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RESEARCH NOTE

Nonlinear Orthotropic Elasticity: Only Six Invariants are Independent

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Abstract It is often declared in the literature that the seven classical invariants used to characterize the strain energy of a compressible orthotropic elastic solid are independent. In this paper, we show that only six of the seven classical invariants are independent, and a syzygy exists between the classical invariants. Consequently, all other sets of seven invariants, proposed in the literature, that are uniquely related to the set of classical invariants, have only six independent invariants.

Keywords Six · Invariants · Independent · Orthotropic · Nonlinear · Elasticity

Mathematics Subject Classification 74B20

1 Introduction

Following the work of Spencer [1], a strain energy function W_e of a compressible elastic solid with two preferred orthogonal directions $\boldsymbol{a}$ and $\boldsymbol{b}$ can be expressed as

$$W_e = W(\boldsymbol{C}, \boldsymbol{a} \otimes \boldsymbol{a}, \boldsymbol{b} \otimes \boldsymbol{b}), \tag{1}$$

where $\boldsymbol{C}$ is the right Cauchy-Green deformation tensor and $\otimes$ denotes the dyadic product. W is an isotropic invariant $\boldsymbol{C}$, $\boldsymbol{a} \otimes \boldsymbol{a}$ and $\boldsymbol{b} \otimes \boldsymbol{b}$, i.e.,

$$W(\boldsymbol{C}, \boldsymbol{a} \otimes \boldsymbol{a}, \boldsymbol{b} \otimes \boldsymbol{b}) = W\big(\boldsymbol{Q}\boldsymbol{C}\boldsymbol{Q}^T, \boldsymbol{Q}(\boldsymbol{a} \otimes \boldsymbol{a})\boldsymbol{Q}^T, \boldsymbol{Q}(\boldsymbol{b} \otimes \boldsymbol{b})\boldsymbol{Q}^T\big) \tag{2}$$

for all proper orthogonal tensors $\boldsymbol{Q}$. It follows that the strain energy function W_e can be expressed as

$$W_e = \hat{W}(I_1, I_2, I_3, I_4, I_5, I_6, I_7), \tag{3}$$

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where

$$I_1 = \mathrm{tr}(\boldsymbol{C}), \qquad I_2 = \frac{I_1^2 - \mathrm{tr}(\boldsymbol{C}^2)}{2}, \qquad I_3 = \det(\boldsymbol{C}), \qquad I_4 = \boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{a},$$
$$I_5 = \boldsymbol{a} \bullet \boldsymbol{C}^2\boldsymbol{a}, \qquad I_6 = \boldsymbol{b} \bullet \boldsymbol{C}\boldsymbol{b}, \qquad I_7 = \boldsymbol{b} \bullet \boldsymbol{C}^2\boldsymbol{b} \tag{4}$$

and tr denotes the trace of a second order tensor. These commonly used classical invariants were proposed by Spencer [1], but he did not mention in his paper [1], that the seven invariants are independent. However, in the past, several authors have declared that the seven invariants are independent (see for example references [2, 3]). In Sect. 2, we show that only six of the classical invariants are independent. In Sect. 3 we show that a syzygy exist between the classical invariants and between a set of invariants recently proposed by Shariff [4]. For various reasons, other sets of seven invariants (see for example reference [5]) have been proposed in the literature to characterize orthotropic elastic solids. Since the classical invariants can be expressed in terms of these sets of seven invariants, hence, only six of the seven invariants in these sets of seven invariants are independent.

2 Non-independent

For a compressible anisotropic elastic material with two non-perpendicular preferred directions $\boldsymbol{a}$ and $\boldsymbol{b}$, the classical invariant set $\{I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9\}$ is commonly used to characterize the strain energy function of an anisotropic elastic solid (see Spencer [1, 6]), where

$$I_8 = \cos(2\phi)\boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{b}, \qquad I_9 = (\boldsymbol{a} \bullet \boldsymbol{b})^2 = \cos^2(2\phi) \tag{5}$$

and 2ϕ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The invariants I_{1-9} are independent [6] with respect to the tensor $\boldsymbol{C}$ and, unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. If we consider the components of $\boldsymbol{C}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ relative to a fixed Cartesian coordinate system, then there are ten independent components (six from $\boldsymbol{C}$ and four from $\boldsymbol{a}$ and $\boldsymbol{b}$) on the right hand-side of (4) and (5). For the set $\{I_{1-9}\}$ to be independent the rank of the corresponding ten by nine *"Jacobian"* matrix should be nine. The set $\{I_{1-9}\}$ is a minimal integrity basis [1, 6], hence all other polynomial invariants can be generated from this set. Since I_9 is fixed for an arbitrary deformation, the strain energy function is generally written in terms of the invariants I_{1-8}. When the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal, then $\cos(2\phi) = 0$, and the strain energy can be characterized using only the seven invariants I_{1-7}. If, for arbitrary $\boldsymbol{a}$ and $\boldsymbol{b}$, we choose the directions of the Cartesian X_1 and X_2 axes to be parallel to $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, we then have, from (4)

