Stability of Homogeneous States with Gradient Damage Models: Size Effects and Shape Effects in the Three-Dimensional Setting

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Abstract Considering a family of gradient-enhanced damage models and taking advantage of its variational formulation, we study the stability of homogeneous states in a full three-dimensional context. We show that gradient terms have a stabilizing effect, but also how those terms induce structural effects. We emphasize the great importance of the type of boundary conditions, the size and the shape of the body on the stability properties of such states.

1 Introduction

Before their failure, quasi-brittle materials exhibit a softening behavior in their mechanical response under uniaxial tests [2]. The modeling of this ultimate stage of degradation within the framework of damage theory is a complicated task. Indeed, the associated boundary-value problem governing the evolution of damage in the sample ceases to be well-posed with local models. Accordingly, regularization techniques must be introduced as those based on the gradient of strain [20], on the gradient of damage [7, 9, 21, 23, 24] or on integral laws [25]. Additional non-local terms remove some pathologies of local models such as failure by localization without any dissipated energy or a part of numerical simulations mesh-dependency. For instance, Pijaudier-Cabot and Benallal [26] and Peerlings et al. [19] showed by means of a wave propagation analysis the benefits of their regularized damage

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model. However, none of these regularization methods have solved the issue of the non-uniqueness of the response which turns out to be an intrinsic property of any softening law. For instance, in the case of regularization by gradient damage terms, it was proved in [3] in a simplified one-dimensional context that there exists a continuum set of branches which are solutions of the damage evolution problem. Therefore, it has become necessary to introduce additional criteria to select physically realistic solutions. In the spirit of Nguyen [18], recent variational approaches [4–6, 13, 17] propose to reinforce usual stationary conditions with a stability condition. Making a full use of the justification of such an energetic approach given by Marigo [14, 15], see also DeSimone et al. [8], Pham and Marigo [22, 23] introduced this stability criterion as one of the three principles (along with irreversibility and energy balance) that governs the evolution of damage in a body. This allowed Pham et al. [24] to study the stability of the homogeneous response[1] of a bar whose displacements at the ends are controlled by a hard device. The main goal of this paper is to generalize this analysis in a full three-dimensional setting. For this purpose, we consider three-dimensional bodies under boundary conditions that are compatible with a homogeneous state. Then, we study whether such a homogeneous state is stable or not. The results depend on the *type* of boundary conditions (soft or hard device), on the *size* of the body and on its *shape*. These results are in agreement with what is commonly observed in the quasi-brittle materials, see [2]. Moreover, we claim, as already shown in Pham et al. [24], that the study of the stability of these states can be used for identifying the different constitutive functions of the model.

The paper is structured as follows. In Sect. 2, we set the main ingredients of the gradient damage model and introduce the stability criterion. In Sect. 3 we start the study of the stability of homogeneous states. We distinguish stress-hardening from stress-softening behaviors, elastic homogeneous states from damaging homogeneous states and different types of boundary conditions depending on whether those are prescribed by a soft device or a hard device, including uniaxial test. Then, the stability of any homogeneous state in the case of stress-hardening materials is obtained. We also prove that elastic homogeneous states are always stable. This allows us to focus on damaging states for stress-softening materials which is the most common case in real-world applications. We easily prove that in such a case, any damaging state is always unstable when the boundary is controlled by a soft device. Section 4 is devoted to the case where the boundary is controlled by a hard device and we show how this leads to the appearance of size effects. More specifically, we prove that a damaging state is stable provided that the size of the body is less than a critical value which depends in general both on the state and the shape of the body. However, it can happen that this critical size becomes infinite which means that the state is stable independently of the size of the body. Moreover, the finiteness of this critical size is conditional on the state and the damage model, but not on the shape of the body. This is illustrated by some examples. In Sect. 5, we consider the case of uniaxial tensile tests of cylinders and show that both size and shape effects are present. In particular, we study the asymptotic case of slender cylinders and obtain the closed-form expression of the critical size. The last section is devoted to a comparison between our stability criterion and the strong ellipticity condition which is generally used for studying the stability of homogeneous state in softening materials.

Throughout the paper, the following notation is used. The vectors and second order tensors are denoted by boldface letters, like $\mathbf{u}$ for the displacement vector and $\boldsymbol{\varepsilon}$ for the strain tensor, while their components are denoted by italic letters, like u_i and ε_{ij}. The fourth order tensors are represented by sans serif letters, like A and S for the rigidity and the compliance tensors. In general, intrinsic notation is used: for instance $\mathsf{A}\boldsymbol{\varepsilon}$ denotes the second order

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[1]The response for which both strain and damage fields are constant in space.

tensor whose ij component is given by $A_{ijkl}\varepsilon_{kl}$ (where here and henceforth the summation convention is implicitly used). The inner product between vectors or between second order tensors is indicated by a dot. Accordingly, one reads $\mathbf{u} \cdot \mathbf{v} = u_i v_i$, $\mathsf{A}\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} = A_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$. The space of $n \times n$ symmetric matrices is denoted by $\mathbb{M}_s^n$. When $\mathbf{v}$ is a displacement field, *i.e.* a vector field of $\mathbb{R}^n$, its associated strain tensor field $\boldsymbol{\varepsilon}(\mathbf{v})$ is the symmetric part of the gradient of $\mathbf{v}$, *i.e.* $2\boldsymbol{\varepsilon}(\mathbf{v}) = \nabla\mathbf{v} + (\nabla\mathbf{v})^T$. In terms of components, we get $2\varepsilon_{ij}(\mathbf{v}) = v_{i,j} + v_{j,i}$ where the comma denotes the partial derivative with respect to the concerned coordinate. $L^2(\Omega)$ denotes the Hilbert space of square integrable functions over the open set Ω of $\mathbb{R}^n$, equipped with its natural norm $\|\cdot\|_0$; $H^1(\Omega)$ denotes the Hilbert space of functions f which are in $L^2(\Omega)$ and whose first weak derivatives $f_{,i}$ are also in $L^2(\Omega)$, equipped with its natural norm $\|\cdot\|_1$:

$$\|f\|_0^2 = \int_\Omega f(\mathbf{x})^2 dx, \qquad \|f\|_1^2 = \|f\|_0^2 + \sum_{i=1}^n \|f_{,i}\|_0^2.$$

For vector-valued or tensor-valued fields, the spaces $L^2(\Omega, \mathbb{R}^n)$, $L^2(\Omega, \mathbb{M}_s^n)$, $H^1(\Omega, \mathbb{R}^n)$ are defined in similar way. For instance, for the vector field $\mathbf{v}$ and the tensor field $\boldsymbol{\varepsilon}$ we set:

$$\|\mathbf{v}\|_0^2 = \int_\Omega \mathbf{v}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) dx, \qquad \|\boldsymbol{\varepsilon}\|_0^2 = \int_\Omega \boldsymbol{\varepsilon}(\mathbf{x}) \cdot \boldsymbol{\varepsilon}(\mathbf{x}) dx, \qquad \|\mathbf{v}\|_1^2 = \|\mathbf{v}\|_0^2 + \|\nabla\mathbf{v}\|_0^2.$$

2 The Gradient Damage Model

2.1 The Non-local Form of the Strain Work

We assume that the damage state at a material point is characterized by a scalar internal variable α growing from 0 to $\alpha_m \le +\infty$, 0 corresponding to the undamaged state and α_m to the full damaged state. This scalar damage variable represents at a macroscale a measure of the presence of defects at a microscale like microvoids or microcracks. But the precise link between microdefects and the effective behavior of the material goes well beyond the scope of this paper and we will follow a phenomenological procedure to set the constitutive equations. More specifically, in our variational approach, we directly postulate the form of the non-local strain work at a material point. In Pham and Marigo [23], it has been shown that for an *elastic isotropic* material with a *scalar* damage variable, up to a change of damage variable, the strain work can be written in the following form:

$$W(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha) = w_1\alpha + \frac{1}{2}\mathsf{A}(\alpha)\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} + \frac{1}{2}w_1\ell(\alpha)^2\nabla\alpha \cdot \nabla\alpha. \tag{1}$$

In (1), w_1 is a material constant which has the dimension of a pressure (*i.e.*, an energy by volume unit) provided that α is dimensionless. $\mathsf{A}(\alpha)$ denotes the stiffness tensor of the material in the damage state α, $\boldsymbol{\varepsilon}$ is the strain tensor and hence $\frac{1}{2}\mathsf{A}(\alpha)\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}$ is the elastic energy density. Because of the damage process, $\alpha \mapsto \mathsf{A}(\alpha)$ is a decreasing function, going from $\mathsf{A}(0) = \mathsf{A}_0$ to $\mathsf{A}(\alpha_m) = 0$, A_0 being the stiffness tensor of the sound material. We also use the "damaged" compliance tensor $\alpha \mapsto \mathsf{S}(\alpha)$ defined as the inverse of $\mathsf{A}(\alpha)$, $\mathsf{S}(\alpha) = \mathsf{A}(\alpha)^{-1}$. The last term in the right hand side of (1), *i.e.*, $\frac{1}{2}w_1\ell(\alpha)^2\nabla\alpha \cdot \nabla\alpha$, corresponds to the non-local contribution which contains the real-valued positive function $\alpha \mapsto \ell(\alpha)$, $\ell(\alpha)$ having the physical dimension of a length. Since the material is isotropic, the stiffness and the

compliance tensors of the material in the damaged state α can read as

$$
\begin{aligned}
A(\alpha)_{ijkl} &= \lambda(\alpha)\delta_{ij}\delta_{kl} + \mu(\alpha)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \\
S(\alpha)_{ijkl} &= -\frac{\nu(\alpha)}{E(\alpha)}\delta_{ij}\delta_{kl} + \frac{1+\nu(\alpha)}{E(\alpha)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})
\end{aligned}
\tag{2}
$$

where $\lambda(\alpha)$, $\mu(\alpha)$, $\nu(\alpha)$ and $E(\alpha)$ are respectively the Lamé coefficients, the Poisson's ratio and the Young modulus of the material in the damaged state α.

Throughout the paper we adopt the following assumptions on the constitutive functions:

Hypothesis 1 *For all* $\alpha \in [0, \alpha_m)$, $A(\alpha) > 0$, $A'(\alpha) < 0$, $S(\alpha) > 0$, $S'(\alpha) > 0$, $\ell(\alpha) > 0$.

2.2 The Hardening and Softening Properties

Like in plasticity, hardening and softening properties play a keyrole in the well-posedness of the problem. These properties deal with the local behavior of the material and only involve the *local part of the strain work*

$$
W_0(\boldsymbol{\varepsilon}, \alpha) = w_1\alpha + \frac{1}{2}A(\alpha)\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}.
\tag{3}
$$

Following [14], the elastic domain $\mathbb{E}(\alpha)$ in which strain is allowed to lie, when damage is spatially locally uniform ($\nabla\alpha = 0$) and equal to α, is given by

$$
\mathbb{E}(\alpha) = \left\{ \boldsymbol{\varepsilon} \in \mathbb{M}_s^n : -\frac{1}{2}A'(\alpha)\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} \le w_1 \right\} = \left\{ \boldsymbol{\varepsilon} \in \mathbb{M}_s^n : \frac{\partial W_0}{\partial\alpha}(\boldsymbol{\varepsilon}, \alpha) \ge 0 \right\}.
\tag{4}
$$

This corresponds to a critical elastic energy release rate criterion. The image of the elastic domain in the space of stresses, namely $\mathbb{E}^*(\alpha)$, is obtained after introducing the Legendre transform $\boldsymbol{\sigma} \mapsto W_0^*(\boldsymbol{\sigma}, \alpha)$ of $\boldsymbol{\varepsilon} \mapsto W_0(\boldsymbol{\varepsilon}, \alpha)$,

$$
W_0^*(\boldsymbol{\sigma}, \alpha) = \sup_{\boldsymbol{\varepsilon}' \in \mathbb{M}_s^n} \left\{ \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}' - W_0(\boldsymbol{\varepsilon}', \alpha) \right\} = \frac{1}{2}S(\alpha)\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - w_1\alpha.
\tag{5}
$$

It follows that

$$
\mathbb{E}^*(\alpha) = \left\{ \boldsymbol{\sigma} \in \mathbb{M}_s^n : \frac{1}{2}S'(\alpha)\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} \le w_1 \right\} = \left\{ \boldsymbol{\sigma} \in \mathbb{M}_s^n : \frac{\partial W_0^*}{\partial\alpha}(\boldsymbol{\sigma}, \alpha) \le 0 \right\}.
\tag{6}
$$

Definition 1 One says that the material behavior is *strain-hardening* when $\alpha \mapsto \mathbb{E}(\alpha)$ is increasing, *stress-hardening* when $\alpha \mapsto \mathbb{E}^*(\alpha)$ is increasing and *stress-softening* when $\alpha \mapsto \mathbb{E}^*(\alpha)$ is decreasing.

These variations of the elastic domains must be understood in the sense of inclusion between sets. For instance, the behavior *strain hardening* means that if α and α' are such that $\alpha < \alpha'$ then $\mathbb{E}(\alpha) \subset \mathbb{E}(\alpha')$. These intrinsic material properties can be deduced from convexity properties of the constitutive function $\alpha \mapsto A(\alpha)$ as it can be seen in the following

Proposition 1 *The behavior is* strain-hardening *when* $\alpha \mapsto A'(\alpha)$ *is an increasing function with respect to* α,

$$
A''(\alpha) > 0.
\tag{7}
$$

The behavior is stress-hardening (resp., *stress-softening*) *when* $\alpha \mapsto S'(\alpha)$ *is decreasing with respect to* α,

$$S''(\alpha) < (\text{resp.} >) 0. \tag{8}$$

Hypothesis 2 *From now on, we only consider brittle materials with a* strain-hardening *behavior.*

2.3 The Stability Criterion

For rate-independent material behaviors, *stability* is a state property of a body at a given time and not a property of the time-evolution problem. Accordingly, this latter is defined by comparing the total energy of a body in the considered state with the total energy that this body should have after a state perturbation, the loading being fixed. More specifically, let us consider a homogeneous body whose reference configuration is the open connected bounded set $\Omega \subset \mathbb{R}^n$. This body is made of a non-local damaging material characterized by the state function (1). We suppose that the set of kinematically admissible displacement is of the form $\mathcal{C}_\mathbf{U} = \mathbf{U} + \mathcal{C}_0$ where $\mathbf{U}$ is a given displacement field and $\mathcal{C}_0$ is a vectorial space. The body is also subjected to a system of external forces whose potential is the linear form $\mathcal{W}_e : \mathcal{C}_\mathbf{U} \mapsto \mathbb{R}$. Considering only damage fields without failure, the convex set of admissible damage field is given by

$$\mathcal{D} = \left\{ \alpha \in H^1(\Omega, \mathbb{R}) : \alpha \geq 0, \sup_\Omega \alpha < \alpha_m \right\},$$

and the convex cone of admissible damage direction fields by

$$\mathcal{D}^+ = \left\{ \alpha \in H^1(\Omega, \mathbb{R}) : \alpha \geq 0 \right\}.$$

The energy of the body in the admissible state $(\mathbf{u}, \alpha) \in \mathcal{C}_U \times \mathcal{D}$ reads as

$$\mathcal{P}(\mathbf{u}, \alpha) = \int_\Omega W\big(\boldsymbol{\varepsilon}(\mathbf{u})(x), \alpha(x), \nabla\alpha(x)\big)dx - \mathcal{W}_e(u), \tag{9}$$

where $\boldsymbol{\varepsilon}(\mathbf{u})$ stands for the symmetric part of the gradient of $\mathbf{u}$. The stability property is then given by

Definition 2 The state $(\mathbf{u}, \alpha) \in \mathcal{C}_\mathbf{U} \times \mathcal{D}$ is stable with respect to a perturbation in the direction $(\mathbf{u}^*, \alpha^*) \in \mathcal{C}_0 \times \mathcal{D}^+$, or, shortly, *stable in the direction* $(\mathbf{u}^*, \alpha^*)$, if there exists $\bar{h} > 0$ such that for all $h \in [0, \bar{h}]$

$$\mathcal{P}\big(\mathbf{u} + h\mathbf{u}^*, \alpha + h\alpha^*\big) \geq \mathcal{P}(\mathbf{u}, \alpha). \tag{10}$$

If $(\mathbf{u}, \alpha)$ is stable in all directions of $\mathcal{C}_0 \times \mathcal{D}^+$, then $(\mathbf{u}, \alpha)$ is *directionally stable*.

