J Elast (2014) 116:27–51
DOI 10.1007/s10659-013-9456-z

Nonlocal Constrained Value Problems for a Linear Peridynamic Navier Equation

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Received: 17 April 2013 / Published online: 23 August 2013

Abstract In this paper, we carry out further mathematical studies of nonlocal constrained value problems for a peridynamic Navier equation derived from linear state-based peridynamic models. Given the nonlocal interactions effected in the model, constraints on the solution over a volume of nonzero measure are natural conditions to impose. We generalize previous well-posedness results that were formulated for very special kernels of nonlocal interactions. We also give a more rigorous treatment to the convergence of solutions to nonlocal peridynamic models to the solution of the conventional Navier equation of linear elasticity as the horizon parameter goes to zero. The results are valid for arbitrary Poisson ratio, which is a characteristic of the state-based peridynamic model.

Keywords Peridynamic model · Nonlocal diffusion · Nonlocal operator · Nonlocal elasticity · Navier equation · Nonlocal Poincare inequality · Well-posedness · Local limit

Mathematics Subject Classification (2000) 45A05 · 45K05 · 47G10 · 74G65 · 74H20 · 74H25

1 Introduction

Recently, there have been much interests in the nonlocal peridynamic (PD) continuum theory introduced first by Silling in [16]. Most of the existing theoretical works on the mathematical foundation of peridynamics [2, 3, 8, 10, 11, 14, 15, 24] have so far largely focused on

The research is supported in part by the U.S. Department of Energy grant DE-SC0005346 and the U.S. NSF grant DMS-1016073.

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the original bond-based PD models where the nonlocal force interaction at a material point is specified as an integral over all pairs of points (bonds) within a spherical neighborhood. However, much of the recent PD type modeling and simulation efforts have been focused on the more general state-based PD models introduced in [18]. This naturally motivates us to establish rigorous mathematical theory for general state-based PD systems.

Indeed, it is well-understood that with the use of only direct bond forces, bond-based PD equations are suitable mostly for special materials systems [16, 18, 20]. For instance, the linear isotropic bond-based PD model corresponds only to linear elastic materials with a Poisson ratio $1/4$. This is not a limitation of PD but rather the nature of simplistic bond-based models. To work with more general materials, a sophisticated state-based PD model was presented in [18] which is an important generalization of bond-based PD models. Silling introduced a deformation operator—the deformation state about an equilibrium— to allow the force interaction at a point to be dependent upon the deformation state, thus leading to the state-based peridynamic theory. The resulting state-based PD system can represent a linear elastic material with a general Poisson ratio. Meanwhile, with the exception of [5], the mathematical theory of state-based PD systems has largely been under-developed. The contribution of [5] is to introduce rigorous variational formulations of a nonlocal state-based PD system, that is, a peridynamic Navier equation, using basic operators of the nonlocal calculus developed in [6]. The paper [5] also demonstrated the well-posedness of constrained value problems over a volume of non-zero measure for a special class of nonlocal interaction kernels. The present work establishes general well-posedness results for nonlocal linear PD Navier equations which are valid under more general conditions on the nonlocal interaction kernels than that previously considered for linear bond-based PD models in [15] and for state-based models in [5].

While the current work serves as a companion paper of [5], the key ingredients used here are significantly different from that in [5] and are technically more involved than that in [15]. Indeed, unlike [5], the current work analyzes models that may incorporate spatial heterogeneity. Moreover, it develops the general theory that gives rise to the well posedness of the PD Navier system in function spaces that may be either L^2 or a subspace properly in between L^2 and H^1. This is because our analysis can be applied to more general interaction kernel which may have stronger singularities (but with finite second moments), while in [5], only square integrable interaction kernels are considered. Technically, and in contrast to [5] where the theory of Hilbert-Schmidt operators associated could be utilized with the standard L^2 space acting as the energy space, in this work we need to provide more detailed characterization of nonlocal operators and their associated energy spaces that arise from the use of more general interaction kernels. For instance, the resulting nonlocal Navier operator could be unbounded in L^2 and its definition may have to be understood in more generalized sense which is elucidated here. In addition, the validity of nonlocal Poincaré-type inequalities for vector fields, which essentially is the coercivity of the PD operator, has to be established carefully. The rigorous analytical set-up also allows us to study the convergence of nonlocal state-based models to its local limit, as the extent of nonlocal interaction gets reduced to zero. By this we mean, not only the convergence of the PD operator to the classical operator as done in [5, 10, 18], but also the convergence of the corresponding solutions for a give forcing term (see [15] for a similar result for the bond-based model). The solution to the PD system may not have additional regularity other than being in the energy space. However, with a careful quantification of the relationship between the solution and the horizon, together with compactness arguments, we are able to prove L^2-convergence of solutions without extra regularity assumptions. Such results are of practical interests since one of the distinct advantages of nonlocal PD models is that they allow for deformation fields which may not have well-defined gradients as in the classical sense.

The paper is organized as follows. We begin the discussion with a brief account of model equations in Sect. 2. In Sect. 3 we give variational formulations of volume constrained problems associated with linear PD models and present the main well-posedness results. In Sect. 4, we examine the local limit.

2 A Linear Peridynamic Solid

We assume that a body of a certain mass density occupies a bounded connected domain Ω in $\mathbb{R}^d$ with sufficiently smooth boundary. According to Silling et al. [18], for constitutively linear, isotropic solid undergoing deformation, the peridynamic model is given by the strain energy density function

$$W(\mathbf{x}) = W_{dil}(\mathbf{x}) + W_{dev}(\mathbf{x}).$$

The functions $W_{dil}(\mathbf{x})$ and $W_{dev}(\mathbf{x})$ are strain energy densities associated with the dilatation and the deviatoric portions of the deformation, respectively. To give a precise expression for the densities, let us first assume that the deformation $\mathbf{y}$ is given by $\mathbf{y}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x})$, where $\mathbf{u}$ is the displacement field. Let

$$e\big\langle \mathbf{x}', \mathbf{x} \big\rangle = \big|\mathbf{y}\big(\mathbf{x}'\big) - \mathbf{y}(\mathbf{x})\big| - \big|\mathbf{x}' - \mathbf{x}\big|$$

measure the change in the length of the bond $\mathbf{x}' - \mathbf{x}$ due to the deformation. Writing $e\langle \mathbf{x}', \mathbf{x}\rangle = s\langle \mathbf{x}', \mathbf{x}\rangle |\mathbf{x}' - \mathbf{x}|$, we notice that the expression $s\langle \mathbf{x}', \mathbf{x}\rangle = \frac{e\langle \mathbf{x}', \mathbf{x}\rangle}{|\mathbf{x}'-\mathbf{x}|}$ is the strain, also called elongation, of the bond $\mathbf{x}' - \mathbf{x}$ at $\mathbf{x}$. Taking into account the collective deformation of a neighborhood of $\mathbf{x}$, one can then introduce the nonlocal dilatation at $\mathbf{x}$ (cf. [5, 18]) as a weighted mean of the strain given by

$$\vartheta(\mathbf{x}) = \frac{d}{m(\mathbf{x})} \int_{B_\delta(\mathbf{x})\cap\Omega} \rho\big(\mathbf{x}' - \mathbf{x}\big)\big|\mathbf{x}' - \mathbf{x}\big|^2 \left(\frac{e\langle \mathbf{x}', \mathbf{x}\rangle}{|\mathbf{x}' - \mathbf{x}|}\right) d\mathbf{x}',$$

where the (locally) integrable function $\tilde{\rho}(\boldsymbol{\xi}) = \rho(\boldsymbol{\xi})|\boldsymbol{\xi}|^2$ measures the interaction strength of the bond $\boldsymbol{\xi}$ between $\mathbf{x}$ and $\mathbf{x}' = \mathbf{x} + \boldsymbol{\xi}$ and the weighted volume $m(\mathbf{x})$ is defined as

$$m(\mathbf{x}) = \int_\Omega \tilde{\rho}\big(\mathbf{x}' - \mathbf{x}\big) d\mathbf{x}' = \int_\Omega \rho\big(\mathbf{x}' - \mathbf{x}\big)\big|\mathbf{x}' - \mathbf{x}\big|^2 d\mathbf{x}'. \tag{1}$$

The strain energy density associated with the nonlocal dilatation is given by

$$W_{dil}(\mathbf{x}) = \frac{k(\mathbf{x})}{2}\big(\vartheta(\mathbf{x})\big)^2,$$

where $k(\mathbf{x})$ is the usual bulk modulus [18].

On the other hand, as observed in [17], the deviatoric part $\frac{e\langle \mathbf{x}', \mathbf{x}\rangle}{|\mathbf{x}'-\mathbf{x}|} - \frac{1}{d}\vartheta(\mathbf{x})$ of the strain $\frac{e\langle \mathbf{x}', \mathbf{x}\rangle}{|\mathbf{x}'-\mathbf{x}|}$ represents the bond strain after subtracting off an isotropic expansion of the neighborhood of $\mathbf{x}$ with dilatation ϑ. This deviatoric part includes not only shear, but also any deformation of the region in a δ neighborhood of $\mathbf{x}$ other than isotropic expansion. The energy density associated with the deviatoric part of the deformation is given by

$$W_{dev}(\mathbf{x}) = \frac{\alpha(\mathbf{x})}{2} \int_{B_\delta(\mathbf{x})\cap\Omega} \rho\big(\mathbf{x}' - \mathbf{x}\big)\big|\mathbf{x}' - \mathbf{x}\big|^2 \left(\frac{e\langle \mathbf{x}', \mathbf{x}\rangle}{|\mathbf{x}' - \mathbf{x}|} - \frac{1}{d}\vartheta(\mathbf{x})\right)^2 d\mathbf{x}',$$

where $\alpha(\mathbf{x})$ is a scalar that is proportional to the shear modulus of the material.

To formulate the linearized peridynamics for solids, a uniform small difference displacement

$$\sup_{|\mathbf{x}'-\mathbf{x}|<\delta} \left|\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})\right| \ll 1$$

is assumed, which leads to the peridynamic strain energy density function $\tilde{W}$ as a second-order (quadratic) approximation of the strain energy W discussed above. The strain energy density $\tilde{W}$ is again a sum of two energy densities associated with the different components of the linearized deformation and is given by

$$\tilde{W}(\mathbf{x}) = \frac{k(\mathbf{x})}{2}\left(\tilde{\vartheta}_{\mathbf{u}}(\mathbf{x})\right)^2 + \frac{\alpha(\mathbf{x})}{2}\int_{\Omega} \rho\left(\mathbf{x}'-\mathbf{x}\right)\left|\mathbf{x}'-\mathbf{x}\right|^2 \left(\frac{\tilde{e}_{\mathbf{u}}\langle\mathbf{x}',\mathbf{x}\rangle}{|\mathbf{x}'-\mathbf{x}|} - \frac{1}{d}\tilde{\vartheta}_{\mathbf{u}}(\mathbf{x})\right)^2 d\mathbf{x}', \tag{2}$$

where $\tilde{e}_{\mathbf{u}}\langle\mathbf{x}',\mathbf{x}\rangle$ is the linearization of the measure of extension of bond with respect to small relative displacement and is given by

$$\tilde{e}_{\mathbf{u}}\left\langle\mathbf{x}',\mathbf{x}\right\rangle = \mathrm{Tr}\left(\mathcal{D}^*\mathbf{u}\right)\left(\mathbf{x}',\mathbf{x}\right)\left|\mathbf{x}'-\mathbf{x}\right| := \left(\mathbf{u}\left(\mathbf{x}'\right) - \mathbf{u}(\mathbf{x})\right)\cdot\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|}. \tag{3}$$

Here $\mathrm{Tr}(\mathcal{D}^*)$ is the trace of a nonlocal gradient operator $\mathcal{D}^*$ defined in [6], namely, $\mathbf{u} \mapsto \mathcal{D}^*\mathbf{u}(\mathbf{x},\mathbf{y}) := (\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) \otimes \alpha(\mathbf{x},\mathbf{y})$ where $\alpha(\mathbf{x},\mathbf{y})$ is an antisymmetric vector field which is taken as $\alpha(\mathbf{x},\mathbf{y}) = \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^2}$ in the present work. This represents a slight modification to the operator specialized in [5] for formulating the peridynamic Navier equations (and adopted in [15] for the bond-based PD systems) where $\alpha(\mathbf{x},\mathbf{y})$ is taken to be a unit vector. The definition adopted here is normalized to have the physically correct unit which is more compatible with the notion of the local gradient and at the same time does not alter the mathematical analysis except for a few notational changes.

As a consequence, the linearized nonlocal dilatation $\tilde{\vartheta}_{\mathbf{u}}(\mathbf{x})$ is given by

$$\tilde{\vartheta}_{\mathbf{u}}(\mathbf{x}) = \mathrm{Tr}\left(\mathcal{D}^*_\omega\mathbf{u}\right)(\mathbf{x}) := \int_{\Omega} \omega\left(\mathbf{x}',\mathbf{x}\right)\frac{\tilde{e}_{\mathbf{u}}\langle\mathbf{x}',\mathbf{x}\rangle}{|\mathbf{x}'-\mathbf{x}|} d\mathbf{x}', \tag{4}$$

where the function

$$\omega\left(\mathbf{x}',\mathbf{x}\right) = \frac{d}{m(\mathbf{x})}\tilde{\rho}\left(\mathbf{x}'-\mathbf{x}\right) = \frac{d}{m(\mathbf{x})}\rho\left(\mathbf{x}'-\mathbf{x}\right)\left|\mathbf{x}'-\mathbf{x}\right|^2, \tag{5}$$

is a weight function, and $\mathrm{Tr}(\mathcal{D}^*_\omega)$ is the trace of the weighted operator [6]

$$\mathcal{D}^*_\omega : \mathbf{u} \mapsto \mathcal{D}^*_\omega(\mathbf{u})(\mathbf{x}) := \int_{\Omega} \mathcal{D}^*(\mathbf{u})(\mathbf{x},\mathbf{y})\omega(\mathbf{y},\mathbf{x})d\mathbf{y}.$$

Note that $\mathcal{D}^*_\omega$ remains exactly the same as the nonlocal weighted gradient defined in [5, 6], although as pointed out earlier, the operator $\mathcal{D}^*$ introduced in this work makes a slight modification to that used in [6] in order to be more compatible in physical units.

We remark that there are other models for linear peridynamic solids such as the one proposed by Aguiar and Fosdick in [1]. In fact, when ρ is radial, the energy density function in (2) is a particular case of the linearly elastic material model studied in [1].