$$I_1 = \mathrm{tr}(\boldsymbol{C}), \qquad I_2 = \frac{I_1^2 - \mathrm{tr}(\boldsymbol{C}^2)}{2}, \qquad I_3 = \det(\boldsymbol{C}), \qquad I_4 = C_{11}, \qquad I_5 = C_{1r}C_{r1},$$
$$I_6 = C_{22}, \qquad I_7 = C_{2r}C_{r2}, \tag{6}$$

where C_{ij} are the Cartesian components of $\boldsymbol{C}$. In this communication all subscripts i and j take the values 1, 2 and 3, unless stated otherwise. Since $\boldsymbol{C}$ has six independent components, there is a relation among the invariants I_{1-7}, which indicates that only six of the seven invariants are independent. We can also show that the seven invariants are not independent without resorting to the Cartesian components of $\boldsymbol{C}$. We do this by writing $\boldsymbol{C}$ in the form

$$\boldsymbol{C} = \sum_{i=1}^{3} \lambda_i^2 \boldsymbol{e}_i \otimes \boldsymbol{e}_i, \tag{7}$$

(λ_i and $\boldsymbol{e}_i$ are the principal value and the principal direction of the right stretch tensor $\boldsymbol{U}$, respectively) and substitute (7) in (4) to obtain the expressions

$$\begin{aligned} &I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \qquad I_3 = (\lambda_1\lambda_2\lambda_3)^2, \\ &I_4 = \lambda_1^2\zeta_1 + \lambda_2^2\zeta_2 + \lambda_3^2\zeta_3, \qquad I_5 = \lambda_1^4\zeta_1 + \lambda_2^4\zeta_2 + \lambda_3^4\zeta_3, \\ &I_6 = \lambda_1^2\xi_1 + \lambda_2^2\xi_2 + \lambda_3^2\xi_3, \qquad I_7 = \lambda_1^4\xi_1 + \lambda_2^4\xi_2 + \lambda_3^4\xi_3, \end{aligned} \tag{8}$$

where $\zeta_i = (\boldsymbol{a} \bullet \boldsymbol{e}_i)^2$, $\xi_i = (\boldsymbol{b} \bullet \boldsymbol{e}_i)^2$,

$$\zeta_3 = 1 - \zeta_1 - \zeta_2 \quad \text{and} \quad \xi_3 = 1 - \xi_1 - \xi_2. \tag{9}$$

The invariant set $SH = \{\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \xi_1, \xi_2\}$ has been used by Shariff [4] to characterize the strain energy of an orthotropic elastic solid. However, there exists a relation between four of the invariants in the set SH, in particular the orthogonal relation,

$$\boldsymbol{a} \bullet \boldsymbol{b} = \sum_{i=1}^{3} (\boldsymbol{a} \bullet \boldsymbol{e}_i)(\boldsymbol{b} \bullet \boldsymbol{e}_i) = 0, \tag{10}$$

taking note that,

$$\begin{aligned} &\boldsymbol{a} \bullet \boldsymbol{e}_1 = \pm\sqrt{\zeta_1}, \qquad \boldsymbol{a} \bullet \boldsymbol{e}_2 = \pm\sqrt{\zeta_2}, \qquad \boldsymbol{b} \bullet \boldsymbol{e}_1 = \pm\sqrt{\xi_1}, \qquad \boldsymbol{b} \bullet \boldsymbol{e}_2 = \pm\sqrt{\xi_2}, \\ &\boldsymbol{a} \bullet \boldsymbol{e}_3 = \pm\sqrt{1 - \zeta_1 - \zeta_2}, \qquad \boldsymbol{b} \bullet \boldsymbol{e}_3 = \pm\sqrt{1 - \xi_1 - \xi_2}. \end{aligned} \tag{11}$$

In view of (10) and (11), only three of the invariants, say, ζ_1, ζ_2 and ξ_1 are independent. Hence, from (8), the seven classical invariants depend on six independent variables (invariants), which suggests that there exists a relationship among the seven invariants. In the next section, we show *syzygies* exist between the seven classical invariants and between the invariants proposed by Shariff [4].

3 Syzygy

Before we prove there is a syzygy between the seven classical invariants, we review a few preliminary concepts given in [6] to facilitate our analysis. In reference [6, p. 246] Spencer stated that:

1. A polynomial invariant is said to be *reducible* if it can expressed as a *polynomial* in other invariants; otherwise, it is said to be *irreducible*.
2. A set of polynomial invariants which has the property that any polynomial invariant can be expressed as a polynomial in members of the given set is called an *integrity basis*.
3. An integrity basis is *minimal* if contains the smallest possible number of members. Clearly, all members of a minimal integrity basis are irreducible.
4. It frequently happens that polynomial relations exist between invariants which do not permit any one invariant to be expressed as a polynomial in the remainder. Such relations are called *syzygies*.