Note that the stability criterion only requires to compare the energy of the tested state $(\mathbf{u}, \alpha)$ with the energy of more damaged states. Indeed, by virtue of the irreversibility of damage, the structure can only evolve to more damaged configurations than its actual state.

3 The Question of the Stability of Homogeneous States

3.1 Homogeneous States and the Different Types of Control of the Boundary Conditions

In order to identify the constitutive function $\alpha \mapsto A(\alpha)$ and the material constant w_1, one often uses uni- or multi-axial tests on cylindrical samples by prescribing boundary conditions compatible with a spatially homogeneous response. Then, the corresponding state of the sample is the homogeneous one $(\mathbf{u}_0, \alpha_0) = (\varepsilon_0 x, \alpha_0)$ with $(\varepsilon_0, \alpha_0) \in \mathbb{M}_s^n \times [0, \alpha_m)$. (Note that we assume that the gradient of $\mathbf{u}_0$ is symmetric. We could add a skew-symmetric part which would correspond to a overall rotation, all the analysis would remain unchanged.) However, the observability of such a state is possible only if this state is directionally stable. The goal of the present section is to study under which conditions this stability property holds. We first remark that the stability of the state generally depends on how one controls the boundary conditions: the same homogeneous state can be stable under displacement controlled (hard device) but unstable under force controlled (soft device). Accordingly, we will discriminate between several types of boundary conditions:

1. *Soft device*. The boundary $\partial\Omega$ of the body is subjected to homogeneous surfaces forces $\boldsymbol{\sigma}_0 \mathbf{n}$ where $\boldsymbol{\sigma}_0 \in \mathbb{M}_s^n$ is a given stress tensor. The associated homogeneous state of the body is then given by $\alpha(\mathbf{x}) = \alpha_0$ and $\mathbf{u}(\mathbf{x}) = S(\alpha_0)\boldsymbol{\sigma}_0 \mathbf{x}$. The corresponding stress state is $\boldsymbol{\sigma}_0$ while the set $\mathcal{C}_0$ of kinematically admissible directions of perturbation and the potential $\mathcal{W}_e$ of the given external forces are

$$\mathcal{C}_0 = H^1(\Omega, \mathbb{R}^n), \qquad \mathcal{W}_e(\mathbf{v}) = \int_{\partial\Omega} \boldsymbol{\sigma}_0 \mathbf{n} \cdot \mathbf{v} \, dS. \tag{11}$$

2. *Hard device*. The boundary $\partial\Omega$ of the body is subjected to displacements $\varepsilon_0 \mathbf{x}$ where $\varepsilon_0 \in \mathbb{M}_s^n$ is a given strain tensor. The associated homogeneous state of the body is then given by $\alpha(\mathbf{x}) = \alpha_0$ and $\mathbf{u}(\mathbf{x}) = \varepsilon_0 \mathbf{x}$. The corresponding stress state is $\boldsymbol{\sigma}_0 = A(\alpha_0)\varepsilon_0$ while the set $\mathcal{C}_0$ of kinematically admissible directions of perturbation and the potential $\mathcal{W}_e$ of the given external forces read

$$\mathcal{C}_0 = H_0^1(\Omega, \mathbb{R}^n), \qquad \mathcal{W}_e(\mathbf{v}) = 0 \tag{12}$$

where $H_0^1(\Omega)$ denotes the subspace of $H^1(\Omega)$ made of fields whose trace over $\partial\Omega$ vanishes.

3. *Uniaxial tensile test*. The body is the cylinder $\Omega = (0, L) \times \Sigma$ with Σ its cross-section. The lateral boundary $(0, L) \times \partial\Sigma$ is free ($\boldsymbol{\sigma}\mathbf{n} = \mathbf{0}$). The ends $x_1 = 0$ and $x_1 = L$ are under mixed boundary conditions: there is no shear ($\sigma_{21} = \sigma_{31} = 0$) and the normal displacement is fixed to a uniform value on each end ($u_1|_{x_1=0} = 0$, $u_1|_{x_1=L} = \varepsilon L$). Then, the associated homogeneous state of the body is given by $\alpha(\mathbf{x}) = \alpha_0$ and $\mathbf{u}(\mathbf{x}) = \varepsilon_0 \mathbf{x}$ with $\varepsilon_0 = \varepsilon \mathbf{e}_1 \otimes \mathbf{e}_1 - \nu(\alpha_0)\varepsilon(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3)$. The corresponding stress state is $\boldsymbol{\sigma}_0 = E(\alpha_0)\varepsilon \mathbf{e}_1 \otimes \mathbf{e}_1$ while the set $\mathcal{C}_0$ of kinematically admissible directions of perturbation and the potential $\mathcal{W}_e$ of the given external forces are

$$\mathcal{C}_0 = \left\{ \mathbf{v} \in H^1(\Omega, \mathbb{R}^n) : v_1 = 0 \text{ on } \{0, L\} \times \Sigma \right\}, \qquad \mathcal{W}_e(\mathbf{v}) = 0. \tag{13}$$

3.2 Non Damaging and Damaging Homogeneous States

Let us consider a homogeneous state $(\mathbf{u}_0, \alpha_0) = (\varepsilon_0 x, \alpha_0)$ with $(\varepsilon_0, \alpha_0) \in \mathbb{M}_s^n \times [0, \alpha_m)$. Let $(\mathbf{v}, \beta) \in \mathcal{C}_0 \times \mathcal{D}^+$ be an admissible direction and h be a small positive number. Expanding

the total energy of the body in the state $(\mathbf{u}_0 + h\upsilon, \alpha_0 + h\beta)$ with respect to h up to the second order leads to

$$\mathcal{P}(\mathbf{u}_0 + h\mathbf{v}, \alpha_0 + h\beta) = \mathcal{P}(\mathbf{u}_0, \alpha_0) + h\mathcal{P}'(\mathbf{u}_0, \alpha_0)(\mathbf{v}, \beta)$$
$$+ \frac{h^2}{2} \mathcal{P}''(\mathbf{u}_0, \alpha_0)(\mathbf{v}, \beta) + o(h^2). \quad (14)$$

In (14), the first order derivative is given by

$$\mathcal{P}'(\mathbf{u}_0, \alpha_0)(\mathbf{v}, \beta) = \int_{\Omega} \boldsymbol{\sigma}_0 \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dx - \mathcal{W}_e(\mathbf{v}) + \int_{\Omega} \left(w_1 - \frac{1}{2} \mathsf{S}'(\alpha_0) \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0 \right) \beta dx, \quad (15)$$

where we used the identity $\mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0 = -\mathsf{A}'(\alpha_0)\boldsymbol{\varepsilon}_0 \cdot \boldsymbol{\varepsilon}_0$ with $\boldsymbol{\sigma}_0 = \mathsf{A}(\alpha_0)\boldsymbol{\varepsilon}_0$. In (15) there is no term involving the non local character of the damage model, *i.e.*, no term containing $\ell(\alpha_0)$ or $\ell'(\alpha_0)$ as the state α_0 is assumed to be spatially homogeneous and hence $\nabla\alpha_0 = \mathbf{0}$. The second order derivative reads as

$$\mathcal{P}''(\mathbf{u}_0, \alpha_0)(\mathbf{v}, \beta) = \int_{\Omega} \left(\mathsf{A}(\alpha_0)\boldsymbol{\varepsilon}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) + w_1 \ell(\alpha_0)^2 \nabla\beta \cdot \nabla\beta + 2\mathsf{A}'(\alpha_0)\boldsymbol{\varepsilon}_0 \cdot \boldsymbol{\varepsilon}(\mathbf{v})\beta \right) dx$$
$$+ \int_{\Omega} \frac{1}{2} \mathsf{A}''(\alpha_0)\boldsymbol{\varepsilon}_0 \cdot \boldsymbol{\varepsilon}_0 \beta^2 dx. \quad (16)$$

Note that it contains terms with the gradient of the direction of damage. Then, a necessary condition for $(\mathbf{u}_0, \alpha_0)$ to be directionally stable is

$$0 \leq \lim_{h \to 0} \frac{1}{h} \left(\mathcal{P}(\mathbf{u}_0 + h\mathbf{v}, \alpha_0 + h\beta) - \mathcal{P}(\mathbf{u}_0, \alpha_0) \right)$$
$$= \mathcal{P}'(\mathbf{u}_0, \alpha_0)(\mathbf{v}, \beta), \quad \forall (\mathbf{v}, \beta) \in \mathcal{C}_0 \times \mathcal{D}^+. \quad (17)$$

Taking $\beta = 0$ in (17) and recalling (15) leads to the variational formulation of the mechanical equilibrium

$$\int_{\Omega} \boldsymbol{\sigma}_0 \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dx = \mathcal{W}_e(\mathbf{v}), \quad \forall \upsilon \in \mathcal{C}_0, \quad (18)$$

which is automatically satisfied by virtue of the definitions of $\mathcal{W}_e$ and $\mathcal{C}_0$ and given the fact that the stress field is uniform. Accordingly, the first order stability condition (17) becomes

$$\mathcal{P}'(\mathbf{u}_0, \alpha_0)(\mathbf{v}, \beta) = \left(w_1 - \frac{1}{2} \mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0 \right) \int_{\Omega} \beta dx \geq 0, \quad (19)$$

where the inequality must hold for any field $\beta \geq 0$. Therefore, the homogeneous state is stable only if the stress satisfies the damage criterion

$$\mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0 \leq 2w_1. \quad (20)$$

This inequality can also be written in terms of the strain tensor and then reads as

$$-\mathsf{A}'(\alpha_0)\boldsymbol{\varepsilon}_0 \cdot \boldsymbol{\varepsilon}_0 \leq 2w_1. \quad (21)$$

A state such that the inequality in (20) or (21) is strict should correspond in an evolution problem to a state where the damage yield criterion is not reached and then for which the

damage rate $\dot{\alpha}$ should vanish. We will call such a state a *non damaging* state. On the other hand, a state such that (20) or (21) is an equality should correspond in an evolution problem to a state where the damage yield criterion is reached. In this case, the damage rate could be positive. We will call such a state a *damaging* state. For further references, this terminology is recalled in the following definition

Definition 3 A homogeneous state $(\boldsymbol{\varepsilon}_0 \mathbf{x}, \alpha_0)$ such that $-\mathsf{A}'(\alpha_0)\boldsymbol{\varepsilon}_0 \cdot \boldsymbol{\varepsilon}_0 < 2w_1$ is called a *non damaging* state, whereas a homogeneous state such that $-\mathsf{A}'(\alpha_0)\boldsymbol{\varepsilon}_0 \cdot \boldsymbol{\varepsilon}_0 = 2w_1$ is called a *damaging* state.

Let us first study the stability of non damaging states.

3.3 Stability of the Non Damaging States

In the case of a non damaging state, since $\mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0 < 2w_1$, the first term in the expansion of the perturbed energy (14) vanishes if and only if the damage direction is $\beta = 0$ (everywhere in Ω). When $\beta \neq 0$ (somewhere in Ω), then the first order term is positive and the state is stable in this direction of perturbation. On the other hand, when $\beta = 0$, then the first order term vanishes and the stability in this direction depends on the sign of the second order term. Calculating the second order derivative for $\beta = 0$, we find $\mathcal{P}''(\mathbf{u}_0, \alpha_0)(\mathbf{v}, 0) = \int_{\Omega} \mathsf{A}(\alpha_0)\boldsymbol{\varepsilon}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\mathbf{v})dx$. Accordingly, the second order term in the expansion (14) is non negative. Moreover, it vanishes if and only if $\boldsymbol{\varepsilon}(\mathbf{v}) = \mathbf{0}$, *i.e.*, if and only if the perturbation corresponds to a rigid motion. Therefore, one can conclude

Proposition 2 *Any non damaging state is directionally stable, independently of the type of boundary conditions and the hardening properties of the material.*

3.4 Damaging State for Stress-Hardening Behavior

Let us consider a damaging state. Since $-\mathsf{A}'(\alpha_0)\boldsymbol{\varepsilon}_0 \cdot \boldsymbol{\varepsilon}_0 = 2w_1$, the first order term in (14) vanishes. Moreover, by virtue of the strain hardening condition (7), with the strain tensor $\boldsymbol{\varepsilon}_0 \in \mathbb{M}_s^n$ is associated a *unique* $\alpha_0 \in [0, \alpha_m)$ such that (21) holds. In other words, the damage state is given by the strain state. The stability of this homogeneous state depends on the sign of the second derivative of the energy. Introducing the stress state $\boldsymbol{\sigma}_0$ rather than the strain state $\boldsymbol{\varepsilon}_0$, $\mathcal{P}''(\mathbf{u}_0, \alpha_0)$ can read as

$$\mathcal{P}''(\mathbf{u}_0, \alpha_0)(\mathbf{v}, \beta) = \int_{\Omega} \mathsf{A}(\alpha_0)\big(\boldsymbol{\varepsilon}(\mathbf{v}) - \beta \mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0\big) \cdot \big(\boldsymbol{\varepsilon}(\mathbf{v}) - \beta \mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0\big)dx$$

$$+ w_1 \ell(\alpha_0)^2 \int_{\Omega} \nabla\beta \cdot \nabla\beta dx - \frac{1}{2}\mathsf{S}''(\alpha_0)\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0 \int_{\Omega} \beta^2 dx, \quad (22)$$

where we used the identity

$$\mathsf{S}''(\alpha_0)\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0 = 2\mathsf{A}(\alpha_0)\mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0 \cdot \mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0 - \mathsf{A}''(\alpha_0)\boldsymbol{\varepsilon}_0 \cdot \boldsymbol{\varepsilon}_0. \quad (23)$$

Accordingly, if the material behavior is stress-hardening, then $\mathsf{S}''(\alpha_0) < 0$ and the second order derivative of the total energy in (22) is the sum of three non negative terms. Moreover, the three terms vanish simultaneously if and only if $\beta = 0$ and $\boldsymbol{\varepsilon}(\mathbf{v}) = \mathbf{0}$. Therefore, we can conclude that

Proposition 3 *Any homogeneous state is directionally stable when the material behavior is stress-hardening, independently of the type of boundary conditions.*

It remains to study the stability of damaging states for stress-softening material and hence we adopt the

Hypothesis 3 *Throughout the remainder of the paper, the material behavior is with stress softening and the homogeneous state is a damaging state.*

In such a case, the type of boundary conditions becomes essential. Let us first remark that damaging states are stable in all directions $(\mathbf{v}, 0)$ where $\mathbf{v} \in \mathcal{C}_0$. Indeed, for such directions, the second derivative reads as $\mathcal{P}''(\mathbf{u}_0, \alpha_0)(\mathbf{v}, 0) = \int_\Omega \mathsf{A}(\alpha_0)\boldsymbol{\varepsilon}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\mathbf{v})dx$ and hence is positive for any $\mathbf{v}$ which is not a rigid motion. Consequently, we only have to consider directions of perturbation such that $\beta \neq 0$. Let us now remark that if the material behavior is stress-softening, then $\mathsf{S}''(\alpha_0) > 0$. Hence, to find the sign of the second order derivative is equivalent to compare with 1 the infimum of the following Rayleigh ratio over $\mathcal{C}_0 \times (\mathcal{D}^+ \setminus \{0\})$:

$$\mathcal{R}(\mathbf{v}, \beta)$$
$$= \frac{\int_\Omega w_1 \ell(\alpha_0)^2 \nabla\beta \cdot \nabla\beta dx + \int_\Omega \mathsf{A}(\alpha_0)(\boldsymbol{\varepsilon}(\mathbf{v}) - \beta\mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0) \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \beta\mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0)dx}{\frac{1}{2}\mathsf{S}''(\alpha_0)\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0 \int_\Omega \beta^2 dx}.$$
$$(24)$$

More specifically, the state will be directionally stable if (*resp.*, only if) $\inf_{\mathcal{C}_0 \times (\mathcal{D}^+ \setminus \{0\})} \mathcal{R} >$ (resp. $\geq$) 1. It is easy to conclude when the forces are controlled on the whole boundary, whereas the other cases are more subtle and require a more detailed analysis.