To complete the section, we summarize the conditions on ρ and give additional notation to be used throughout the paper. We assume that ρ satisfies the following minimal conditions:

$$\begin{cases} \rho = \rho(\boldsymbol{\xi}) \geq 0, \rho(\boldsymbol{\xi}) = \rho(-\boldsymbol{\xi}), \tilde{\rho}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2\rho(\boldsymbol{\xi}) \in L^1_{loc}(\mathbb{R}^d);\ \text{there exist} \\ \text{positive constants } \delta \text{ and } \lambda \text{ satisfying } B_\lambda(\mathbf{0}) \subset \text{supp}(\rho) \subset B_\delta(\mathbf{0}). \end{cases} \tag{H}$$

The condition on ρ is said to be minimal as it allows more general ρ than the usually assumed radially symmetric form used in many existing works for the well-posedness studies of PD models and related scalar nonlocal diffusion models, see for example, [14, 15]. The generality of allowing ρ to be even but possibly non-radial is not only of theoretical interest but will also help us studying anisotropic models in the future. The function ρ is not necessarily locally integrable; rather, we assume that it has finite second moments, which is a less restrictive condition that allows for a much broader set of nonlocal interaction kernels, all of which correspond to materials with finite elastic moduli. It is also a necessary condition to make the weighted volume $m(\mathbf{x})$ well defined. In addition to locally integrable even functions, functions $\rho(|\boldsymbol{\xi}|)$ that are comparable to $|\boldsymbol{\xi}|^{-d-2s}$ for some $s \in (0, 1)$ near the origin, also satisfy (H). A kernel not included in the above class is the radial function $\rho(|\boldsymbol{\xi}|) = -|\boldsymbol{\xi}|^{-d}\ln(|\boldsymbol{\xi}|)$ for $|\boldsymbol{\xi}| \leq 1$ and 0 otherwise with $d \geq 2$.

The constant δ in condition (H) measures the extent of material points interactions and is called the *horizon*. In principle, this quantity is material dependent. More discussions on the effect of ρ and δ on the materials properties can be found in [23]. The non-degeneracy condition in assumption (H), namely $B_\lambda(\mathbf{0}) \subset \text{Supp}(\rho)$, is needed to establish a nonlocal Poincaré-type inequality that will be used in determining the solvability of the PD equations, and it is automatically satisfied if ρ is radial and non-degenerate. An elementary but important consequence of this is that there are positive constants m_0 and M such that $0 < m_0 = \min_{\mathbf{x}\in\Omega} m(\mathbf{x}) \leq \max_{\mathbf{x}\in\Omega} m(\mathbf{x}) = M < \infty$. As we proceed, additional necessary conditions on ρ will be provided to establish special properties of interests.

For other materials dependent properties associated with PD models, we suppose that $0 < \alpha_0 = \min_{\mathbf{x}\in\Omega} \alpha(\mathbf{x}) \leq \max_{\mathbf{x}\in\Omega} \alpha(\mathbf{x}) = A < \infty$, $0 < k_0 = \min_{\mathbf{x}\in\Omega} k(\mathbf{x}) \leq \max_{\mathbf{x}\in\Omega} k(\mathbf{x}) = K < \infty$. As for the domain Ω, we take it to be smooth enough to satisfy an interior cone condition, that is, there exists a positive constant angle $\theta > 0$ such that for any $\mathbf{x} \in \bar{\Omega}$, $\bar{\Omega} \cap B_\lambda(\mathbf{x})$ contains a spherical cone $C^\theta_\lambda(\mathbf{x}, \mathbf{e_x})$ with its apex at $\mathbf{x}$, radius λ, aperture angle 2θ and around an axis in the direction of a unit vector $\mathbf{e_x}$.

3 Variational Principle

To simplify the discussion, unless otherwise noted, we refer to the linearized state-based PD model simply as the PD model in the sequel. The potential energy under the external force density function $\mathbf{b}(\mathbf{x})$ is then given by

$$\begin{aligned} \mathbf{E}(\mathbf{u}) &= \int_\Omega \tilde{W}_\mathbf{u}(\mathbf{x})d\mathbf{x} - \int_\Omega \mathbf{u}\cdot\mathbf{b}\,d\mathbf{x} \\ &= \int_\Omega \frac{\alpha(\mathbf{x})}{2}\int_\Omega \tilde{\rho}(\mathbf{x}'-\mathbf{x})\Big(\text{Tr}\big(\mathcal{D}^*\mathbf{u}\big)\big(\mathbf{x}',\mathbf{x}\big) - \frac{1}{d}\text{Tr}\big(\mathcal{D}^*_\omega\mathbf{u}\big)(\mathbf{x})\Big)^2 d\mathbf{x}'\,d\mathbf{x} \\ &\quad + \int_\Omega \left\{\frac{k(\mathbf{x})}{2}\big(\text{Tr}\big(\mathcal{D}^*_\omega\mathbf{u}\big)(\mathbf{x})\big)^2 - \mathbf{u}\cdot\mathbf{b}\right\}d\mathbf{x}. \end{aligned} \tag{6}$$

The energy space is the set $\mathcal{S}(\Omega)$ of all square integrable vector fields $\mathbf{u}$ defined on Ω such that the total strain energy is finite. It is clearly the natural space to study the optimization problem and one contribution of this paper is the characterization and close study of this space for a wide class of weight functions. Our first result in that direction is given by the following proposition which provides an equivalent, but simpler, characterization of the space.

Proposition 1 *The energy space $\mathcal{S}(\Omega)$ is given by*

$$\mathcal{S}(\Omega)=\left\{\mathbf{u}\in L^2(\Omega;\mathbb{R}^d):\int_\Omega\int_\Omega\tilde{\rho}(\mathbf{x}'-\mathbf{x})\big(\mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x})\big)^2d\mathbf{x}'d\mathbf{x}<\infty\right\}.$$

Proof Let us denote the set of vector fields on the right hand side by $\mathcal{E}(\Omega)$ for the moment. The objective is to show that $\mathcal{S}(\Omega)=\mathcal{E}(\Omega)$. Pick $\mathbf{u}\in\mathcal{E}(\Omega)$. We begin by writing the strain energy density as

$$\begin{aligned}\tilde{W}_{\mathbf{u}}(\mathbf{x})=&\left(\frac{k(\mathbf{x})}{2}-\frac{\alpha(\mathbf{x})m(\mathbf{x})}{2d^2}\right)\big(\mathrm{Tr}(\mathcal{D}_\omega^*\mathbf{u})(\mathbf{x})\big)^2\\&+\frac{\alpha(\mathbf{x})}{2}\int_\Omega\tilde{\rho}(\mathbf{x}'-\mathbf{x})\big(\mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x})\big)^2d\mathbf{x}'.\end{aligned}$$

Then it follows from the above that

$$\tilde{W}_{\mathbf{u}}(\mathbf{x})\le\frac{k(\mathbf{x})}{2}\big(\mathrm{Tr}(\mathcal{D}_\omega^*\mathbf{u})(\mathbf{x})\big)^2+\frac{\alpha(\mathbf{x})}{2}\int_\Omega\tilde{\rho}(\mathbf{x}'-\mathbf{x})\big(\mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x})\big)^2d\mathbf{x}'. \tag{7}$$

Now we note that for all $\mathbf{u}\in\mathcal{E}(\Omega)$ and for each $\mathbf{x}\in\Omega$, applying the Cauchy-Schwarz inequality, we have

$$\mathrm{Tr}(\mathcal{D}_\omega^*\mathbf{u})(\mathbf{x})\le\frac{d}{m(\mathbf{x})^{1/2}}\left(\int_\Omega\tilde{\rho}(\mathbf{x}'-\mathbf{x})\big|\mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x})\big|^2d\mathbf{x}'\right)^{1/2}.$$

Then from the above we obtain that

$$\begin{aligned}\int_\Omega\tilde{W}_{\mathbf{u}}(\mathbf{x})d\mathbf{x}\le&\int_\Omega\frac{k(\mathbf{x})}{2}\big(\mathrm{Tr}(\mathcal{D}_\omega^*\mathbf{u})(\mathbf{x})\big)^2d\mathbf{x}\\&+\int_\Omega\frac{\alpha(\mathbf{x})}{2}\int_\Omega\tilde{\rho}(\mathbf{x}'-\mathbf{x})\big(\mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x})\big)^2d\mathbf{x}'d\mathbf{x}\\\le&\left(\frac{d^2K}{2m_0}+\frac{A}{2}\right)\int_\Omega\int_\Omega\tilde{\rho}(\mathbf{x}'-\mathbf{x})\big(\mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x})\big)^2d\mathbf{x}'d\mathbf{x}<\infty.\end{aligned}$$

This establishes the inclusion $\mathcal{E}(\Omega)\subset\mathcal{S}(\Omega)$. Let us show the inclusion in the other direction. Let $\mathbf{u}\in\mathcal{S}(\Omega)$. Then

$$\int_\Omega\big(\mathrm{Tr}(\mathcal{D}_\omega^*\mathbf{u})(\mathbf{x})\big)^2d\mathbf{x}<\infty,$$

and

$$\int_\Omega\int_\Omega\tilde{\rho}(\mathbf{x}'-\mathbf{x})\left(\mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x})-\frac{\mathrm{Tr}(\mathcal{D}_\omega^*\mathbf{u})(\mathbf{x})}{d}\right)^2d\mathbf{x}'d\mathbf{x}<\infty.$$

It follows that

$$\begin{aligned}&\int_{\Omega}\int_{\Omega}\tilde{\rho}\big(\mathbf{x}'-\mathbf{x}\big)\big(\mathrm{Tr}\big(\mathcal{D}^{*}\mathbf{u}\big)\big(\mathbf{x}',\mathbf{x}\big)\big)^{2}d\mathbf{x}'d\mathbf{x}\\&=\int_{\Omega}\left\{\int_{\Omega}\tilde{\rho}\big(\mathbf{x}'-\mathbf{x}\big)\left(\mathrm{Tr}\big(\mathcal{D}^{*}\mathbf{u}\big)\big(\mathbf{x}',\mathbf{x}\big)-\frac{\mathrm{Tr}(\mathcal{D}_{\omega}^{*}\mathbf{u})(\mathbf{x})}{d}\right)^{2}d\mathbf{x}'+\frac{1}{d^{2}}\big(\mathrm{Tr}\big(\mathcal{D}_{\omega}^{*}\mathbf{u}\big)(\mathbf{x})\big)^{2}\right\}d\mathbf{x},\\&\leq\left(\frac{2}{a_{0}}+\frac{2M}{d^{2}k_{0}}\right)\int_{\Omega}\tilde{W}_{\mathbf{u}}(\mathbf{x})d\mathbf{x}<\infty,\end{aligned}$$

demonstrating that $\mathbf{u}\in\mathcal{E}(\Omega)$. □

Remark 1 From the proof of the above proposition we infer the following inequality: for any $\mathbf{u}\in L^{2}(\Omega;\mathbb{R}^{d})$,

$$\begin{aligned}\int_{\Omega}\int_{\Omega}\tilde{\rho}\big(\mathbf{x}'-\mathbf{x}\big)\big(\mathrm{Tr}\big(\mathcal{D}^{*}\mathbf{u}\big)\big(\mathbf{x}',\mathbf{x}\big)\big)^{2}d\mathbf{x}'d\mathbf{x}&\leq\left(\frac{2}{a_{0}}+\frac{2M}{d^{2}k_{0}}\right)\int_{\Omega}\tilde{W}_{\mathbf{u}}(\mathbf{x})d\mathbf{x}\\&\leq\left(\frac{2}{a_{0}}+\frac{2M}{d^{2}k_{0}}\right)\left(\frac{A}{2}+\frac{d^{2}K}{2m_{0}}\right)\int_{\Omega}\int_{\Omega}\tilde{\rho}\big(\mathbf{x}'-\mathbf{x}\big)\big(\mathrm{Tr}\big(\mathcal{D}^{*}\mathbf{u}\big)\big(\mathbf{x}',\mathbf{x}\big)\big)^{2}d\mathbf{x}'d\mathbf{x}.\end{aligned}$$

For ρ radial, the set $\mathcal{S}(\Omega)$ is shown in [15] to be a Hilbert space with inner product

$$(\mathbf{u},\mathbf{w})_{s}=(\mathbf{u},\mathbf{w})+\big((\mathbf{u},\mathbf{w})\big),\tag{8}$$

where $(\cdot,\cdot)$ is the standard L^{2}-inner product and $((\cdot,\cdot))$ is defined as

$$\big((\mathbf{u},\mathbf{v})\big)=\int_{\Omega}\int_{\Omega}\tilde{\rho}\big(\mathbf{x}'-\mathbf{x}\big)\mathrm{Tr}\big(\mathcal{D}^{*}\mathbf{u}\big)\big(\mathbf{x}',\mathbf{x}\big)\mathrm{Tr}\big(\mathcal{D}^{*}\mathbf{v}\big)\big(\mathbf{x}',\mathbf{x}\big)\,d\mathbf{x}'d\mathbf{x}.\tag{9}$$

A slight modification of the argument used to prove [15, Theorem 2.1] can also be used to prove the completeness of $\mathcal{S}(\Omega)$ for any ρ satisfying the minimal conditions stated in the previous section.

We will use the notation $\|\mathbf{u}\|$ to denote the L^{2} norm of $\mathbf{u}$ and $|\mathbf{u}|_{s}$ to denote the seminorm $\sqrt{((\mathbf{u},\mathbf{u}))}$ of $\mathbf{u}$ in $\mathcal{S}(\Omega)$ and naturally we will use the notation $\|\cdot\|_{s}$ to denote the norm on $\mathcal{S}(\Omega)$:

$$\|\mathbf{u}\|_{s}^{2}=\|\mathbf{u}\|_{L^{2}}^{2}+|\mathbf{u}|_{s}^{2}.$$

Note that for smooth $\mathbf{u}$, say $\mathbf{u}\in C^{2}(\overline{\Omega};\mathbb{R}^{d})$,

$$\mathrm{Tr}\big(\mathcal{D}^{*}\mathbf{u}\big)\big(\mathbf{x}',\mathbf{x}\big)\approx e(\nabla\mathbf{u})\quad\text{for all }\mathbf{x}',\mathbf{x}\in\Omega\text{ with small }\big|\mathbf{x}'-\mathbf{x}\big|,$$

where $e(\nabla\mathbf{u})=(\nabla\mathbf{u}+\nabla\mathbf{u}^{T})/2$ is the symmetrized gradient. Thus one may think of the norm $|\mathbf{u}|_{s}^{2}$ as a nonlocal approximation of the local seminorm $\|e(\nabla\mathbf{u})\|_{L^{2}(\Omega)}^{2}$.

Our goal is to study the solvability of the nonlocal peridynamic Navier equation as a nonlocal constrained value problem with the constraints imposed over a volume having non-zero measure. To that end, we give a class of subspaces of the energy space $\mathcal{S}(\Omega)$ in which we expect the solution to belong.

