = g(\lambda_1, \lambda_2, \lambda_3), \quad \text{for all } a > 0. \tag{6.21}$$

In fact, we have:

Remark 6.5 All functions $F \mapsto W(F) = W_1(\|\operatorname{dev}_3 \log U\|^2)$ are invariant under the scaling: $F \mapsto aF, a > 0$.

Let us consider the substitution $(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{\lambda}_3) = (a\lambda_1, a\lambda_2, a\lambda_3)$, for all $a > 0$. For the derivatives, we deduce

$$\frac{\partial}{\partial \widetilde{\lambda}_i} g(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{\lambda}_3) = \frac{1}{a} \frac{\partial}{\partial \lambda_i} g(\lambda_1, \lambda_2, \lambda_3),$$

$$\frac{\partial^2}{\partial \widetilde{\lambda}_i \partial \widetilde{\lambda}_j} g(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{\lambda}_3) = \frac{1}{a^2} \frac{\partial^2}{\partial \lambda_i \partial \lambda_j} g(\lambda_1, \lambda_2, \lambda_3). \tag{6.22}$$

Hence, for the function g corresponding to our energy $F \mapsto e^{k\|\operatorname{dev}_3 \log U\|^2}$, the inequalities in Dacorogna's criterion are also invariant under scaling. More generally:

Remark 6.6

(i) Let $F \mapsto W(F) = W_1(\|\operatorname{dev}_3 \log U\|^2)$ be a function on $\mathrm{GL}^+(3)$. Then, the inequalities in Dacorogna's criterion in terms of the corresponding function $g : \mathbb{R}_+ \to \mathbb{R}$ are invariant under scaling.

(ii) For all functions $F \mapsto W(F)$ (for instance for functions $F \mapsto W(F) = W_{\mathrm{iso}}(\frac{F}{\det F^{1/3}})$) which are invariant under scaling, the inequalities in Dacorogna's criterion in terms of the corresponding function $g : \mathbb{R}_+ \to \mathbb{R}$ are invariant under scaling.

Therefore, if the function g does not satisfy the requested inequalities in Dacorogna's criterion in a point $(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)})$, then it also does not satisfy them in the point $(\widetilde{\lambda}_1^{(0)}, \widetilde{\lambda}_2^{(0)}, \widetilde{\lambda}_3^{(0)}) = (a\lambda_1^{(0)}, a\lambda_2^{(0)}, a\lambda_3^{(0)})$ for arbitrary $a > 0$. In the following we will exploit this insight.

In the previous subsections we have proved that there exists a point in which the function $F \mapsto e^{\|\operatorname{dev}_3 \log U\|^2}$ looses the LH-ellipticity, namely in

$$\lambda_1^{(0)} = e^{11}, \qquad \lambda_2^{(0)} = e^7, \qquad \lambda_3^{(0)} = e^{-1} \tag{6.23}$$

for compressible materials and in

$$\lambda_1^{(0)} = e^4, \qquad \lambda_2^{(0)} = e^{-4}, \qquad \lambda_3^{(0)} = 1 \tag{6.24}$$

in the case of incompressible materials. In view of the scaling invariance discussed above, we have

Lemma 6.7 *If the function* $g : \mathbb{R}_+^3 \to \mathbb{R}$, $g(\lambda_1, \lambda_2, \lambda_3) := e^{\frac{k}{3}[\log^2 \frac{\lambda_1}{\lambda_2} + \log^2 \frac{\lambda_2}{\lambda_3} + \log^2 \frac{\lambda_3}{\lambda_1}]}$ *is not elliptic in a point* $P^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)})$, *then it is not elliptic in all points* $P(\lambda_1, \lambda_2, \lambda_3)$,

---

[29]This means $e^{k\|\operatorname{dev}_3 \log(aU)\|^2} = e^{k\|\operatorname{dev}_3 \log U\|^2}$ for all $a > 0$.

Fig. 16 $F \mapsto e^{k\|\operatorname{dev}_3 \log U\|^2}$ is not LH-elliptic in the domain $(0, y) \times (0, y) \times (0, y), y > 0$
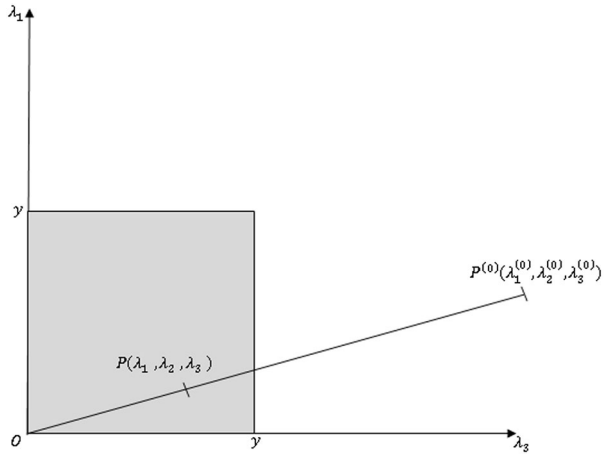
$(\lambda_1, \lambda_2, \lambda_3) \neq (0, 0, 0)$ *belonging to the line* $O P^{(0)}$, *where* $O = (0, 0, 0)$. *In other words, the ellipticity domain is invariant under scaling.*

More general:

Remark 6.8 Let $F \mapsto W(F)$ be an invariant under scaling function (for instance $F \mapsto W(F) = W_{\mathrm{iso}}(\frac{F}{\det F^{1/3}})$ or $F \mapsto W(F) = W_1(\|\operatorname{dev}_3 \log U\|^2)$) defined on $\mathrm{GL}^+(3)$. If the corresponding function $g : \mathbb{R}^3_+ \to \mathbb{R}$ is not elliptic in a point $P^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)})$, then it is not elliptic in all points $P(\lambda_1, \lambda_2, \lambda_3)$, $(\lambda_1, \lambda_2, \lambda_3) \neq (0, 0, 0)$ belonging to the line $O P^{(0)}$, where $O = (0, 0, 0)$. In other words, the ellipticity domain of a function invariant under scaling function will be invariant under scaling.

Proposition 6.9 *The energy* $F \mapsto e^{\|\operatorname{dev}_3 \log U\|^2}$, $F \in \mathrm{GL}^+(3)$ *cannot be LH-elliptic in any cube like domain* $(0, y) \times (0, y) \times (0, y), y > 0$.

Proof If the point $P^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)})$ given by (6.23) belongs to the domain $(0, y) \times (0, y) \times (0, y)$ then we have nothing more to prove. If $P^{(0)}(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}) \notin (0, y) \times (0, y) \times (0, y)$, then there is a point $P(\lambda_1, \lambda_2, \lambda_3)$, $(\lambda_1, \lambda_2, \lambda_3) \neq (0, 0, 0)$ belonging to the line $O P$ and $P(\lambda_1, \lambda_2, \lambda_3) \in (0, y) \times (0, y) \times (0, y)$ (see Fig. 16). For instance the point $\left(\frac{\lambda_1^{(0)}}{a}, \frac{\lambda_2^{(0)}}{a}, \frac{\lambda_3^{(0)}}{a}\right)$, where $a > \max_{i=1,2,3}\left\{\frac{\lambda_i^{(1)}}{y}\right\}$. In view of Lemma 6.7 the proof is complete. $\square$

Proposition 6.10 *The energy* $F \mapsto e^{\|\operatorname{dev}_3 \log U\|^2}$, $F \in \mathrm{GL}^+(3)$ *is not LH-elliptic in any cube like domain* $(x, \infty) \times (x, \infty) \times (x, \infty), x > 0$.

Proof The proof is similar to the proof of the previous proposition, because for $b < \min_{i=1,2,3}\left\{\frac{\lambda_i^{(0)}}{x}\right\}$, the point $\left(\frac{\lambda_1^{(0)}}{b}, \frac{\lambda_2^{(0)}}{b}, \frac{\lambda_3^{(0)}}{b}\right) \in O P$ belongs also to the domain $(x, \infty) \times (x, \infty) \times (x, \infty)$ (see Fig. 17). $\square$

We already can prove this more general result:

Proposition 6.11 *Let* $F \mapsto W(F)$ *be a function defined on* $\mathrm{GL}^+(3)$ *which is invariant under scaling. If the corresponding function* $g : \mathbb{R}^3_+ \to \mathbb{R}$ *is not elliptic in a point* $P^{(0)} =$

Fig. 17 $F \mapsto e^{k\|\operatorname{dev}_3 \log U\|^2}$ is not LH-elliptic in the domain $(x,\infty) \times (x,\infty) \times (x,\infty),\, x > 0$
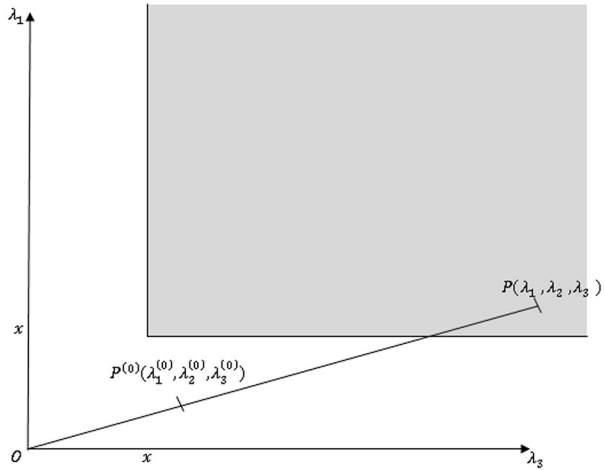
$(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)})$, *then there are no cube-like domains* $(0, y)^3,\, y > 0,$ *or* $(x,\infty)^3,\, x > 0,$ *on which g is elliptic.*

On the other hand, the energy $F \mapsto e^{k\|\operatorname{dev}_3 \log U\|^2}$ is invariant under inversion,[30] i.e.,

$$e^{k\|\operatorname{dev}_3 \log U\|^2} = e^{k\|\operatorname{dev}_3 \log U^{-1}\|^2} \quad \Leftrightarrow \quad g(\lambda_1, \lambda_2, \lambda_3) = g\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}\right),$$

$$\text{for all } (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}_+^3. \tag{6.25}$$

However, Dacorogna's ellipticity criterion is not invariant under inversion. This is the reason why Proposition 6.10 does not follow directly from Proposition 6.9 using the invariance under inversion.

Remark 6.12 Looking back to the quadratic Hencky energy $F \mapsto W_B(F) := \|\operatorname{dev}_3 \log U\|^2$ considered by Bruhns et al. [262][31] and to the corresponding function $g_B : \mathbb{R}_+^3 \to \mathbb{R}$, $g_B(\lambda_1, \lambda_2, \lambda_3) := \frac{1}{3}\left[\log^2 \frac{\lambda_1}{\lambda_2} + \log^2 \frac{\lambda_2}{\lambda_3} + \log^2 \frac{\lambda_3}{\lambda_1}\right]$, we remark:

- g_B is separately convex (see Proposition 5.9 and Corollary 5.10) only for those U such that the eigenvalues μ_1, μ_2, μ_3 of $\operatorname{dev}_3 \log U$ are smaller than $\frac{2}{3}$, i.e, if and only if the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of U are such that

$$\lambda_1^2 \le e^2 \lambda_2 \lambda_3, \qquad \lambda_2^2 \le e^2 \lambda_3 \lambda_1, \qquad \lambda_3^2 \le e^2 \lambda_1 \lambda_2. \tag{6.26}$$

- g_B always satisfies the BE-inequalities.
- Numerical computations give us reasons to believe that there exists a number $a_B > 0$ such that $F \mapsto \|\operatorname{dev}_3 \log U\|^2$ is LH-elliptic in the domain (invariant under scaling)

$$\widetilde{\mathcal{E}}\left(W_B, \text{LH}, U, \frac{4}{3}\right) := \left\{U \in \mathrm{PSym}(3)\,\big|\,\|\operatorname{dev}_3 \log U\|^2 < a_B < \frac{4}{3}\right\}.$$
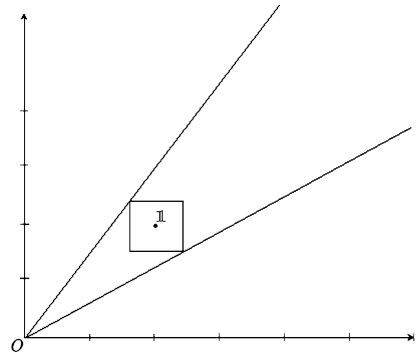
---

[30] The invariance under inversion of an energy W is the tension-compression symmetry $W(F) = W(F^{-1})$.

[31] Hutchinson and Neale [116] have considered the energy $\|\operatorname{dev}_3 \log U\|^N$ for $0 < N \le 1$.