3.5 Case of a Soft Device

The surface forces being $\boldsymbol{\sigma}_0\mathbf{n}$, the homogeneous state is $(\mathsf{S}(\alpha_0)\boldsymbol{\sigma}_0\mathbf{x}, \alpha_0)$ with α_0 the unique solution of $\mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0 = w_1$. The set of admissible displacement directions is $\mathcal{C}_0 = H^1(\Omega, \mathbb{R}^3)$. Therefore, considering the direction $(\mathbf{v}, \beta) = (\mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0\mathbf{x}, 1)$ which belongs to $\mathcal{C}_0 \times \mathcal{D}^+$, one immediately obtains that the Rayleigh ratio vanishes:

$$\mathcal{R}\big(\mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0\mathbf{x}, 1\big) = 0. \qquad (25)$$

We can then conclude that

Proposition 4 *Any damaging state in the case of a stress-softening material is unstable when the boundary conditions are prescribed by a soft device.*

4 Size Effects on the Stability of the Damaging States in the Case of a Hard Device

4.1 Setting of the Problem

At each point $\mathbf{x}$ of the boundary $\partial\Omega$, the displacement is prescribed to $\boldsymbol{\varepsilon}_0\mathbf{x}$. The homogeneous state is $(\boldsymbol{\varepsilon}_0\mathbf{x}, \alpha_0)$ with α_0 given by (21) and the stress state is $\boldsymbol{\sigma}_0 = \mathsf{A}(\alpha_0)\boldsymbol{\varepsilon}_0$. Since the displacements at the boundary are prescribed, the space of perturbation directions of the

displacements is $H_0^1(\Omega, \mathbb{R}^n)$ and it does not contain the direction $\mathsf{S}(\alpha_0)\boldsymbol{\sigma}_0\mathbf{x}$. Therefore, we can not conclude as easily as in the case of a control by a soft device. In fact, we show that the stability of the homogeneous state essentially depends on the size of the domain.

For studying these size effects, we consider families of domains such that all the members of a given family share the same shape and only differ by their size. More precisely, for a given domain Ω_1 we associate the family of homothetic domains $\{\Omega_L = L\Omega_1\}_{L>0}$. To compare the Rayleigh ratio of the different members of the family, it is more convenient to work with spaces of perturbation directions that are independent of the size of the domain. From this perspective, one performs the change of coordinates $\mathbf{x} \mapsto \mathbf{y} = \mathbf{x}/L$ which maps the domain Ω_L into the domain Ω_1. In the same way, one transports the directions of perturbation $(\mathbf{v}^L, \beta^L)$ of $H_0^1(\Omega_L, \mathbb{R}^n) \times H^1(\Omega_L, \mathbb{R}^+)$ into $(\mathbf{v}, \beta)$ of $\mathcal{C}_0 \times \mathcal{D}^+ = H_0^1(\Omega_1, \mathbb{R}^n) \times H^1(\Omega_1, \mathbb{R}^+)$ by using

$$\mathbf{v}^L(\mathbf{x}) = L\mathbf{v}(\mathbf{y}), \qquad \beta^L(\mathbf{x}) = \beta(\mathbf{y}). \tag{26}$$

Accordingly, the Rayleigh ratio associated with the domain Ω_L is denoted by $\mathcal{R}_L$ and is defined on the fixed space $\mathcal{C}_0 \times (\mathcal{D}^+ \setminus \{0\})$ by

$$\mathcal{R}_L(\mathbf{v}, \beta) = \frac{w_1 \frac{\ell_0^2}{L^2} \int_{\Omega_1} \nabla\beta \cdot \nabla\beta \, dy + \int_{\Omega_1} \mathsf{A}_0(\boldsymbol{\varepsilon}(\mathbf{v}) - \beta\mathbf{e}_0) \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \beta\mathbf{e}_0) dy}{\frac{1}{2}\mathsf{S}_0''\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0 \int_{\Omega_1} \beta^2 dy}, \tag{27}$$

where we have used the condensed notations

$$\begin{aligned} \mathsf{A}_0 = \mathsf{A}(\alpha_0), \qquad \mathsf{S}_0'' = \mathsf{S}''(\alpha_0), \qquad \boldsymbol{\sigma}_0 = \mathsf{A}(\alpha_0)\boldsymbol{\varepsilon}_0, \\ \ell_0 = \ell(\alpha_0), \qquad \mathbf{e}_0 = \mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0. \end{aligned} \tag{28}$$

Note that the size of the domain appears explicitly in the definition of the Rayleigh ratio and that for a given direction $(\mathbf{v}, \beta)$ the Rayleigh ratio is a decreasing function of L. Let us call

$$\rho_L(\boldsymbol{\varepsilon}_0, \Omega_1) = \inf_{\mathcal{C}_0 \times (\mathcal{D}^+ \setminus \{0\})} \mathcal{R}_L \tag{29}$$

and let us first prove that the infimum is reached and positive.

It is clear that the infimum is non-negative and finite. Let $(\mathbf{v}_n, \beta_n)$ be a minimizing sequence. Owing to the homogeneity of the ratio, we can choose β_n such that $\int_{\Omega_1} \beta_n^2 dx = 1$. Since the numerator converges to the infimum, β_n and $\mathbf{v}_n$ are bounded in $H^1(\Omega_1)$. Hence, there exists a subsequence that weakly converges in $H^1(\Omega_1)$ and strongly in $L^2(\Omega_1)$ to $(\mathbf{v}_\infty, \beta_\infty)$ in $\mathcal{C}_0 \times \mathcal{D}^+$ (which is weakly closed in $H^1(\Omega_1, \mathbb{R}^n) \times H^1(\Omega_1)$). We deduce that $\int_{\Omega_1} \beta_\infty^2 dx = 1$. Then, by virtue of the weakly lower semi-continuity of the numerator of $\mathcal{R}_L$, we obtain

$$\mathcal{R}_L(\mathbf{v}_\infty, \beta_\infty) \leq \lim_{n\to\infty} \mathcal{R}_L(\mathbf{v}_n, \beta_n) = \inf_{\mathcal{C}_0 \times (\mathcal{D}^+ \setminus \{0\})} \mathcal{R}_L.$$

Therefore the infimum is a minimum. If the minimum was 0, β_∞ should be a positive constant and $\boldsymbol{\varepsilon}(\mathbf{v}_\infty)$ should be equal to $\beta_\infty \mathbf{e}_0$. Integrating over Ω_1 and taking into account that $\mathbf{v}_\infty = \mathbf{0}$ on $\partial\Omega_1$ we should get $\mathbf{0} = \mathbf{e}_0 \int_{\Omega_1} \beta_\infty dy$ which is impossible since $\mathbf{e}_0 \neq \mathbf{0}$ and hence $\rho_L(\boldsymbol{\varepsilon}_0, \Omega_1) > 0$.

For a given domain Ω_1 and a given strain tensor $\boldsymbol{\varepsilon}_0$, it is clear from (27) that $L \mapsto \rho_L(\boldsymbol{\varepsilon}_0, \Omega_1)$ is a decreasing function of L. Therefore, the homogeneous states have more chance to be stable for small domains than for large ones. Let us first study the stability for small domains before to consider the full range of sizes.

4.2 Case of Small Domains

Let us prove the following fundamental result:

Proposition 5 *Since* $\lim_{L \to 0} \rho_L(\boldsymbol{\varepsilon}_0, \Omega_1) = \frac{\mathsf{A}(\alpha_0)\mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0 \cdot \mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0}{\frac{1}{2}\mathsf{S}''(\alpha_0)\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0} > 1$, *there exists for each family of homothetic domains a positive* (*possibly infinite*) *critical size, say* $L_c(\boldsymbol{\varepsilon}_0, \Omega_1)$, *below which the damaging state* $(\boldsymbol{\varepsilon}_0 \mathbf{x}, \alpha_0)$ *is directionally stable.*

Proof To simplify the notation, we do not make explicit the dependence on $\boldsymbol{\varepsilon}_0$ and Ω_1 of any quantity. Let us call $(\mathbf{v}_L, \beta_L)$ the minimizer of $\mathcal{R}_L$ and let us use the condensed notation (28). Then the following relations hold:

$$\rho_L = \mathcal{R}_L(\mathbf{v}_L, \beta_L), \tag{30}$$

$$1 = \int_{\Omega_1} \beta_L^2 dy, \tag{31}$$

$$0 = \int_{\Omega_1} \mathsf{A}_0\big(\boldsymbol{\varepsilon}(\mathbf{v}_L) - \beta_L \mathbf{e}_0\big) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dy, \quad \forall \mathbf{v} \in \mathcal{C}_0, \tag{32}$$

where the last equation is the stationary condition: $\mathcal{R}'_L(\mathbf{v}_L, \beta_L)(\mathbf{v}, 0) = 0$, $\forall \mathbf{v} \in \mathcal{C}_0$. Taking $\mathbf{v} = \mathbf{v}_L$ in (32) and inserting into (30) leads to

$$\frac{1}{2}\mathsf{S}''_0 \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0 \, \rho_L = w_1 \frac{\ell_0^2}{L^2} \int_{\Omega_1} \nabla \beta_L \cdot \nabla \beta_L dy + \mathsf{A}_0 \mathbf{e}_0 \cdot \mathbf{e}_0 - \int_{\Omega_1} \beta_L \mathsf{A}_0 \mathbf{e}_0 \cdot \boldsymbol{\varepsilon}(\mathbf{v}_L) dy. \tag{33}$$

By virtue of the minimality of ρ_L, we have for any $L > 0$

$$0 < \rho_L \leq \mathcal{R}_L(0, 1) = \frac{\mathsf{A}_0 \mathbf{e}_0 \cdot \mathbf{e}_0}{\frac{1}{2}\mathsf{S}''_0 \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0}, \tag{34}$$

hence the sequence $L \mapsto \rho_L$ is bounded. Since ρ_L is a decreasing function of L, we deduce that the sequence $L \mapsto \rho_L$ converges to some $\rho_0 \geq 0$ when L goes to 0. From (34), we deduce that the sequences $L \mapsto \mathbf{v}_L$ and $L \mapsto \beta_L$ are bounded in $H^1(\Omega_1)$. Therefore, (up to extracting a subsequence) β_L converges weakly in $H^1(\Omega_1)$, strongly in $L^2(\Omega_1)$ and almost everywhere to some β_*. Similarly, $\mathbf{v}_L$ converges weakly in $H^1(\Omega_1, \mathbb{R}^n)$ and strongly in $L^2(\Omega_1, \mathbb{R}^n)$ to some $\mathbf{v}_*$. Passing to the limit in (31) and (32) gives

$$1 = \int_{\Omega_1} \beta_*^2 dy, \tag{35}$$

$$0 = \int_{\Omega_1} \mathsf{A}_0\big(\boldsymbol{\varepsilon}(\mathbf{v}_*) - \beta_* \mathbf{e}_0\big) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dy, \quad \forall v \in \mathcal{C}_0. \tag{36}$$

Moreover, since for any $L > 0$, β_L is non negative *a.e.*, its limit β_* is also non-negative *a.e.* The inequality (34) shows in addition that $\|\nabla \beta_L\|_{L^2(\Omega_1)} \leq CL$ where C is a positive constant (independent of L). By virtue of the lower semi-continuity of the norm, we get $\|\nabla \beta_*\|_{L^2(\Omega_1)} = 0$ which means that β_* is a positive constant over Ω_1. Accordingly, (36) reads

$$\int_{\Omega_1} \mathsf{A}_0 \boldsymbol{\varepsilon}(\mathbf{v}_*) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dy = \beta_* \mathsf{A}_0 \mathbf{e}_0 \cdot \int_{\Omega_1} \boldsymbol{\varepsilon}(\mathbf{v}) dy = 0, \quad \forall \mathbf{v} \in \mathcal{C}_0. \tag{37}$$

Taking $\mathbf{v} = \mathbf{v}_*$ and using the positivity of A_0, we deduce that $\boldsymbol{\varepsilon}(\mathbf{v}_*) = \mathbf{0}$ and hence $\mathbf{v}_* = \mathbf{0}$. Using (33) and (34) gives the following bounds for ρ_L

$$\mathsf{A}_0\mathbf{e}_0 \cdot \mathbf{e}_0 - \int_{\Omega_1} \beta_L \mathsf{A}_0\mathbf{e}_0 \cdot \boldsymbol{\varepsilon}(\mathbf{v}_L)dy \leq \frac{1}{2}\mathsf{S}_0''\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0\, \rho_L \leq \mathsf{A}_0\mathbf{e}_0 \cdot \mathbf{e}_0. \tag{38}$$

Since $\boldsymbol{\varepsilon}(\mathbf{v}_L)$ is bounded in $L^2(\Omega_1)$, $\boldsymbol{\varepsilon}(\mathbf{v}_L)$ converges weakly to $\boldsymbol{\varepsilon}(\mathbf{v}_*) = \mathbf{0}$ in $L^2(\Omega_1)$ and β_L converges strongly to β_* in $L^2(\Omega_1)$. Hence, we have

$$\lim_{L \to 0} \int_{\Omega_1} \beta_L \mathsf{A}_0\mathbf{e}_0 \cdot \boldsymbol{\varepsilon}(\mathbf{v}_L)dy = 0.$$

Inserting into (38) gives the desired result

$$\lim_{L \to 0} \rho_L = \rho_0 := \frac{\mathsf{A}_0\mathbf{e}_0 \cdot \mathbf{e}_0}{\frac{1}{2}\mathsf{S}_0''\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0}.$$

Note that this limit is independent of the shape of the family of domains. Now, making use of the identity (23) we deduce from the strain hardening assumption $\frac{1}{2}\mathsf{A}''(\alpha_0)\boldsymbol{\varepsilon}_0 \cdot \boldsymbol{\varepsilon}_0 > 0$ that $\rho_0 > 1$. As $L \mapsto \rho_L$ is a decreasing function, we conclude that there exists a critical size $L_c > 0$ such that the state is stable if $L < L_c$ and unstable if $L > L_c$. Of course, if $L_c = +\infty$, then the state is stable independently of the size of the domain. $\qquad\square$

4.3 Case of Large Domains

We have shown in the previous subsection that a damaging state is directionally stable for sufficiently "small" structures. It remains to see whether this stability property holds for the full range of sizes or whether there exists a finite critical length beyond which the homogeneous state is no more stable. To answer this question it is sufficient to determine the limit of ρ_L when L goes to ∞. Indeed, by virtue of Proposition 5, we have

Corollary 1 *Depending on whether* $\lim_{L \to \infty} \rho_L(\boldsymbol{\varepsilon}_0, \Omega_1)$ *is less or greater than* 1, *we are in one of the two following situations*:

1. *If* $\lim_{L \to \infty} \rho_L(\boldsymbol{\varepsilon}_0, \Omega_1) \geq 1$, *then the damaging homogeneous state is directionally stable, independently of the size of the domain*;
2. *If* $\lim_{L \to \infty} \rho_L(\boldsymbol{\varepsilon}_0, \Omega_1) < 1$, *then the damaging homogeneous state is directionally stable when the size of the domain is less than a* (*positive and finite*) *critical value* $L_c(\boldsymbol{\varepsilon}_0, \Omega_1)$ *and unstable otherwise*.