Given an open subset $\Theta\subset\Omega$ such that $|\Theta|>0$, one may define

$$\mathcal{V}_{\Theta}=\big\{\mathbf{u}\in\mathcal{S}(\Omega):\mathbf{u}=0\text{ on }\Theta\big\}.$$

Then $\mathcal{V}_\Theta$ is a closed subspace of $\mathcal{S}(\Omega)$. Indeed, if $\mathbf{u}_n \in \mathcal{V}_\Theta$ and $\mathbf{u}_n \to \mathbf{u}$ in $\mathcal{S}(\Omega)$, then clearly $\mathbf{u}_n \to \mathbf{u}$ in $L^2(\Omega; \mathbb{R}^d)$, implying that $\mathbf{u} = 0$ on Θ, and thus $\mathbf{u} \in \mathcal{V}_\Theta$. It follows then that $\mathcal{V}_\Theta$ is a Hilbert space with the inner product $(\cdot, \cdot)_s$ given by (8).

A special case is when $\Theta = \Omega \setminus \Omega_\delta$, where $\Omega_\delta = \{\mathbf{x} \in \Omega : \mathrm{dist}(\mathbf{x}, \partial\Omega) > \delta\}$, the set of points in Ω that are a δ-distance away from the boundary, where δ is the horizon. This enables us to impose constraints ("nonlocal boundary conditions") on a boundary layer of thickness δ, mimicking the local Dirichlet boundary value problems. We denote this energy space by $V_0^\delta = \mathcal{V}_{\Omega\setminus\Omega_\delta}$, the set of elements of $\mathcal{S}(\Omega)$ that vanish on a boundary layer of thickness δ. For further discussions on nonlocal constrained value problems over a non-measure zero volume, we refer to [7].

3.1 Nonlocal Poincaré-Type Inequality

Another important ingredient for our analysis is a Poincaré-type inequality.

We begin with a characterization of vector fields $\mathbf{u}$ such that $\mathbf{u}$ is nonzero and yet $|\mathbf{u}|_s = 0$. For this, let Π denote the set of infinitesimally rigid displacements given by

$$\Pi = \left\{\mathbf{u} : \mathbf{u} = \mathbb{Q}\mathbf{x} + \mathbf{b}, \mathbb{Q} \in \mathbb{R}^{d\times d}, \mathbb{Q}^T = -\mathbb{Q}, \mathbf{b} \in \mathbb{R}^d\right\}.$$

Next, we provide a nonlocal means of identifying elements of Π, as done in [5, 13] for various conditions on ρ. Replicating the argument used in [5, Lemma 2], one can prove the following more general characterization as well.

Lemma 1 *Suppose that* ρ *satisfies* (H) *and* $\tilde{\rho}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2\rho(\boldsymbol{\xi})$. *Then*

$$\mathbf{u} \in \Pi \quad \Longleftrightarrow \quad \int_\Omega\int_\Omega \rho\big(\mathbf{x}' - \mathbf{x}\big)\big|\big(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})\big) \cdot \big(\mathbf{x}' - \mathbf{x}\big)\big|^2 d\mathbf{x}' d\mathbf{x} = 0.$$

The above lemma together with the equivalence of the total strain energy with the semi-norm yields the following corollary.

Corollary 1 *Suppose that* ρ *satisfies* (H). *Then for all* $\mathbf{u} \in \mathcal{S}(\Omega)$,

$$\int_\Omega \tilde{W}_{\mathbf{u}}(\mathbf{x}) d\mathbf{x} \geq 0, \quad \textit{with equality holding if and only if } \mathbf{u} \in \Pi.$$

We also need the following lemma in the proof of the next result.

Lemma 2 *If* ρ *satisfies assumption* (*H*) *and* Ω *is sufficiently smooth, then there exists a constant* $\gamma > 0$ *such that*

$$\mathbb{A}(\mathbf{x}) := \int_\Omega \rho(\mathbf{y} - \mathbf{x})(\mathbf{y} - \mathbf{x}) \otimes (\mathbf{y} - \mathbf{x}) d\mathbf{y} \geq \gamma\mathbb{I}$$

in the sense of quadratic forms, uniformly in $\mathbf{x} \in \Omega$.

Proof Denoting the unit sphere in $\mathbb{R}^d$ by $\mathbb{S}^{d-1}$ and the characteristic function on Ω by $\chi_\Omega = \chi_\Omega(\mathbf{x})$, we define a function $\psi : \overline{\Omega} \times \mathbb{S}^{d-1}$ by

$$\psi(\mathbf{x}, \mathbf{v}) = \int_{B(\mathbf{0},\lambda)} \chi_\Omega(\boldsymbol{\xi} + \mathbf{x})\rho(\boldsymbol{\xi})|\boldsymbol{\xi} \cdot \mathbf{v}|^2 d\boldsymbol{\xi}.$$

Then for any $(\mathbf{x}, \mathbf{v}) \in \overline{\Omega} \times \mathbb{S}^{d-1}$, since the domain Ω satisfies the interior cone condition, we have at most a zero measure subset in $C^{\theta}_{\lambda}(\mathbf{x}, \mathbf{e_x})$ which is perpendicular to $\mathbf{v}$, thus,

$$\mathbf{v}^T \mathbb{A}(\mathbf{x})\mathbf{v} \geq \psi(\mathbf{x}, \mathbf{v}) > 0.$$

The proof of the lemma will be complete if we show that $\psi(\mathbf{x}, \mathbf{v})$ is a continuous function on $\overline{\Omega} \times \mathbb{S}^{d-1}$. Then, we may simply take

$$\gamma = \min_{(\mathbf{x},\mathbf{v})\in\overline{\Omega}\times\mathbb{S}^{d-1}} \psi(\mathbf{x}, \mathbf{v}) > 0.$$

To that end, let us pick $(\mathbf{x}, \mathbf{v})$ and $(\mathbf{y}, \mathbf{w})$ both in $\overline{\Omega} \times \mathbb{S}^{d-1}$. Then we have

$$\begin{aligned}\left|\psi(\mathbf{x}, \mathbf{v}) - \psi(\mathbf{y}, \mathbf{w})\right| \leq &\int_{B(\mathbf{0},\delta)} \chi_{\Omega}(\boldsymbol{\xi} + \mathbf{x})\rho(\boldsymbol{\xi})\left||\boldsymbol{\xi}\cdot\mathbf{v}|^2 - |\boldsymbol{\xi}\cdot\mathbf{w}|^2\right|d\boldsymbol{\xi} \\ &+ \int_{B(\mathbf{0},\delta)} \left|\chi_{\Omega}(\boldsymbol{\xi} + \mathbf{x}) - \chi_{\Omega}(\boldsymbol{\xi} + \mathbf{y})\right|\rho(\boldsymbol{\xi})|\boldsymbol{\xi}\cdot\mathbf{w}|^2 d\boldsymbol{\xi} = I_1 + I_2.\end{aligned}$$

For the first term I_1, it is estimated as

$$I_1 = \int_{B(\mathbf{0},\delta)} \chi_{\Omega}(\boldsymbol{\xi} + \mathbf{x})\rho(\boldsymbol{\xi})\left||\boldsymbol{\xi}\cdot\mathbf{v}|^2 - |\boldsymbol{\xi}\cdot\mathbf{w}|^2\right|d\boldsymbol{\xi} \leq 2|\mathbf{v} - \mathbf{w}| \int_{B(\mathbf{0},\delta)} \rho(\boldsymbol{\xi})|\boldsymbol{\xi}|^2 d\boldsymbol{\xi}.$$

Define the set $\Omega(\mathbf{x}, \boldsymbol{y}) = B(\mathbf{0}, \delta) \cap (\overline{\Omega} - \{\mathbf{x}\} \setminus \overline{\Omega} - \{\mathbf{y}\}) \cup (\overline{\Omega} - \{\mathbf{y}\} \setminus \overline{\Omega} - \{\mathbf{x}\})$. We note that $|\chi_{\Omega}(\boldsymbol{\xi} + \mathbf{x}) - \chi_{\Omega}(\boldsymbol{\xi} + \mathbf{y})| = 1$ if $\boldsymbol{\xi} \in \Omega(\mathbf{x}, \boldsymbol{y})$ and 0 otherwise. The second term I_2 can be estimated as

$$I_2 = \int_{B(\mathbf{0},\delta)} \left|\chi_{\Omega}(\boldsymbol{\xi} + \mathbf{x}) - \chi_{\Omega}(\boldsymbol{\xi} + \mathbf{y})\right|\rho(\boldsymbol{\xi})|\boldsymbol{\xi}\cdot\mathbf{w}|^2 d\boldsymbol{\xi} \leq \int_{\Omega(\mathbf{x},\mathbf{y})} \rho(\boldsymbol{\xi})|\boldsymbol{\xi}|^2 d\boldsymbol{\xi}.$$

Given the smoothness of the domain, we obtain that, as $\mathbf{x} - \mathbf{y} \to 0$, the measures of the set $\overline{\Omega} - \{\mathbf{x}\} \setminus \overline{\Omega} - \{\mathbf{y}\}$ and $\overline{\Omega} - \{\mathbf{y}\} \setminus \overline{\Omega} - \{\mathbf{x}\}$ goes to 0, and therefore $|\Omega(\mathbf{x}, \mathbf{y})| \to 0$. Now using the absolute continuity of the integral, if $|\mathbf{x} - \mathbf{y}| \to 0$, we have

$$\int_{\Omega(\mathbf{x},\mathbf{y})} \tilde{\rho}(\boldsymbol{\xi})d\boldsymbol{\xi} = \int_{\Omega(\mathbf{x},\mathbf{y})} \rho(\boldsymbol{\xi})|\boldsymbol{\xi}|^2 d\boldsymbol{\xi} \to 0.$$

Thus, when $(\mathbf{x}, \mathbf{v}) \to (\mathbf{y}, \mathbf{w})$ in $\overline{\Omega} \times \mathbb{S}^{d-1}$, $|\psi(\mathbf{x}, \mathbf{v}) - \psi(\mathbf{y}, \mathbf{w})| \to 0$ as well. This establishes the continuity of $\psi(\mathbf{x}, \mathbf{v})$ on $\overline{\Omega} \times \mathbb{S}^{d-1}$. □

Finally, we state and prove the nonlocal Poincaré inequality that will be used to demonstrate the coercivity of the energy. We note that it is established under more general assumptions on spaces and nonlocal kernels than the ones presented before, say, in [15].

Proposition 2 *Suppose that* ρ *satisfies* (H) *and* V *is a closed subspace of* $L^2(\Omega; \mathbb{R}^d)$ *such that* $V \cap \Pi = \{0\}$. *Then, there exists* $C = C(\rho, V, \Omega)$ *such that*

$$\|\mathbf{u}\|^2_{L^2} \leq C \int_{\Omega}\int_{\Omega} \tilde{\rho}\big(\mathbf{x}' - \mathbf{x}\big)\left|\big(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})\big)\cdot\big(\mathbf{x}' - \mathbf{x}\big)\right|^2 d\mathbf{x}'d\mathbf{x}, \quad \forall \mathbf{u} \in V.$$

As a consequence, by taking $\kappa = \sqrt{C}|\mathrm{diam}(\Omega)|$,

$$\|\mathbf{u}\|_s \leq \kappa|\mathbf{u}|_s, \quad \textit{for all } \mathbf{u} \in V \cap \mathcal{S}(\Omega).$$

Proof The proof adapts the proof of the standard Poincaré inequality. Suppose the conclusion of the proposition is false. Then, there exist $\mathbf{u}_n \in L^2$ such that, for all n, $\|\mathbf{u}_n\| = 1$ and

$$\int_\Omega \int_\Omega \tilde{\rho}(\mathbf{x}' - \mathbf{x}) \left| (\mathbf{u}_n(\mathbf{x}') - \mathbf{u}_n(\mathbf{x})) \cdot (\mathbf{x}' - \mathbf{x}) \right|^2 d\mathbf{x}' d\mathbf{x} \to 0 \quad \text{as } n \to \infty.$$

We claim that

$$\|\mathbf{u}_n\|_{L^2} \to 0 \quad \text{as } n \to \infty,$$

thus resulting in a desired contradiction. We now prove the claim. Let $\mathbf{u}$ be the weak limit (up to a subsequence) of the sequence $\mathbf{u}_n$. Then, $\mathbf{u} \in V$, since V is a closed subspace of $L^2(\Omega; \mathbb{R}^d)$. The aim is to show that $\mathbf{u} \in V$ is an infinitesimal rigid displacement, and therefore $\mathbf{u} = \mathbf{0}$. To that end, let us define the bounded linear map L on $L^2(\Omega; \mathbb{R}^d)$ by

$$L\mathbf{w}(\mathbf{x}) = -2 \int_\Omega \rho(\mathbf{x}' - \mathbf{x})(\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x})(\mathbf{w}(\mathbf{x}') - \mathbf{w}(\mathbf{x})) d\mathbf{x}'. \tag{10}$$

For $\boldsymbol{\phi} \in C_c^\infty(\Omega; \mathbb{R}^d)$, define the sequence

$$J_n = \int_\Omega L\boldsymbol{\phi}(\mathbf{x}) \cdot \mathbf{u}_n(\mathbf{x})\, d\mathbf{x}. \tag{11}$$

Then it follows from (11) together with (10) by interchanging $\mathbf{x}' \to \mathbf{x}$ that

$$\begin{aligned} J_n &= \int_\Omega L\mathbf{u}_n(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x})\, d\mathbf{x} \\ &= -2 \int_\Omega \int_\Omega \tilde{\rho}(\mathbf{x}' - \mathbf{x}) \left((\mathbf{u}_n(\mathbf{x}') - \mathbf{u}_n(\mathbf{x})) \cdot (\mathbf{x}' - \mathbf{x}) \right) \left((\mathbf{x}' - \mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x}) \right) d\mathbf{x}' d\mathbf{x}. \end{aligned}$$

Now, applying the Cauchy-Schwartz inequality, on the one hand, we get

$$\begin{aligned} J_n &\le 2 \left[\int_\Omega \int_\Omega \tilde{\rho}(\mathbf{x}' - \mathbf{x}) \left| (\mathbf{u}_n(\mathbf{x}') - \mathbf{u}_n(\mathbf{x})) \cdot (\mathbf{x}' - \mathbf{x}) \right|^2 d\mathbf{x}' d\mathbf{x} \right]^{\frac{1}{2}} \\ &\quad \times \left[\int_\Omega \int_\Omega |\boldsymbol{\phi}(\mathbf{x})|^2 \tilde{\rho}(\mathbf{x}' - \mathbf{x}) d\mathbf{x}' d\mathbf{x} \right]^{\frac{1}{2}} \\ &\le C \left[\int_\Omega \int_\Omega \tilde{\rho}(\mathbf{x}' - \mathbf{x}) \left| (\mathbf{u}_n(\mathbf{x}') - \mathbf{u}_n(\mathbf{x})) \cdot (\mathbf{x}' - \mathbf{x}) \right|^2 d\mathbf{x}' d\mathbf{x} \right]^{\frac{1}{2}} \\ &\quad \times \left[\int_\Omega |\boldsymbol{\phi}(\mathbf{x})|^2 d\mathbf{x} \right]^{1/2} \big(\text{by condition } (H)\big) \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned}$$

On the other hand because $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in $L^2(\Omega; \mathbb{R}^d)$, as $n \to \infty$,

$$J_n \to \int_\Omega L\boldsymbol{\phi}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\, d\mathbf{x}.$$

We conclude that for all $\boldsymbol{\phi} \in C_c^\infty(\Omega;\mathbb{R}^d)$,

$$\int_\Omega L\mathbf{u}(\mathbf{x})\cdot\boldsymbol{\phi}d\mathbf{x} = \int_\Omega L\boldsymbol{\phi}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\,d\mathbf{x} = 0.$$

This implies that, for almost all $\mathbf{x}\in\Omega$, $L\mathbf{u}(\mathbf{x}) = 0$. After multiplying this last equation by $\mathbf{u}$ and integrating it, we obtain that

$$\int_\Omega\int_\Omega \tilde{\rho}\big(\mathbf{x}'-\mathbf{x}\big)\big|\big(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x})\big)\cdot\big(\mathbf{x}'-\mathbf{x}\big)\big|^2 d\mathbf{x}'d\mathbf{x} = 0.$$

But by Lemma 1, this happens only when $\mathbf{u}$ is an infinitesimal rigid displacement.