Fig. 18 A section along the first diagonal along the line containing the origin O and $\mathbb{1}$ of the LH-ellipticity domain of the energy function $F \mapsto e^{\| \operatorname{dev}_3 \log U \|^2}$ if it is LH-elliptic in a box $(x, y) \times (x, y) \times (x, y)$

- The results already obtained in [39] cannot be used for the energy $W_B(F) = \| \operatorname{dev}_3 \log U \|^2 = \| \log U \|^2 - \frac{1}{3} [\operatorname{tr}(\log U)]^2$, because they are applicable only for energies $W_H(F) = \mu \| \log U \|^2 + \frac{\lambda}{2} [\operatorname{tr}(\log U)]^2$ for which $\mu, \lambda \geq 0$.

Remark 6.13 As expected, the above properties are improved by considering the exponentiated Hencky energy $F \mapsto W(F) = e^{k \| \operatorname{dev}_3 \log U \|^2}$. The corresponding function $g : \mathbb{R}^3_+ \to \mathbb{R}$, $g(\lambda_1, \lambda_2, \lambda_3) := e^{\frac{k}{3} \left[\log^2 \frac{\lambda_1}{\lambda_2} + \log^2 \frac{\lambda_2}{\lambda_3} + \log^2 \frac{\lambda_3}{\lambda_1} \right]}$ is such that:

- g is separately convex everywhere if $k > \frac{3}{16}$.
- g always satisfies the BE-inequalities.
- Numerical computations give us reasons to believe that there is a number $a > 0$, $a > \frac{4}{3} > a_B > 0$, such that $F \mapsto e^{k \| \operatorname{dev}_3 \log U \|^2}$ is LH-elliptic in the domain $\| \operatorname{dev}_3 \log U \|^2 < a$. The approximative value which we observed is $a = 27$, i.e., $U \in \mathcal{E}(W_{eH}, LH, U, 27)$, where

$$\mathcal{E}(W_{eH}^{iso}, LH, U, 27) := \left\{ U \in \operatorname{PSym}(3) \mid \| \operatorname{dev}_3 \log U \|^2 \leq 27 \right\}. \tag{6.27}$$

Of course, $\mathcal{E}(W_{eH}^{iso}, LH, U, 27)$ contains a neighborhood of $\mathbb{1}$. Rephrasing the remark of Ogden [182, page 409], "the question whether a constitutive inequality [LH, TSS-I, TSTS-M, BSTS, BSS, etc.] holds for all deformations of a compressible solid is open. The applicability of elastic theory outside [a bounded domain in stretch space] is itself questionable because, for example, there may exist yield surfaces beyond which permanent deformation occurs."

*Remark 6.14* The major open problems in this respect are:

- Do there exist numbers $x, y > 0$ such that the energy function $F \mapsto e^{\| \operatorname{dev}_3 \log U \|^2}$ is LH-elliptic in $(x, y) \times (x, y) \times (x, y)$ and $\mathbb{1} \in (x, y) \times (x, y) \times (x, y)$? If true, then in view of Lemma 6.7 the function $F \mapsto e^{\| \operatorname{dev}_3 \log U \|^2}$ is LH-elliptic in the domain given by Fig. 18. In the three-dimensional representation it is a cone with the angle in the origin.
- Is the function $F \mapsto e^{\| \operatorname{dev}_3 \log U \|^2}$ elliptic in a ball containing $\mathbb{1}$? If true, then in view of Lemma 6.7 the function $F \mapsto e^{\| \operatorname{dev}_3 \log U \|^2}$ is LH-elliptic in the domain given by Fig. 19. In the two-dimensional representation this domain is a corner domain but in the three-dimensional representation it is the interior of an infinite cone (not necessarily circular) with the angle in the origin (see Fig. 20).
- In fact it is enough to check where the energy function $F \mapsto e^{\| \operatorname{dev}_3 \log U \|^2}$ looses the ellipticity in all planes $\lambda_i = 1$, meaning planes π_i containing the point $(1, 1, 1)$ and orthogonal

Fig. 19 A section along the first diagonal along the line containing the origin O and $\mathbb{1}$ of the LH-ellipticity domain of the energy function
$F \mapsto e^{\|\operatorname{dev}_3 \log U\|^2}$ if it is LH-elliptic in a ball
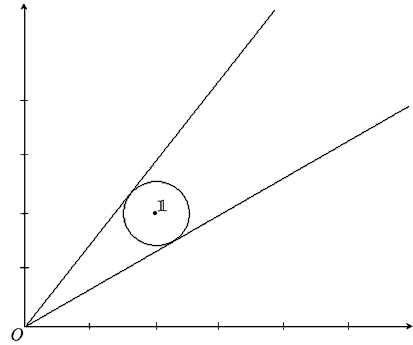
Fig. 20 A section along the first diagonal along the line containing the origin O and $\mathbb{1}$ of the domain $\|\operatorname{dev}_3 \log U\|^2 < 1$. This indicates that, similar to TSTS-M$^+$, ellipticity might be controlled by the distortional energy
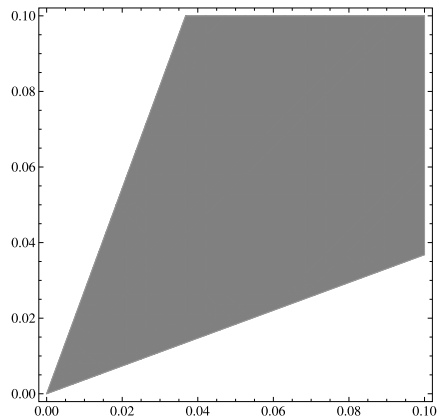
to the axes $O\lambda_i$, respectively. In view of the symmetry in λ_i, it is enough to see what happens in the plane $\pi_1 : \lambda_1 = 1$.

7 Summary and Open Problems

To summarize, in the present paper:

- We have proved that the planar exponentiated Hencky strain energy function

$$F \mapsto W_{\mathrm{eH}}(F) := \widehat{W}_{\mathrm{eH}}(U) := \begin{cases} \dfrac{\mu}{k} e^{k\|\operatorname{dev}_2 \log U\|^2} + \dfrac{\kappa}{2\widehat{k}} e^{\widehat{k}(\operatorname{tr}(\log U))^2} & \text{if } \det F > 0, \\ +\infty & \text{if } \det F \leq 0 \end{cases} \quad (7.1)$$

  is *rank-one convex* for $\mu > 0, \kappa > 0, k \geq \frac{1}{4}$ and $\widehat{k} \geq \frac{1}{8}$;
- We have shown that the exponentiated volumetric energy function

$$F \mapsto \frac{\kappa}{2\widehat{k}} e^{\widehat{k}(\operatorname{tr}(\log U))^2}, \quad F \in \mathrm{GL}^+(n) \quad (7.2)$$

  is *rank-one convex* w.r.t. F for the volumetric strain parameter $\widehat{k} \geq \frac{1}{m^{(m+1)}}$. ($m = 2 : \widehat{k} \geq \frac{1}{8}$, $m = 3 : \widehat{k} \geq \frac{1}{81}$);

- We have shown that for all distortional strain stiffening parameters $k > 0$ the energy function

$$F \mapsto \frac{\mu}{k} e^{k \| \operatorname{dev}_3 \log U \|^2}, \quad F \in \mathrm{GL}^+(3) \tag{7.3}$$

is *not rank-one convex*;
- Numerical tests suggest that the LH-ellipticity domain of the distortional energy function $F \mapsto \frac{\mu}{k} e^{k \| \operatorname{dev}_3 \log U \|^2}$, $F \in \mathrm{GL}^+(3)$, with $k \geq \frac{3}{16}$ (the necessary condition for separate convexity (SC) of $e^{k \| \operatorname{dev}_3 \log U \|^2}$ in 3D) is an extremely large cone

$$\mathcal{E}(W_{\mathrm{eH}}, \mathrm{LH}, U, 27) = \left\{ U \in \mathrm{PSym}(3) \,\big|\, \| \operatorname{dev}_3 \log U \|^2 < 27 \right\}; \tag{7.4}$$

- We have proved that the energy function

$$F \mapsto \frac{\mu}{k} e^{k \| \log U \|^2}, \quad F \in \mathrm{GL}^+(n), \quad n = 2, 3 \tag{7.5}$$

is *not rank-one convex*;
- We have shown that the true-stress-true-strain relation is invertible for the family of energies W_{eH};
- The monotonicity of the Cauchy stress tensor, as a function of $\log V$, for our family of exponentiated Hencky energies is true in certain domains of bounded distortions

$$\mathcal{E}\left(W_{\mathrm{eH}}, \mathrm{TSTS\text{-}M}^+, \tau_{\mathrm{eH}}, \frac{2}{3} \sigma_{\mathbf{y}}^2 \right) := \left\{ \tau \in \mathrm{Sym}(3) \,\big|\, \| \operatorname{dev}_3 \tau \|^2 \leq \frac{2}{3} \sigma_{\mathbf{y}}^2 \right\}, \tag{7.6}$$

superficially similar to the observed ellipticity domains $\mathcal{E}(W_{\mathrm{eH}}, \mathrm{TSTS\text{-}M}^+, \tau_{\mathrm{eH}}, \frac{2}{3} \sigma_{\mathbf{y}}^2)$;
- For all exponentiated energies KSTS-M$^+$, KSTS-I, TSTS-I, TSS-I conditions are satisfied everywhere;
- For $n = 3$ among the family W_{eH} we have singled out a special ($k = \frac{2}{3} \widehat{k}$) three parameter subset

$$W_{\mathrm{eH}}^{\sharp}(\log V) = \frac{1}{2k} \left\{ \frac{E}{1+v} e^{k \| \operatorname{dev}_3 \log V \|^2} + \frac{E}{2(1-2v)} e^{\frac{2}{3} k (\operatorname{tr}(\log V))^2} \right\}$$

such that uniaxial tension leads to no lateral contraction if and only if $v = 0$, as in linear elasticity.

In forthcoming papers [170, 171, 177] our geodesic invariants

$$\text{"the magnitude-of-dilatation":} \quad K_1^2 = \left| \operatorname{tr}(\log U) \right|^2 = |\log \det U|^2$$
$$= |\log \det V|^2 = |\log \det F|^2,$$
$$\text{"the magnitude-of-distortion":} \quad K_2^2 = \| \operatorname{dev}_3 \log U \|^2 = \| \operatorname{dev}_3 \log V \|^2,$$

as basic ingredients of idealized isotropic strain energies will be motivated in detail. As already stated in the introduction, it can be shown that [170, 171, 177]

$$\operatorname{dist}_{\mathrm{geod}}^2 \left((\det F)^{1/n} \cdot \mathbb{1}, \mathrm{SO}(n) \right) = \operatorname{dist}_{\mathrm{geod}, \mathbb{R}_+ \cdot \mathbb{1}}^2 \left((\det F)^{1/n} \cdot \mathbb{1}, \mathbb{1} \right)$$
$$= |\log \det F|^2 = \widetilde{W}_{\mathrm{H}}^{\mathrm{vol}}(\det U),$$

$$\text{dist}^2_{\text{geod}}\left(\frac{F}{(\det F)^{1/n}}, \text{SO}(n)\right) = \text{dist}^2_{\text{geod,SL}(n)}\left(\frac{F}{(\det F)^{1/n}}, \text{SO}(n)\right)$$

$$= \|\text{dev}_n \log U\|^2 = \widetilde{W}_{\text{H}}^{\text{iso}}\left(\frac{U}{\det U^{1/n}}\right),$$

where $\text{dist}^2_{\text{geod},\mathbb{R}_+\cdot\mathbb{1}}$ and $\text{dist}^2_{\text{geod,SL}(n)}$ are the canonical left invariant geodesic distances on the Lie-group $\text{SL}(n)$ and on the group $\mathbb{R}_+ \cdot \mathbb{1}$, respectively (see [170, 171, 177]). For this investigation new mathematical tools had to be discovered [132, 177] also having consequences for the classical polar decomposition [118, 119].

Hence, using this terminology, in the present paper we have shown rank-one convexity of

$$W_{\text{eH}}(F) := \frac{\mu}{k}e^{k\,\text{dist}^2_{\text{geod,SL}(2)}\left(\frac{F}{\det F^{1/2}}, \text{SO}(2)\right)} + \frac{\kappa}{2\widehat{k}}e^{\widehat{k}\,\text{dist}^2_{\text{geod},\mathbb{R}_+\cdot\mathbb{1}}\left(\det F^{1/2}\cdot\mathbb{1}, \text{SO}(2)\right)}. \tag{7.7}$$

Our W_{eH} formulation ignores at first sight yield surfaces and other aspects of a theory of plasticity. Yet, our investigation on the ellipticity conditions in 3D suggests a relation between loss of ellipticity conditions and permanent deformations [173].