Let us determine the asymptotic behavior of the Rayleigh ratio when the characteristic size L of the structure goes to $+\infty$. Note first that the limit exists because ρ_L is a decreasing function of L bounded from below by 0. By definition, we have

$$\rho_L \leq \mathcal{R}_L(\mathbf{v}, \beta), \quad \forall(\mathbf{v}, \beta) \in \mathcal{C}_0 \times \mathcal{D}^+. \tag{39}$$

For a fixed direction $(\mathbf{v}, \beta)$, passing to the limit when $L \to +\infty$, we obtain

$$\rho_\infty \leq \mathcal{R}_\infty(\mathbf{v}, \beta), \quad \forall(\mathbf{v}, \beta) \in \mathcal{C}_0 \times \mathcal{D}^+, \tag{40}$$

where $\mathcal{R}_\infty$ is defined as the "limit" Rayleigh ratio

$$\mathcal{R}_\infty(\mathbf{v}, \beta) = \frac{\int_{\Omega_1} \mathsf{A}_0(\boldsymbol{\varepsilon}(\mathbf{v}) - \beta\mathbf{e}_0) \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \beta\mathbf{e}_0)dy}{\frac{1}{2}\mathsf{S}_0''\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0 \int_{\Omega_1} \beta^2 dy}, \tag{41}$$

which involves no more gradient damage term. By taking the infimum over $\mathcal{C}_0 \times \mathcal{D}^+$ in (40), we deduce that $\rho_\infty \leq \inf_{\mathcal{C}_0 \times \mathcal{D}^+} \mathcal{R}_\infty$. Moreover, since the contribution of gradient damage terms to the Rayleigh ratio is non negative, we have conversely

$$\rho_L = \mathcal{R}_L(\mathbf{v}_L, \beta_L) \geq \mathcal{R}_\infty(\mathbf{v}_L, \beta_L) \geq \inf_{\mathcal{C}_0 \times \mathcal{D}^+} \mathcal{R}_\infty \tag{42}$$

and hence finally

$$\rho_\infty = \inf_{\mathcal{C}_0 \times \mathcal{D}^+} \mathcal{R}_\infty. \tag{43}$$

Note that we are not ensured that the infimum is reached and we have to consider minimizing sequences. The value of ρ_∞ can be derived by following the method proposed by [12] to calculate the relaxation of a double-well elastic potential. This requires the introduction of some preliminary definitions.

Let $\mathbb{S}^n$ be the unit sphere of $\mathbb{R}^n$, *i.e.* $\mathbb{S}^n = \{\mathbf{k} \in \mathbb{R}^n : |\mathbf{k}| = 1\}$. With $\mathbf{k} \in \mathbb{S}^n$ we associate the n-dimension subspace $V(\mathbf{k})$ of $\mathbb{M}_s^n$ by

$$V(\mathbf{k}) = \left\{ \mathbf{k} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{k} : \mathbf{v} \in \mathbb{R}^n \right\}. \tag{44}$$

Let $\delta(\boldsymbol{\varepsilon}_0)$ be the real number defined by

$$\delta(\boldsymbol{\varepsilon}_0) = \max_{\{\mathbf{k} \,:\, |\mathbf{k}|=1\}} \mathsf{A}_0\boldsymbol{\xi}(\mathbf{k}) \cdot \boldsymbol{\xi}(\mathbf{k}), \tag{45}$$

where $\boldsymbol{\xi}(\mathbf{k}) \in V(\mathbf{k})$ is obtained by solving the following minimization problem (which depends only on $\boldsymbol{\varepsilon}_0$):

$$\boldsymbol{\xi}(\mathbf{k}) = \underset{\boldsymbol{\xi} \in V(\mathbf{k})}{\operatorname{argmin}} \mathsf{A}_0(\mathbf{e}_0 - \boldsymbol{\xi}) \cdot (\mathbf{e}_0 - \boldsymbol{\xi}). \tag{46}$$

Following the method proposed by Kohn [12], let us prove the

Lemma 1 *The infimum of $\mathcal{R}_\infty$ over $\mathcal{C}_0 \times (\mathcal{D}^+ \setminus \{0\})$ is independent of Ω_1 and is given by*

$$\rho_\infty(\boldsymbol{\varepsilon}_0) = \frac{\mathsf{A}(\alpha_0)\mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0 \cdot \mathsf{S}'(\alpha_0)\boldsymbol{\sigma}_0 - \delta(\boldsymbol{\varepsilon}_0)}{\frac{1}{2}\mathsf{S}''(\alpha_0)\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0}. \tag{47}$$

Proof By homogeneity, we can assume that β is such that $\int_{\Omega_1} \beta^2 dy = 1$. Accordingly, setting

$$\Delta(\mathbf{v}, \beta) = \int_{\Omega_1} \left(\mathsf{A}_0\boldsymbol{\varepsilon}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) - 2\beta\mathsf{A}_0\mathbf{e}_0 \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \right)dy$$

one has to prove that

$$-\delta(\boldsymbol{\varepsilon}_0) = \inf_{\substack{\beta \geq 0 \\ \int_{\Omega_1} \beta^2 dy=1}} \left\{ \min_{\mathbf{v} \in H_0^1(\Omega_1, \mathbb{R}^n)} \Delta(\mathbf{v}, \beta) \right\}. \tag{48}$$

It is possible to prove directly that the infimum does not depend on Ω_1 and hence to take for Ω_1 the cube $(0, 2\pi)^n$. Moreover, the Dirichlet conditions for $\mathbf{v}$ can be replaced by periodic conditions without changing the infimum, see [12, Lemma 2.1]. By density, the infimum can be approached by piecewise constant damage fields β. Accordingly, we consider β such that

$$\beta(\mathbf{y}) = \sum_{i=1}^N \beta_i \chi_i(\mathbf{y}), \quad \beta_i \geq 0, \chi_i(\mathbf{y}) \in \{0, 1\}, \quad \chi_i(\mathbf{y})\chi_j(\mathbf{y}) = 0 \quad \text{if } j \neq i, \quad \sum_i \chi_i(\mathbf{y}) = 1,$$

with

$$\int_{\Omega_1} \chi_i \, dy = \theta_i \operatorname{meas}(\Omega_1), \quad \sum_{i=1}^N \beta_i^2 \theta_i \operatorname{meas}(\Omega_1) = 1.$$

Thus, the χ_i's are characterization functions, the β_i's are the corresponding constant values of β and the θ_i's are the volume fractions of the domains where β is constant. Any β in $\mathcal{D}^+$ can be approached by such piecewise constant functions and hence we can construct a minimizing sequence by such functions. For such a β, $\Delta(\mathbf{v}, \beta)$ becomes

$$\Delta(\mathbf{v}, \beta) = \int_{\Omega_1} \left(\mathsf{A}_0 \boldsymbol{\varepsilon}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) - 2 \sum_{i=1}^N \chi_i \beta_i \mathsf{A}_0 \mathbf{e}_0 \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \right) dy$$

and the problem is equivalent to finding the relaxation of an elastic potential which contains N wells located at $\boldsymbol{\xi}_i = \beta_i \mathbf{e}_0$. In [12] the relaxation of a double-well potential is performed by a procedure which cannot be extended in general to more than two wells. However, in our case, since all the wells are proportional to $\mathbf{e}_0$, we can still follow the procedure proposed in [12, Proposition 8.1] and based on Fourier analysis. Taking the Fourier transform of the N periodic characteristic functions χ_i, i.e.,

$$\chi_i(\mathbf{y}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \hat{\chi}_i(\mathbf{k}) \exp^{i\mathbf{k} \cdot \mathbf{y}},$$

and minimizing $\Delta(\mathbf{v}, \sum_{i=1} \beta_i \chi_i)$ over all the periodic $\mathbf{v}$, one gets

$$\min_{\mathbf{v}} \Delta\left(\mathbf{v}, \sum_{i=1}^N \beta_i \chi_i\right) = - \sum_{\mathbf{k} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}} \sum_{i,j=1}^N \beta_i \beta_j \hat{\chi}_i(\mathbf{k}) \overline{\hat{\chi}_j(\mathbf{k})} \mathsf{A}_0 \boldsymbol{\xi}(\mathbf{k}) \cdot \boldsymbol{\xi}(\mathbf{k}),$$

where $\overline{z}$ denotes the complex conjugate of z and $\boldsymbol{\xi}(\mathbf{k})$ is given by (46). Since

$$\sum_{\mathbf{k} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}} \sum_{i,j=1}^N \beta_i \beta_j \hat{\chi}_i(\mathbf{k}) \overline{\hat{\chi}_j(\mathbf{k})} = \int_{\Omega_1} \left(\sum_{i=1}^N \beta_i \left(\chi_i(\mathbf{y}) - \theta_i \right) \right)^2 dy$$

$$= 1 - \frac{1}{\operatorname{meas}(\Omega_1)} \left(\int_{\Omega_1} \beta(\mathbf{y}) dy \right)^2,$$

one immediately obtains

$$\inf_{\beta} \min_{\mathbf{v}} \Delta(\mathbf{v}, \beta) \geq - \max_{\{\mathbf{k}: |\mathbf{k}|=1\}} \mathsf{A}_0 \boldsymbol{\xi}(\mathbf{k}) \cdot \boldsymbol{\xi}(\mathbf{k}) = -\delta(\boldsymbol{\varepsilon}_0).$$

To obtain the converse inequality, one first chooses $\mathbf{k}$ as a maximizer of $\mathsf{A}_0 \boldsymbol{\xi}(\mathbf{k}) \cdot \boldsymbol{\xi}(\mathbf{k})$ over $\mathbb{S}^n$ and one then approaches $\mathbf{k}$ by $\tilde{\mathbf{k}}/\|\tilde{\mathbf{k}}\|$ with $\tilde{\mathbf{k}} \in \mathbb{Z}^n$. For β we construct a minimizing sequence $\{\beta_m\}_{m \in \mathbb{N}}$ such that

$$\beta_m(\mathbf{y}) = f_m(\mathbf{k} \cdot \mathbf{y}) \quad \text{with } f_m(\theta) = \begin{cases} \frac{\sqrt{m}}{(2\pi)^{n/2}} & \text{if } 0 < \theta < \frac{2\pi}{m}, \\ 0 & \text{otherwise.} \end{cases}$$

Accordingly, $\int_{\Omega_1} \beta_m^2(\mathbf{y}) dy = 1$ and $\lim_{m \to \infty} \int_{\Omega_1} \beta_m(\mathbf{y}) dy = 0$. We finally obtain

$$\lim_{m \to \infty} \min_{\mathbf{v}} \Delta(\mathbf{v}, \beta_m) = -\delta(\boldsymbol{\varepsilon}_0),$$

which completes the proof. $\qquad\square$

Since ρ_∞ does not depend on the shape of the reference domain Ω_1 but only on $\boldsymbol{\varepsilon}_0$, we get that, if $\rho_\infty(\boldsymbol{\varepsilon}_0) \geq 1$, then $L_c(\boldsymbol{\varepsilon}_0, \Omega_1) = +\infty, \forall \Omega_1$. Note, however, that if $\rho_\infty(\boldsymbol{\varepsilon}_0) < 1$, then the critical length (is finite and) depends in general both on $\boldsymbol{\varepsilon}_0$ and Ω_1. The following Proposition gives the cases where $\rho_\infty(\boldsymbol{\varepsilon}_0) = 0$ and hence cases where the damaging state is necessarily unstable for sufficiently large structures:

Proposition 6 *When the damaging homogeneous state is such that one of the two following properties holds*

(i) $\mathbf{e}_0 := \mathsf{S}'(\alpha_0)\mathsf{A}(\alpha_0)\boldsymbol{\varepsilon}_0$ *is a rank one tensor of* $\mathbb{M}_s^n$,
(ii) $\mathbf{e}_0 := \mathsf{S}'(\alpha_0)\mathsf{A}(\alpha_0)\boldsymbol{\varepsilon}_0$ *is a rank two tensor of* $\mathbb{M}_s^n$ *and its nonzero eigenvalues have opposite signs,*

then $\rho_\infty(\boldsymbol{\varepsilon}_0) = 0$ and hence $0 < L_c(\boldsymbol{\varepsilon}_0, \Omega_1) < +\infty$ for all Ω_1. Conversely, $\rho_\infty(\boldsymbol{\varepsilon}_0) = 0$ only if $\mathbf{e}_0$ satisfies (i) or (ii).

Proof It is clear from (45)–(47) that $\rho_\infty(\boldsymbol{\varepsilon}_0) = 0$ if and only if $\mathbf{e}_0$ belongs to $V(\mathbf{k})$ for some $\mathbf{k}$.

If $\mathbf{e}_0$ is a rank one tensor of $\mathbb{M}_s^n$, then it can be written $\mathbf{e}_0 = \gamma \mathbf{k} \otimes \mathbf{k}$ with $\mathbf{k} \in \mathbb{S}^n$ and hence $\mathbf{e}_0 \in V(\mathbf{k})$.

If $\mathbf{e}_0$ is a rank-two tensor of $\mathbb{M}_s^n$ with its two non zero eigenvalues of opposite signs, then $\mathbf{e}_0$ can read as $\mathbf{e}_0 = \gamma_1 \mathbf{k}_1 \otimes \mathbf{k}_1 - \gamma_2 \mathbf{k}_2 \otimes \mathbf{k}_2$ with $\gamma_1 > 0$, $\gamma_2 > 0$, $\mathbf{k}_1$ and $\mathbf{k}_2 \in \mathbb{S}^n$, $\mathbf{k}_1 \cdot \mathbf{k}_2 = 0$. Setting

$$\mathbf{k} = \sqrt{\frac{\gamma_1}{\gamma_1 + \gamma_2}}\mathbf{k}_1 + \sqrt{\frac{\gamma_2}{\gamma_1 + \gamma_2}}\mathbf{k}_2, \qquad \mathbf{v} = \sqrt{\gamma_1 + \gamma_2}\left(\frac{\sqrt{\gamma_1}}{2}\mathbf{k}_1 - \frac{\sqrt{\gamma_2}}{2}\mathbf{k}_2\right),$$

one easily checks that $\mathbf{e}_0 = \mathbf{k} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{k}$.

If $\mathbf{e}_0 = \mathbf{k} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{k}$ with $\mathbf{k} \in \mathbb{S}^n$ and $\mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, then $\mathbf{e}_0$ is a rank one or a rank two tensor according to whether or not $\mathbf{v}$ is parallel to $\mathbf{k}$. When $\mathbf{v}$ is not parallel to $\mathbf{k}$, then $\mathbf{v}$ can read as $\mathbf{v} = v\mathbf{k} + v_*\mathbf{k}_*$ with $v_* \neq 0$ and $\mathbf{k}_* \in \mathbb{S}^n$, $\mathbf{k} \cdot \mathbf{k}_* = 0$. Therefore, $\mathbf{e}_0$ can be represented by the following matrix in the basis $(\mathbf{k}, \mathbf{k}_*)$:

$$\mathbf{e}_0 = \begin{pmatrix} 2v & v_* \\ v_* & 0 \end{pmatrix}_{(\mathbf{k}, \mathbf{k}_*)}.$$

The product of its two eigenvalues being $-v_*^2$, these eigenvalues have opposite signs. $\qquad\square$

When the damaging state does not satisfy one of the two cases above, one cannot conclude without giving more detailed information on the model. As an illustrative example, we will consider in the next subsection the case of spherical homogeneous states with a particular class of isotropic brittle materials.