We next show that the weak limit is actually a strong limit. We do this by showing that $\|\mathbf{u}_n\|\to 0$ strongly in $L^2(\Omega;\mathbb{R}^d)$. We begin by observing that the operator L defined (10) can be written as $L\mathbf{u}(\mathbf{x}) = \mathbb{K}*\overline{\mathbf{u}}(\mathbf{x}) + \mathbb{A}(x)\mathbf{u}(\mathbf{x})$, where $\overline{\mathbf{u}}$ is the zero extension of $\mathbf{u}$, $\mathbb{K}(\boldsymbol{\xi}) = -2\rho(\boldsymbol{\xi})\boldsymbol{\xi}\otimes\boldsymbol{\xi}$,

$$\mathbb{A}(\mathbf{x}) = -\int_\Omega \mathbb{K}\big(\mathbf{x}'-\mathbf{x}\big)d\mathbf{x}',\quad \text{and}\quad (\mathbb{K}*\overline{\mathbf{u}})_j = \sum_{i=1}^d\int_\Omega K_{ij}\big(\mathbf{x}'-\mathbf{x}\big)u_i\big(\mathbf{x}'\big)d\mathbf{x}'.$$

For any $\mathbf{x}\in\Omega$, by Lemma 2, we have $\mathbb{A}(\mathbf{x})\ge\gamma\mathbb{I}$ for some $\gamma>0$ in the sense of quadratic forms. Now proceeding to prove the lemma, we have

$$(L\mathbf{u}_n,\mathbf{u}_n) = \int_\Omega\int_\Omega \tilde{\rho}\big(\mathbf{x}'-\mathbf{x}\big)\big|\big(\mathbf{u}_n\big(\mathbf{x}'\big)-\mathbf{u}_n(\mathbf{x})\big)\cdot\big(\mathbf{x}'-\mathbf{x}\big)\big|^2 d\mathbf{x}'d\mathbf{x}\to 0 \quad \text{as } n\to\infty.$$

Since $\mathbf{u}_n \rightharpoonup 0$ weakly in $L^2(\Omega;\mathbb{R}^d)$ we use compactness of the convolution operator to obtain

$$\mathbb{K}*\overline{\mathbf{u}}_n \to 0 \quad \text{strongly in } L^2\big(\Omega;\mathbb{R}^d\big).$$

Therefore,

$$\gamma\lim_{n\to\infty}\int_\Omega|\mathbf{u}_n|^2d\mathbf{x} \le \lim_{n\to\infty}\big(\hat{\mathbb{A}}(\mathbf{x})\mathbf{u}_n,\mathbf{u}_n\big) + (\mathbb{K}*\overline{\mathbf{u}}_n,\mathbf{u}_n) = \lim_{n\to\infty}(L\mathbf{u}_n,\mathbf{u}_n) = 0,$$

completing the proof of the proposition. □

Remark 2 We remark that the requirement that V is a closed subspace of $L^2(\Omega;\mathbb{R}^d)$ can not be waived for the non-local Poincaré inequality, even if V is a proper subset of $L^2(\Omega;\mathbb{R}^d)$ that does not include nontrivial rigid displacements. Indeed, the inequality, for example, does not hold when $V = \{\mathbf{u}\in H^1(\Omega;\mathbb{R}^d) : \mathbf{u}|_\Gamma = 0, \emptyset\neq\Gamma\subset\partial\Omega\}$. For $d=1$, we present the counterexample given in [9, 12]. Given $0<s<1/2$, there are functions $f_n\in C^\infty([0,1])$ such that $f_n = 0$ on $[0,\frac{1}{2^n}]$, $\|f_n\|_{L^2[0,1]}\to 1$, as $n\to\infty$, while

$$\int_0^1\int_0^1\frac{|f_n(x')-f_n(x)|^2}{|x'-x|^{1+2s}}dx'dx \longrightarrow 0, \quad \text{as } n\to\infty.$$

Note that in this case $\rho(\xi) = |\xi|^{-1-2s}$, and satisfies condition (H).

From the nonlocal Poincaré inequality, if V is a closed subspace of $L^2(\Omega;\mathbb{R}^d)$ such that $V\cap\Pi = \{0\}$, then the inner product space $(V\cap\mathcal{S}(\Omega), ((\cdot,\cdot)))$ is a Hilbert space and that the seminorm $|\cdot|_s$ defines an equivalent norm in the subspace. Hereafter $\mathcal{V}_s$ denotes the generic subspace $V \cap \mathcal{S}(\Omega)$ with its dual denoted by $\mathcal{V}'_s$. We note that a special case is given by $\mathcal{V}_s = \mathcal{V}_\Theta$ for some measure non-zero set $\Theta \subset \Omega$ or, in particular, we may take $\mathcal{V}_\Theta = V_0^\delta$ which, as defined before, contains elements of $\mathcal{S}(\Omega)$ that vanish on a boundary layer of Ω with thickness δ.

3.2 Lax-Milgram

Let us define the bilinear form $B : \mathcal{S}(\Omega) \times \mathcal{S}(\Omega) \to \mathbb{R}$ by

$$B(\mathbf{u},\mathbf{v}) = \int_\Omega \Bigg(\bigg(k(\mathbf{x}) - \frac{\alpha(\mathbf{x})m(\mathbf{x})}{d^2}\bigg)\mathrm{Tr}\big(\mathcal{D}_\omega^*\mathbf{u}\big)(\mathbf{x})\mathrm{Tr}\big(\mathcal{D}_\omega^*\mathbf{v}\big)(\mathbf{x}) + \alpha(\mathbf{x})\int_\Omega \tilde{\rho}\big(\mathbf{x}'-\mathbf{x}\big)\mathrm{Tr}\big(\mathcal{D}^*\mathbf{u}\big)\big(\mathbf{x}',\mathbf{x}\big)\mathrm{Tr}\big(\mathcal{D}^*\mathbf{v}\big)\big(\mathbf{x}',\mathbf{x}\big)d\mathbf{x}'\Bigg)d\mathbf{x}. \tag{12}$$

It is not difficult to see, from (6) and (12), that for any $\mathbf{u} \in \mathcal{S}(\Omega)$,

$$B(\mathbf{u},\mathbf{u}) = 2\int_\Omega \tilde{W}_{\mathbf{u}}(\mathbf{x})d\mathbf{x}.$$

Lemma 3 (Energy estimates) *There exist positive constants c and C such that, for any $\mathbf{u},\mathbf{v} \in \mathcal{V}_s$,*

$$B(\mathbf{u},\mathbf{v}) \le C|\mathbf{u}|_s|\mathbf{v}|_s, \qquad B(\mathbf{u},\mathbf{u}) \ge c|\mathbf{u}|_s^2.$$

Proof The lemma follows from Proposition 2 and the estimate (7) which holds for all $\mathbf{u} \in \mathcal{S}(\Omega)$ and all $\mathbf{x} \in \Omega$. □

The above energy estimates say that $B = B(\mathbf{u},\mathbf{v})$ is a continuous and coercive bilinear form on $\mathcal{V}_s$. Thus it induces the nonlocal peridynamic Navier operator $\mathcal{L} : \mathcal{S}(\Omega) \to \mathcal{S}'(\Omega)$ which is a bounded linear operator defined by

$$B(\mathbf{u},\mathbf{v}) = \big\langle -\mathcal{L}(\mathbf{u}),\mathbf{v}\big\rangle \quad \text{for any } \mathbf{u},\mathbf{v} \in \mathcal{V}_s. \tag{13}$$

More discussions on the operator $\mathcal{L} = \mathcal{L}(\mathbf{u})$ will be given in later sections.

As discussed in [6], it is possible to pose variational problems in $\mathcal{V}_s$. A standard application of Lax-Milgram theorem yields the following result.

Theorem 1 *Assume that ρ satisfies (H). For a given $\mathbf{b} \in \mathcal{V}'_s$, there exists a unique $\mathbf{u} \in \mathcal{V}_s$ such that $B(\mathbf{u},\mathbf{v}) = \langle \mathbf{b},\mathbf{v}\rangle$, for all $\mathbf{v} \in \mathcal{V}_s$ and satisfies the apriori estimate,*

$$|\mathbf{u}|_s \le c|\mathbf{b}|_{\mathcal{V}'_s}.$$

Moreover,

$$\mathbf{u} = \arg\min_{\mathbf{v}\in\mathcal{V}_s}\big(\mathbf{E}(\mathbf{v}) - \langle \mathbf{b},\mathbf{v}\rangle\big),$$

where $\mathbf{E}(\mathbf{v})$ is the potential energy defined in (6).

3.3 Existence of Minimizer for Problems with Sign-Changing Kernels

In earlier sections, we have established the theory for nonlocal volume constrained value problems for kernels that are always non-negative. In relation to the study of materials stability within the nonlocal peridynamic theory, necessary conditions for the existence of minimizers of variational problems associated with peridynamic models have been studied when the kernels are allowed to have sign-changes. For the linear state-based PD model with an integrable kernel $\rho = \rho(\boldsymbol{\xi})$, since the potential energy is quadratic, the second variation of the energy functional leads to a quadratic form

$$\int_{\Omega} \tilde{W}_{\mathbf{v}}(\mathbf{x}) d\mathbf{x} = (-\mathcal{L}\mathbf{v}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{V}_{\Theta},$$

where the operator $\mathcal{L}$ is defined in (13) and has the form, according to [17],

$$\begin{aligned} \mathcal{L}(\mathbf{u})(\mathbf{x}) = & \int_{\Omega} \big(\alpha(\mathbf{x}) + \alpha(\mathbf{x}')\big) \frac{\rho(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2} (\mathbf{x}'-\mathbf{x})(\mathbf{x}'-\mathbf{x}) \cdot \big(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})\big) d\mathbf{x}' \\ & + \int_{\Omega} \tau(\mathbf{x}') \rho(\mathbf{x}-\mathbf{x}')(\mathbf{x}'-\mathbf{x}) \left(\int_{\Omega} \rho(\mathbf{x}'-\mathbf{z})(\mathbf{z}-\mathbf{x}') \cdot \big(\mathbf{u}(\mathbf{z}) - \mathbf{u}(\mathbf{x}')\big) d\mathbf{z} \right) d\mathbf{x}' \\ & + \int_{\Omega} \tau(\mathbf{x}) \rho(\mathbf{x}-\mathbf{x}')(\mathbf{x}'-\mathbf{x}) \left(\int_{\Omega} \rho(\mathbf{x}-\mathbf{z})(\mathbf{z}-\mathbf{x}) \cdot \big(\mathbf{u}(\mathbf{z}) - \mathbf{u}(\mathbf{x})\big) d\mathbf{z} \right) d\mathbf{x}', \end{aligned} \quad (14)$$

where $\tau(\mathbf{x}) = \frac{d^2 k(\mathbf{x})}{m(\mathbf{x})^2} - \frac{\alpha(\mathbf{x})}{m(\mathbf{x})}$. Then we have the following lemma.

Lemma 4 *If $\rho = \rho(\boldsymbol{\xi})$ is integrable, a necessary condition for the existence of a minimizer for the potential energy is that the matrix function*

$$\begin{aligned} \mathbb{P}(\mathbf{x}) := & \int_{\Omega} \big(\alpha(\mathbf{x}) + \alpha(\mathbf{x}')\big) \frac{\rho(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2} (\mathbf{x}'-\mathbf{x}) \otimes (\mathbf{x}'-\mathbf{x}) d\mathbf{x}' \\ & + \tau(\mathbf{x}) \int_{\Omega} \rho(\mathbf{x}'-\mathbf{x})(\mathbf{x}'-\mathbf{x}) d\mathbf{x}' \otimes \int_{\Omega} \rho(\mathbf{x}'-\mathbf{x})(\mathbf{x}'-\mathbf{x}) d\mathbf{x}' \end{aligned}$$

is positive definite for all $\mathbf{x} \in \Omega \setminus \overline{\Theta}$.