Consider the right-handed set of vectors $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{n}\}$. Note that

$$\boldsymbol{C}\boldsymbol{a} = (\boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{a})\boldsymbol{a} + (\boldsymbol{b} \bullet \boldsymbol{C}\boldsymbol{a})\boldsymbol{b} + (\boldsymbol{n} \bullet \boldsymbol{C}\boldsymbol{a})\boldsymbol{n}. \tag{12}$$

Hence

$$\boldsymbol{a} \bullet \boldsymbol{C}^2\boldsymbol{a} = (\boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{a})^2 + (\boldsymbol{b} \bullet \boldsymbol{C}\boldsymbol{a})^2 + (\boldsymbol{n} \bullet \boldsymbol{C}\boldsymbol{a})^2. \tag{13}$$

For simplicity of notations, we let $I_{ab} = \boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{b}$, $I_{an} = \boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{n}$, $I_{bn} = \boldsymbol{b} \bullet \boldsymbol{C}\boldsymbol{n}$, $I_n = \boldsymbol{n} \bullet \boldsymbol{C}\boldsymbol{n}$ and $I_{nn} = \boldsymbol{n} \bullet \boldsymbol{C}^2\boldsymbol{n}$. From (13), we have

$$I_{an}^2 = I_5 - I_4^2 - I_{ab}^2. \tag{14}$$

Similarly, it can be easily shown that

$$I_{bn}^2 = I_7 - I_6^2 - I_{ab}^2 \tag{15}$$

and

$$I_{nn} = I_{an}^2 + I_{bn}^2 + I_n^2. \tag{16}$$

From the relation

$$I_1 = \operatorname{tr}(\boldsymbol{C}) = \boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{a} + \boldsymbol{b} \bullet \boldsymbol{C}\boldsymbol{b} + \boldsymbol{n} \bullet \boldsymbol{C}\boldsymbol{n}, \tag{17}$$

we have

$$I_n = I_1 - I_4 - I_6. \tag{18}$$

From the above equations, we have

$$I_{nn} = I_5 + I_7 - 2I_{ab}^2 + I_1^2 - 2I_1(I_4 + I_6) + 2I_4I_6. \tag{19}$$

From the relation

$$\operatorname{tr}\big(\boldsymbol{C}^2\big) = \boldsymbol{a} \bullet \boldsymbol{C}^2\boldsymbol{a} + \boldsymbol{b} \bullet \boldsymbol{C}^2\boldsymbol{b} + \boldsymbol{n} \bullet \boldsymbol{C}^2\boldsymbol{n} \tag{20}$$

we get

$$I_1^2 - 2I_2 = \operatorname{tr}\big(\boldsymbol{C}^2\big) = I_5 + I_7 + I_{nn}. \tag{21}$$

Substituting equation (19) into (21) we have the relation

$$I_{ab}^2 = I_2 + I_4I_6 + I_5 + I_7 - I_1(I_4 + I_6). \tag{22}$$

We note that relation (22) was also obtained by Merodio & Ogden [3]. Holzapfel & Ogden [2] showed that

$$I_4I_6I_n - I_4I_{bn}^2 - I_6I_{an}^2 - I_nI_{ab}^2 + 2I_{ab}I_{an}I_{bn} = I_3. \tag{23}$$

We then have

$$\big(I_3 - I_4I_6I_n + I_4I_{bn}^2 + I_6I_{an}^2 + I_nI_{ab}^2\big)^2 = 4I_{ab}^2I_{an}^2I_{bn}^2. \tag{24}$$

In view of (14), (15), (18) and (22), it is clear that (24) shows a syzygy between the classical invariants I_{1-7}.

In view of (8), we can easily show from (24) that a syzygy exists between the invariants $\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \xi_1$ and ξ_2. If we let $\lambda_1 = \lambda_2 = \lambda_3 = 1$ a syzygy exist between the invariants ζ_1, ζ_2, ξ_1 and ξ_2. Alternatively, we can also show that a syzygy exists among the invariants ζ_1, ζ_2, ξ_1 and ξ_2 from (10), where we have

$$2(\boldsymbol{a} \bullet \boldsymbol{e}_1)(\boldsymbol{b} \bullet \boldsymbol{e}_1)(\boldsymbol{a} \bullet \boldsymbol{e}_2)(\boldsymbol{b} \bullet \boldsymbol{e}_2) = \zeta_3\xi_3 - \zeta_1\xi_1 - \zeta_2\xi_2. \tag{25}$$

Hence, we have

$$4\zeta_1\xi_1\zeta_2\xi_2 = \big([1 - \zeta_1 - \zeta_2][1 - \xi_1 - \xi_2] - \zeta_1\xi_1 - \zeta_2\xi_2\big)^2 \tag{26}$$

which shows a syzygy between the invariants ζ_1, ζ_2, ξ_1 and ξ_2.

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