Let us finish this paper with some conjectures stemming from our unsuccessful attempts in this direction:

Conjecture 7.1 *For n=2,3 the energy* $F \mapsto \frac{\mu}{k}e^{k\|\text{dev}_n \log U\|^2}$, $k > \frac{3}{16}$ *is rank-one convex in a set which contains the large cone*

$$\mathcal{E}(W_{\text{eH}}, \text{LH}, U, 27) = \left\{U \in \text{PSym}(3) \,|\, \|\text{dev}_3 \log U\|^2 < 27\right\}. \tag{7.8}$$

Moreover, it would be interesting to know the rank-one convex and quasiconvex envelope of the energy $F \mapsto \frac{\mu}{k}e^{k\|\text{dev}_n \log U\|^2}$, $k > \frac{3}{16}$.

Conjecture 7.2 *For $n = 3$ there is no elastic energy expression*

$$W = W\left(K_2^2\right) = W\left(\|\text{dev}_3 \log U\|^2\right) \tag{7.9}$$

such that $F \mapsto W(\|\text{dev}_3 \log U\|^2)$ *is Legendre-Hadamard elliptic in* $\text{GL}^+(3)$, *i.e., over the entire deformation range.*

Conjecture 7.3 *For $n = 2, 3$ there is no elastic energy expression*

$$W = W\left(K_2^2\right) = W\left(\|\text{dev}_n \log U\|^2\right) \tag{7.10}$$

such that $F \mapsto W(\|\text{dev}_n \log U\|^2)$ *satisfies the TSTS-M$^+$ condition in* $\text{GL}^+(n)$, *i.e., over the entire deformation range.*

A further open problem is to find an energy $F \mapsto W(\|\text{dev}_3 \log U\|^2, [\text{tr}(\log U)]^2)$ such that the BSS-I condition is satisfied. In a future contribution we will discuss the application of the family $W_{\text{eH}}(F)$ to the description of large strain rubber elasticity for Treloar's classical data.

Appendix

A.1 Some Useful Identities

- $\operatorname{tr}(B^{-1} X B) = \operatorname{tr}(X)$ for any invertible matrix B.
- $\operatorname{dev}_n(B^{-1} X B) = B^{-1} X B - \frac{1}{n} \operatorname{tr}(B^{-1} X B) = B^{-1}(\operatorname{dev}_n X) B$ for any invertible matrix B.
- $\|\operatorname{dev}_n X\|^2 = \|X - \frac{1}{n} \operatorname{tr} X \cdot \mathbb{1}\|^2 = \|X\|^2 + \frac{1}{n^2}(\operatorname{tr} X)^2 \|\mathbb{1}\|^2 - \frac{2}{n} \operatorname{tr} X \langle X, I \rangle = \|X\|^2 - \frac{1}{n}(\operatorname{tr} X)^2$.
- The norm of the deviator in $\mathbb{R}^{n \times n}$:

$$\left\| \operatorname{dev}_n \begin{pmatrix} \xi_1 & 0 & \cdots & 0 \\ 0 & \xi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_n \end{pmatrix} \right\|^2 = \sum_{i=1}^n \xi_i^2 - \frac{1}{n}\left(\sum_{i=1}^n \xi_i\right)^2 = \frac{n-1}{n}\sum_{i=1}^n \xi_i^2 - \frac{2}{n}\sum_{i,j=1,i<j}^n \xi_i \xi_j$$

$$= \frac{1}{n}\left[(n-1)\sum_{i=1}^n \xi_i^2 - 2\sum_{i,j=1,i<j}^n \xi_i \xi_j\right] \tag{A.1}$$

$$= \frac{1}{n}\sum_{i,j=1,i<j}^n \left(\xi_i^2 - 2\xi_i\xi_j + \xi_j^2\right) = \frac{1}{n}\sum_{i,j=1,i<j}^n (\xi_i - \xi_j)^2.$$

- From [165, page 200] we have: $\frac{\|X\|^p}{z^\alpha}$ is convex in (X,z) if $\frac{\alpha+1}{\alpha} \geq \frac{p}{p-1} \Leftrightarrow p \geq \alpha + 1$.
- $\log U = \sum_{i=1}^n \log \lambda_i N_i \otimes N_i$, where N_i are the eigenvectors of U and λ_i are the eigenvalues of U.
- $\log U = (U - \mathbb{1}) - \frac{1}{2}(U - \mathbb{1})^2 + \frac{1}{3}(U - \mathbb{1})^3 - \cdots$, convergent for $\|U - \mathbb{1}\| < 1$.
- $\log V = \sum_{i=1}^n \log \widehat{\lambda}_i \widehat{N}_i \otimes \widehat{N}_i$, where $\widehat{N}_i$ are the eigenvectors of V and $\widehat{\lambda}_i$ are the eigenvalues of V.
- $\log V = (V - \mathbb{1}) - \frac{1}{2}(V - \mathbb{1})^2 + \frac{1}{3}(V - \mathbb{1})^3 - \cdots$, convergent for $\|V - \mathbb{1}\| < 1$.
-

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}^{-1} = \frac{1}{F_{11}F_{22} - F_{12}F_{21}} \begin{pmatrix} F_{22} & -F_{12} \\ -F_{21} & F_{11} \end{pmatrix} \quad \Rightarrow \quad \|F^{-1}\|^2 \overset{n=2}{=} \frac{1}{(\det F)^2}\|F\|^2.$$

-

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \qquad U^2 = F^T F = \begin{pmatrix} F_{11}^2 + F_{21}^2 & F_{11}F_{12} + F_{21}F_{22} \\ F_{11}F_{12} + F_{21}F_{22} & F_{12}^2 + F_{22}^2 \end{pmatrix}.$$

The eigenvalues of U^2 are:

$$\mu_1 = \frac{1}{2}\left(F_{11}^2 + F_{12}^2 + F_{21}^2 + F_{22}^2 - \sqrt{(F_{11}^2 + F_{12}^2 + F_{21}^2 + F_{22}^2)^2 - 4(F_{12}F_{21} - F_{11}F_{22})^2}\right)$$

$$= \frac{1}{2}\left(\|F\|^2 - \sqrt{\|F\|^4 - 4(\det F)^2}\right),$$

$$\mu_2 = \frac{1}{2}\left(F_{11}^2 + F_{12}^2 + F_{21}^2 + F_{22}^2 + \sqrt{(F_{11}^2 + F_{12}^2 + F_{21}^2 + F_{22}^2)^2 - 4(F_{12}F_{21} - F_{11}F_{22})^2}\right)$$

$$= \frac{1}{2}\left(\|F\|^2 + \sqrt{\|F\|^4 - 4(\det F)^2}\right).$$

The principal stretches of F, i.e., the eigenvalues of $U = \sqrt{F^T F}$, which are the same as the eigenvalues of $V = \sqrt{F F^T}$, are $\lambda_1(F) = \sqrt{\mu_1}$, $\lambda_2(F) = \sqrt{\mu_2}$.

- Taking the pure stretch under shear stress $F_1 = \left(\begin{smallmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} & 0 \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and the simple glide $F_2 = \left(\begin{smallmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, the corresponding rates $L_1(t) = \frac{\mathrm{d}}{\mathrm{d}t}F_1 \cdot F_1^{-1} \neq \frac{\mathrm{d}}{\mathrm{d}t}F_2 \cdot F_2^{-1} = L_2(t)$ are different, as is $\log U_1(t) \neq \log\sqrt{F_2^T F_2} = \log U_2(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\log U_1 \neq \frac{\mathrm{d}}{\mathrm{d}t}\log U_2(t)$. However, $D_1(t) = \mathrm{sym}\,L_1(t) = \mathrm{sym}\,L_2(t) = D_2(t)$. This shows that $\frac{\mathrm{d}}{\mathrm{d}t}\log U(t) = D(t)$ is true only for coaxial families $U(t)$.

A.2 Vallée's Formula

Lemma A.1 (Vallée's formula[32] (see also [130, 210, 256, 257])) *Let us consider $S \in$ Sym(3) and let $\Psi : \mathrm{Sym}(3) \to \mathbb{R}$ be a differentiable isotropic scalar value function. We define $W(S) = \Psi(\exp(S))$. Then, the following chain rules hold*:

$$D_S\big[\Psi\big(\exp(S)\big)\big] = \exp(S) \cdot D\Psi\big(\exp(S)\big), \qquad D_S W(S) = D\Psi\big(\exp(S)\big) \cdot \exp(S),$$
$$D_C \Psi(C) = DW(\log C) \cdot C^{-1}, \qquad C \cdot D_C \Psi(C) = DW(\log C), \tag{A.2}$$

while it is generally not true that $D_C[\log C].H = \langle C^{-1}, H \rangle$.

Proof Let us first remark that

$$\exp(X + H)$$

$$= \mathbb{1} + (X + H) + \frac{1}{2}(X + H)^2 + \frac{1}{6}(X + H)^3 + \cdots$$

$$= \mathbb{1} + (X + H) + \frac{1}{2}\big(X^2 + XH + HX + H^2\big)$$

$$+ \frac{1}{6}\big(X^3 + XHX + HXX + H^2X + X^2H + XH^2 + HXH + H^3\big) + \cdots$$

---

[32]In [257] Vallée et al. have given a proof without using a Taylor expansion.

$$= \mathbb{1} + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \cdots + H + \frac{1}{2}(XH + HX) + \frac{1}{6}(XXH + XHX + HXX)$$

$$= \exp(X) + \underbrace{H + \frac{1}{2}(XH + HX) + \frac{1}{6}(XXH + XHX + HXX) + \cdots}_{D(\exp(X)).H}. \tag{A.3}$$

Further we consider the Taylor expansion of the function $\Psi(\exp(S))$

$$\Psi(\exp(S + H))$$
$$= \Psi(\exp(S) + D(\exp(S)).H + \cdots)$$
$$= \Psi(\exp(S)) + \langle D\Psi(\exp(S)), D(\exp(S)).H \rangle + \cdots$$
$$= \Psi(\exp(S)) + \left\langle D\Psi(\exp(S)), H + \frac{1}{2}(SH + HS) \right\rangle$$
$$\quad + \left\langle D\Psi(\exp(S)), \frac{1}{6}(SSH + SHS + HSS) + \cdots \right\rangle + \cdots$$
$$= \Psi(\exp(S)) + \langle D\Psi(\exp(S)), H \rangle + \frac{1}{2}\langle D\Psi(\exp(S)), SH + HS \rangle$$
$$\quad + \frac{1}{6}\langle D\Psi(\exp(S)), SSH + SHS + HSS \rangle + \cdots$$
$$= \Psi(\exp(S)) + \langle D\Psi(\exp(S)), H \rangle + \frac{1}{2}\left[\langle S^T D\Psi(\exp(S)), H \rangle + \langle D\Psi(\exp(S))S^T, H \rangle\right]$$
$$\quad + \frac{1}{6}\left[S^T S^T \langle D\Psi(\exp(S)), H \rangle + \langle S^T D\Psi(\exp(S))S^T, H \rangle\right.$$
$$\quad \left. + \langle D\Psi(\exp(S))S^T S^T, H \rangle\right] + \cdots. \tag{A.4}$$

Since $S \in \mathrm{Sym}(3)$, it follows

$$\Psi(\exp(S + H)) = \Psi(\exp(S)) + \langle D\Psi(\exp(S)), H \rangle$$
$$\quad + \frac{1}{2}\left[\langle SD\Psi(\exp(S)), H \rangle + \langle D\Psi(\exp(S))S, H \rangle\right]$$
$$\quad + \frac{1}{6}\left[SS\langle D\Psi(\exp(S)), H \rangle + \langle SD\Psi(\exp(S))S, H \rangle\right.$$
$$\quad \left. + \langle D\Psi(\exp(S))SS, H \rangle\right] + \cdots. \tag{A.5}$$

On the other hand, since $D\Psi$ is a isotropic tensor function and obvious $\exp(S)$ is also isotropic, we have that $D\Psi(\exp(S))$ is also a isotropic tensor function and therefore it holds

$$D\Psi(\exp(S)) \cdot S = S \cdot D\Psi(\exp(S)). \tag{A.6}$$

Therefore,

$$\Psi(\exp(S + H))$$
$$= \Psi(\exp(S)) + \langle D\Psi(\exp(S)), H \rangle + \langle D\Psi(\exp(S))S, H \rangle$$

$$+ \frac{1}{2} \langle D\Psi(\exp(S)) S^2, H \rangle + \cdots$$

$$= \Psi(\exp(S)) + \left\langle D\Psi(\exp(S)) \left[\mathbb{1} + S + \frac{1}{2} S^2 + \cdots \right], H \right\rangle$$

$$= \Psi(\exp(S)) + \langle D\Psi(\exp(S)) \cdot \exp(S), H \rangle. \tag{A.7}$$

Using again the isotropy of $D\Psi(\exp(S))$, we obtain

$$\Psi(\exp(S + H)) = \Psi(\exp(S)) + \langle \exp(S) \cdot D\Psi(\exp(S)), H \rangle + \cdots. \tag{A.8}$$