Proof To prove the lemma, we pick $\mathbf{x}_0 \in \Omega \setminus \overline{\Theta}$. Choose $\epsilon > 0$ small so that $B(\mathbf{x}_0, \epsilon) \subset \Omega$. Associated to $\mathbf{x}_0$, we define the sequence of functions

$$\mathbf{v}_{\epsilon}(\mathbf{x}) = \frac{\mathbf{a}}{\sqrt{|B(\mathbf{x}_0, \epsilon)|}} \chi_{B(\mathbf{x}_0,\epsilon)}(\mathbf{x}).$$

Then $\mathbf{v}_{\epsilon} \in V_{\Theta}$. Moreover $\mathbf{v}_{\epsilon}$ weakly converges to 0 in $L^2(\Omega; \mathbb{R}^d)$. Writing $\mathcal{L}(\mathbf{v}) = L_1(\mathbf{v}) + L_2(\mathbf{v}) + L_3(\mathbf{v})$, where

$$L_1(\mathbf{v}) = \int_{\Omega} \big(\alpha(\mathbf{x}) + \alpha(\mathbf{x}')\big) \frac{\rho(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2} (\mathbf{x}'-\mathbf{x})(\mathbf{x}'-\mathbf{x}) \cdot \big(\mathbf{v}(\mathbf{x}') - \mathbf{v}(\mathbf{x})\big) d\mathbf{x}',$$

$$L_2(\mathbf{v}) = \int_{\Omega} \tau(\mathbf{x}') \rho(\mathbf{x}-\mathbf{x}')(\mathbf{x}'-\mathbf{x}) \left(\int_{\Omega} \rho(\mathbf{x}'-\mathbf{z})(\mathbf{z}-\mathbf{x}') \cdot \big(\mathbf{v}(\mathbf{z}) - \mathbf{v}(\mathbf{x}')\big) d\mathbf{z} \right) d\mathbf{x}'$$

and

$$L_3(\mathbf{v}) = \int_\Omega \tau(\mathbf{x})\rho\big(\mathbf{x}-\mathbf{x}'\big)\big(\mathbf{x}'-\mathbf{x}\big)\bigg(\int_\Omega \rho(\mathbf{x}-\mathbf{z})(\mathbf{z}-\mathbf{x})\cdot\big(\mathbf{v}(\mathbf{z})-\mathbf{v}(\mathbf{x})\big)d\mathbf{z}\bigg)d\mathbf{x}',$$

we proceed to compute the limits of $L_i(\mathbf{v}_\epsilon)$, for $i = 1, 2, 3$. The limit of the part that involves L_1 can be computed as

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_\Omega -L_1(\mathbf{v}_\epsilon)\cdot\mathbf{v}_\epsilon(\mathbf{x})d\mathbf{x}\\
&\quad= \lim_{\epsilon\to 0}\int_\Omega\int_\Omega \big(\alpha(\mathbf{x})+\alpha\big(\mathbf{x}'\big)\big)\frac{\rho(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\big(\mathbf{x}'-\mathbf{x}\big)\big(\mathbf{x}'-\mathbf{x}\big)\cdot\mathbf{v}_\epsilon(\mathbf{x}), \mathbf{v}_\epsilon(\mathbf{x}))d\mathbf{x}'d\mathbf{x}\\
&\quad= \lim_{\epsilon\to 0}\frac{1}{|B(\mathbf{x}_0,\epsilon)|}\int_{B(\mathbf{x}_0,\epsilon)}\int_\Omega \big(\alpha(\mathbf{x})+\alpha\big(\mathbf{x}'\big)\big)\frac{\rho(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\big|\big(\mathbf{x}'-\mathbf{x}\big)\cdot\mathbf{a}\big|^2 d\mathbf{x}'d\mathbf{x}\\
&\quad= \int_\Omega \big(\alpha(\mathbf{x}_0)+\alpha\big(\mathbf{x}'\big)\big)\frac{\rho(\mathbf{x}'-\mathbf{x}_0)}{|\mathbf{x}'-\mathbf{x}_0|^2}\big|\big(\mathbf{x}'-\mathbf{x}_0\big)\cdot\mathbf{a}\big|^2 d\mathbf{x}'.
\end{aligned}$$

The first equality follows from the fact that $\mathbf{v}_\epsilon$ weakly converges to 0, and as a consequence where we have used

$$\int_\Omega \big(\alpha(\mathbf{x})+\alpha\big(\mathbf{x}'\big)\big)\frac{\rho(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\big(\mathbf{x}'-\mathbf{x}\big)\big(\mathbf{x}'-\mathbf{x}\big)\cdot(\mathbf{v}_\epsilon\big(\mathbf{x}'\big)d\mathbf{x}' \to 0 \quad \text{strongly in } L^2.$$

We now work on the other terms. Let us introduce the functions

$$\Lambda(\mathbf{x}) = \int_\Omega \rho(\mathbf{x}-\mathbf{z})(\mathbf{z}-\mathbf{x})d\mathbf{z} \quad \text{and} \quad \psi_\epsilon(\mathbf{x}) = \int_\Omega \rho(\mathbf{x}-\mathbf{z})(\mathbf{z}-\mathbf{x})\cdot\mathbf{v}_\epsilon(\mathbf{z})d\mathbf{z}.$$

Observe that Λ is a continuous vector function and that $\psi_\epsilon(\mathbf{x}) \to 0$ strongly in $L^2(\Omega)$ as $\epsilon \to 0$. On one hand, it follows that

$$\int_\Omega -L_2(\mathbf{v}_\epsilon)(\mathbf{x})\cdot\mathbf{v}_\epsilon(\mathbf{x})d\mathbf{x} = \int_\Omega \tau\big(\mathbf{x}'\big)\psi_\epsilon\big(\mathbf{x}'\big)d\mathbf{x}' - \int_\Omega \tau\big(\mathbf{x}'\big)\psi_\epsilon\big(\mathbf{x}'\big)\Lambda\big(\mathbf{x}'\big)\cdot\mathbf{v}_\epsilon\big(\mathbf{x}'\big)d\mathbf{x}' \longrightarrow 0,$$

as $\epsilon \to 0$. On the other hand, again writing

$$\int_\Omega -L_3(\mathbf{v}_\epsilon)(\mathbf{x})\cdot\mathbf{v}_\epsilon(\mathbf{x})d\mathbf{x} = -\int_\Omega \tau(\mathbf{x})\psi_\epsilon(\mathbf{x})\Lambda(\mathbf{x})\cdot\mathbf{v}_\epsilon(\mathbf{x})d\mathbf{x} + \int_\Omega \tau(\mathbf{x})\big|\Lambda(\mathbf{x})\cdot\mathbf{v}_\epsilon(\mathbf{x})\big|^2 d\mathbf{x},$$

we see that the first term goes to 0 as $\epsilon \to 0$, while the second term

$$\int_\Omega \tau(\mathbf{x})\big|\Lambda(\mathbf{x})\cdot\mathbf{v}_\epsilon(\mathbf{x})\big|^2 d\mathbf{x} \to \tau(\mathbf{x}_0)\big|\Lambda(\mathbf{x}_0)\cdot\mathbf{a}\big|^2$$

as $\epsilon \to 0$, thus proving the lemma. □

Remark 3 For homogeneous and isotropic materials, the necessity of the positive definiteness of $\mathbb{P}(\mathbf{x})$ for the existence of minimizers of bond-based models has been formally recognized in Silling's original work [16]. The same condition also appears in [18] and [21] without accounting for the complication due to boundary. In [16, 18, 21], the matrix $\mathbb{P}(x)$ is actually a constant matrix and it has the same form for both the bond-based and state based models. Our analysis, on the other hand, is valid for more general cases that involve inhomogeneities and anisotropies. We should mention that for scalar nonlocal models with sign changing kernel similar necessary conditions are established in [14].

3.4 The Nonlocal Operator

With the introduction of the nonlocal Navier operator $\mathcal{L}$ via the bilinear form, Theorem 1 can be interpreted as saying that for any $\mathbf{b} \in \mathcal{V}_s'$, the equation $-\mathcal{L}(\mathbf{u}) = \mathbf{b}$ is uniquely solvable in $\mathcal{V}_s$ and, as a consequence, the operator $\mathcal{L}$ restricted on $\mathcal{V}_s$

$$-\mathcal{L} : \mathcal{V}_s \to \mathcal{V}_s'$$

is an isomorphism. In addition, using the continuous embedding of $L^2(\Omega; \mathbb{R}^d)$ into $\mathcal{V}_s'$, we see that the restriction of inverse operator $\mathcal{L}^{-1}$ to $L^2(\Omega; \mathbb{R}^d)$ is linear and bounded:

$$(-\mathcal{L})^{-1} : L^2\big(\Omega; \mathbb{R}^d\big) \to \mathcal{V}_s$$

and satisfies the inequality $|\mathcal{L}^{-1}(\mathbf{b})|_s \leq C\|\mathbf{b}\|_{L^2(\Omega;\mathbb{R}^d)}$.

In the remaining part of this section, we will give a closed explicit form of the operator $\mathcal{L}$. We first consider an important special case.

Lemma 5 *If $\rho = \rho(\boldsymbol{\xi}) \in L^1_{1oc}(\mathbb{R}^d)$, then the operator $\mathcal{L}$ is a bounded operator on $L^2(\Omega; \mathbb{R}^d)$ and is given by, for any $\mathbf{u} \in L^2(\Omega; \mathbb{R}^d)$,*

$$\mathcal{L}(\mathbf{u}) = \int_\Omega \mathbb{C}\big(\mathbf{x}', \mathbf{x}\big)\big(\mathbf{u}\big(\mathbf{x}'\big) - \mathbf{u}(\mathbf{x})\big)d\mathbf{x}',$$

where the matrix function $\mathbb{C}(\mathbf{x}', \mathbf{x}) = \boldsymbol{\Phi}_1(\mathbf{x}', \mathbf{x}) + \boldsymbol{\Phi}_2(\mathbf{x}', \mathbf{x})$, with

$$\boldsymbol{\Phi}_1\big(\mathbf{x}', \mathbf{x}\big) = \big(\alpha(\mathbf{x}) + \alpha\big(\mathbf{x}'\big)\big)\frac{\rho(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2}\big(\mathbf{x}' - \mathbf{x}\big) \otimes \big(\mathbf{x}' - \mathbf{x}\big),$$

$$\tau(\mathbf{x}) = \frac{d^2 k(\mathbf{x})}{m(\mathbf{x})^2} - \frac{\alpha(\mathbf{x})}{m(\mathbf{x})},$$

and

$$\begin{aligned}\boldsymbol{\Phi}_2\big(\mathbf{x}', \mathbf{x}\big) = \int_\Omega \Big(&\tau(\mathbf{z})\rho(\mathbf{z} - \mathbf{x})\rho\big(\mathbf{x}' - \mathbf{z}\big)(\mathbf{x} - \mathbf{z}) \otimes \big(\mathbf{z} - \mathbf{x}'\big) \\ &- \tau\big(\mathbf{x}'\big)\rho\big(\mathbf{x}' - \mathbf{x}\big)\rho\big(\mathbf{z} - \mathbf{x}'\big)\big(\mathbf{x} - \mathbf{x}'\big) \otimes \big(\mathbf{x}' - \mathbf{z}\big) \\ &+ \tau(\mathbf{x})\rho(\mathbf{z} - \mathbf{x})\rho\big(\mathbf{x}' - \mathbf{x}\big)(\mathbf{x} - \mathbf{z}) \otimes \big(\mathbf{x} - \mathbf{x}'\big)\Big)d\mathbf{z}.\end{aligned}$$

Proof The results follows from Schur's test, that is, a special form of Young's inequality, by first verifying that $\mathbb{C} \in L^\infty(\Omega, L^1(\Omega)) \cap L^1(\Omega, L^\infty(\Omega))$ under the condition $\rho = \rho(\boldsymbol{\xi}) \in L^1_{1oc}(\mathbb{R}^d)$. □

For any ρ satisfying (H), because of the potential singularity that may appear, we should understand the formula for $\mathcal{L}$ in Lemma 5 in a proper way. To that end, let us introduce the notation $\rho_\epsilon(\boldsymbol{\xi}) = \rho(\boldsymbol{\xi})\chi_{\{\boldsymbol{\xi}:\rho(\boldsymbol{\xi})\leq 1/\epsilon\}}(\boldsymbol{\xi})$, and define $\mathrm{Tr}(\mathcal{D}^{\epsilon,*}\mathbf{u})$ and $\mathrm{Tr}(\mathcal{D}^{\epsilon,*}_\omega\mathbf{u})$ the same way as $\mathrm{Tr}(\mathcal{D}^*\mathbf{u})$ and $\mathrm{Tr}(\mathcal{D}^*_\omega\mathbf{u})$ in (3) and (4) but with ρ replaced by ρ_ϵ and $m(\mathbf{x})$, $\omega(\mathbf{x}', \mathbf{x})$, $\tau(\mathbf{x})$ modified accordingly to $m_\epsilon(\mathbf{x})$, $\omega_\epsilon(\mathbf{x}', \mathbf{x})$, $\tau_\epsilon(\mathbf{x})$. Then, we have the operator

$$\mathcal{L}_\epsilon(\mathbf{u})(\mathbf{x}) = \int_\Omega \big(\alpha(\mathbf{x}) + \alpha\big(\mathbf{x}'\big)\big)\frac{\rho_\epsilon(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2}\big(\mathbf{x}' - \mathbf{x}\big)\big(\mathbf{x}' - \mathbf{x}\big) \cdot \big(\mathbf{u}\big(\mathbf{x}'\big) - \mathbf{u}(\mathbf{x})\big)d\mathbf{x}'$$

$$+\int_{\Omega}\tau_\epsilon(\mathbf{x}')\rho_\epsilon(\mathbf{x}-\mathbf{x}')(\mathbf{x}'-\mathbf{x})\left(\int_{\Omega}\rho_\epsilon(\mathbf{x}'-\mathbf{z})(\mathbf{z}-\mathbf{x}')\cdot(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}'))d\mathbf{z}\right)d\mathbf{x}'$$

$$+\int_{\Omega}\tau_\epsilon(\mathbf{x})\rho_\epsilon(\mathbf{x}-\mathbf{x}')(\mathbf{x}'-\mathbf{x})\left(\int_{\Omega}\rho_\epsilon(\mathbf{x}-\mathbf{z})(\mathbf{z}-\mathbf{x})\cdot(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}))d\mathbf{z}\right)d\mathbf{x}'.$$

The following lemma summarizes the property of $m(\mathbf{x})$ and the relationship between the functions $m_\epsilon(\mathbf{x})$ and $m(\mathbf{x})$.