4.4 Stability of Spherical Homogeneous States for a Class of Isotropic Brittle Materials

Let us consider the following damage law:

$$\mathsf{A}(\alpha) = a(\alpha)\mathsf{A}^0 \quad \text{with } \mathsf{A}^0_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{49}$$

$$\mathsf{S}(\alpha) = s(\alpha)\mathsf{S}^0 \quad \text{with } \mathsf{S}^0_{ijkl} = -\frac{v}{E}\delta_{ij}\delta_{kl} + \frac{1+v}{2E}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{50}$$

where $a : [0, \alpha_m] \to \mathbb{R}^+, \alpha \mapsto a(\alpha)$ is a decreasing twice differentiable dimensionless function such that $a(0) = 1$, $a(\alpha_m) = 0$ and $a' < 0$. In (50), s denotes the compliance factor, *i.e.*, $s = 1/a$. Accordingly, parameters (λ, μ) are the Lamé coefficients, v the Poisson's ratio and E the Young modulus of the sound material. Moreover, the Poisson's ratio does not change when damage grows. We will only consider cases with spatial dimension $n = 2$ or 3. Hence, the positivity of stiffness and compliance tensors is equivalent to

$$\mu > 0, \qquad n\lambda + 2\mu > 0, \qquad E > 0, \qquad -1 < v < \frac{1}{n-1}. \tag{51}$$

Relations between stiffness and compliance coefficients read as

$$2\mu = \frac{E}{1+v}, \qquad \lambda = \frac{vE}{(1+v)(1-(n-1)v)}, \tag{52}$$

while the strain-hardening and stress-softening conditions are equivalent to $a'' > 0$ and $s'' > 0$.

Displacements are prescribed at the boundary of the body so that the homogeneous strain and stress states are spherical tensors:

$$\boldsymbol{\varepsilon}_0 = \varepsilon\mathbf{I}, \qquad \boldsymbol{\sigma}_0 = (n\lambda + 2\mu)a(\alpha_0)\varepsilon\mathbf{I}, \tag{53}$$

where n is the spatial dimension, ε is a given real number such that the homogeneous state be a damaging state, *i.e.*, $\varepsilon \geq \varepsilon_c$ with

$$\varepsilon_c = \sqrt{\frac{2w_1}{-(n\lambda + 2\mu)a'(0)}}. \tag{54}$$

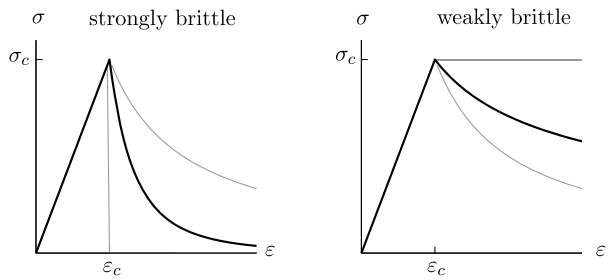The damage state α_0 depends only on ε and is given by

$$\alpha_0 = (a')^{-1}\left(a'(0)\frac{\varepsilon_c^2}{\varepsilon^2}\right). \tag{55}$$

The relation between spherical stress σ and the spherical strain ε is then

$$\sigma = (n\lambda + 2\mu)a(\alpha_0)\varepsilon. \tag{56}$$

Fig. 1 Spherical stress versus spherical strain response in the case of a strongly brittle material (*left*) or a weakly brittle material (*right*)

Providing this choice of loading and damage law, a direct calculation gives

$$
\begin{cases}
\mathsf{A}(\alpha_0)\mathbf{e}_0 \cdot \mathbf{e}_0 = n(n\lambda + 2\mu)s'(\alpha_0)^2 a(\alpha_0)^3 \varepsilon^2, \\
\delta(\boldsymbol{\varepsilon}_0) = \frac{(n\lambda + 2\mu)^2}{\lambda + 2\mu} s'(\alpha_0)^2 a(\alpha_0)^3 \varepsilon^2, \\
\frac{1}{2}\mathsf{S}''(\alpha_0)\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0 = \frac{1}{2} n(n\lambda + 2\mu)s''(\alpha_0)a(\alpha_0)^2 \varepsilon^2.
\end{cases}
\tag{57}
$$

The unique calculation which is not straightforward is that of $\delta(\boldsymbol{\varepsilon}_0)$. To obtain the expression above, one starts from (45)–(46) and uses the special form (49) of A_0. Then, introducing $\mathbf{v}(\mathbf{k})$ as the optimal vector of $\mathbb{R}^n$, *i.e.*, such that $\boldsymbol{\xi}(\mathbf{k}) = \mathbf{k} \otimes \mathbf{v}(\mathbf{k}) + \mathbf{v}(\mathbf{k}) \otimes \mathbf{k}$, we get

$$
\mathbf{v}(\mathbf{k}) = \underset{\mathbf{v}\in\mathbb{R}^n}{\mathrm{argmin}}\big((\lambda + \mu)(\mathbf{v} \cdot \mathbf{k})^2 + \mu\mathbf{v} \cdot \mathbf{v} - (n\lambda + 2\mu)s'(\alpha_0)a(\alpha_0)\varepsilon\mathbf{v} \cdot \mathbf{k}\big).
$$

A direct calculation gives $\mathbf{v}(\mathbf{k}) = \frac{n\lambda + 2\mu}{2(\lambda + 2\mu)}s'(\alpha_0)a(\alpha_0)\varepsilon\mathbf{k}$, then it turns out that $\mathsf{A}_0\boldsymbol{\xi}(\mathbf{k}) \cdot \boldsymbol{\xi}(\mathbf{k})$ is independent of $\mathbf{k}$ and one finally gets the expression (57) for $\delta(\boldsymbol{\varepsilon}_0)$.

We deduce from (47) and (57) that $\rho_\infty(\varepsilon)$ reads

$$
\rho_\infty(\boldsymbol{\varepsilon}_0) = \frac{(n-1)(1-(n-1)\nu)}{n(1-(n-2)\nu)} \frac{2s'(\alpha_0)^2}{s(\alpha_0)s''(\alpha_0)}
\tag{58}
$$

and the fact that it is greater or less than 1 depends on the strain state ε, the spatial dimension n, the Poisson's ratio ν and the compliance function s. Let us study these dependencies in two examples.

Example 1 Let us first consider a family of *strongly brittle materials* in the sense of [24] such that the maximal damage value α_m and the stiffness function $\alpha \mapsto a(\alpha)$ are given by

$$
\alpha_m = 1, \qquad a(\alpha) = (1-\alpha)^q, \quad q > 1,
\tag{59}
$$

where the exponent q must be greater than 1 so that the strain hardening condition is satisfied. The condition of stress softening is then automatically satisfied. Relations (55) and (56) between α_0, σ and ε for a damaging state read as

$$
\alpha_0 = 1 - \left(\frac{\varepsilon_c}{\varepsilon}\right)^{\frac{2}{q-1}}, \quad \sigma = \sigma_c\left(\frac{\varepsilon_c}{\varepsilon}\right)^{\frac{q+1}{q-1}}, \quad \sigma_c = (n\lambda + 2\mu)\varepsilon_c,
$$

see Fig. 1 (left). To compare $\rho_\infty(\boldsymbol{\varepsilon}_0)$ with 1, let us discriminate according to the spatial dimension n.

Fig. 2 For a strongly brittle material, the gray area depicts the range of exponent q and Poisson's ratio ν for which $L_c(\varepsilon_0, \Omega_1) < +\infty$, *i.e.*, the critical size of the domain below which the damaging homogeneous *spherical* state is directionally stable is finite. *Left*: $n = 2$; *Right*: $n = 3$
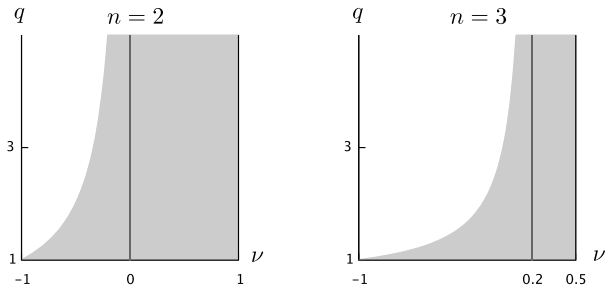
For dimension $n = 2$, we find that

$$\rho_\infty(\varepsilon_0) = \frac{(1 - \nu)q}{q + 1}$$

and hence $L_c(\varepsilon_0, \Omega_1) < +\infty$ when $\nu q + 1 > 0$. This situation corresponds to the gray area in Fig. 2 (left).

For dimension $n = 3$, we find that

$$\rho_\infty(\varepsilon_0) = \frac{4(1 - 2\nu)q}{3(1 - \nu)(q + 1)}$$

and hence $L_c(\varepsilon_0, \Omega_1) < +\infty$ when $(1 - 5\nu)q < 3(1 - \nu)$. This situation corresponds to the gray area in Fig. 2 (right).

Example 2 Let us now consider a family of *weakly brittle materials* in the sense of [24]:

$$\alpha_m = +\infty, \qquad a(\alpha) = \frac{1}{(1 + \alpha)^p}, \qquad p > 1, \tag{60}$$

where the maximal damage value is infinite and the exponent p must be greater than 1 so that the behavior be with softening. The relations (55) and (56) between α_0, σ and ε for a damaging state read as

$$\alpha_0 = \left(\frac{\varepsilon}{\varepsilon_c}\right)^{\frac{2}{p+1}} - 1, \qquad \sigma = \sigma_c \left(\frac{\varepsilon_c}{\varepsilon}\right)^{\frac{p-1}{p+1}}, \qquad \sigma_c = (n\lambda + 2\mu)\varepsilon_c,$$

see Fig. 1 (right). To compare $\rho_\infty(\varepsilon_0)$ with 1, we discriminate again according to the spatial dimension.

For dimension $n = 2$, we find that

$$\rho_\infty(\varepsilon_0) = \frac{(1 - \nu)p}{p - 1}$$

and thus $L_c(\varepsilon_0, \Omega_1) < +\infty$ when $\nu p > 1$. This situation corresponds to the gray area in Fig. 3 (left).

For dimension $n = 3$, we find that

$$\rho_\infty(\varepsilon_0) = \frac{4(1 - 2\nu)p}{3(1 - \nu)(p - 1)}$$

and thus $L_c(\varepsilon_0, \Omega_1) < +\infty$ when $(1 - 5\nu)p + 3(1 - \nu) < 0$. This situation corresponds to the gray area in Fig. 3 (right).

Fig. 3 For a weakly brittle material, the gray area depicts the range of exponent q and Poisson's ratio ν for which $L_c(\boldsymbol{\varepsilon}_0, \Omega_1) < +\infty$, *i.e.*, the critical size of the domain below which the damaging homogeneous *spherical* state is directionally stable is finite. *Left*: $n = 2$; *Right*: $n = 3$
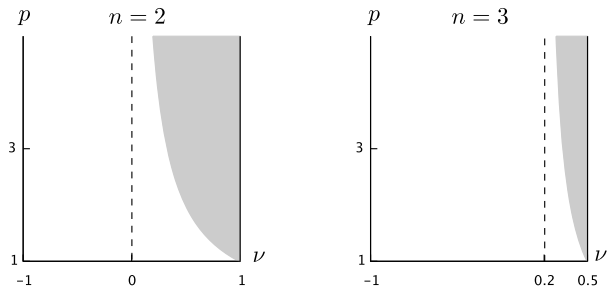
5 Size and Shape Effects in the Stability of the Homogeneous Damaging States in the Case of a Uniaxial Tensile Test

5.1 Setting of the Problem

We consider now the third type of boundary conditions, *i.e.*, those which corresponds to a uniaxial tensile test (the most common experimental test), see Sect. 3. The reference domain Ω_1 is the three dimensional cylinder of length 1 and cross-section Σ_1, *i.e.* $\Omega_1 = (0, 1) \times \Sigma_1$. Other domains are obtained by homothety, $\Omega_L = L\Omega_1$, $L > 0$. Using the change of coordinates $\mathbf{x} \mapsto \mathbf{y} = \mathbf{x}/L$ and the mappings (26), we can consider that all perturbation fields are defined on Ω_1 and that the sets of kinematically admissible displacement perturbations and admissible damage perturbations are independent of L and given by

$$\mathcal{C}_0 = \left\{ \mathbf{v} \in H^1(\Omega_1, \mathbb{R}^3) : v_1 = 0 \text{ on } \{0, 1\} \times \Sigma_1 \right\}, \qquad \mathcal{D}^+ = \left\{ \beta \in H^1(\Omega_1) : \beta \geq 0 \right\}.$$

The constitutive property of the material is assumed to be isotropic and the damage law is such that the Poisson's ratio remains constant and equal to $\nu \in (-1, 1/2)$. Thus the stiffness and compliance tensors are given by (49)–(50):

$$\mathsf{A}(\alpha) = a(\alpha)\mathsf{A}^0, \quad \mathsf{S}(\alpha) = s(\alpha)\mathsf{S}^0 \quad \text{with } a > 0, a' < 0, a'' > 0, s = 1/a, s'' > 0.$$

Accordingly, the damaging homogeneous state is $(\boldsymbol{\varepsilon}_0 \mathbf{y}, \alpha_0)$ where the strain tensor $\boldsymbol{\varepsilon}_0$ and the damage state α_0 are given by

$$\boldsymbol{\varepsilon}_0 = \varepsilon \mathbf{e}_1 \otimes \mathbf{e}_1 - \nu \varepsilon \mathbf{e}_2 \otimes \mathbf{e}_2 - \nu \varepsilon \mathbf{e}_3 \otimes \mathbf{e}_3, \qquad a'(\alpha_0) = a'(0)\frac{\varepsilon_c^2}{\varepsilon^2}, \qquad \varepsilon \geq \varepsilon_c = \sqrt{\frac{2w_1}{-a'(0)E}}.$$

The associated homogeneous stress tensor is uniaxial and reads as

$$\boldsymbol{\sigma}_0 = a(\alpha_0)E\varepsilon \mathbf{e}_1 \otimes \mathbf{e}_1.$$

The Rayleigh ratio $\mathcal{R}_L$ associated with the domain Ω_L and defined on the fixed space $\mathcal{C}_0 \times (\mathcal{D}^+ \setminus \{0\})$ is still given by (27) if we use the condensed notations (28). By virtue of the particular forms of the damage law and of the specific properties of the uniaxial test, some quantities can be easily calculated. Indeed, we have

$$\begin{aligned} \mathbf{e}_0 &= -\frac{a'(\alpha_0)}{a(\alpha_0)}\boldsymbol{\varepsilon}_0, \qquad \mathsf{A}_0\mathbf{e}_0 = -a'(\alpha_0)E\varepsilon \mathbf{e}_1 \otimes \mathbf{e}_1, \\ \mathsf{A}_0\mathbf{e}_0 \cdot \mathbf{e}_0 &= \frac{a'(\alpha_0)^2}{a(\alpha_0)}E\varepsilon^2, \qquad \mathsf{S}_0''\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0 = s''(\alpha_0)a(\alpha_0)^2 E\varepsilon^2. \end{aligned} \tag{61}$$

The minimum of $\mathcal{R}_L$ over $\mathcal{C}_0 \times (\mathcal{D}^+ \setminus \{0\})$ depends *a priori* on L, ε and Σ_1. Thus, let us set

$$\rho_L(\varepsilon, \Sigma_1) = \min_{\mathcal{C}_0 \times (\mathcal{D}^+ \setminus \{0\})} \mathcal{R}_L.$$

The homogeneous state is directionally stable depending on whether $\rho_L(\varepsilon, \Sigma_1)$ is less or greater than 1. Since only a part of the displacements are controlled on the boundary, we are in an intermediate situation between a soft device and a hard device. However, it turns out that the present case is quite similar to a hard device as it is proved in the following proposition.