We recall that we simultaneously have

$$\Psi(\exp(S + H)) = \Psi(\exp(S)) + \langle D_S\Psi(\exp(S)), H \rangle + \cdots, \tag{A.9}$$

for all $H \in \mathrm{Sym}(3)$. Thus, we deduce

$$\begin{aligned} \langle D_S\Psi(\exp(S)), H \rangle &= \langle \exp(S) \cdot D\Psi(\exp(S)), H \rangle, \\ \langle D_S W(S), H \rangle &= \langle \exp(S) \cdot D\Psi(\exp(S)), H \rangle. \end{aligned} \tag{A.10}$$

Choosing $S = \log C$, the relations (A.2)$_3$ also results and the proof is complete. $\qquad \square$

A.3 LH-Ellipticity for Functions of the Type $F \mapsto h(\det F)$

We consider a function $h : \mathbb{R} \to \mathbb{R}$ and we analyze when the function $F \mapsto h(\det F)$ is LH-elliptic as a function of F, $F \in \mathbb{R}^{3 \times 3}$. We recall that

$$D(\det F).H = \det F \cdot \mathrm{tr}(H F^{-1}) = \langle \mathrm{Cof}\, F, H \rangle. \tag{A.11}$$

Using the first Frechét-formal derivative, we compute the derivative

$$D(h(\det F)).(H, H) = h'(\det F) \cdot \langle \mathrm{Cof}\, F, H \rangle, \tag{A.12}$$

and the second derivative will be

$$\begin{aligned} D^2(h(\det F)).(H, H) \\ &= h''(\det F) \cdot \langle \mathrm{Cof}\, F, H \rangle^2 + h'(\det F) \langle D(\mathrm{Cof}\, F).H, H \rangle \\ &= h''(\det F) \cdot \langle \mathrm{Cof}\, F, H \rangle^2 + h'(\det F) \{ \langle \langle \mathrm{Cof}\, F, H \rangle F^{-T}, H \rangle \\ &\quad + \det F \langle -F^{-T} H^T F^{-T}, H \rangle \}, \\ &= h''(\det F) \cdot \langle \mathrm{Cof}\, F, H \rangle^2 + h'(\det F) \det F \{ \langle F^{-T}, H \rangle^2 \\ &\quad - \langle F^{-T} H^T F^{-T}, H \rangle \}. \end{aligned} \tag{A.13}$$

Hence, for $\xi, \eta \in \mathbb{R}^3$ we have

$$\begin{aligned} D^2(h(\det F)).(\xi \otimes \eta, \xi \otimes \eta)) \\ &= h''(\det F) \cdot \langle \mathrm{Cof}\, F, (\xi \otimes \eta) \rangle^2 + h'(\det F) \det F \{ \langle F^{-T}, (\xi \otimes \eta) \rangle^2 \end{aligned}$$

$$- \left\langle F^{-T}(\xi \otimes \eta)^T F^{-T}, (\xi \otimes \eta) \right\rangle \}. \tag{A.14}$$

On the other hand

$$\left\langle F^{-T}, (\xi \otimes \eta) \right\rangle^2 - \left\langle F^{-T}(\xi \otimes \eta)^T F^{-T}, (\xi \otimes \eta) \right\rangle$$

$$= \left\langle \mathbb{1}, F^{-1}(\xi \otimes \eta) \right\rangle^2 - \left\langle \left(\eta \otimes F^{-1}\xi \right), \left(F^{-1}\xi \otimes \eta \right) \right\rangle$$

$$= \left\langle \mathbb{1}, F^{-1}(\xi \otimes \eta) \right\rangle^2 - \left\langle \left(F^{-1}\xi \otimes \eta \right)^T, \left(F^{-1}\xi \otimes \eta \right) \right\rangle = \left\langle F^{-1}\xi, \eta \right\rangle^2 - \left\langle F^{-1}\xi, \eta \right\rangle^2 = 0.$$

This leads to the surprising simplification

$$D^2\big(h(\det F)\big).(\xi \otimes \eta, \xi \otimes \eta) = h''(\det F) \cdot \left\langle \mathrm{Cof}\, F, (\xi \otimes \eta) \right\rangle^2. \tag{A.15}$$

In conclusion, $F \mapsto h(\det F)$ is LH-elliptic if and only if $t \mapsto h(t)$ is convex since $\langle \mathrm{Cof}\, F, (\xi \otimes \eta) \rangle^2$ is positive.

From [55, page 213] we know more:

Proposition A.2 *Let* $W : \mathbb{R}^{n \times n} \to \mathbb{R}$ *be quasiaffine but not identically constant and* $h : \mathbb{R} \to \mathbb{R}$ *be such that* $W(F) = h(\det F)$. *Then*

$$W\, polyconvex \quad \Leftrightarrow \quad W\, quasiconvex \quad \Leftrightarrow \quad W\, rank\, one\, convex \quad \Leftrightarrow \quad h\, convex. \tag{A.16}$$

A.4 Convexity for Functions of the Type $t \mapsto \xi((\log t)^2)$

We consider a generic function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ and we find a characterization of the convexity for the function $t \mapsto \xi((\log t)^2)$. In the following let ζ denote the function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$, $\zeta(t) = (\log t)^2$. We deduce

$$\frac{\mathrm{d}}{\mathrm{d}t} \xi\big((\log t)^2\big) = \xi'\big((\log t)^2\big) 2 \frac{1}{t} \log t,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \xi\big((\log t)^2\big) = 2 \frac{\mathrm{d}}{\mathrm{d}t} \left(\xi'\big((\log t)^2\big) 2 \frac{1}{t} \log t \right)$$

$$= 4 \xi''\big((\log t)^2\big) \frac{1}{t^2} (\log t)^2 - 2 \xi'\big((\log t)^2\big) \frac{1}{t^2} \log t + 2 \xi'\big((\log t)^2\big) \frac{1}{t^2}$$

$$= 2 \frac{1}{t^2} \big[2 \xi''\big((\log t)^2\big)(\log t)^2 + \xi'\big((\log t)^2\big)(1 - \log t) \big], \tag{A.17}$$

where $\xi' = \frac{d\xi}{d\zeta}$. Hence, the function $t \mapsto \xi((\log t)^2)$ is

- convex on $[1, \infty)$ as a function of t if and only if $2 \frac{d^2\xi(\zeta)}{d\zeta^2} \zeta + \frac{d\xi(\zeta)}{d\zeta}(1 - \sqrt{\zeta}) \geq 0$, for all $\zeta \in \mathbb{R}_+$.
- convex on $(0, 1)$ as a function of t if and only if $2 \frac{d^2\xi(\zeta)}{d\zeta^2} \zeta + \frac{d\xi(\zeta)}{d\zeta}(1 + \sqrt{\zeta}) \geq 0$, for all $\zeta \in \mathbb{R}_+$.

A.5 Connecting $\mathrm{dev}_3 \log U$ with $\mathrm{dev}_2 \log U$

For $U^\sharp \in \mathrm{GL}(2)$, we define the lifted quantity

$$U = \begin{pmatrix} U^\sharp & & 0 \\ & & 0 \\ 0 & 0 & (\det U^\sharp)^{1/2} \end{pmatrix} \in \mathrm{GL}(3). \tag{A.18}$$

We remark that

$$\det \begin{pmatrix} U^\sharp & & 0 \\ & & 0 \\ 0 & 0 & (\det U^\sharp)^{1/2} \end{pmatrix} = \det U^\sharp (\det U^\sharp)^{1/2} = (\det U^\sharp)^{3/2}, \tag{A.19}$$

which implies $(\det U)^{1/3} = [\det U^\sharp (\det U^\sharp)^{1/2}]^{1/3} = [(\det U^\sharp)^{3/2}]^{1/3} = (\det U^\sharp)^{1/2}$. Moreover, we obtain

$$\mathrm{dev}_3 \log U = \log \frac{U}{\det U^{1/3}} = \log \frac{U}{(\det U^\sharp)^{1/2}} = \log \begin{pmatrix} \frac{U^\sharp}{(\det U^\sharp)^{1/2}} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \log \frac{U^\sharp}{(\det U^\sharp)^{1/2}} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{dev}_2 \log U^\sharp & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.20}$$

In general, for $A^\sharp \in \mathbb{R}^{2\times 2}$ and $\alpha \in \mathbb{R}$ we have

$$\left\| \mathrm{dev}_3 \begin{pmatrix} A^\sharp & 0 \\ & & 0 \\ 0 & 0 & \alpha \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} A^\sharp & 0 \\ & & 0 \\ 0 & 0 & \alpha \end{pmatrix} \right\|^2 - \frac{1}{3} \left[\mathrm{tr} \begin{pmatrix} A^\sharp & 0 \\ & & 0 \\ 0 & 0 & \alpha \end{pmatrix} \right]^2$$

$$= \|A^\sharp\|^2 + \alpha^2 - \frac{1}{3}\big[\mathrm{tr}(A^\sharp) + \alpha\big]^2$$

$$= \|A^\sharp\|^2 - \frac{1}{3}\big[\mathrm{tr}(A^\sharp)\big]^2 - \frac{2}{3}\alpha\,\mathrm{tr}(A^\sharp) - \frac{1}{3}\alpha^2 + \alpha^2$$

$$= \|\mathrm{dev}_2 A^\sharp\|^2 - \frac{2}{3}\alpha\,\mathrm{tr}(A^\sharp) + \frac{2}{3}\alpha^2. \tag{A.21}$$

Thus

$$\left\| \mathrm{dev}_3 \begin{pmatrix} A^\sharp & 0 \\ & & 0 \\ 0 & 0 & \alpha \end{pmatrix} \right\|^2 = \|\mathrm{dev}_2 A^\sharp\|^2 \tag{A.22}$$

if and only if $\alpha = 0$ or $\alpha = \mathrm{tr}(A^\sharp)$. Hence, we deduce $\|\mathrm{dev}_3 \log U\|^2 = \|\mathrm{dev}_2 \log U^\sharp\|^2$, for U of the form (A.18). Since $U^\sharp \in \mathrm{PSym}(2)$, we can assume that $U^\sharp = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda_1, \lambda_2 \in \mathbb{R}_+$. Then, the lifted quantity U lies in $\mathrm{PSym}(3)$ and $U = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & (\lambda_1\lambda_2)^{1/2} \end{pmatrix}$.

The next problem is if for a given deformation $\varphi^\sharp = (\varphi_1^\sharp, \varphi_2^\sharp) : \mathbb{R}^2 \to \mathbb{R}^2$ such that $U^\sharp = \sqrt{(\nabla\varphi^\sharp)^T \nabla\varphi^\sharp}$ we can construct an ansatz $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $U = \sqrt{\nabla\varphi^T \nabla\varphi}$,

where U is the lifted quantity associated to $U^\sharp$. For this it is necessary to have $\varphi = (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2), x_3\alpha(x_1, x_2))$ and $\alpha_{,x_1} = 0, \alpha_{,x_2} = 0$. Checking the compatibility equation we see that this is possible if and only if $\det \nabla\varphi^\sharp = K = const.$, which implies $\varphi_{3,x_3} = K$. In the incompressible case $\det \nabla\varphi = 1$, an appropriate ansatz is therefore

$$\varphi(x_1, x_2, x_3) = \left(\varphi_1^\sharp(x_1, x_2), \varphi_2^\sharp(x_1, x_2), x_3\right), \tag{A.23}$$

since

$$U^2 = \nabla\varphi^T \nabla\varphi = \begin{pmatrix} (\nabla\varphi^\sharp)^T \nabla\varphi^\sharp & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} (\nabla\varphi^\sharp)^T \nabla\varphi^\sharp & & 0 \\ & & 0 \\ 0 & 0 & \det[(\nabla\varphi^\sharp)^T \nabla\varphi^\sharp] \end{pmatrix}$$

$$= \begin{pmatrix} (U^\sharp)^2 & 0 \\ & & 0 \\ 0 & 0 & (\det[(U^\sharp)^{1/2}])^2 \end{pmatrix} = \begin{pmatrix} U^\sharp & 0 \\ & & 0 \\ 0 & 0 & (\det U^\sharp)^{1/2} \end{pmatrix}^2, \tag{A.24}$$

with $\det U^\sharp = 1$.