Lemma 6 *Suppose* ρ *satisfies* (H). *Then the functions* $m(\mathbf{x})$ *and* $m_\epsilon(\mathbf{x})$ *are continuous function on* $\overline{\Omega}$. *Moreover*, $\min_{\mathbf{x}\in\Omega} m(\mathbf{x}) > 0$,

$$\|m_\epsilon - m\|_{L^\infty(\Omega)} \to 0, \quad \textit{as } \epsilon \to 0,$$

and as a consequence there exists ϵ_0 *small such that*

$$\sup_{0<\epsilon\le\epsilon_0}\min_{\mathbf{x}\in\Omega} m_\epsilon(\mathbf{x}) \ge \frac{1}{2}\min_{\mathbf{x}\in\Omega} m(\mathbf{x}) > 0.$$

Proof The proof of the uniform convergence follows from the estimate

$$\sup_{\mathbf{x}\in\Omega}\left|\int_{\Omega}\left(\rho_\epsilon(\mathbf{x}'-\mathbf{x})-\rho(\mathbf{x}'-\mathbf{x})\right)\left|\mathbf{x}'-\mathbf{x}\right|^2 d\mathbf{x}'\right| \le \int_{\{\boldsymbol{\xi}:\rho(\boldsymbol{\xi})>1/\epsilon\}}\rho(\boldsymbol{\xi})|\boldsymbol{\xi}|^2 d\boldsymbol{\xi} \to 0$$

as $\epsilon \to 0$. The other statements are not difficult to prove. □

Proposition 3 *The operator* $\mathcal{L}$ *is the weak* limit of the sequence of operators* $\mathcal{L}_\epsilon$. *That is, for any* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathcal{S}(\Omega)$,

$$\lim_{\epsilon\to 0}\int_{\Omega}-\mathcal{L}_\epsilon(\mathbf{u})(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})d\mathbf{x} = \langle -\mathcal{L}(\mathbf{u}), \mathbf{v}\rangle.$$

Proof Using the notation specified above,

$$\int_{\Omega}-\mathcal{L}_\epsilon\mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})d\mathbf{x} = \int_{\Omega}\left(\left(k(\mathbf{x})-\frac{\alpha(\mathbf{x})m_\epsilon(\mathbf{x})}{d^2}\right)\mathrm{Tr}\left(\mathcal{D}_\omega^{\epsilon,*}\mathbf{u}\right)(\mathbf{x})\mathrm{Tr}\left(\mathcal{D}_\omega^{\epsilon,*}\mathbf{v}\right)(\mathbf{x})\right.$$

$$\left.+\alpha(\mathbf{x})\int_{\Omega}\tilde{\rho}_\epsilon(\mathbf{x}'-\mathbf{x})\mathrm{Tr}\left(\mathcal{D}^*\mathbf{u}\right)(\mathbf{x}',\mathbf{x})\mathrm{Tr}\left(\mathcal{D}^*\mathbf{v}\right)(\mathbf{x}',\mathbf{x})d\mathbf{x}'\right)d\mathbf{x}.$$

The convergence to the appropriate limit of the second term is easy to show since $\rho_\epsilon(\boldsymbol{\xi}) \to \rho(\boldsymbol{\xi})$ as $\epsilon \to 0$ and that $\rho_\epsilon(\boldsymbol{\xi}) \le \rho(\boldsymbol{\xi})$. Let us focus on the first term. Note that for any $\mathbf{u}, \mathbf{v} \in \mathcal{S}(\Omega)$, using Lemma 6, we have both $\mathrm{Tr}(\mathcal{D}_\omega^{\epsilon,*}\mathbf{u})$ and $\mathrm{Tr}(\mathcal{D}_\omega^{\epsilon,*}\mathbf{v})$ uniformly bounded in $L^2(\Omega)$, thus $\mathrm{Tr}(\mathcal{D}_\omega^{\epsilon,*}\mathbf{u})\mathrm{Tr}(\mathcal{D}_\omega^{\epsilon,*}\mathbf{v})$ is uniformly bounded in $L^1(\Omega)$. Now for all $\mathbf{x}\in\Omega$, as $\epsilon \to 0$, we have

$$\mathrm{Tr}\left(\mathcal{D}_\omega^{\epsilon,*}\mathbf{u}\right)(\mathbf{x})\mathrm{Tr}\left(\mathcal{D}_\omega^{\epsilon,*}\mathbf{v}\right)(\mathbf{x}) \to \mathrm{Tr}\left(\mathcal{D}_\omega^{*}\mathbf{u}\right)(\mathbf{x})\mathrm{Tr}\left(\mathcal{D}_\omega^{*}\mathbf{v}\right)(\mathbf{x}).$$

Then applying Lebesgue dominated theorem, we conclude that

$$\int_{\Omega}\left(k(\mathbf{x})-\frac{\alpha(\mathbf{x})m_\epsilon(\mathbf{x})}{d^2}\right)\mathrm{Tr}\left(\mathcal{D}_\omega^{\epsilon,*}\mathbf{u}\right)(\mathbf{x})\mathrm{Tr}\left(\mathcal{D}_\omega^{\epsilon,*}\mathbf{v}\right)(\mathbf{x})d\mathbf{x}$$

$$\to \int_{\Omega}\left(k-\frac{\alpha m(\mathbf{x})}{d^2}\right)\mathrm{Tr}\left(\mathcal{D}_\omega^{*}\mathbf{u}\right)(\mathbf{x})\mathrm{Tr}\left(\mathcal{D}_\omega^{*}\mathbf{v}\right)(\mathbf{x})d\mathbf{x},$$

which completes the proof of the lemma. □

Theorem 2 *Suppose that* ρ *satisfies* (H). *Assume that* $k(\mathbf{x})$ *and* $\alpha(\mathbf{x})$ *are at least Lipschitz continuous on* Ω. *Suppose also that* $\mathbf{u} \in C_c^\infty(\Omega;\mathbb{R}^d)$, *and that the horizon* $\delta \ll h = \mathrm{dist}(\partial\Omega, \mathrm{Supp}(\mathbf{u}))$. *Then the following is true.*

1. $\sup_{\epsilon>0}\sup_{\mathbf{x}\in\Omega}|\mathcal{L}_\epsilon(\mathbf{u})(\mathbf{x})| \le C$, *where the constant* $0 < C = C(\rho,\mathbf{u},\Omega)$,
2. $\mathcal{L}_\epsilon(\mathbf{u}) \to \mathcal{L}(\mathbf{u})$ *strongly in* $L^2(\Omega;\mathbb{R}^d)$, *thus, for almost all* $\mathbf{x}\in\Omega$,

$$\mathcal{L}(\mathbf{u})(\mathbf{x}) = \lim_{\epsilon\to 0}\mathcal{L}_\epsilon(\mathbf{u})(\mathbf{x}). \tag{15}$$

We thus may write the closed formula for $\mathcal{L}(\mathbf{u})$ *as in* (14) *for almost all* $\mathbf{x}\in\Omega$, *but understood in the sense of* (15).

Proof To show the bounds in (1), suppose that $\mathbf{u} \in C_c^\infty(\Omega;\mathbb{R}^d)$. Assume that the horizon $\delta < h/8$. When $\mathbf{x} \in \Omega^c_{3h/4}$, then $B(\mathbf{x},\delta) \subset \Omega^c_{7h/8}$. Moreover, for any $\mathbf{x}' \in B(\mathbf{x},\delta)$, the ball $B(\mathbf{x}',\delta) \subset \Omega^c_h$. As a result, $\mathcal{L}_\epsilon(\mathbf{u})(\mathbf{x}) = 0$.

When $\mathbf{x} \in \Omega_{3h/4}$, $B(\mathbf{x},\delta) \subset \Omega_{h/2}$ and for any $\mathbf{x}' \in B(\mathbf{x},\delta)$, $B(\mathbf{x}',\delta) \subset \Omega_{h/4}$. In this case, $m_\epsilon(\mathbf{x}) = m_\epsilon(\mathbf{x}') = m_\epsilon$, a constant, and we can write $\mathcal{L}_\epsilon(\mathbf{u})(\mathbf{x})$ as

$$\begin{aligned}
\mathcal{L}_\epsilon(\mathbf{u})(\mathbf{x}) = &\int_{B(\mathbf{x},\delta)} \big(\alpha(\mathbf{x}) + \alpha(\mathbf{x}')\big)\frac{\rho_\epsilon(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{x}'-\mathbf{x})(\mathbf{x}'-\mathbf{x})\cdot\big(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})\big)d\mathbf{x}' \\
&+ \int_{B(\mathbf{x},\delta)} \tau_\epsilon(\mathbf{x}')\rho_\epsilon(\mathbf{x}-\mathbf{x}')(\mathbf{x}'-\mathbf{x}) \\
&\times\left(\int_{B(\mathbf{x}',\delta)} \rho_\epsilon(\mathbf{x}'-\mathbf{z})(\mathbf{z}-\mathbf{x}')\cdot\big(\mathbf{u}(\mathbf{z}) - \mathbf{u}(\mathbf{x}')\big)d\mathbf{z}\right)d\mathbf{x}' \\
&+ \int_{B(\mathbf{x},\delta)} \tau_\epsilon(\mathbf{x})\rho_\epsilon(\mathbf{x}-\mathbf{x}')(\mathbf{x}'-\mathbf{x}) \\
&\times\left(\int_{B(\mathbf{x},\delta)} \rho_\epsilon(\mathbf{x}-\mathbf{z})(\mathbf{z}-\mathbf{x})\cdot\big(\mathbf{u}(\mathbf{z}) - \mathbf{u}(\mathbf{x})\big)d\mathbf{z}\right)d\mathbf{x}'.
\end{aligned}$$

Noting that the third term is zero, after a change of variable to the first two terms we get

$$\begin{aligned}
\mathcal{L}_\epsilon(\mathbf{u})(\mathbf{x}) &= \int_{B(0,\delta)} \big(\alpha(\mathbf{x}) + \alpha(\mathbf{x}+\mathbf{z})\big)\frac{\rho_\epsilon(\mathbf{z})}{|\mathbf{z}|^2}(\mathbf{z}\otimes\mathbf{z})\big(\mathbf{u}(\mathbf{x}+\mathbf{z}) - \mathbf{u}(\mathbf{x})\big)d\mathbf{z} \\
&\quad + \int_{B(0,\delta)} \tau_\epsilon(\mathbf{x}+\mathbf{z})\rho_\epsilon(\mathbf{z})\mathbf{z}\left(\int_{B(0,\delta)} \rho_\epsilon(\mathbf{w})\mathbf{w}\cdot\big(\mathbf{u}(\mathbf{x}+\mathbf{z}+\mathbf{w}) - \mathbf{u}(\mathbf{x}+\mathbf{z})\big)d\mathbf{w}\right)d\mathbf{z} \\
&= J_1^\epsilon(\mathbf{x}) + J_2^\epsilon(\mathbf{x}).
\end{aligned}$$

Now let us bound $J_1^\epsilon(\mathbf{x})$ and $J_2^\epsilon(\mathbf{x})$ uniformly in ϵ and $\mathbf{x}$. First,

$$\begin{aligned}
J_1^\epsilon(\mathbf{x}) &= \int_{B(0,\delta)} \big(\alpha(\mathbf{x}) + \alpha(\mathbf{x}+\mathbf{z})\big)\frac{\rho_\epsilon(\mathbf{z})}{|\mathbf{z}|^2}(\mathbf{z}\otimes\mathbf{z})\big(\mathbf{u}(\mathbf{x}+\mathbf{z}) - \mathbf{u}(\mathbf{x})\big)d\mathbf{z} \\
&= \int_{B(0,\delta)} \big(\alpha(\mathbf{x}+\mathbf{z}) - \alpha(\mathbf{x})\big)\frac{\rho_\epsilon(\mathbf{z})}{|\mathbf{z}|^2}(\mathbf{z}\otimes\mathbf{z})\big(\mathbf{u}(\mathbf{x}+\mathbf{z}) - \mathbf{u}(\mathbf{x})\big)d\mathbf{z}
\end{aligned}$$

$$+ 2\alpha(\mathbf{x}) \int_{B(0,\delta)} \frac{\rho_\epsilon(\mathbf{z})}{|\mathbf{z}|^2} \big\langle D^2(\mathbf{u})\big(\zeta(\mathbf{x},\mathbf{z})\big)\mathbf{z}\otimes\mathbf{z}, \mathbf{z}\big\rangle \mathbf{z}\, d\mathbf{z},$$

implying that

$$\big|J_1^\epsilon(\mathbf{x})\big| \le m\big(\|\nabla\alpha\|_{L^\infty(\Omega)}\|\nabla\mathbf{u}\|_{L^\infty(\Omega)} + \|\alpha\|_{L^\infty(\Omega)}\big\|D^2(\mathbf{u})\big\|_{L^\infty(\Omega)}\big).$$

Similarly, using the mean value theorem, we may rewrite the second integral as

$$\begin{aligned}
J_2^\epsilon(\mathbf{x}) &= \int_{B(0,\delta)} \tau_\epsilon(\mathbf{x}+\mathbf{z})\rho_\epsilon(\mathbf{z})\mathbf{z}\left(\int_{B(0,\delta)} \frac{\rho_\epsilon(\mathbf{w})}{|\mathbf{w}|}\int_0^{|\mathbf{w}|}\left\langle \nabla\mathbf{u}\left(\mathbf{x}+\mathbf{z}+t\frac{\mathbf{w}}{|\mathbf{w}|}\right)\mathbf{w}, \mathbf{w}\right\rangle dt\, d\mathbf{w}\right) d\mathbf{z}\\
&= \int_{B(0,\delta)} \tau_\epsilon(\mathbf{x}+\mathbf{z})\rho_\epsilon(\mathbf{z})\mathbf{z}\left(\int_{B(0,\delta)} \frac{\rho_\epsilon(\mathbf{w})}{|\mathbf{w}|}\int_0^{|\mathbf{w}|}\left\langle\left(\nabla\mathbf{u}\left(\mathbf{x}+\mathbf{z}+t\frac{\mathbf{w}}{|\mathbf{w}|}\right)\right.\right.\right.\\
&\quad \left.\left.\left. - \nabla\mathbf{u}\left(\mathbf{x}+t\frac{\mathbf{w}}{|\mathbf{w}|}\right)\right)\mathbf{w}, \mathbf{w}\right\rangle dt\, d\mathbf{w}\right) d\mathbf{z}\\
&\quad + \int_{B(0,\delta)} \big(\tau_\epsilon(\mathbf{x}+\mathbf{z}) - \tau_\epsilon(\mathbf{x})\big)\rho_\epsilon(\mathbf{z})\mathbf{z}\left(\int_{B(0,\delta)} \frac{\rho_\epsilon(\mathbf{w})}{|\mathbf{w}|}\right.\\
&\quad \left. \times \int_0^{|\mathbf{w}|}\left\langle \nabla\mathbf{u}\left(\mathbf{x}+t\frac{\mathbf{w}}{|\mathbf{w}|}\right)\mathbf{w}, \mathbf{w}\right\rangle dt\, d\mathbf{w}\right) d\mathbf{z}
\end{aligned}$$

from which we obtain that

$$\big|J_2^\epsilon(\mathbf{x})\big| \le \big(\big\|m_\epsilon^2\tau_\epsilon\big\|_{L^\infty}\big\|D^2\mathbf{u}\big\|_{L^\infty} + \big\|m_\epsilon^2\nabla\tau_\epsilon\big\|_{L^\infty}\|D\mathbf{u}\|_{L^\infty}\big).$$

That completes the proof of Part (1) after observing that $m_\epsilon^2\tau_\epsilon(\mathbf{x}) = k(\mathbf{x}) + \alpha(\mathbf{x})m_\epsilon(\mathbf{x})/d^2$ and both $\|m_\epsilon^2\tau_\epsilon\|_{L^\infty}$ and $\|m_\epsilon^2\nabla\tau_\epsilon\|_{L^\infty}$ are uniformly bounded in ϵ.