5.2 Size Effects

Proposition 7 *The minimum $\rho_L(\varepsilon, \Sigma_1)$ of the Rayleigh ratio is a decreasing function of L such that*

$$\lim_{L \to 0} \rho_L(\varepsilon, \Sigma_1) = \frac{1}{1 - \frac{a(\alpha_0)a''(\alpha_0)}{2a'(\alpha_0)^2}} > 1. \tag{62}$$

Therefore, the damaging homogeneous state is directionally stable provided that the size of the cylinder is small enough.

Proof The proof is essentially the same as the one of Proposition 5 and we simply emphasize the differences. The notations are unchanged, we still call $(\mathbf{v}_L, \beta_L)$ the minimizer of $\mathcal{R}_L$ which still satisfy (30)–(32). Note that in this case $\mathbf{v}_L$ is not unique, but is determined up to a rigid motion compatible with the boundary conditions, *i.e.* up to a transversal translation and a rotation around the x_1-axis. This indeterminacy can be fixed by considering the quotient of $\mathcal{C}_0$ by these rigid motions. Then, we deduce that the sequences $L \mapsto \rho_L$, $L \mapsto \mathbf{v}_L$ and $L \mapsto \beta_L$ are bounded and converge (for the relevant topology) respectively to $\rho_0 \geq 0$, $\mathbf{v}_* \in \mathcal{C}_0$ and a positive constant β_*. Passing to the limit in (32) gives

$$\int_{\Omega_1} \mathsf{A}_0 \boldsymbol{\varepsilon}(\mathbf{v}_*) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dy = \beta_* \mathsf{A}_0 \mathbf{e}_0 \cdot \int_{\Omega_1} \boldsymbol{\varepsilon}(\mathbf{v}) dy, \quad \forall \mathbf{v} \in \mathcal{C}_0. \tag{63}$$

Since $\mathsf{A}_0 \mathbf{e}_0 \cdot \boldsymbol{\varepsilon}(\mathbf{v}) = -a'(\alpha_0)E\varepsilon v_{1,1}$, we still have $\mathsf{A}_0 \mathbf{e}_0 \cdot \int_{\Omega_1} \boldsymbol{\varepsilon}(\mathbf{v}) dy = 0$, $\forall \mathbf{v} \in \mathcal{C}_0$, and hence $\boldsymbol{\varepsilon}(\mathbf{v}_*) = \mathbf{0}$. The end of the proof is unchanged, one obtains

$$\rho_0 = \frac{\mathsf{A}_0 \mathbf{e}_0 \cdot \mathbf{e}_0}{\frac{1}{2}\mathsf{S}_0'' \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0}.$$

By using (61), one finally gets (62). □

To know if the stability of the homogeneous state holds for any size of the cylinder, one must compare $\rho_\infty(\varepsilon, \Sigma_1) := \lim_{L \to \infty} \rho_L(\varepsilon, \Sigma_1)$ with 1. Following the same procedure as in the previous section, one easily obtains

$$\rho_\infty(\varepsilon, \Sigma_1) = \inf_{\mathcal{C}_0 \times (\mathcal{D}^+ \setminus \{0\})} \mathcal{R}_\infty,$$

where $\mathcal{R}_\infty$ is still given by (41) (provided that one uses (61) for the definition of the state dependent quantities). However, since $\mathcal{C}_0 \supset H_0^1(\Omega, \mathbb{R}^3)$, one has merely $\inf_{H_0^1(\Omega, \mathbb{R}^3) \times (\mathcal{D}^+ \setminus \{0\})} \mathcal{R}_\infty \geq \inf_{\mathcal{C}_0 \times (\mathcal{D}^+ \setminus \{0\})} \mathcal{R}_\infty$ and one can only use Lemma 1 to obtain an upper bound for $\rho_\infty(\varepsilon, \Sigma_1)$. More specifically, one gets

Lemma 2 *When the size L of the domain tends to ∞, the limit value $\rho_\infty(\varepsilon, \Sigma_1)$ of the minimum of the Rayleigh ratio is bounded from above as follows*:

$$\rho_\infty(\varepsilon, \Sigma_1) \leq \frac{1 - \frac{(1+\nu)(1-2\nu)}{1-\nu}}{1 - \frac{a(\alpha_0)a''(\alpha_0)}{2a'(\alpha_0)^2}} \quad \text{if } \nu \leq 0,$$

$$\rho_\infty(\varepsilon, \Sigma_1) \leq \frac{\nu^2}{1 - \frac{a(\alpha_0)a''(\alpha_0)}{2a'(\alpha_0)^2}} \quad \text{if } \nu \geq 0. \tag{64}$$

Proof By virtue of Lemma 1, one has

$$\rho_\infty(\varepsilon, \Sigma_1) \leq \frac{\mathsf{A}_0 \mathbf{e}_0 \cdot \mathbf{e}_0 - \max_{\mathbf{k} \in \mathbb{S}^3} \mathsf{A}_0 \boldsymbol{\xi}(\mathbf{k}) \cdot \boldsymbol{\xi}(\mathbf{k})}{\frac{1}{2} \mathsf{S}_0'' \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0}.$$

It remains to determine $\max_{\mathbf{k} \in \mathbb{S}^3} \mathsf{A}_0 \boldsymbol{\xi}(\mathbf{k}) \cdot \boldsymbol{\xi}(\mathbf{k})$. Let us first determine $\boldsymbol{\xi}(\mathbf{k})$ for $\mathbf{k} \in \mathbb{S}^3$. By its definition (46), $\boldsymbol{\xi}(\mathbf{k})$ is the element of $V(\mathbf{k})$ which minimizes $\mathsf{A}_0(\mathbf{e}_0 - \boldsymbol{\xi}) \cdot (\mathbf{e}_0 - \boldsymbol{\xi})$ over all the $\boldsymbol{\xi}$ of $V(\mathbf{k})$. Accordingly, $\boldsymbol{\xi}(\mathbf{k})$ can read as $\boldsymbol{\xi}(\mathbf{k}) = \mathbf{k} \otimes \mathbf{v}(\mathbf{k}) + \mathbf{v}(\mathbf{k}) \otimes \mathbf{k}$ with $\mathbf{v}(\mathbf{k}) \in \mathbb{R}^3$ given by

$$\mathbf{v}(\mathbf{k}) = \operatorname*{argmin}_{\mathbf{v} \in \mathbb{R}^3} \left\{ (\lambda + \mu)(\mathbf{v} \cdot \mathbf{k})^2 + \mu \mathbf{v} \cdot \mathbf{v} + \frac{a'(\alpha_0)}{a(\alpha_0)} E\varepsilon \, k_1 v_1 \right\}.$$

We immediately deduce that the condition of optimality for $\mathbf{v}(\mathbf{k})$ reads as

$$(\lambda + \mu)\mathbf{v}(\mathbf{k}) \cdot \mathbf{k} \, \mathbf{k} + \mu \mathbf{v}(\mathbf{k}) + \frac{a'(\alpha_0)}{2a(\alpha_0)} E\varepsilon \, k_1 \mathbf{e}_1 = 0,$$

from which one easily obtains $\mathbf{v}(\mathbf{k})$

$$\mathbf{v}(\mathbf{k}) = \frac{a'(\alpha_0)}{a(\alpha_0)} \frac{E\varepsilon}{2\mu} \left(\frac{\lambda + \mu}{\lambda + 2\mu} k_1^2 \, \mathbf{k} - k_1 \mathbf{e}_1 \right).$$

Therefore, one gets the following expression of $\mathsf{A}_0 \boldsymbol{\xi}(\mathbf{k}) \cdot \boldsymbol{\xi}(\mathbf{k})$:

$$\mathsf{A}_0 \boldsymbol{\xi}(\mathbf{k}) \cdot \boldsymbol{\xi}(\mathbf{k}) = \frac{2a'(\alpha_0)^2}{a(\alpha_0)} E\varepsilon^2 (1 + \nu) k_1^2 \left(1 - \frac{k_1^2}{2(1-\nu)} \right),$$

which is a strictly concave function of k_1^2 that one has to maximize over the closed interval $[0, 1]$. The maximum is reached for $k_1^2 = 1$ or $k_1^2 = 1 - \nu$ depending on whether ν is negative or positive. After elementary calculations, one finally obtains (64). $\qquad\square$

Note that, when $\nu = 0$, Lemma 2 gives $\rho_\infty(\varepsilon, \Sigma_1) = 0$, which a result is in agreement with Proposition 6, $\mathbf{e}_0$ being then of rank one. Accordingly, when the Poisson's ratio vanishes, the homogeneous state is stable if and only if the size of the cylinder is small enough, the critical length $L_c(\varepsilon, \Sigma_1)$ depending *a priori* on the damage state and on the shape of the cross section. Since ρ_∞ depends continuously on the Poisson's ratio, this property holds true in some interval of $\nu = 0$. The fact that $\rho_\infty(\varepsilon, \Sigma_1) < 1$ (and hence that there exists a finite critical size beyond which the homogeneous state is unstable) for any $\nu \in (-1, +1/2)$ can depend on the damage model. Let us consider for instance the case of the family of strongly

Fig. 4 For a strongly brittle material, the gray area depicts the range of exponent q and Poisson's ratio ν for which the critical size of the domain below which the damaging homogeneous *uniaxial* state is directionally stable is necessarily finite
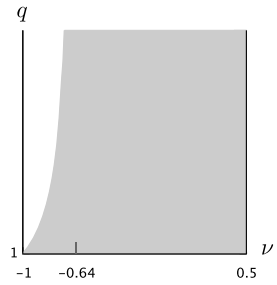
brittle materials of the example 1 above, see (59). Inserting into (64) gives

$$\rho_\infty(\varepsilon, \Sigma_1) \leq \frac{4\nu^2 q}{(1-\nu)(q+1)} \quad \text{if } \nu \leq 0, \qquad \rho_\infty(\varepsilon, \Sigma_1) \leq \frac{2\nu^2 q}{(q+1)} \quad \text{if } \nu \geq 0.$$

Accordingly, one necessarily has $L_c(\varepsilon, \Sigma_1) < +\infty$ (and that for every $\varepsilon \geq \varepsilon_c$ and every Σ_1) when the constitutive parameters q and ν satisfy the following inequalities

$$\left((1-\nu-4\nu^2)q + 1 - \nu \geq 0 \quad \text{and} \quad \nu < 0\right) \quad \text{or} \quad \nu \geq 0.$$

This corresponds to the gray area in Fig. 4. It appears that the above estimate of $\rho_\infty(\varepsilon, \Sigma_1)$ is not sufficient to ensure that $L_c(\varepsilon, \Sigma_1) < +\infty$ in the full range of material parameters. It turns out that this property can depend on the shape of the cross-section as it is explained in the next subsection.

5.3 Shape Effects

5.3.1 General Estimates

Throughout this subsection, Greek indices run from 2 to 3 whereas Latin indices run from 1 to 3. We propose here to study the influence of the *slenderness* of the family of cylinders on the stability of the homogeneous state. The slenderness parameter η is defined as the ratio of the diameter of the cross-section by the length of the cylinder, the diameter being defined as the smallest disk which contains the cross-section. The inverse $1/\eta$ of the slenderness is the *thinness*. The cylinder of unit length and slenderness η is denoted by Ω_1^η and its cross-section by Σ_1^η instead of Ω_1 and Σ_1 as it was made in the previous sections. Accordingly, $\Omega_1^\eta = (0, 1) \times \Sigma_1^\eta$ and $\Omega_L^\eta = L\Omega_1^\eta$.

To study the influence of η we use the mapping $\mathbf{y} = (y_1, y_2, y_3) \mapsto \mathbf{z} = (y_1, y_2/\eta, y_3/\eta)$ which transforms the cross-section Σ_1^η into its homothetic one Σ_1^1 of diameter 1, and the cylinder Ω_1^η onto $\Omega_1^1 = (0, 1) \times \Sigma_1^1$. The origin of the coordinates is chosen so that $(0, 0)$ is the center of Σ_1^1. The area of Σ_1^1 and its geometrical second moments will be denoted by $|\Sigma_1^1|$ and $I_{\alpha\beta}^1$:

$$|\Sigma_1^1| = \int_{\Sigma_1^1} dz_2 dz_3, \qquad \int_{\Sigma_1^1} z_\alpha dz_2 dz_3 = 0, \qquad I_{\alpha\beta}^1 = \int_{\Sigma_1^1} z_\alpha z_\beta dz_2 dz_3. \qquad (65)$$

Moreover, with the displacement field $\hat{\mathbf{v}}$ defined on Ω_1^η we associate the "rescaled" displacement field $\mathbf{v}$ defined on Ω_1^1 by

$$\hat{v}_1(\mathbf{y}) = \frac{a'(\alpha_0)}{a(\alpha_0)} \varepsilon v_1(\mathbf{z}), \qquad \hat{v}_\alpha(\mathbf{y}) = \frac{1}{\eta} \frac{a'(\alpha_0)}{a(\alpha_0)} \varepsilon v_\alpha(\mathbf{z}), \qquad \alpha \in \{2, 3\}.$$

Accordingly, the Rayleigh ratio which governs the stability of the homogeneous state of Ω_L^η can be seen as the following functional $\mathcal{R}_L^\eta$ defined on the fixed set $\mathcal{C}_1 \times \mathcal{D}_1^+$ by

$$\mathcal{R}_L^\eta(\mathbf{v}, \beta) = \frac{\mathcal{N}_L^\eta(\mathbf{v}, \beta)}{1 - \frac{a(\alpha_0)a''(\alpha_0)}{2a'(\alpha_0)^2}}, \tag{66}$$

with

$$\mathcal{N}_L^\eta(\mathbf{v}, \beta) = 1 + \int_{\Omega_1^1} \left(\frac{d_0^2}{L^2} \beta_{,1}^2 + \frac{\lambda + 2\mu}{E} \varepsilon_{11}(\mathbf{v})^2 + 2\beta\varepsilon_{11}(\mathbf{v}) \right) dz$$

$$+ \frac{1}{\eta^2} \int_{\Omega_1^1} \left(\frac{d_0^2}{L^2} \beta_{,\alpha} \beta_{,\alpha} + \frac{2\lambda}{E} \varepsilon_{11}(\mathbf{v})\varepsilon_{\alpha\alpha}(\mathbf{v}) + \frac{4\mu}{E} \varepsilon_{\alpha 1}(\mathbf{v})\varepsilon_{\alpha 1}(\mathbf{v}) \right) dz$$

$$+ \frac{1}{\eta^4} \int_{\Omega_1^1} \left(\frac{\lambda}{E} \varepsilon_{\alpha\alpha}(\mathbf{v})\varepsilon_{\beta\beta}(\mathbf{v}) + \frac{2\mu}{E} \varepsilon_{\alpha\beta}(\mathbf{v})\varepsilon_{\alpha\beta}(\mathbf{v}) \right) dz \tag{67}$$

and

$$d_0^2 = \frac{w_1 a(\alpha_0)\ell(\alpha_0)^2}{E\varepsilon^2 a'(\alpha_0)^2}. \tag{68}$$