References

1. Adamov, A.A.: Comparative analysis of the two-constant generalizations of Hooke's law for isotropic elastic materials at finite strains. J. Appl. Mech. Tech. Phys. **42**(5), 890–897 (2001)
2. Al-Mohy, A., Higham, N., Relton, S.: Computing the Fréchet derivative of the matrix logarithm and estimating the condition number. SIAM J. Sci. Comput. **35**(4), C394–C410 (2013)
3. Alexander, H.: The tensile instability of an inflated cylindrical membrane as affected by an axial load. Int. J. Mech. Sci. **13**(2), 87–95 (1971)
4. Altenbach, H., Eremeyev, V., Lebedev, L., Rendón, L.A.: Acceleration waves and ellipticity in thermoelastic micropolar media. Arch. Appl. Mech. **80**(3), 217–227 (2010)
5. Anand, L.: On H. Hencky's approximate strain energy function for moderate deformations. J. Appl. Mech. **46**, 78–82 (1979)
6. Anand, L.: Moderate deformations in extension-torsion of incompressible isotropic elastic materials. J. Mech. Phys. Solids **34**, 293–304 (1986)
7. Annaidh, A.N., Destrade, M., Gilchrist, M.D., Murphy, J.G.: Deficiencies in numerical models of anisotropic nonlinearly elastic materials. Biomech. Model. Mechanobiol. **12**(4), 781–791 (2012)
8. Arghavani, J., Auricchio, F., Naghdabadi, R.: A finite strain kinematic hardening constitutive model based on Hencky strain: general framework, solution algorithm and application to shape memory alloys. Int. J. Plast. **27**(6), 940–961 (2011)
9. Aron, M.: On certain deformation classes of compressible Hencky materials. Math. Mech. Solids **11**, 467–478 (2006)
10. Aubert, G.: Necessary and sufficient conditions for isotropic rank-one convex functions in dimension 2. J. Elast. **39**(1), 31–46 (1995)
11. Aubert, G., Tahraoui, R.: Conditions nécessaires de faible fermeture et de 1-rang convexité en dimension 3. Rend. Circ. Mat. Palermo **34**(3), 460–488 (1985)
12. Aubert, G., Tahraoui, R.: Sur la faible fermeture de certains ensembles de contraintes en élasticité non-linéaire plane. Arch. Ration. Mech. Anal. **97**(1), 33–58 (1987)
13. Baker, M., Ericksen, J.L.: Inequalities restricting the form of the stress-deformation relation for isotropic elastic solids and Reiner-Rivlin fluids. J. Wash. Acad. Sci. **44**, 33–35 (1954)
14. Ball, J.M.: Constitutive inequalities and existence theorems in nonlinear elastostatics. In: Knops, R.J. (ed.) Herriot Watt Symposion: Nonlinear Analysis and Mechanics, vol. 1, pp. 187–238. Pitman, London (1977)
15. Ball, J.M.: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Ration. Mech. Anal. **63**, 337–403 (1977)
16. Ball, J.M.: Differentiability properties of symmetric and isotropic functions. Duke Math. J. **51**(3), 699–728 (1984)

17. Ball, J.M.: Some open problems in elasticity. In: Newton, P., et al. (eds.) Geometry, Mechanics, and Dynamics, pp. 3–59. Springer, New York (2002)

18. Balzani, D., Neff, P., Schröder, J., Holzapfel, G.A.: A polyconvex framework for soft biological tissues. Adjustment to experimental data. Int. J. Solids Struct. **43**(20), 6052–6070 (2006)

19. Balzani, D., Schröder, J., Gross, D., Neff, P.: Modeling of Anisotropic Damage in Arterial Walls Based on Polyconvex Stored Energy Functions. In: Owen, D.R.J., Onate, E., Suarez, B. (eds.) Computational Plasticity VIII, Fundamentals and Applications, Part 2, pp. 802–805. CIMNE, Barcelona (2005)

20. Balzani, D., Schröder, J., Neff, P., Holzapfel, G.A.: Materially stable constitutive equations for arterial walls based on polyconvex energies—application to damage modeling and residual stresses. J. Biomech. **39**, 409 (2006)

21. Batra, R.C.: On the coincidence of the principal axes of stress and strain in isotropic elastic bodies. Lett. Appl. Eng. Sci. **3**, 435–439 (1975)

22. Batra, R.C.: Deformation produced by a simple tensile load in an isotropic elastic body. J. Elast. **6**(1), 109–111 (1976)

23. Batra, R.C., dell'Isola, F., Vidoli, S.: A second-order solution of Saint-Venant's problem for a piezo-electric circular bar using Signorini's perturbation method. J. Elast. **52**(1), 75–90 (1998)

24. Bažant, Z.P.: Easy-to-compute tensors with symmetric inverse approximating Hencky finite strain and its rate. J. Eng. Mater. Trans. ASME **120**, 131–136 (1998)

25. Beatty, M.: Topics in finite elasticity: hyperelasticity of rubber, elastomers, and biological tissues—with examples. Appl. Mech. Rev. **40**(12), 1699–1734 (1987)

26. Beatty, M.F.: Introduction to nonlinear elasticity. In: Carroll, M.M., Hayes, M.A. (eds.) Nonlinear Effects in Fluids and Solids, pp. 13–112. Springer, Berlin (1996)

27. Becker, G.F.: The finite elastic stress-strain function. Am. J. Sci. **XLVI**, 337–356 (1893). https://www.uni-due.de/imperia/md/content/mathematik/ag_neff/becker_latex_new1893.pdf

28. Bell, J.F.: Mechanics of Solids: Volume 1: The Experimental Foundations of Solid Mechanics. Handbuch der Physik. Springer, Berlin (1973)

29. Bernstein, D.S.: Matrix Mathematics. Princeton University Press, New Jersey (2009)

30. Bertram, A., Böhlke, T., Šilhavý, M.: On the rank 1 convexity of stored energy functions of physically linear stress-strain relations. J. Elast. **86**, 235–243 (2007)

31. Bhatia, R., Holbrook, J.: Riemannian geometry and matrix geometric means. Linear Algebra Appl. **413**(2), 594–618 (2006)

32. Bîrsan, M., Neff, P., Lankeit, J.: Sum of squared logarithms: an inequality relating positive definite matrices and their matrix logarithm. J. Inequal. Appl. **2013**(1), 168 (2013)

33. Blume, J.: On the form of the inverted stress-strain law for isotropic hyperelastic solids. Int. J. Non-Linear Mech. **27**(3), 413–421 (1992)

34. Böhlke, T., Bertram, A.: On ellipticity of finite isotropic linear elastic laws. Preprint der Fakultät für Maschinenbau, Otto-von-Guericke-Universität Magdeburg (1), 1–18 (2002)

35. Borwein, J.M., Vanderwerff, J.D.: Convex Functions. Constructions, Characterizations and Counterexamples. Cambridge University Press, Cambridge (2010)

36. Bruhns, O.T.: Die Berücksichtigung einer isotropen Werkstoffverfestigung bei der elastisch-plastischen Blechbiegung mit endlichen Formänderungen. Ing.-Arch. **39**(1), 63–72 (1970)

37. Bruhns, O.T.: Elastoplastische Scheibenbiegung bei endlichen Formänderungen. Z. Angew. Math. Mech. **51**, T101–T103 (1971)

38. Bruhns, O.T., Thermann, K.: Elastisch-plastische Biegung eines Plattenstreifens bei endlichen Formänderungen. Ing.-Arch. **38**(3), 141–152 (1969)

39. Bruhns, O.T., Xiao, H., Mayers, A.: Constitutive inequalities for an isotropic elastic strain energy function based on Hencky's logarithmic strain tensor. Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci. **457**, 2207–2226 (2001)

40. Bruhns, O.T., Xiao, H., Mayers, A.: Finite bending of a rectangular block of an elastic Hencky material. J. Elast. **66**(3), 237–256 (2002)

41. Bruhns, O.T., Xiao, H., Meyers, A.: Hencky's elasticity model with the logarithmic strain measure: a study on Poynting effect and stress response in torsion of tubes and rods. Arch. Mech. **52**(4), 489–509 (2000)

42. Buliga, M.: Four applications of majorization to convexity in the calculus of variations. Linear Algebra Appl. **429**(7), 1528–1545 (2008)

43. Cai, Z., Starke, G.: First-order system least squares for the stress-displacement formulation: linear elasticity. SIAM J. Numer. Anal. **41**(2), 715–730 (2003)

44. Cai, Z., Starke, G.: Least-squares methods for linear elasticity. SIAM J. Numer. Anal. **42**(2), 826–842 (2004)

45. Carroll, M.M.: Controllable states of stress for compressible elastic solids. J. Elast. **3**(1), 57–61 (1973)

46. Carroll, M.M.: Must elastic materials be hyperelastic? Math. Mech. Solids **14**(4), 369–376 (2009)

47. Chen, Y.C.: Stability and bifurcation of homogeneous deformations of a compressible elastic body under pressure load. Math. Mech. Solids **1**(1), 57–72 (1996)
48. Ciarlet, P.G.: Three-Dimensional Elasticity, 1st edn. Studies in Mathematics and Its Applications, vol. 1. Elsevier, Amsterdam (1988)
49. Cleja-Ţigoiu, S.: Yield criteria in anisotropic finite elasto-plasticity. Arch. Mech. **57**(2–3), 81–102 (2005)
50. Cleja-Ţigoiu, S.: Consequences of the dissipative restrictions in finite anisotropic elasto-plasticity. Int. J. Plast. **19**(11), 1917–1964 (2003)
51. Criscione, J.C.: Direct tensor expression for natural strain and fast, accurate approximation. Compos. Struct. **80**(25), 1895–1905 (2002)
52. Criscione, J.C., Humphrey, J.D., Douglas, A.S., Hunter, W.C.: An invariant basis for natural strain which yields orthogonal stress response terms in isotropic hyperelasticity. J. Mech. Phys. Solids **48**(12), 2445–2465 (2000)
53. Curnier, A., Rakotomanana, L.: Generalized strain and stress measures, critical survey and new results. Eng. Trans. (Warsaw) **39**, 461–538 (1991)
54. Dacorogna, B.: Necessary and sufficient conditions for strong ellipticity of isotropic functions in any dimension. Discrete Contin. Dyn. Syst. **1**(2), 257–263 (2001)
55. Dacorogna, B.: Direct Methods in the Calculus of Variations, 2nd edn. Applied Mathematical Sciences, vol. 78. Springer, Berlin (2008)
56. Dacorogna, B., Koshigoe, H.: On the different notions of convexity for rotationally invariant functions. Ann. Fac. Sci. Toulouse **2**, 163–184 (1993)
57. Dacorogna, B., Marcellini, P.: Implicit Partial Differential Equations. Birkhäuser, Boston (1999)
58. Dacorogna, B., Maréchal, P.: A note on spectrally defined polyconvex functions. In: Carozza, M., et al. (eds.) Proceedings of the Workshop "New Developments in the Calculus of Variations", pp. 27–54. Edizioni Scientifiche Italiane, Napoli (2006)
59. Davis, C.: All convex invariant functions of Hermitian matrices. Arch. Math. **8**, 276–278 (1957)
60. de Boer, R.: Die elastisch-plastische Biegung eines Plattenstreifens aus inkompressiblem Werkstoff bei endlichen Formänderungen. Ing.-Arch. **36**(3), 145–154 (1967)
61. de Boer, R., Bruhns, O.T.: Zur Berechnung der Eigenspannungen bei einem durch endliche Biegung verformten inkompressiblen Plattenstreifen. Acta Mech. **8**(3–4), 146–159 (1969)
62. dell'Isola, F., Ruta, G.C., Batra, R.C.: A second-order solution of Saint-Venant's problem for an elastic pretwisted bar using Signorini's perturbation method. J. Elast. **49**(2), 113–127 (1997)
63. dell'Isola, F., Ruta, G.C., Batra, R.C.: Generalized Poynting effects in predeformed prismatic bars. J. Elast. **50**(2), 181–196 (1998)
64. Destrade, M., Murphy, J., Saccomandi, G.: Simple shear is not so simple. Int. J. Non-Linear Mech. **47**(2), 210–214 (2012)
65. Diani, J., Gilormini, P.: Combining the logarithmic strain and the full-network model for a better understanding of the hyperelastic behavior of rubber-like materials. J. Mech. Phys. Solids **53**(11), 2579–2596 (2005)
66. Dłużewski, P.: Anisotropic hyperelasticity based upon general strain measures. J. Elast. **60**(2), 119–129 (2000)
67. Dłuzewski, P., Jurczak, G., Antúnez, H.: Logarithmic measure of strains in finite element modelling of anisotropic deformations of elastic solids. Comput. Assist. Mech. Eng. Sci. **10**, 69–79 (2003)
68. Dłużewski, P., Traczykowski, P.: Numerical simulation of atomic positions in quantum dot by means of molecular statics. Arch. Mech. **55**(5–6), 393–406 (2003)
69. Dui, G.S.: Some basis-free formulae for the time rate and conjugate stress of logarithmic strain tensor. J. Elast. **83**(2), 113–151 (2006)
70. Ebbing, V., Schröder, J., Neff, P.: Approximation of anisotropic elasticity tensors at the reference state with polyconvex energies. Arch. Appl. Mech. **79**, 651–657 (2009)
71. Edelstein, W., Fosdick, R.: A note on non-uniqueness in linear elasticity theory. Z. Angew. Math. Phys. **19**(6), 906–912 (1968)
72. Ernst, E.: Ellipticity loss in isotropic elasticity. J. Elast. **51**(3), 203–211 (1998)
73. Fiala, Z.: Time derivative obtained by applying the Riemannian manifold of Riemannian metrics to kinematics of continua. C. R., Méc. **332**(2), 97–102 (2004)
74. Fiala, Z.: Geometry of finite deformations, linearization, and incremental deformations under initial stress/strain. In: Proceedings of International Conference Engineering Mechanics 2008 (2008). 20 pp.
75. Fiala, Z.: Logarithmic strain in 1 versus 3(2) dimensions. In: Proceedings of International Conference Engineering Mechanics 2010 (2010) 11 pp.
76. Fiala, Z.: Geometrical setting of solid mechanics. Ann. Phys. **326**(8), 1983–1997 (2011)
77. Fitzgerald, J.E.: A tensorial Hencky measure of strain and strain rate for finite deformations. J. Appl. Phys. **51**, 5111–5115 (1980)