To prove Part (2), we first notice that by uniform bounded convergence theorem, $\mathcal{L}_\epsilon(\mathbf{u})(\mathbf{x}) \to L_0(\mathbf{u})(\mathbf{x})$ strongly in $L^2(\Omega;\mathbb{R}^d)$ as $\epsilon \to 0$ where

$$L_0(\mathbf{u})(\mathbf{x}) = \lim_{\epsilon\to 0}\mathcal{L}_\epsilon(\mathbf{u})(\mathbf{x}), \quad \text{for } \mathbf{x}\in\Omega.$$

Then for any $\mathbf{v}\in\mathcal{S}(\Omega)$, we have

$$\big\langle -L_0(\mathbf{u}), \mathbf{v}\big\rangle = \big(-L_0(\mathbf{u}), \mathbf{v}\big) = \lim_{\epsilon\to 0}\big(-\mathcal{L}_\epsilon(\mathbf{u}), \mathbf{v}\big) = \big\langle -\mathcal{L}(\mathbf{u}), \mathbf{v}\big\rangle,$$

demonstrating that $\mathcal{L}(\mathbf{u})$ is in fact a vector function in $L^2(\Omega;\mathbb{R}^d)$ and agrees with $L_0(\mathbf{u})(\mathbf{x})$ for almost all $\mathbf{x}\in\Omega$. □

4 The Limiting Behavior for Vanishing Non-locality

When the horizon is sufficiently close to zero, the nonlocal PD Navier operator approximates the classical Navier operator of arbitrary Poisson ratio. Such a statement has been verified with different degrees of generality and rigor in earlier works [6, 18, 19]. Except for simpler bond based nonlocal models [8, 15], the results are derived for operators defined in the whole space, that is, without taking into account the effect of bounded domain and the approaches are either based on Taylor expansion or Fourier transforms, see for example, [10, 24]. The aim of this section is to implement the approach used in [15] to prove the convergence of nonlocal-to-local for the state-based model for radial kernels under minimal assumptions on the regularity of solutions. By this we mean, not only we show that the sequence of PD Navier operators, parametrized by the horizon, approaches the classical Navier operator as the horizon vanishes, as shown in [5, 10, 18], but also, for a given right hand side, the associated sequence of solutions to the nonlocal problem converges to the corresponding solution to the classical problem. To do this, we carefully study the dependence of the nonlocal PD solutions on the horizon. Our analysis, unlike previous works, allows spatial inhomogeneities while focusing on volume constrained problems.

To that end, let ρ_1 be a nonnegative nonincreasing radial function satisfying

$$\rho_1\big(|\boldsymbol{\xi}|\big) > 0, \quad \text{near } \boldsymbol{\xi} = 0, \quad \mathrm{Supp}(\rho_1) \subset B(0,1), \quad \text{and} \quad \int_{B(0,1)} \rho_1\big(|\boldsymbol{\xi}|\big) d\boldsymbol{\xi} = 1.$$

We denote

$$\tilde{\rho}_\delta\big(|\boldsymbol{\xi}|\big) = \delta^{-d}\rho_1\big(\delta^{-1}|\boldsymbol{\xi}|\big), \qquad \rho_\delta\big(|\boldsymbol{\xi}|\big) = |\boldsymbol{\xi}|^{-2}\tilde{\rho}_\delta\big(|\boldsymbol{\xi}|\big).$$

Given $\mathbf{b} \in L^2(\Omega; \mathbb{R}^d)$, we would like to study the limiting behavior of the solution to the nonlocal equation as the measure of nonlocality $\delta \to 0$.

We have shown that the equation for any $\delta > 0$,

$$\begin{cases} -\mathcal{L}^\delta(\mathbf{u}) = \mathbf{b} & \mathbf{x} \in \Omega \\ \mathbf{u} = 0 & \text{on } \Omega \setminus \Omega_\delta, \end{cases}$$

has a unique solution $\mathbf{u}_\delta \in V_0^\delta = V_0(\Omega_\delta)$ where we have denoted $\mathcal{L}^\delta$ as the peridynamic operator corresponding to ρ_δ, and as before, $\Omega_\delta = \{\mathbf{x} \in \Omega : \mathrm{dist}(\mathbf{x}, \partial\Omega) > \delta\}$.

Similar as before, we define $\mathrm{Tr}(\mathcal{D}^{\delta,*}\mathbf{u})$ and $\mathrm{Tr}(\mathcal{D}^{\delta,*}_\omega\mathbf{u})$ the same way as $\mathrm{Tr}(\mathcal{D}^*\mathbf{u})$ and $\mathrm{Tr}(\mathcal{D}^*_\omega\mathbf{u})$ in (3) and (4) but with ρ being replaced by ρ_δ, and $m(\mathbf{x})$, $\omega(\mathbf{x}', \mathbf{x})$ and $\tau(\mathbf{x})$ being modified accordingly with the corresponding modified quantities being denoted by $m_\delta(\mathbf{x})$, $\omega_\delta\big(\mathbf{x}', \mathbf{x}\big)$ and $\tau_\delta(\mathbf{x})$. We also use $B_\delta(\mathbf{u}, \mathbf{v})$ to denote the same bilinear form as $B(\mathbf{u}, \mathbf{v})$ defined in (12) but with the above modified operators and coefficients. Let us first discuss the convergence of operators. The following proposition states that the above sequence of nonlocal operators converge to a well known local operator.

Proposition 4 *Assume that $k(\mathbf{x}), \alpha(\mathbf{x})$ are nonnegative bounded functions. Then for all $\mathbf{w}, \mathbf{v} \in C^1(\overline{\Omega}; \mathbb{R}^d)$ we have*

$$\big\langle \mathcal{L}^\delta \mathbf{w}, \mathbf{v} \big\rangle \longrightarrow \langle \mathcal{L}_0 \mathbf{w}, \mathbf{v} \rangle \quad \text{as } \delta \to 0,$$

where the bilinear form $\langle \mathcal{L}_0 \mathbf{w}, \mathbf{v} \rangle$ is given by

$$\langle \mathcal{L}_0 \mathbf{w}, \mathbf{v} \rangle = \int_\Omega \big[\mu(\mathbf{x})\nabla\mathbf{w}(\mathbf{x}) \cdot \nabla\mathbf{v}(\mathbf{x}) + \big(\mu(\mathbf{x}) + \lambda(\mathbf{x})\big)\mathrm{div}\,\mathbf{w}(\mathbf{x})\mathrm{div}\,\mathbf{v}(\mathbf{x})\big] d\mathbf{x}$$

with

$$\mu(\mathbf{x}) = \frac{\alpha(\mathbf{x})}{d(d+2)}, \quad \text{and} \quad \lambda(\mathbf{x}) = \left(\frac{1}{d(d+2)} - \frac{1}{d^2}\right)\alpha(\mathbf{x}) + k(\mathbf{x}).$$

Proof Recall that

$$\big\langle \mathcal{L}^\delta \mathbf{w}, \mathbf{v} \big\rangle = B_\delta(\mathbf{w}, \mathbf{v}) = I_\delta^1 + I_\delta^2,$$

where

$$I_\delta^1 = \int_{\Omega\setminus\Omega_\delta}\Bigg[\left(k(\mathbf{x}) - \frac{\alpha(\mathbf{x})m_\delta(\mathbf{x})}{d^2}\right)\mathrm{Tr}\big(\mathcal{D}_\omega^{\delta,*}\mathbf{w}\big)(\mathbf{x})\mathrm{Tr}\big(\mathcal{D}_\omega^{\delta,*}\mathbf{v}\big)(\mathbf{x})$$
$$+\alpha(\mathbf{x})\int_\Omega \tilde\rho_\delta\big(|\mathbf{x}'-\mathbf{x}|\big)\mathrm{Tr}\big(\mathcal{D}^{\delta,*}\mathbf{w}\big)\big(\mathbf{x}',\mathbf{x}\big)\mathrm{Tr}\big(\mathcal{D}^{\delta,*}\mathbf{v}\big)\big(\mathbf{x}',\mathbf{x}\big)d\mathbf{x}'\Bigg]d\mathbf{x},$$

and $I_\delta^2 = \langle\mathcal{L}^\delta\mathbf{w},\mathbf{v}\rangle - I_\delta^1$. Now, on the one hand, using the inequalities,

$$\sup_{\mathbf{x}\in\Omega}\big|m_\delta(\mathbf{x})\big| \le 1, \qquad \sup_{\delta>0,\mathbf{x}\in\Omega}\big|\mathrm{Tr}\big(\mathcal{D}_\omega^{\delta,*}\mathbf{w}\big)(\mathbf{x})\mathrm{Tr}\big(\mathcal{D}_\omega^{\delta,*}\mathbf{v}\big)(\mathbf{x})\big| \le d^2\|\nabla\mathbf{w}\|_{L^\infty}\|\nabla\mathbf{v}\|_{L^\infty}$$

and $|\mathrm{Tr}(\mathcal{D}^{\delta,*}\mathbf{w})(\mathbf{x}',\mathbf{x})\mathrm{Tr}(\mathcal{D}^{\delta,*}\mathbf{v})(\mathbf{x}',\mathbf{x})| \le |\mathbf{x}'-\mathbf{x}|^2\|\nabla\mathbf{w}\|_{L^\infty}\|\nabla\mathbf{v}\|_{L^\infty}$, we obtain that $I_\delta^1 \to 0$, as $\delta\to 0$.

On the other hand, using the fact that when $\mathbf{x}\in\Omega_\delta$, $\tau_\delta(\mathbf{x}) = \tau(\mathbf{x}) = k(\mathbf{x})d^2 - \alpha(\mathbf{x})$, it is not difficult to show that as $\delta\to 0$,

$$\int_{\Omega_\delta}\left(k(\mathbf{x}) - \frac{\alpha(\mathbf{x})m_\delta(\mathbf{x})}{d^2}\right)\mathrm{Tr}\big(\mathcal{D}_\omega^{\delta,*}\mathbf{w}\big)(\mathbf{x})\mathrm{Tr}\big(\mathcal{D}_\omega^{\delta,*}\mathbf{v}\big)(\mathbf{x})d\mathbf{x}$$
$$\longrightarrow \int_\Omega \frac{\tau(\mathbf{x})}{d^2}\big(\mathrm{div}\mathbf{w}(\mathbf{x})\big)\big(\mathrm{div}\mathbf{v}(\mathbf{x})\big)d\mathbf{x}.$$

Moreover, as $\delta\to 0$,

$$\int_{\Omega_\delta}\alpha(\mathbf{x})\int_{B(\mathbf{x},\delta)}\tilde\rho_\delta\big(|\mathbf{x}'-\mathbf{x}|\big)\mathrm{Tr}\big(\mathcal{D}^{\delta,*}\mathbf{w}\big)\big(\mathbf{x}',\mathbf{x}\big)\mathrm{Tr}\big(\mathcal{D}^{\delta,*}\mathbf{v}\big)\big(\mathbf{x}',\mathbf{x}\big)d\mathbf{x}'$$
$$= \int_{\Omega_\delta}\alpha(\mathbf{x})\int_{B(\mathbf{x},\delta)}\tilde\rho_\delta\big(|\boldsymbol{\xi}|\big)\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\big(\mathbf{w}(\mathbf{x}+\boldsymbol{\xi})-\mathbf{w}(\mathbf{x})\big)\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\big(\mathbf{v}(\mathbf{x}+\boldsymbol{\xi})-\mathbf{v}(\mathbf{x})\big)d\boldsymbol{\xi}d\mathbf{x}$$
$$\longrightarrow \sum_{i,j,k,l=1}^d\int_\Omega\alpha(\mathbf{x})\frac{\partial\mathbf{w}_i}{\partial x_j}\frac{\partial\mathbf{v}_l}{\partial x_k}\int_{B(0,1)}\frac{\rho(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2}\xi_i\xi_j\xi_l\xi_k d\boldsymbol{\xi}$$
$$= \int_\Omega\frac{\alpha(\mathbf{x})}{d(d+2)}\big[\nabla\mathbf{w}(\mathbf{x})\cdot\nabla\mathbf{v}(\mathbf{x}) + 2\mathrm{div}\,\mathbf{w}(\mathbf{x})\mathrm{div}\,\mathbf{v}(\mathbf{x})\big]d\mathbf{x}.$$

Adding up the two terms, we conclude that, as $\delta\to 0$,

$$\big\langle\mathcal{L}^\delta\mathbf{w},\mathbf{v}\big\rangle \longrightarrow \int_\Omega\Bigg[\frac{\alpha(\mathbf{x})}{d(d+2)}\nabla\mathbf{w}(\mathbf{x})\cdot\nabla\mathbf{v}(\mathbf{x})$$
$$+\left(\frac{2\alpha(\mathbf{x})}{d(d+2)} + \frac{\tau(\mathbf{x})}{d^2}\right)\mathrm{div}\,\mathbf{w}(\mathbf{x})\mathrm{div}\,\mathbf{v}(\mathbf{x})\Bigg]d\mathbf{x}.$$

The right hand side is precisely

$$\langle\mathcal{L}_0\mathbf{w},\mathbf{v}\rangle = \int_\Omega\big[\mu(\mathbf{x})\nabla\mathbf{w}(\mathbf{x})\cdot\nabla\mathbf{v}(\mathbf{x}) + \big(\mu(\mathbf{x})+\lambda(\mathbf{x})\big)\mathrm{div}\,\mathbf{w}(\mathbf{x})\mathrm{div}\,\mathbf{v}(\mathbf{x})\big]d\mathbf{x},$$

proving the proposition. □

Theorem 3 *Assume that* $k(\mathbf{x})$ *and* $\alpha(\mathbf{x})$ *are smooth functions (at least* C^1*). Then for* $\mathbf{w} \in C_c^\infty(\Omega;\mathbb{R}^d)$, $\mathcal{L}^\delta \mathbf{w}$ *is uniformly bounded in* $L^\infty(\Omega;\mathbb{R}^d)$*, and that for almost all* $\mathbf{x} \in \Omega$,

$$\mathcal{L}^\delta \mathbf{w}(\mathbf{x}) \longrightarrow \mathcal{L}_0 \mathbf{w}, (\mathbf{x})$$

as $\delta \to 0$, *where* $\mathcal{L}_0$ *is the operator*:

$$\mathcal{L}_0\mathbf{w}(\mathbf{x}) = \operatorname{div}\big(\mu(\mathbf{x})\nabla\mathbf{w}(\mathbf{x})\big) + \nabla\big(\big(\mu(\mathbf{x})+\lambda(\mathbf{x})\big)\operatorname{div}\mathbf{w}(\mathbf{x})\big),$$

with $\mu(\mathbf{x})$, *and* $\lambda(\mathbf{x})$ *as defined in Proposition 4.*