In the set $\mathcal{C}_1$ of admissible displacements, we introduce three integral constraints so that the transversal translation and the axial rotation of the cylinder are fixed:

$$\mathcal{C}_1 = \left\{ \mathbf{v} \in H^1\left(\Omega_1^1, \mathbb{R}^3\right) : v_1 = 0 \text{ on } \{0,1\} \times \Sigma_1^1, \right.$$

$$\left. \int_{\Omega_1^1} v_2 dz = \int_{\Omega_1^1} v_3 dz = \int_{\Omega_1^1} (z_3 v_2 - z_2 v_3) dz = 0 \right\}.$$

The admissible damage fields are normalized and $\mathcal{D}_1^+$ reads as

$$\mathcal{D}_1^+ = \left\{ \beta \in H^1\left(\Omega_1^1\right) : \beta \geq 0, \int_{\Omega_1^1} \beta^2 dz = 1 \right\}.$$

Let $(\mathbf{v}_L^\eta, \beta_L^\eta)$ be a minimizer of $\mathcal{R}_L^\eta$ and hence of $\mathcal{N}_L^\eta$ over $\mathcal{C}_1 \times \mathcal{D}_1^+$ (we know that such a minimizer exists). The optimality condition (32) becomes

$$0 = \int_{\Omega_1^1} \left(\frac{\lambda + 2\mu}{E} \varepsilon_{11}\left(\mathbf{v}_L^\eta\right) + \beta_L^\eta \right) \varepsilon_{11}(\mathbf{v}) dz$$

$$+ \frac{1}{\eta^2} \int_{\Omega_1^1} \left(\frac{\lambda}{E} \varepsilon_{11}\left(\mathbf{v}_L^\eta\right)\varepsilon_{\alpha\alpha}(\mathbf{v}) + \frac{\lambda}{E} \varepsilon_{\alpha\alpha}\left(\mathbf{v}_L^\eta\right)\varepsilon_{11}(\mathbf{v}) + \frac{4\mu}{E} \varepsilon_{\alpha 1}\left(\mathbf{v}_L^\eta\right)\varepsilon_{\alpha 1}(\mathbf{v}) \right) dz$$

$$+ \frac{1}{\eta^4} \int_{\Omega_1^1} \left(\frac{\lambda}{E} \varepsilon_{\alpha\alpha}\left(\mathbf{v}_L^\eta\right)\varepsilon_{\beta\beta}(\mathbf{v}) + \frac{2\mu}{E} \varepsilon_{\alpha\beta}\left(\mathbf{v}_L^\eta\right)\varepsilon_{\alpha\beta}(\mathbf{v}) \right) dz. \tag{69}$$

After introducing the symmetric tensor field $\boldsymbol{\varepsilon}_L^\eta$ and the vector field $\mathbf{g}_L^\eta$,

$$\boldsymbol{\varepsilon}_L^\eta = \begin{pmatrix} \varepsilon_{11}(\mathbf{v}_L^\eta) & \frac{1}{\eta}\varepsilon_{1\beta}(\mathbf{v}_L^\eta) \\ \frac{1}{\eta}\varepsilon_{\alpha 1}(\mathbf{v}_L^\eta) & \frac{1}{\eta^2}\varepsilon_{\alpha\beta}(\mathbf{v}_L^\eta) \end{pmatrix}, \qquad \mathbf{g}_L^\eta = \begin{pmatrix} \beta_{L,1}^\eta \\ \frac{1}{\eta}\beta_{L,\alpha}^\eta \end{pmatrix}, \tag{70}$$

the minimum N_L^η of $\mathcal{N}_L^\eta$ can read as

$$N_L^\eta := \mathcal{N}_L^\eta(\mathbf{v}_L^\eta, \beta_L^\eta) = 1 + \int_{\Omega_1^1} \left(\frac{\mathsf{A}^0}{E} \boldsymbol{\varepsilon}_L^\eta \cdot \boldsymbol{\varepsilon}_L^\eta + \frac{d_0^2}{L^2} \mathbf{g}_L^\eta \cdot \mathbf{g}_L^\eta \right) dz + 2 \int_{\Omega_1^1} \beta_L^\eta \varepsilon_{L11}^\eta dz, \qquad (71)$$

where A^0 is the stiffness tensor of the sound material. Since $\mathcal{N}_L^\eta(\mathbf{v} = 0, \beta = \sqrt{|\Sigma_1^1|}) = 1$, we have $N_L^\eta \leq 1$. Owing to the positivity of A^0, there exists a positive constant κ (which depends only on ν) such that $\mathsf{A}^0 \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} \geq \kappa E \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}$. By Cauchy-Schwarz inequality and using the normalization of β_L^η, we get

$$\int_{\Omega_1^1} \beta_L^\eta \boldsymbol{\varepsilon}_{L11}^\eta dz \geq -\|\beta_L^\eta\|_0 \|\varepsilon_{L11}^\eta\|_0 = -\|\varepsilon_{L11}^\eta\|_0.$$

Therefore, we have the inequalities

$$1 + \kappa \|\boldsymbol{\varepsilon}_L^\eta\|_0^2 + \frac{d_0^2}{L^2} \|\mathbf{g}_L^\eta\|_0^2 - 2\|\varepsilon_{L11}^\eta\|_0 \leq N_L^\eta \leq 1,$$

from which we deduce the following estimates,

$$\begin{aligned} \|\varepsilon_{11}(\mathbf{v}_L^\eta)\|_0 &\leq C, \quad \|\varepsilon_{\alpha1}(\mathbf{v}_L^\eta)\|_0 \leq C\eta, \quad \|\varepsilon_{\alpha\beta}(\mathbf{v}_L^\eta)\|_0 \leq C\eta^2, \\ \|\beta_{L,1}^\eta\|_0 &\leq CL, \quad \|\beta_{L,\alpha}^\eta\|_0 \leq C\eta L, \end{aligned} \qquad (72)$$

where C stands for a generic positive constant which depends only on ν. Since these estimates hold in the whole range of L and η, they are useful to study the asymptotic cases of very slender or very thin cylinders. However, we will consider here only slender cylinders.

5.3.2 Case of Slender Cylinders: $\eta \ll 1$

The length L is fixed and we study the behavior of $(\mathbf{v}_L^\eta, \beta_L^\eta)$ when η goes to 0. Once this asymptotic behavior will be found for any given L, we will study the dependence on L and in particular the asymptotic behavior when L goes to infinity. Note that this procedure is different from the one followed to study the size effects. Indeed, in this latter case, the slenderness η was fixed and we studied the dependence of $(\mathbf{v}_L^\eta, \beta_L^\eta)$ on L.

Let us prove the following:

Proposition 8 *For a given L, when η goes to 0, the minimum of the Rayleigh ratio ρ_L^η tends to a limit given by*

$$\lim_{\eta \to 0} \rho_L^\eta = \rho_L^0 := \frac{N_L^0}{1 - \frac{a(\alpha_0)a''(\alpha_0)}{2a'(\alpha_0)^2}},$$

$$N_L^0 = \min_{\beta \in (\mathcal{D}^+ \setminus \{0\})} \frac{\frac{d_0^2}{L^2} \int_0^1 \beta'(\zeta)^2 d\zeta + (\int_0^1 \beta(\zeta)d\zeta)^2}{\int_0^1 \beta(\zeta)^2 d\zeta}, \qquad (73)$$

where $\mathcal{D}^+ = \{\beta \in H^1(0,1) : \beta \geq 0\}$.

Proof The proof is divided into several steps.

1. *Asymptotic behavior of* β_L^η. We deduce from (72) that β_L^η is bounded in $H^1(\Omega_1^1)$ and hence (up to extracting a subsequence) weakly converges in $H^1(\Omega_1^1)$ and strongly in $L^2(\Omega_1^1)$ to some $\beta_L^0 \in \mathcal{D}_1^+$. Furthermore, since $\|\beta_{L,\alpha}^\eta\|_0 \leq C\eta L$, $\beta_{L,\alpha}^\eta$ converges strongly to 0 in $L^2(\Omega_1^1)$ and hence β_L^0 is a non-negative function of only z_1 which can be seen as an element of $\mathcal{D}^+$ such that its $L^2(0,1)$ norm is equal to $1/|\Sigma_1^1|$.

2. *Asymptotic behavior of* $\mathbf{v}_L^\eta$. To a large extent, this step roots at the basis of the construction of the asymptotic theory of linearly elastic beams. Since this procedure is now well known, we merely recall the main lines of the proofs and the interested reader is invited to refer to Geymonat et al. [10, 11], Marigo and Meunier [16] for more details.

We deduce from (72) that $\mathbf{v}_L^\eta$ is bounded in $H^1(\Omega_1^1, \mathbb{R}^3)$. Hence, one can extract a subsequence which weakly converges to some $\mathbf{v}_L^0 \in \mathcal{C}_1$. Furthermore, since $\|\varepsilon_{\alpha 1}(\mathbf{v}_L^\eta)\|_0 \leq C\eta$ and $\|\varepsilon_{\alpha\beta}(\mathbf{v}_L^\eta)\|_0 \leq C\eta^2$, $\mathbf{v}_L^0$ satisfies $\varepsilon_{\alpha 1}(\mathbf{v}_L^0) = \varepsilon_{\alpha\beta}(\mathbf{v}_L^0) = 0$ and hence is a Bernoulli-Navier displacement field. Accordingly, $\mathbf{v}_L^0$ can read as

$$\mathbf{v}_L^0(\mathbf{z}) = \big(V_{L1}(z_1) - z_\alpha V_{L\alpha}'(z_1)\big)\mathbf{e}_1 + V_{L\alpha}(z_1)\mathbf{e}_\alpha. \tag{74}$$

Moreover, in order that $\mathbf{v}_L^0$ be in $\mathcal{C}_1$, the fields V_{Li} must satisfy

$$V_{L1} \in H_0^1(0,1), \;\; V_{L\alpha} \in H^2(0,1) \text{ with}$$
$$V_{L\alpha}'(0) = V_{L\alpha}'(1) = 0, \quad \int_0^1 V_{L\alpha}(\zeta)d\zeta = 0. \tag{75}$$

From the estimates, we also get that $\varepsilon_{\alpha\beta}(\mathbf{v}_L^\eta)/\eta^2$ weakly converges in $L^2(\Omega_1^1)$ to $\varepsilon_{\alpha\beta}(\mathbf{v}_L^*)$ where $\mathbf{v}_L^*$ does not belong in general to $\mathcal{C}_1$ but merely to $L^2((0,1), H^1(\Sigma_1^1))$, *i.e.*, the space of functions f which are in $L^2(\Omega_1^1)$ and whose tangential derivatives $f_{,\alpha}$ are also in $L^2(\Omega_1^1)$. Accordingly, $\mathbf{v}_L^*$ does not satisfy in general the boundary conditions at $z_1 = 0$ and $z_1 = 1$.

3. *Determination of* $\mathbf{v}_L^0$ *and* $\mathbf{v}_L^*$. Multiplying (69) by η^2 and passing to the limit as $\eta \to 0$ leads to

$$0 = \int_{\Omega_1^1} \big(\lambda\big(\varepsilon_{11}(\mathbf{v}_L^0) + \boldsymbol{\varepsilon}_{\alpha\alpha}(\mathbf{v}_L^*)\big)\varepsilon_{\beta\beta}(\mathbf{v}) + 2\mu\boldsymbol{\varepsilon}_{\alpha\beta}(\mathbf{v}_L^*)\varepsilon_{\alpha\beta}(\mathbf{v})\big)dz, \quad \forall\mathbf{v} \in \mathcal{C}_1. \tag{76}$$

This equation is nothing but the variational formulation of the de Saint Venant's problems of stretching and bending which give $\mathbf{v}_L^*$ in terms of $\mathbf{v}_L^0$. More specifically, one obtains that the plane components of the stress must vanish, *i.e.*,

$$\lambda\big(\varepsilon_{11}(\mathbf{v}_L^0) + \boldsymbol{\varepsilon}_{\gamma\gamma}(\mathbf{v}_L^*)\big)\delta_{\alpha\beta} + 2\mu\boldsymbol{\varepsilon}_{\alpha\beta}(\mathbf{v}_L^*) = 0 \quad \text{in } \Omega_1^1,$$

which gives in particular

$$\boldsymbol{\varepsilon}_{\alpha\alpha}(\mathbf{v}_L^*) = -\frac{\lambda}{\lambda + \mu}\varepsilon_{11}(\mathbf{v}_L^0) \quad \text{in } \Omega_1^1. \tag{77}$$

Taking for $\mathbf{v}$ in (69) a Bernoulli-Navier displacement field, *i.e.*,

$$\mathbf{v}(\mathbf{z}) = \big(V_1(z_1) - z_\alpha V_\alpha'(z_1)\big)\mathbf{e}_1 + V_\alpha(z_1)\mathbf{e}_\alpha, \tag{78}$$

and passing to the limit as $\eta \to 0$ leads to

$$\int_{\Omega_1^1} \left(\frac{\lambda + 2\mu}{E}\varepsilon_{11}(\mathbf{v}_L^0) + \frac{\lambda}{E}\boldsymbol{\varepsilon}_{\alpha\alpha}(\mathbf{v}_L^*) + \beta_L^0\right)\varepsilon_{11}(\mathbf{v})dz = 0. \tag{79}$$

Inserting (77) into (79) and remarking that $\lambda + 2\mu - \lambda^2/(\lambda + \mu) = E$, one finally obtains the problem which gives $\mathbf{v}_L^0$ in terms of β_L^0:

$$\int_{\Omega_1^1} \left(\varepsilon_{11}(\mathbf{v}_L^0) + \beta_L^0\right)\varepsilon_{11}(\mathbf{v})dz = 0, \tag{80}$$

where the equality holds for any Bernoulli-Navier displacement in $\mathcal{C}_1$. Using (65), (74), (78) and the fact that β_L^0 depends only on z_1, (80) becomes

$$\int_0^1 \left|\Sigma_1^1\right|\left(V_{L1}^{0'}(z_1) + \beta_L^0(z_1)\right)V_1'(z_1)dz_1 + \int_0^1 I_{\alpha\beta}^1 V_{L\alpha}^{0''}(z_1)V_\beta''(z_1)dz_1 = 0$$

and holds for any $V_1 \in H_0^1(0, 1)$ and any $V_\alpha \in H^2(0, 1)$ such that $V_\alpha'(0) = V_\alpha'(1) = 0$ and $\int_0^1 V_\alpha(\zeta)d\zeta = 0$. One immediately deduces that $V_{L\alpha} = 0$ (no bending) and that $V_{L1}^{0'}(z_1) + \beta_L^0(z_1) = \text{constant} = \int_0^1 \beta_L^0(\zeta)d\zeta$. Finally one gets

$$\mathbf{v}_L^0(\mathbf{z}) = \left(z_1 \int_0^1 \beta_L^0(\zeta)d\zeta - \int_0^{z_1} \beta_L^0(\zeta)d\zeta\right)\mathbf{e}_1. \tag{81}$$

4. *Lower bound of* $\liminf_{\eta \to 0} \rho_L^\eta$. By virtue of (69), the minimum of $\mathcal{N}_L^\eta$ can read as

$$\mathcal{N}_L^\eta(\mathbf{v}_L^\eta, \beta_L^\eta) = 1 + \frac{d_0^2}{L^2}\int_{\Omega_1^1} \beta_{L,1}^\eta \beta_{L,1}^\eta dz + \frac{d_0^2}{\eta^2 L^2}\int_{\Omega_1^1} \beta_{L,\alpha}^\eta \beta_{L,\alpha}^\eta dz + \int_{\Omega_1^1} \beta_L^\eta \varepsilon_{11}(\mathbf{v}_L^\eta)dz. \tag{82}$$

Let us examine the limit of each term in the right hand side of (82). By lower semi-continuity of the $L^2(\Omega_1^1)$ norm, one has

$$\lim_{\eta \to 0}\int_{\Omega_1^1} \beta_{L,1}^\eta \beta_{L,1}^\eta dz \geq \int_{\Omega_1^1} \beta_{L,1}^0 \beta_{L,1}^0 dz = \left|\Sigma_1^1\right|\int_0^1 \beta_L^{0'}(\zeta)^2 d\zeta.$$

By the positivity of the norm, one has

$$\lim_{\eta \to 0}\frac{1}{\eta^2}\int_{\Omega_1^1} \beta_{L,\alpha}^\eta \beta_{L,\alpha}^\eta dz \geq 0.$$

By virtue of the weak convergence of $\varepsilon_{11}(\mathbf{v}_L^\eta)$ and of the strong convergence of β_L^η in $L^2(\Omega_1^1)$, after using (81) and recalling that $\left|\Sigma_1^1\right|\int_0^1 \beta_L^0(\zeta)^2 d\zeta = 1$, one gets

$$\lim_{\eta \to 0}\int_{\Omega_1^1} \beta_L^\eta \varepsilon_{11}(\mathbf{v}_L^\eta)dz = \int_{\Omega_1^1} \beta_L^0 \varepsilon_{11}(\mathbf{v}_L^0)dz = \left|\Sigma_1^1\right|\left(\int_0^1 \beta_L^0(\zeta)d\zeta\right)^2 - 1.$$

Since all these estimates hold for any convergent subsequence, one has obtained

$$\liminf_{\eta \to 0}\mathcal{N}_L^\eta(\mathbf{v}_L^\eta, \beta_L^\eta) \geq \left|\Sigma_1^1\right|\frac{d_0^2}{L^2}\int_0^1 \beta_L^{0'}(\zeta)^2 d\zeta + \left|\Sigma_1^1\right|\left(\int_0^1 \beta_L^0(\zeta)d\zeta\right)^2.$$

Since β_L^0 is an element of $\mathcal{D}^+$ with a $L^2(0, 1)$ norm equals to $1/\sqrt{\left|\Sigma_1^1\right|}$, one concludes that

$$\liminf_{\eta \to 0}\rho_L^\eta \geq \rho_L^0,$$

with ρ_L^0 given by (73).