78. Flory, P.J.: Thermodynamic relations for high elastic materials. Trans. Faraday Soc. **57**, 829–838 (1961)
79. Fosdick, R., Piccioni, M.D., Puglisi, G.: A note on uniqueness in linear elastostatics. J. Elast. **88**(1), 79–86 (2007)
80. Fosdick, R., Šilhavý, M.: Generalized Baker-Ericksen inequalities. J. Elast. **85**(1), 39–44 (2006)
81. Fosdick, R., Volkmann, E.: Normality and convexity of the yield surface in nonlinear plasticity. Q. Appl. Math. **51**, 117–127 (1993)
82. Freed, A.D.: Natural strain. J. Eng. Mater. Technol. **117**(4), 379–385 (1995)
83. Freed, A.D.: Hencky strain and logarithmic rates in Lagrangian analysis. Int. J. Eng. Sci. **81**, 135–145 (2014)
84. Fu, Y.B., Ogden, R.W.: Nonlinear Elasticity: Theory and Applications, vol. 281. Cambridge University Press, Cambridge (2001)
85. Fung, Y.C.: Inversion of a class of nonlinear stress-strain relationships of biological soft tissues. J. Biomech. Eng. **101**(1), 23–27 (1979)
86. Gao, X.L.: Finite deformation elasto-plastic solution for the pure bending problem of a wide plate of elastic linear-hardening material. Int. J. Solids Struct. **31**(10), 1357–1376 (1994)
87. Gao, X.L.: Finite deformation continuum model for single-walled carbon nanotubes. Int. J. Solids Struct. **40**(26), 7329–7337 (2003)
88. Gao, X.L., Atluri, S.N.: An exact finite deformation elasto-plastic solution for the outside-in free eversion problem of a tube of elastic linear-hardening material. IMA J. Appl. Math. **58**(3), 259–275 (1997)
89. Gearing, B.P., Anand, L.: Notch-sensitive fracture of polycarbonate. Int. J. Solids Struct. **41**(3), 827–845 (2004)
90. Germain, S., Scherer, M., Steinmann, P.: On inverse form finding for anisotropic hyperelasticity in logarithmic strain space. Int. J. Struct. Chang. Solids - Mech. Appl. **2**(2), 1–16 (2010)
91. Ghiba, I.D., Müller, B., Neff, P., Starke, G.: On the stress-strain invertibility in nonlinear elasticity (2015, in preparation)
92. Glüge, R., Kalisch, J.: Graphical representations of the regions of rank-one-convexity of some strain energies. Tech. Mech. **32**, 227–237 (2012)
93. Gurtin, M.E., Spear, K.: On the relationship between the logarithmic strain rate and the stretching tensor. Int. J. Solids Struct. **19**(5), 437–444 (1983)
94. Hanin, M., Reiner, M.: On isotropic tensor-functions and the measure of deformation. Z. Angew. Math. Phys. **7**(5), 377–393 (1956)
95. Hartig, E.: Der Elastizitätsmodul des geraden Stabes als Funktion der spezifischen Beanspruchung. Z. Civilingenieur **XXXIX**, 113–138 (1893). https://www.uni-due.de/imperia/md/content/mathematik/ag_neff/hartig_elastizitaetsmodul.pdf
96. Hartmann, S., Neff, P.: Polyconvexity of generalized polynomial type hyperelastic strain energy functions for near incompressibility. Int. J. Solids Struct. **40**(11), 2767–2791 (2003)
97. Henann, D.L., Anand, L.: Fracture of metallic glasses at notches: effects of notch-root radius and the ratio of the elastic shear modulus to the bulk modulus on toughness. Acta Mater. **57**(20), 6057–6074 (2009)
98. Henann, D.L., Anand, L.: A large strain isotropic elasticity model based on molecular dynamics simulations of a metallic glass. J. Elast. **104**(1–2), 281–302 (2011)
99. Hencky, H.: Zur Theorie plastischer Deformationen und der hierdurch im Material hervorgerufenen Nachspannungen. Z. Angew. Math. Mech. **4**(4), 323–334 (1924)
100. Hencky, H.: Über die Form des Elastizitätsgesetzes bei ideal elastischen Stoffen. Z. Tech. Phys. **9**, 215–220 (1928). https://www.uni-due.de/imperia/md/content/mathematik/ag_neff/hencky1928.pdf (see also the technical translation NASA TT-21602)
101. Hencky, H.: Das Superpositionsgesetz eines endlich deformierten relaxationsfähigen elastischen Kontinuums und seine Bedeutung für eine exakte Ableitung der Gleichungen für die zähe Flüssigkeit in der Eulerschen Form. Ann. Phys. **2**, 617–630 (1929). https://www.uni-due.de/imperia/md/content/mathematik/ag_neff/hencky_superposition1929.pdf
102. Hencky, H.: Welche Umstände bedingen die Verfestigung bei der bildsamen Verformung von festen isotropen Körpern? Z. Phys. **55**, 145–155 (1929). https://www.uni-due.de/imperia/md/content/mathematik/ag_neff/hencky1929.pdf
103. Hencky, H.: The law of elasticity for isotropic and quasi-isotropic substances by finite deformations. J. Rheol. **2**, 169–176 (1931). https://www.uni-due.de/imperia/md/content/mathematik/ag_neff/henckyjrheology31.pdf
104. Hencky, H.: The elastic behavior of vulcanized rubber. Rubber Chem. Technol. **6**(2), 217–224 (1933). https://www.uni-due.de/imperia/md/content/mathematik/ag_neff/hencky_vulcanized_rubber.pdf

105. Hill, R.: A theory of the yielding and plastic flow of anisotropic metals. Proc. R. Soc., Math. Phys. Eng. Sci. **193**(1033), 281–297 (1948)
106. Hill, R.: The Mathematical Theory of Plasticity. Clarendon, Oxford (1950)
107. Hill, R.: On constitutive inequalities for simple materials—I. J. Mech. Phys. Solids **16**(4), 229–242 (1968)
108. Hill, R.: Constitutive inequalities for isotropic elastic solids under finite strain. Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. **314**(1519), 457–472 (1970)
109. Hill, R.: Aspects of invariance in solid mechanics. Adv. Appl. Mech. **18**, 1–75 (1978)
110. Hill, R.: On the theory of plane strain in finitely deformed compressible materials. Math. Proc. Camb. Philos. Soc. **86**, 161–178 (1979)
111. Hocine, B., Chevalier, L., Idjeri, M.: A constitutive model for isotropic rubber-like materials based on the logarithmic strain invariants approach (2014, submitted)
112. Hoger, A.: The material time derivative of logarithmic strain. Int. J. Solids Struct. **22**(9), 1019–1032 (1986)
113. Hoger, A.: The stress conjugate to logarithmic strain. Int. J. Solids Struct. **23**(12), 1645–1656 (1987)
114. Horgan, C., Murphy, J.: Constitutive modeling for moderate deformations of slightly compressible rubber. J. Rheol. **53**, 153 (2009)
115. Horgan, C., Murphy, J.: On the volumetric part of strain-energy functions used in the constitutive modeling of slightly compressible solid rubbers. Int. J. Solids Struct. **46**(16), 3078–3085 (2009)
116. Hutchinson, J.W., Neale, K.W.: Finite strain J_2-deformation theory. In: Carlson, D.E., Shield, R.T. (eds.) Proceedings of the IUTAM Symposium on Finite Elasticity, pp. 237–247. Nijhoff, Dordrecht (1982). https://www.uni-due.de/imperia/md/content/mathematik/ag_neff/hutchinson_ellipticity80.pdf
117. Imbert, A.: Recherches théoriques et expérimentales sur l'élasticité du Caoutchouc. University-de Lyon, Ph.D.-thesis, 1880, https://www.uni-due.de/imperia/md/content/mathematik/ag_neff/imbert_rubber.pdf
118. Jog, C.S.: Foundations and Applications of Mechanics: Continuum Mechanics, vol. 1. CRC Press, Boca Raton (2002)
119. Jog, C.S.: On the explicit determination of the polar decomposition in n-dimensional vector spaces. J. Elast. **66**(2), 159–169 (2002)
120. Jog, C.S.: The explicit determination of the logarithm of a tensor and its derivatives. J. Elast. **93**(2), 141–148 (2008)
121. Jog, C.S., Patil, K.D.: Conditions for the onset of elastic and material instabilities in hyperelastic materials. Arch. Appl. Mech. **83**, 1–24 (2013)
122. Johnson, B., Hoger, A.: The dependence of the elasticity tensor on residual stress. J. Elast. **33**(2), 145–165 (1993)
123. Jones, D.F., Treloar, L.R.G.: The properties of rubber in pure homogeneous strain. J. Phys. D, Appl. Phys. **8**(11), 1285 (1975)
124. Kakavas, P.A.: Prediction of the nonlinear Poisson function using large volumetric strains estimated from a finite hyperelastic material law. Polym. Eng. Sci. **40**(6), 1330–1333 (2000)
125. Kearsley, E.A.: Asymmetric stretching of a symmetrically loaded elastic sheet. Int. J. Solids Struct. **22**(2), 111–119 (1986)
126. Kijowski, J., Magli, G.: Relativistic elastomechanics as a Lagrangian field theory. J. Geom. Phys. **9**(3), 207–223 (1992)
127. Knowles, J.K., Sternberg, E.: On the ellipticity of the equations of nonlinear elastostatics for a special material. J. Elast. **5**(3–4), 341–361 (1975)
128. Knowles, J.K., Sternberg, E.: On the failure of ellipticity of the equations for finite elastostatic plane strain. Arch. Ration. Mech. Anal. **63**(4), 321–336 (1976)
129. Knowles, J.K., Sternberg, E.: On the failure of ellipticity and the emergence of discontinuous deformation gradients in plane finite elastostatics. J. Elast. **8**(4), 329–379 (1978)
130. Kochkin, A.P.: Stress-strain dependence in the nonlinear theory of elasticity. Indian J. Pure Appl. Math. **17**(4), 564–579 (1986)
131. Krawietz, A.: A comprehensive constitutive inequality in finite elastic strain. Arch. Ration. Mech. Anal. **58**, 127–149 (1975)
132. Lankeit, J., Neff, P., Nakatsukasa, Y.: The minimization of matrix logarithms: on a fundamental property of the unitary polar factor. Linear Algebra Appl. **449**(0), 28–42 (2014)
133. Lehmann, Th., Liang, H.: The stress conjugate to logarithmic strain $\log V$. Z. Angew. Math. Mech. **73**(12), 357–363 (1993)
134. Lehmich, S., Neff, P., Lankeit, J.: On the convexity of the function $C \to f(\det C)$ on positive definite matrices. Math. Mech. Solids **19**, 369–375 (2014)
135. Lewis, A.S.: Convex analysis on the Hermitian matrices. SIAM J. Optim. **6**(1), 164–177 (1996)
136. Lewis, A.S.: The mathematics of eigenvalue optimization. Math. Program., Ser. B **97**(1–2), 155–176 (2003)