Proof Given $\mathbf{w} \in C_c^\infty(\Omega;\mathbb{R}^d)$, the estimate $\sup_{\delta>0}\|\mathcal{L}^\delta(\mathbf{w})\|_{L^\infty(\Omega)} \le C$ can be proved in the same way as part (1) of Theorem 2. To show the convergence, we recall that for sufficiently small δ, the function $\mathcal{L}^\delta\mathbf{w} \in L^2(\Omega;\mathbb{R}^d)$ and that for almost all $\mathbf{x} \in \Omega$, it is given by

$$\begin{aligned}
\mathcal{L}^\delta(\mathbf{w})(\mathbf{x}) = &\int_\Omega \big(\alpha(\mathbf{x})+\alpha(\mathbf{x}')\big)\frac{\rho^\delta(|\mathbf{x}'-\mathbf{x}|)}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{x}'-\mathbf{x})(\mathbf{x}'-\mathbf{x})\cdot\big(\mathbf{w}(\mathbf{x}')-\mathbf{w}(\mathbf{x})\big)d\mathbf{x}'\\
&+\int_\Omega \tau^\delta(\mathbf{x}')\rho^\delta(\mathbf{x}-\mathbf{x}')(\mathbf{x}'-\mathbf{x})\left(\int_\Omega \rho^\delta(\mathbf{x}'-\mathbf{z})(\mathbf{z}-\mathbf{x}')\cdot\big(\mathbf{w}(\mathbf{z})-\mathbf{w}(\mathbf{x}')\big)d\mathbf{z}\right)d\mathbf{x}'\\
&+\int_\Omega \tau^\delta(\mathbf{x})\rho^\delta(\mathbf{x}-\mathbf{x}')(\mathbf{x}'-\mathbf{x})\left(\int_\Omega \rho^\delta(\mathbf{x}-\mathbf{z})(\mathbf{z}-\mathbf{x})\cdot\big(\mathbf{w}(\mathbf{z})-\mathbf{w}(\mathbf{x})\big)d\mathbf{z}\right)d\mathbf{x}',
\end{aligned}$$

understood in the PV sense. To see where it converges, pick $\mathbf{x} \in \Omega$. Choose δ sufficiently small so that $B(\mathbf{x},2\delta) \subset \Omega$. Then for all $\mathbf{x}' \in B(\mathbf{x},\delta)$, $B(\mathbf{x}',\delta) \subset B(\mathbf{x},2\delta) \subset \Omega$. Therefore, $m^\delta(\mathbf{x}) = m^\delta(\mathbf{x}') = 1$, and as a result

$$\begin{aligned}
\mathcal{L}^\delta(\mathbf{w})(\mathbf{x}) = &\int_{B(\mathbf{x},\delta)} \big(\alpha(\mathbf{x})+\alpha(\mathbf{x}')\big)\frac{\rho^\delta(|\mathbf{x}'-\mathbf{x}|)}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{x}'-\mathbf{x})(\mathbf{x}'-\mathbf{x})\cdot\big(\mathbf{w}(\mathbf{x}')-\mathbf{w}(\mathbf{x})\big)d\mathbf{x}'\\
&+\int_{B(\mathbf{x},\delta)} \tau(\mathbf{x}')\rho^\delta(\mathbf{x}-\mathbf{x}')(\mathbf{x}'-\mathbf{x})\left(\int_{B(\mathbf{x}',\delta)} \rho^\delta(\mathbf{x}'-\mathbf{z})(\mathbf{z}-\mathbf{x}')\right.\\
&\left.\cdot\big(\mathbf{w}(\mathbf{z})-\mathbf{w}(\mathbf{x}')\big)d\mathbf{z}\right)d\mathbf{x}'\\
&+\int_{B(\mathbf{x},\delta)} \tau(\mathbf{x})\rho^\delta(\mathbf{x}-\mathbf{x}')(\mathbf{x}'-\mathbf{x})\left(\int_{B(\mathbf{x}',\delta)} \rho^\delta(\mathbf{x}-\mathbf{z})(\mathbf{z}-\mathbf{x})\right.\\
&\left.\cdot\big(\mathbf{w}(\mathbf{z})-\mathbf{w}(\mathbf{x})\big)d\mathbf{z}\right)d\mathbf{x}'.
\end{aligned}$$

Noting that the third term vanishes, the proof of the theorem can then be completed by using Taylor expansion and a careful accounting of terms as a function of δ. Indeed, we obtain that, as $\delta \to 0$,

$$\begin{aligned}
&\int_{B(\mathbf{x},\delta)} \big(\alpha(\mathbf{x})+\alpha(\mathbf{x}')\big)\frac{\rho^\delta(|\mathbf{x}'-\mathbf{x}|)}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{x}'-\mathbf{x})(\mathbf{x}'-\mathbf{x})\cdot\big(\mathbf{w}(\mathbf{x}')-\mathbf{w}(\mathbf{x})\big)d\mathbf{x}'\\
&\quad\longrightarrow \operatorname{div}\big(\mu(\mathbf{x})\big(\nabla\mathbf{w}(\mathbf{x})\big) + \nabla\big(2\mu(\mathbf{x})\operatorname{div}\mathbf{w}(\mathbf{x})\big)\big)
\end{aligned}$$

and

$$\int_{B(\mathbf{x},\delta)} \tau(\mathbf{x}')\rho^{\delta}(\mathbf{x}-\mathbf{x}')(\mathbf{x}'-\mathbf{x})\left(\int_{B(\mathbf{x}',\delta)} \rho^{\delta}(\mathbf{x}'-\mathbf{z})(\mathbf{z}-\mathbf{x}')\cdot(\mathbf{w}(\mathbf{z})-\mathbf{w}(\mathbf{x}'))d\mathbf{z}\right)d\mathbf{x}'$$
$$\longrightarrow \nabla\left(\frac{\tau(\mathbf{x})}{d^2}\mathrm{div}\mathbf{w}(\mathbf{x})\right).$$

Note that $\tau(\mathbf{x}) = d^2k(\mathbf{x}) - \alpha(\mathbf{x})$, since $m(\mathbf{x}) = 1$. Combining the above, we get the conclusion of the theorem. □

Next we study the behavior of the sequence of solutions as $\delta \to 0$. Recall that for all $\mathbf{w} \in V_0^{\delta}$,

$$\langle -\mathcal{L}^{\delta}\mathbf{u}_{\delta}, \mathbf{w}\rangle = (\mathbf{b}, \mathbf{w}). \tag{16}$$

To study the convergence properties of the solutions $\mathbf{u}_{\delta}$, we first obtain estimates that are uniform in δ via a nonlocal Poincaré-type inequality. The inequality we proved in Proposition 2 is not, however, good enough to offer precise estimates that show the explicit dependence of the constant on δ as the constant depends on the subspace $V_0(\Omega_{\delta})$, hence on δ. Rather we use the sharper version that is proved in [15]. We state the statement of the sharper Poincaré-type inequality but refer to [15] for the proof.

Proposition 5 *There exists δ_0 and $C(\delta_0)$ such that for all $\delta \in (0, \delta_0]$,*

$$\|\mathbf{u}\|^2_{L^2(\Omega)} \le C(\delta_0)\int_{\Omega}\int_{\Omega} \tilde{\rho}_{\delta}(|\mathbf{x}'-\mathbf{x}|)\left|\mathrm{Tr}(\mathcal{D}^*\mathbf{u})(\mathbf{x}',\mathbf{x})\right|^2 d\mathbf{x}'d\mathbf{x} \quad \forall \mathbf{u} \in V_0(\Omega_{\delta}).$$

Another ingredient we need is the following compactness result [13, Theorem 5.1] (see also [4, Theorem 4] or [22, Theorem 1.2] for functions) which we state here as a lemma.

Lemma 7 *Suppose that $\mathbf{u}_{\delta}$ is a bounded sequence in $L^2(\Omega;\mathbb{R}^d)$ with compact support in Ω. If*

$$\sup_{\delta>0}\int_{\Omega}\int_{\Omega} \tilde{\rho}_{\delta}(|\mathbf{x}'-\mathbf{x}|)\left|\mathrm{Tr}(\mathcal{D}^*\mathbf{u}_{\delta})(\mathbf{x}',\mathbf{x})\right|^2 d\mathbf{x}'d\mathbf{x} < \infty,$$

then $\mathbf{u}_{\delta}$ is precompact in $L^2(\Omega;\mathbb{R}^d)$. Moreover, any limit point $\mathbf{u} \in W_0^{1,2}(\Omega;\mathbb{R}^d)$.

We are now in a position to establish the convergence of the solutions $\mathbf{u}_{\delta}$ to $\mathbf{u}$ of the classical Navier-Lamé system.

Theorem 4 *Suppose that $\alpha(\mathbf{x})$ and $k(\mathbf{x})$ are positive $C^1(\overline{\Omega})$ functions. Then for any $\mathbf{b} \in L^2(\Omega;\mathbb{R}^d)$ the sequence of solutions $\mathbf{u}_{\delta}$ converges strongly in $L^2(\Omega;\mathbb{R}^d)$ to $\mathbf{u} \in W_0^{1,2}(\Omega;\mathbb{R}^d)$, where $\mathbf{u}$ is the unique weak solution of the (conventional) Navier system*

$$\begin{cases} -\mathcal{L}_0\mathbf{u}(\mathbf{x}) = \mathbf{b}(\mathbf{x}) & a.e \quad \mathbf{x} \in \Omega \\ \mathbf{u}(\mathbf{x}) = 0 & on\ \partial\Omega, \end{cases}$$

where $\mathcal{L}_0\mathbf{u}$ is as defined in Proposition 4.

Proof We begin the proof by obtaining some uniform estimates. Plugging in $\mathbf{u}_\delta$ in place of $\mathbf{w}$ in (16) we obtain that

$$\begin{aligned}\int_\Omega\int_\Omega \frac{\rho_\delta(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\left|(\mathbf{x}'-\mathbf{x})\cdot\left(\mathbf{u}_\delta(\mathbf{x}')-\mathbf{u}_\delta(\mathbf{x})\right)\right|^2 d\mathbf{x}'d\mathbf{x} \\ \leq \left(\frac{2}{\alpha_0}+\frac{2}{d^2k_0}\right)\left\langle -\mathcal{L}^\delta\mathbf{u}_\delta,\mathbf{u}_\delta\right\rangle \\ \leq \left(\frac{2}{\alpha_0}+\frac{2}{d^2k_0}\right)(\mathbf{b},\mathbf{u}_\delta) \\ \leq \left(\frac{2}{\alpha_0}+\frac{2}{d^2k_0}\right)\|\mathbf{b}\|_{L^2(\Omega)}\|\mathbf{u}_\delta\|_{L^2(\Omega)},\end{aligned}$$

where $0<\alpha_0=\min_{\mathbf{x}\in\Omega}\alpha(\mathbf{x})$, and $0<k_0=\min_{\mathbf{x}\in\Omega}k(\mathbf{x})$. Applying the Poincaré-type inequality, Proposition 5, we obtain that

$$\|\mathbf{u}_\delta\|_{L^2}\leq \nu\|\mathbf{b}\|_{L^2},$$

where ν is independent of δ. Combining the above two estimates, we obtain that

$$\int_\Omega\int_\Omega \frac{\rho_\delta(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\left|(\mathbf{x}'-\mathbf{x})\cdot\left(\mathbf{u}_\delta(\mathbf{x}')-\mathbf{u}_\delta(\mathbf{x})\right)\right|^2 d\mathbf{x}'d\mathbf{x}\leq C,$$

where the constant C is independent of δ.

With these uniform estimates at hand, we may apply Lemma 7 to conclude that the sequence $\{\mathbf{u}_\delta\}$ is precompact in $L^2(\Omega;\mathbb{R}^d)$, and any limit point $\mathbf{u}\in W_0^{1,2}(\Omega;\mathbb{R}^d)$. Let us show that any limit point indeed solves the Navier system and is therefore unique and thus the entire sequence actually strongly converge to the unique solution $\mathbf{u}$.

Let $\mathbf{w}\in C_c^\infty(\Omega;\mathbb{R}^d)$. Then since the operators are self adjoint and $\mathcal{L}^\delta\mathbf{w}\in L^\infty(\Omega;\mathbb{R}^d)$ (see Proposition 3), we may rewrite (16) as

$$\left(-\mathcal{L}^\delta\mathbf{w},\mathbf{u}_\delta\right)=(\mathbf{b},\mathbf{w}). \tag{17}$$

Using Proposition 4, the facts that $\mathbf{u}_\delta\to\mathbf{u}$ strongly in $L^2(\Omega;\mathbb{R}^d)$, $\mathbf{u}_\delta=0$ outside of Ω_δ and $\Omega_\delta\subset\operatorname{supp}(\mathbf{w})$ as $\delta\to 0$, we obtain from (17) that for all $\mathbf{w}\in C_c^\infty(\Omega;\mathbb{R}^d)$,

$$\int_\Omega -\mathcal{L}_0\mathbf{w}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})=\int_\Omega \mathbf{b}(\mathbf{x})\cdot\mathbf{w}(\mathbf{x})d\mathbf{x}$$

as $\delta\to 0$, verifying that $\mathbf{u}$ solves the Navier system. □

5 Conclusion

In this work we have analyzed a nonlocal constrained value problem of a linear state-based peridynamic model via a constraint imposed on the solution in a volume of nonzero measure. Our main contribution is showing the well-posedness of the model and the convergence of its solutions to the solution of its local PDE limit as the horizon of nonlocal interactions diminishes to zero. These results are established for very general kernels and under minimal regularity assumptions on the solutions, which essentially give a unified theory of that developed in earlier analysis of both bond-based and state-based PD models. They also reveal that while the state-based models correspond to more general materials class than bond-based models, the basic energy spaces for the associated variational problems remain the same for both models. Moreover, they are of close relevance to materials modeling, given the increasing trend in using state-based models as the more general choice in PD based numerical simulations. In addition, for interaction kernels that may change sign, the stability matrix function is shown rigorously to be related to the existence of minimizers. Furthermore, by allowing inhomogeneities in the material parameters, our analysis is also applicable to models that are more general and more realistic in nature.

The work presented here can also serve as the basis to study other linear peridynamic models [1] as well as more general nonlinear and time-dependent nonlocal models in the future.

Acknowledgements The authors would like to thank M. Gunzburger, R. Lehoucq, S. Silling, R. Lipton and E. Emmrich, for helpful discussions on the subject. They also would like to thank the reviewers for their valuable comments and suggestions, and for bringing additional references such as [1] to our attention.

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