5. *Upper bound of* $\limsup_{\eta \to 0} \rho_L^{\eta}$. Let $\bar{\beta}_L$ be a minimizer (such a minimizer exists because one can prove that the infimum is reached in the same way as for $\mathcal{N}_L^{\eta}$) giving N_L^0 in (73). By homogeneity, for every $k > 0$, $k\bar{\beta}_L$ is also a minimizer and hence we can choose k so that the $L^2(0, 1)$ norm of $\mathcal{R}_L^{\eta}$ is equal to $1/\sqrt{|\Sigma_1^1|}$. Therefore, $\bar{\beta}_L$ can be seen as an element of $\mathcal{D}_1^+$. Let $\bar{\mathbf{v}}_L^{\eta}$ be the unique element of $\mathcal{C}_1$ which minimizes $\mathcal{N}_L^{\eta}(., \bar{\beta}_L)$ over $\mathcal{C}_1$. It is easy to check that $\bar{\mathbf{v}}_L^{\eta}$ satisfies the same estimates (72) as $\mathbf{v}_L^{\eta}$. Following the same steps as for $\mathbf{v}_L^{\eta}$, one obtains that $\bar{\mathbf{v}}_L^{\eta}$ weakly converges to the Bernoulli-Navier displacement $\bar{\mathbf{v}}_L^0$ given by

$$\bar{\mathbf{v}}_L^0(\mathbf{z}) = \left(z_1 \int_0^1 \bar{\beta}_L(\zeta)d\zeta - \int_0^{z_1} \bar{\beta}_L(\zeta)d\zeta \right)\mathbf{e}_1. \tag{83}$$

Moreover, since the limit is unique (for a given minimizer $\bar{\beta}_L$), *all* the sequence $\bar{\mathbf{v}}_L^{\eta}$ converges to $\bar{\mathbf{v}}_L^0$. By virtue of the optimality of $\bar{\mathbf{v}}_L^{\eta}$, $\mathcal{N}_L^{\eta}(\bar{\mathbf{v}}_L^{\eta}, \bar{\beta}_L)$ can read as

$$\mathcal{N}_L^{\eta}(\bar{\mathbf{v}}_L^{\eta}, \bar{\beta}_L) = 1 + |\Sigma_1^1| \frac{d_0^2}{L^2} \int_0^1 \bar{\beta}_L'(\zeta)^2 d\zeta + \int_{\Omega_1^1} \bar{\beta}_L \varepsilon_{11}(\bar{\mathbf{v}}_L^{\eta})dz.$$

Passing to the limit in the last term in the right hand side above, one obtains

$$\lim_{\eta \to 0} \int_{\Omega_1^1} \bar{\beta}_L \varepsilon_{11}(\mathbf{v}_L^{\eta})dz = \int_{\Omega_1^1} \bar{\beta}_L \varepsilon_{11}(\bar{\mathbf{v}}_L^0)dz = |\Sigma_1^1| \left(\int_0^1 \bar{\beta}_L(\zeta)d\zeta \right)^2 - 1$$

and hence

$$\lim_{\eta \to 0} \mathcal{N}_L^{\eta}(\bar{\mathbf{v}}_L^{\eta}, \bar{\beta}_L) = N_L^0.$$

Since $\mathcal{N}_L^{\eta}(\mathbf{v}_L^{\eta}, \beta_L^{\eta}) \leq \mathcal{N}_L^{\eta}(\bar{\mathbf{v}}_L^{\eta}, \bar{\beta}_L)$, by passing to the limit one gets $\lim_{\eta \to 0} \mathcal{N}_L^{\eta}(\mathbf{v}_L^{\eta}, \beta_L^{\eta}) \leq N_L^0$ for any convergent subsequence. Therefore, one has

$$\limsup_{\eta \to 0} \rho_L^{\eta} \leq \rho_L^0.$$

Comparing with the lower bound, we obtain the desired result, $\lim_{\eta \to 0} \rho_L^{\eta} = \rho_L^0$. The proof is complete. $\square$

Equipped with this characterization of the asymptotic behavior of the Rayleigh ratio, it becomes easy to conclude on the stability of damaging states for slender cylinders.

Proposition 9 *For very slender cylinders, the damaging state characterized by the axial strain ε in a uniaxial tensile test is directionally stable if and only if the length L of the cylinder is less than the critical value $L_c(\varepsilon)$. This latter depends only on ε and is given by*

$$L_c(\varepsilon) = \frac{s'(\alpha_0)^{5/2}}{s(\alpha_0)s''(\alpha_0)^{3/2}} 2\pi \ell(\alpha_0) \quad with \; a'(\alpha_0) = -\frac{2w_1}{E\varepsilon^2}. \tag{84}$$

Proof It suffices to calculate ρ_L^0 and to compare it with 1. To obtain ρ_L^0, we use [24, Proposition A.2] where the minimization of the Rayleigh ratio involved in (73) is made. One gets

$$N_L^0 = \begin{cases} 1 & \text{if } 0 < L \leq \pi d_0, \\ (\frac{\pi d_0}{L})^{2/3} & \text{if } L > \pi d_0. \end{cases} \tag{85}$$

Then, using (68) and (73), the critical length is obtained by equaling ρ_L^0 to 1. $\square$

Remark 1 This result is consistent with the one obtained in [24, Proposition 3.4] in the one dimensional setting. This means that, as expected, slender cylinders behave like one-dimensional bars. Note however that we have obtained (73) and hence (84) by passing to the limit as η goes to 0, at *given* L. Therefore, from the practical viewpoint, this result is relevant only when the diameter of the cylinder is (much) smaller than the characteristic length $\ell(\alpha_0)$ of the material.

6 Comparison of Directional Stability with Strong Ellipticity

In this last section, we investigate the link between our definition of directional stability and the strong ellipticity condition [1]. By essence, this latter one, which requires a strict positivity condition for the second order derivative of the total strain work, makes sense only for the underlying local damage model, *i.e.*, for the model without the gradient damage terms. More specifically, since the *underlying local model* is defined by

$$W_0(\boldsymbol{\varepsilon}, \alpha) = w_1 \alpha + \frac{1}{2}\mathsf{A}(\alpha)\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}, \tag{86}$$

the condition of strong ellipticity is satisfied at the given state $(\boldsymbol{\varepsilon}_0, \alpha_0)$ if and only if the following inequality holds,

$$W_0''(\boldsymbol{\varepsilon}_0, \alpha_0)(\mathbf{k} \odot \mathbf{v}, \beta) > 0, \qquad \forall \mathbf{k} \in \mathbb{S}^n, \ \forall (\mathbf{v}, \beta) \in \mathbb{R}^n \times \mathbb{R} \setminus \big\{(\mathbf{0}, 0)\big\}, \qquad \text{(SE)}$$

where $\mathbf{k} \odot \mathbf{v} = \mathbf{k} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{k}$. Using the condensed notation (28), the second order derivative reads

$$W_0''(\boldsymbol{\varepsilon}_0, \alpha_0)(\xi, \beta) = \mathsf{A}_0(\xi - \beta \mathbf{e}_0) \cdot (\xi - \beta \mathbf{e}_0) - \frac{1}{2}\mathsf{S}_0'' \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0 \beta^2. \tag{87}$$

Therefore, we first deduce that if the material has a stress-hardening behavior, then the strong ellipticity condition is automatically fulfilled for any state, because $\mathsf{A}_0 > 0$ and $\mathsf{S}_0'' < 0$. Now let us consider a stress-softening behavior, *i.e.*, the case $\mathsf{S}_0'' > 0$. Since by virtue of the positivity of A_0, the inequality (SE) is satisfied when $\beta = 0$, we can consider only the cases $\beta \neq 0$ and hence, by homogeneity, the cases $\beta = 1$. Accordingly, the strong ellipticity condition (SE) is satisfied if and only if

$$\min_{(\mathbf{k}, \mathbf{v}) \in \mathbb{S}^n \times \mathbb{R}^n} \mathsf{A}_0(\mathbf{k} \odot \mathbf{v} - \mathbf{e}_0) \cdot (\mathbf{k} \odot \mathbf{v} - \mathbf{e}_0) > \frac{1}{2}\mathsf{S}_0'' \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0. \tag{88}$$

Note that the minimization above admits a solution since it is performed over a compact set. For a given $\mathbf{k}$, let $\mathbf{v}(\mathbf{k})$ be the minimizer over $\mathbf{v} \in \mathbb{R}^n$. By virtue of the strict convexity of $\mathbf{v} \mapsto \mathsf{A}_0(\mathbf{k} \odot \mathbf{v} - \mathbf{e}_0) \cdot (\mathbf{k} \odot \mathbf{v} - \mathbf{e}_0)$, there exists a unique minimizer. The optimality condition reads

$$\mathsf{A}_0\big(\mathbf{k} \odot \mathbf{v}(\mathbf{k}) - \mathbf{e}_0\big) \cdot (\mathbf{k} \odot \mathbf{v}) = \mathbf{0}, \quad \forall \mathbf{v} \in \mathbb{R}^n. \tag{89}$$

Then, introducing $\boldsymbol{\xi}(\mathbf{k}) = \mathbf{k} \odot \mathbf{v}(\mathbf{k})$ and setting $\mathbf{v} = \mathbf{v}(\mathbf{k})$ in (89) lead to

$$\mathsf{A}_0\big(\boldsymbol{\xi}(\mathbf{k}) - \mathbf{e}_0\big) \cdot \big(\boldsymbol{\xi}(\mathbf{k}) - \mathbf{e}_0\big) = \mathsf{A}_0 \mathbf{e}_0 \cdot \mathbf{e}_0 - \mathsf{A}_0 \boldsymbol{\xi}(\mathbf{k}) \cdot \boldsymbol{\xi}(\mathbf{k}).$$

Therefore, (88) becomes

$$\mathsf{A}_0 \mathbf{e}_0 \cdot \mathbf{e}_0 - \max_{\mathbf{k} \in \mathbb{S}^n} \mathsf{A}_0 \boldsymbol{\xi}(\mathbf{k}) \cdot \boldsymbol{\xi}(\mathbf{k}) > \frac{1}{2}\mathsf{S}_0'' \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_0. \tag{90}$$

Comparing with Corollary 1 and Lemma 1, we have thus proved

Proposition 10 *For a stress-softening behavior, a damaging state is directionally stable under displacement controlled and independently of the size and the shape of the domain if and only the strong ellipticity condition holds at this state.*

It appears that the strong ellipticity condition (SE) is a particular case of directional stability. Equation (SE) is made to study the stability of homogeneous states only in the cases where the size of the domain is much larger than the characteristic length of the material. (SE) is, by nature, unable to give the critical size under which the homogeneous is stable (except, of course, in the case where this critical size is infinite). Moreover, (SE) is unable to discriminate between the different types of boundary conditions. In conclusion, we can affirm that our directional stability criterion is more general and richer.

7 Concluding Remarks

A stability analysis based on the selection of unilateral local minima of the total energy has been carried out for a class of gradient damage models. We have studied in which situations a homogeneous state can be stable and hence can be observed in experimental tests. Let us see how such homogeneous tests can be useful in practice to identify the damage law of a stress-softening material. Note that the measurement of homogeneous states gives only access to a part of the damage law. Indeed, these states depend only on the parameter w_1 and on the stiffness function $\alpha \mapsto \mathsf{A}(\alpha)$, but not on the state function $\alpha \mapsto \ell(\alpha)$ involved in the gradient damage terms. On the other hand, non-homogeneous states such as localized damage response depend on all the damage law and are only accessible by numerical computations.

We now first summarize how one can observe homogeneous responses. It turns out for a stress-softening material that the stability of a damaging homogeneous state depends essentially on how the boundary is controlled. When the surface forces are prescribed on the whole boundary by a soft device, the state is necessarily unstable. Accordingly, to observe such a state, one must control the displacements on all or a part of the boundary. It appears that the more the displacements are controlled, the more the state has a chance to be stable. In the case where the displacements are controlled on the whole boundary by a hard device, the state is necessarily stable for small enough samples. Moreover, the fact that the stability holds for any size is independent of the shape of the sample and depends only on the material (and on the state). As shown in the examples of Sect. 4, Poisson's ratio seems to have a significant influence on the stability. It also appears that spherical strain or stress states have more chance to be stable than uniaxial ones, even though this property has to be confirmed by a more thorough analysis. Regarding the shape effects, those latter play an important role in uniaxial tests, but a better understanding of this dependency needs also further investigations.

Finally, let us give more insights on how one can have access to the state function $\alpha \mapsto \ell(\alpha)$. As explained in Pham et al. [24] in a one-dimensional setting, detecting the stability loss of the homogeneous state when one changes the size of the sample gives information on the state function $\alpha \mapsto \ell(\alpha)$. More specifically, as shown in Sect. 5 for very slender cylinders, the critical size beyond which the homogeneous state becomes unstable involves all the damage law and is proportional to $\ell(\alpha)$. Accordingly, the measurement of this critical size provides the expected complementary information. In fact, it was shown

in Pham et al. [24] that this measurement, together with the homogeneous stress-strain response, is sufficient to identify all the constitutive functions and parameters, at least in a one-dimensional setting. However, to extend this result to the three-dimensional case for which the critical size $L_c(\boldsymbol{\varepsilon}_0, \Omega_1)$ cannot be obtained in a closed form, such detection experiments of the stability loss of homogeneous states should be coupled with numerical computations.

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