137. Lewis, A.S., Overton, M.L.: Eigenvalue optimization. Acta Numer. **5**, 149–190 (1996)
138. Ludwik, P.: Elemente der technologischen Mechanik. Springer, Berlin (1909). https://www.uni-due.de/imperia/md/content/mathematik/ag_neff/ludwik.pdf
139. Lurie, A.I.: Non-linear Theory of Elasticity. Elsevier, Amsterdam (1990)
140. Marsden, J.E., Hughes, J.R.: Mathematical Foundations of Elasticity. Prentice-Hall, Englewood Cliffs (1983)
141. Marshall, A.W., Olkin, I., Arnold, B.C.: Inequalities: Theory of Majorization and Its Applications. Springer Series in Statistics. Springer, Berlin (2011)
142. Martin, R., Neff, P.: Some remarks on monotonicity of primary matrix functions on the set of symmetric matrices. Arch. Appl. Mech. (2015, accepted)
143. Marzano, S.: An interpretation of Baker-Ericksen inequalities in uniaxial deformation and stress. Meccanica **18**(4), 233–235 (1983)
144. Merodio, J., Neff, P.: A note on tensile instabilities and loss of ellipticity for a fiber-reinforced nonlinearly elastic solid. Arch. Mech. **58**(3), 293–303 (2006)
145. Merrill, G.: Biographical memoir George Ferdinand Becker. Mem. Nat. Acad. Sci., XXI (1927) (see http://en.wikipedia.org/wiki/George_Ferdinand_Becker)
146. Meyers, A.: On the consistency of some Eulerian strain rates. Z. Angew. Math. Mech. **79**(3), 171–177 (1999)
147. Meyers, A., Xiao, H., Bruhns, O.T.: Choice of objective rate in single parameter hypoelastic deformation cycles. Compos. Struct. **84**(17), 1134–1140 (2006)
148. Miehe, C., Apel, N., Lambrecht, M.: Anisotropic additive plasticity in the logarithmic strain space: modular kinematic formulation and implementation based on incremental minimization principles for standard materials. Comput. Methods Appl. Mech. Eng. **191**(47), 5383–5425 (2002)
149. Miehe, C., Méndez Diez, J., Göktepe, S., Schänzel, L.M.: Coupled thermoviscoplasticity of glassy polymers in the logarithmic strain space based on the free volume theory. Int. J. Solids Struct. **48**(13), 1799–1817 (2011)
150. Miehe, C., Göktepe, S., Méndez Diez, J.: Finite viscoplasticity of amorphous glassy polymers in the logarithmic strain space. Int. J. Solids Struct. **46**, 181–202 (2009)
151. Mielke, A.: Finite elastoplasticity, Lie groups and geodesics on SL(d). In: Newton, P. (ed.) Geometry, Mechanics and Dynamics. Volume in Honour of the 60th Birthday of J.E. Marsden, pp. 61–90. Springer, Berlin (2002)
152. Mielke, A.: Necessary and sufficient conditions for polyconvexity of isotropic functions. J. Convex Anal. **12**(2), 291–314 (2005)
153. Mihai, A., Goriely, A.: Positive or negative Poynting effect? The role of adscititious inequalities in hyperelastic materials. Proc. R. Soc. Lond. **467**(2136), 3633–3646 (2011)
154. Mihai, A., Goriely, A.: Numerical simulation of shear and the Poynting effects by the finite element method: an application of the generalised empirical inequalities in non-linear elasticity. Int. J. Non-Linear Mech. **49**, 1–14 (2013)
155. Moakher, M.: Means and averaging in the group of rotations. SIAM J. Matrix Anal. Appl. **24**(1), 1–16 (2002)
156. Moakher, M.: A differential geometric approach to the geometric mean of symmetric positive-definite matrices. SIAM J. Matrix Anal. Appl. **26**(3), 735–747 (2005)
157. Moon, H., Truesdell, C.: Interpretation of adscititious inequalities through the effects pure shear stress produces upon an isotropic elastic solid. Arch. Ration. Mech. Anal. **55**(1), 1–17 (1974)
158. Mooney, M.: A theory of large elastic deformation. J. Appl. Phys. **11**(9), 582–592 (1940)
159. Mott, P.H., Roland, C.M.: Limits to Poisson's ratio in isotropic materials. Phys. Rev. B **80**(13), 132104 (2009), 4 pp.
160. Müller, Ch., Bruhns, O.T.: A thermodynamic finite-strain model for pseudoelastic shape memory alloys. Int. J. Plast. **22**(9), 1658–1682 (2006)
161. Murnaghan, F.D.: The compressibility of solids under extreme pressures. In: Theodore von Karman Anniversary Volume, Calif. Inst. Techn., Pasadena, pp. 121–136. Univ. of Berkeley Press, Berkeley (1941). https://www.uni-due.de/imperia/md/content/mathematik/ag_neff/murnaghan.pdf
162. Murnaghan, F.D.: The compressibility of media under extreme pressures. Proc. Natl. Acad. Sci. USA **30**(9), 244 (1944)
163. Murphy, J.: Linear isotropic relations in finite hyperelasticity: some general results. J. Elast. **86**(2), 139–154 (2007)
164. Nadai, A.: Plastic behavior of metals in the strain-hardening range. Part I. J. Appl. Phys. **8**(3), 205–213 (1937)
165. Neff, P.: Mathematische Analyse multiplikativer Viskoplastizität. Ph.D. thesis, Technische Universität Darmstadt. Shaker Verlag, Aachen, 2000. ISBN:3-8265-7560-1, https://www.uni-due.de/~hm0014/Download_files/cism_convexity08.pdf

166. Neff, P.: Some results concerning the mathematical treatment of finite plasticity. In: Deformation and Failure in Metallic Materials, pp. 251–274. Springer, Berlin (2003)
167. Neff, P.: Critique of "Two-dimensional examples of rank-one convex functions that are not quasiconvex" by M.K. Benaouda and J.J. Telega. Ann. Pol. Math. **86**(2), 193–195 (2005)
168. Neff, P.: A finite-strain elastic-plastic Cosserat theory for polycrystals with grain rotations. Int. J. Eng. Sci. **44**, 574–594 (2006)
169. Neff, P., Eidel, B., Martin, R.: The axiomatic deduction of the quadratic Hencky strain energy by Heinrich Hencky (a new translation of Hencky's original German articles) (2014). arXiv:1402.4027
170. Neff, P., Eidel, B., Martin, R.: Geometry of logarithmic strain measures in solid mechanics. The Hencky energy is the squared geodesic distance of the deformation gradient to SO(n) in any left-invariant, right-O(n)-invariant Riemannian metric on GL(n) (2015, in preparation)
171. Neff, P., Eidel, B., Osterbrink, F., Martin, R.: The Hencky strain energy $\|\log U\|^2$ measures the geodesic distance of the deformation gradient to SO(3) in the canonical left-invariant Riemannian metric on GL(3). PAMM **13**(1), 369–370 (2013)
172. Neff, P., Eidel, B., Osterbrink, F., Martin, R.: A Riemannian approach to strain measures in nonlinear elasticity. C. R. Acad. Sci. **342**, 254–257 (2014)
173. Neff, P., Ghiba, I.D.: The exponentiated Hencky-logarithmic strain energy. Part III: Coupling with idealized isotropic finite strain plasticity. Contin. Mech. Thermodyn., the special issue in honour of D.J. Steigmann (2015, to appear). arXiv:1409.7555
174. Neff, P., Jeong, J., Fischle, A.: Stable identification of linear isotropic Cosserat parameters: bounded stiffness in bending and torsion implies conformal invariance of curvature. Acta Mech. **211**(3–4), 237–249 (2010)
175. Neff, P., Lankeit, J., Madeo, A.: On Grioli's minimum property and its relation to Cauchy's polar decomposition. Int. J. Eng. Sci. **80**, 207–217 (2014)
176. Neff, P., Münch, I., Martin, R.: Rediscovering G.F. Becker's early axiomatic deduction of a multiaxial nonlinear stress-strain relation based on logarithmic strain. Math. Mech. Solids (2014, to appear). arXiv:1403.4675v2, doi:10.1177/1081286514542296
177. Neff, P., Nakatsukasa, Y., Fischle, A.: A logarithmic minimization property of the unitary polar factor in the spectral norm and the Frobenius matrix norm. SIAM J. Matrix Anal. Appl. **35**, 1132–1154 (2014)
178. Norris, A.: Eulerian conjugate stress and strain. J. Mech. Mater. Struct. **3**(2), 243–260 (2008)
179. Ogden, R.W.: Compressible isotropic elastic solids under finite strain-constitutive inequalities. Q. J. Mech. Appl. Math. **23**(4), 457–468 (1970)
180. Ogden, R.W.: On constitutive relations for elastic and plastic materials. Ph.D. Thesis, Cambridge University (1970)
181. Ogden, R.W.: Inequalities associated with the inversion of elastic stress-deformation relations and their implications. Math. Proc. Camb. Philos. Soc. **81**, 313–324 (1977)
182. Ogden, R.W.: Non-linear Elastic Deformations, 1st edn. Mathematics and Its Applications. Ellis Horwood, Chichester (1983)
183. Ogden, R.W., Saccomandi, G., Sgura, I.: Fitting hyperelastic models to experimental data. Comput. Mech. **34**(6), 484–502 (2004)
184. Onaka, S.: Equivalent strain in simple shear deformation described by using the Hencky strain. Philos. Mag. Lett. **90**(9), 633–639 (2010)
185. Onaka, S.: Appropriateness of the Hencky equivalent strain as the quantity to represent the degree of severe plastic deformation. Mater. Trans. **53**(8), 1547–1548 (2012)
186. Ortiz, M., Radovitzky, R.A., Repetto, E.A.: The computation of the exponential and logarithmic mappings and their first and second linearizations. Int. J. Numer. Methods Eng. **52**(12), 1431–1441 (2001)
187. Panov, A.D., Shumaev, V.V.: Using the logarithmic strain measure for solving torsion problems. Mech. Solids **47**(1), 71–78 (2012)
188. Pennec, X.: Left-invariant Riemannian elasticity: a distance on shape diffeomorphisms? In: 1st MICCAI Workshop on Mathematical Foundations of Computational Anatomy: Geometrical, Statistical and Registration Methods for Modeling Biological Shape Variability, pp. 1–13 (2006)
189. Pennec, X., Stefanescu, R., Arsigny, V., Fillard, P., Ayache, N.: Riemannian elasticity: a statistical regularization framework for non-linear registration. In: Medical Image Computing and Computer-Assisted Intervention—-MICCAI 2005, pp. 943–950. Springer, Berlin (2005)
190. Perić, D., Owen, D.R.J., Honnor, M.E.: A model for finite strain elasto-plasticity based on logarithmic strains: computational issues. Comput. Methods Appl. Mech. Eng. **94**(1), 35–61 (1992)
191. Plešek, J., Kruisová, A.: Formulation, validation and numerical procedures for Hencky's elasticity model. Comput. Struct. **84**, 1141–1150 (2006)
192. Poirier, J.-P., Tarantola, A.: A logarithmic equation of state. Phys. Earth Planet. Inter. **109**, 1–8 (1998)
193. Poisson, S.D.: Traité de Mécanique, vol. 2 (1811)

194. Raoult, A.: Non-polyconvexity of the stored energy function of a St.Venant-Kirchhoff material. Apl. Mat. **6**, 417–419 (1986)
195. Reinhardt, W.D., Dubey, R.N.: Application of objective rates in mechanical modeling of solids. J. Appl. Mech. **63**(3), 692–698 (1996)
196. Richter, H.: Das isotrope Elastizitätsgesetz. Z. Angew. Math. Mech. **28**(7–8), 205–209 (1948). https://www.uni-due.de/imperia/md/content/mathematik/ag_neff/richter_isotrop_log.pdf
197. Richter, H.: Verzerrungstensor, Verzerrungsdeviator und Spannungstensor bei endlichen Formänderungen. Z. Angew. Math. Mech. **29**(3), 65–75 (1949). https://www.uni-due.de/imperia/md/content/mathematik/ag_neff/richter_deviator_log.pdf
198. Richter, H.: Zum Logarithmus einer Matrix. Arch. Math. **2**, 360–363 (1950). https://www.uni-due.de/imperia/md/content/mathematik/ag_neff/richter_log.pdf
199. Richter, H.: Zur Elastizitätstheorie endlicher Verformungen. Math. Nachr. **8**, 65–73 (1952). https://www.uni-due.de/imperia/md/content/mathematik/ag_neff/richter_endliche_verzerrungen52.pdf
200. Rivlin, R.: Large elastic deformations of isotropic materials. IV. Further developments of the general theory. Philos. Trans. R. Soc. Lond. Ser. A, Math. Phys. Sci. **241**(835), 379–397 (1948)
201. Rivlin, R.: Restrictions on the strain-energy function for an elastic material. Math. Mech. Solids **9**(2), 131–139 (2004)
202. Rivlin, R.S.: Some restrictions on constitutive equations. In: Domingos, J.J., Nina, M.N.R., Whitelaw, J.H. (eds.) Proc. Int. Symp. on the Foundations of Continuum Thermodynamics, pp. 229–258. Macmillan, London (1974)
203. Rolph, W.D., Bathe, K.J.: On a large strain finite element formulation for elasto-plastic analysis. In: Constitutive Equations: Macro and Computational Aspects, Winter Annual Meeting, pp. 131–147. ASME, New York (1984)
204. Rosakis, P.: Ellipticity and deformations with discontinuous gradients in finite elastostatics. Arch. Ration. Mech. Anal. **109**(1), 1–37 (1990)
205. Rosakis, P.: Characterization of convex isotropic functions. J. Elast. **49**, 257–267 (1998)
206. Rougée, P.: Mécanique des Grandes Transformations. Mathématiques et Applications, vol. 25. Springer, Berlin (1997)
207. Rougée, P.: An intrinsic Lagrangian statement of constitutive laws in large strain. Compos. Struct. **84**(17), 1125–1133 (2006)
208. Sansour, C.: On the dual variable of the logarithmic strain tensor, the dual variable of the Cauchy stress tensor, and related issues. Int. J. Solids Struct. **38**(50), 9221–9232 (2001)
209. Sansour, C.: On the physical assumptions underlying the volumetric-isochoric split and the case of anisotropy. Eur. J. Mech. A, Solids **27**(1), 28–39 (2008)
210. Sansour, C., Kollmann, F.G.: On theory and numerics of large viscoplastic deformation. Comput. Methods Appl. Mech. Eng. **146**(3), 351–369 (1997)
211. Sawyers, K.N., Rivlin, R.: Instability of an elastic material. Int. J. Solids Struct. **9**(5), 607–613 (1973)
212. Sawyers, K.N., Rivlin, R.: A note on the Hadamard criterion for an incompressible isotropic elastic material. Mech. Res. Commun. **5**, 211–224 (1978)
213. Sawyers, K.N., Rivlin, R.: On the speed of propagation of waves in a deformed compressible elastic material. Z. Angew. Math. Phys. **29**, 245–251 (1978)
214. Scheidler, M.: Time rates of generalized strain tensors Part I: Component formulas. Mech. Mater. **11**(3), 199–210 (1991)
215. Scheidler, M.: Time rates of generalized strain tensors, Part II: Approximate basis-free formulas. Mech. Mater. **11**(3), 211–219 (1991)
216. Schröder, J., Neff, P.: Invariant formulation of hyperelastic transverse isotropy based on polyconvex free energy functions. Int. J. Solids Struct. **40**(2), 401–445 (2003)
217. Schröder, J., Neff, P.: Poly, quasi and rank-one convexity in mechanics. In: CISM-Course Udine. Springer, Berlin (2009)
218. Schröder, J., Neff, P.: On the construction of polyconvex transversely isotropic free energy functions. In: IUTAM Symposium on Computational Mechanics of Solid Materials at Large Strains, 21 August 2001 (2001). Organizer: C. Miehe, University Stuttgart
219. Schröder, J., Neff, P., Balzani, D.: A variational approach for materially stable anisotropic hyperelasticity. Int. J. Solids Struct. **42**(15), 4352–4371 (2005)
220. Schröder, J., Neff, P., Ebbing, V.: Anisotropic polyconvex energies on the basis of crystallographic motivated structural tensors. J. Mech. Phys. Solids **56**(12), 3486–3506 (2008)
221. Schwarz, A., Schröder, J., Starke, G.: A modified least-squares mixed finite element with improved momentum balance. Int. J. Numer. Methods Eng. **81**(3), 286–306 (2010)
222. Sendova, T., Walton, J.R.: On strong ellipticity for isotropic hyperelastic materials based upon logarithmic strain. Int. J. Non-Linear Mech. **40**(2–3), 195–212 (2005)

223. Seth, B.R.: Generalized strain measure with applications to physical problems. In: Rainer, M.A. (ed.) Second-Order Effects in Elasticity, Plasticity and Fluid Dynamics, pp. 162–172. Pergamon, Oxford (1961)
224. Shrivastava, S., Ghosh, C., Jonas, J.: A comparison of the von Mises and Hencky equivalent strains for use in simple shear experiments. Philos. Mag. Lett. **92**(7), 779–786 (2012)
225. Sidoroff, R.: Sur les restrictions à imposer à l'énergie de déformation d'un matériau hyperélastique. C. R. Acad. Sci. Paris **279**, 379–382 (1974)
226. Šilhavý, M.: The Mechanics and Thermomechanics of Continuous Media. Springer, Berlin (1997)
227. Šilhavý, M.: Convexity conditions for rotationally invariant functions in two dimensions. In: Sequeira, et al. (eds.) Applied Nonlinear Analysis. Kluwer Academic, New York (1999)
228. Šilhavý, M.: On isotropic rank one convex functions. Proc. R. Soc. Edinb. **129**, 1081–1105 (1999)
229. Šilhavý, M.: Rank 1 convex hulls of isotropic functions in dimension 2 by 2. Math. Bohem. **126**(2), 521–529 (2001)
230. Šilhavý, M.: Convexity Conditions for Rotationally Invariant Functions in Two Dimensions. Springer, Berlin (2002)
231. Šilhavý, M.: Monotonicity of rotationally invariant convex and rank 1 convex functions. Proc. R. Soc. Edinb. **132**, 419–435 (2002)
232. Šilhavý, M.: An O(n) invariant rank 1 convex function that is not polyconvex. Theor. Appl. Mech. **28**, 325–336 (2002)
233. Šilhavý, M.: On SO(n)-invariant rank 1 convex functions. J. Elast. **71**, 235–246 (2003)
234. Silling, S.A.: Numerical studies of loss of ellipticity near singularities in an elastic material. J. Elast. **19**(3), 213–239 (1988)
235. Simo, J.C.: Algorithms for static and dynamic multiplicative plasticity that preserve the classical return mapping schemes of the infinitesimal theory. Comput. Methods Appl. Mech. Eng. **99**(1), 61–112 (1992)
236. Simpson, H., Spector, S.: On copositive matrices and strong ellipticity for isotropic elastic materials. Arch. Ration. Mech. Anal. **84**(1), 55–68 (1983)
237. Simpson, H., Spector, S.: On bifurcation in finite elasticity: buckling of a rectangular rod. J. Elast. **92**(3), 277–326 (2008)
238. Skrzypek, J., Wróblewski, A.: Application of logarithmic strains to changing principal directions via progressing transformations. J. Struct. Mech. **13**(3–4), 283–299 (1985)
239. Smith, C.W., Wootton, R.J., Evans, K.E.: Interpretation of experimental data for Poisson's ratio of highly nonlinear materials. Exp. Mech. **39**(4), 356–362 (1999)
240. Starke, G.: An adaptive least-squares mixed finite element method for elasto-plasticity. SIAM J. Numer. Anal. **45**(1), 371–388 (2007)
241. Starke, G., Schwarz, A., Schröder, J.: Analysis of a modified first-order system least squares method for linear elasticity with improved momentum balance. SIAM J. Numer. Anal. **49**(3), 1006–1022 (2011)
242. Tabor, D.: The bulk modulus of rubber. Polymer **35**(13), 2759–2763 (1994)
243. Tanner, R., Tanner, E.: Heinrich Hencky: a rheological pioneer. Rheol. Acta **42**(1–2), 93–101 (2003)
244. Tarantola, A.: Elements for Physics: Quantities, Qualities, and Intrinsic Theories. Springer, Heidelberg (2006)
245. Tarantola, A.: Stress and strain in symmetric and asymmetric elasticity (2009). Preprint arXiv:0907.1833
246. https://www.uni-due.de/mathematik/ag_neff/neff_hencky/treloarjonesdata.zip
247. Treloar, L.R.G.: Stress-strain data for vulcanised rubber under various types of deformation. Trans. Faraday Soc. **40**, 59–70 (1944)
248. Treloar, L.R.G.: The elasticity and related properties of rubbers. Rep. Prog. Phys. **36**(7), 755 (1973)
249. Treloar, L.R.G.: The Physics of Rubber Elasticity. Oxford University Press, London (1975)
250. Truesdell, C.: Das ungelöste Hauptproblem der endlichen Elastizitätstheorie. Z. Angew. Math. Mech. **36**(3–4), 97–103 (1956)
251. Truesdell, C.: The Mechanical Foundations of Elasticity and Fluid Dynamics, vol. 8. Gordon & Breach, New York (1966)
252. Truesdell, C., Moon, H.: Inequalities sufficient to ensure semi-invertibility of isotropic functions. J. Elast. **5**(34), 183–189 (1975)
253. Truesdell, C., Noll, W.: The non-linear field theories of mechanics. In: Flügge, S. (ed.) Handbuch der Physik, vol. III/3. Springer, Heidelberg (1965)
254. Truesdell, C., Toupin, R.: The classical field theories. In: Flügge, S. (ed.) Handbuch der Physik, vol. III/1. Springer, Heidelberg (1960)
255. Truesdell, C., Toupin, R.: Static grounds for inequalities in finite strain of elastic materials. Arch. Ration. Mech. Anal. **12**(1), 1–33 (1963)

256. Vallée, C.: Lois de comportement élastique isotropes en grandes déformations. Int. J. Eng. Sci. **16**(7), 451–457 (1978)
257. Vallée, C., Fortuné, D., Lerintiu, C.: On the dual variable of the Cauchy stress tensor in isotropic finite hyperelasticity. C. R., Méc. **336**(11), 851–855 (2008)
258. Wang, Y., Aron, M.: A reformulation of the strong ellipticity conditions for unconstrained hyperelastic media. J. Elast. **44**(1), 89–96 (1996)
259. Weber, G., Anand, L.: Finite deformation constitutive equations and a time integration procedure for isotropic, hyperelastic-viscoplastic solids. Comput. Methods Appl. Mech. Eng. **79**(2), 173–202 (1990)
260. Wilber, J.P., Criscione, J.C.: The Baker-Ericksen inequalities for hyperelastic models using a novel set of invariants of Hencky strain. Int. J. Solids Struct. **42**(5–6), 1547–1559 (2005)
261. Xiao, H.: Hencky strain and Hencky model: extending history and ongoing tradition. Multidiscipline Mod. Mat. Struct. **1**(1), 1–52 (2005)
262. Xiao, H., Bruhns, O.T., Meyers, A.: Logarithmic strain, logarithmic spin and logarithmic rate. Acta Mech. **124**(1–4), 89–105 (1997)
263. Xiao, H., Bruhns, O.T., Meyers, A.: Existence and uniqueness of the integrable-exactly hypoelastic equation $\tau^\circ = \lambda(\text{tr}\,D)\text{I} + 2\mu D$ and its significance to finite inelasticity. Acta Mech. **138**(1–2), 31–50 (1999)
264. Xiao, H., Bruhns, O.T., Meyers, A.: Elastoplasticity beyond small deformations. Acta Mech. **182**(1–2), 31–111 (2006)
265. Xiao, H., Chen, L.S.: Hencky's elasticity model and linear stress-strain relations in isotropic finite hyperelasticity. Acta Mech. **157**(1–4), 51–60 (2002)
266. Zhang, Y., Li, H., Xiao, H.: Further study of rubber-like elasticity: elastic potentials matching biaxial data. Appl. Math. Mech. **35**, 13–24 (2014)
267. Zhilin, P.A., Altenbach, H., Ivanova, E.A., Krivtsov, A.: Material strain tensor. In: Altenbach, H., Forest, S., Krivtsov, A. (eds.) Generalized Continua as Models for Materials, pp. 321–331. Springer, Berlin (2013)
268. Zimmermann, J., Stommel, M.: The mechanical behavior of rubber under hydrostatic compression and the effect on the results of finite element analyses. Arch. Appl. Mech. **83**(2), 293–302 (2013)
269. Zubov, L.M., Rudev, A.N.: Necessary and sufficient criteria for ellipticity of the equilibrium equations of a non-linearly elastic medium. J. Appl. Math. Mech. **59**(2), 197–208 (1995)
270. Zubov, L.M., Rudev, A.N.: A criterion for the strong ellipticity of the equilibrium equations of an isotropic nonlinearly elastic material. J. Appl. Math. Mech. **75**, 432–446 (2011)