J Elast (2012) 108:193–207
DOI 10.1007/s10659-011-9364-z

Mechanical Quadrature Methods and Extrapolation Algorithms for Boundary Integral Equations with Linear Boundary Conditions in Elasticity

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Received: 29 March 2011 / Published online: 26 November 2011

Abstract By potential theory, elastic problems with linear boundary conditions are converted into boundary integral equations (BIEs) with logarithmic and Cauchy singularity. In this paper, a mechanical quadrature method (MQMs) is presented to deal with the logarithmic and the Cauchy singularity simultaneously for solving the boundary integral equations. The convergence and stability are proved based on Anselone's collective compact and asymptotical compact theory. Furthermore, an asymptotic expansion with odd powers of errors is presented, which possesses high accuracy order $O(h^3)$. Using h^3-Richardson extrapolation algorithms (EAs), the accuracy order of the approximation can be greatly improved to $O(h^5)$, and an a posteriori error estimate can be obtained for constructing a self-adaptive algorithm. The efficiency of the algorithm is illustrated by examples.

Keywords Mechanical quadrature method · Asymptotic expansion · Extrapolation algorithm · A posteriori error estimate · Elasticity

Mathematics Subject Classification (2000) 65N25 · 65N38

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1 Introduction

Consider elastic equations with linear boundary conditions: to find a non-zero displacement $\bar{u} = (\bar{u}_1, \bar{u}_2)^T$ in the domain Ω and on the boundary Γ satisfying

$$\begin{cases} \sigma_{ij,j} = \mu \Delta \bar{u} + (\lambda + \mu) \text{graddiv } \bar{u} = 0, & \text{in } \Omega, \\ p = (\sigma_{1j} n_j, \sigma_{2j} n_j)^T = -c\bar{u} + \bar{g}, & \text{on } \Gamma, i, j = 1, 2, \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded, simply connected domain with a smooth boundary Γ, σ_{ij} is the stress tensor, λ and μ are Láme constants, $p = (p_1, p_2)^T$ is a traction vector, $n = (n_1, n_2)$ is a unit outward normal on Γ, $c = \text{diag}(c_{11}, c_{22})$ is a constant matrix with $c_{ii} > 0$, $i = 1, 2$, and the function $\bar{g} = (\bar{g}_1, \bar{g}_2)^T$ is assumed given and continuous on Γ. Following vector computational rules, the repeated subscripts imply the summation from 1 to 2.

By means of potential theory, (1) are converted into the following boundary integral equations [2, 4] (BIEs):

$$\gamma_{ij}(y)\bar{u}_j(y) + \int_\Gamma k^*_{ij}(y, x)\bar{u}_j(x)ds_x = \int_\Gamma h^*_{ij}(y, x)p_j(x)ds_x, \tag{2}$$

where $\gamma_{ij}(y) = \delta_{ij}/2$ when $y = (y_1, y_2)$ is on a smooth part of the boundary Γ with the Kronecker delta δ_{ij}, and

$$\begin{cases} h^*_{ij} = \frac{1}{8\pi\mu(1-\nu)}[-(3-4\nu)\delta_{ij} \ln r + r_{.i} r_{.j}], \\ k^*_{ij} = \frac{1}{4\pi(1-\nu)r}[\frac{\partial r}{\partial n}((1-2\nu)\delta_{ij} + 2r_{.i} r_{.j}) \\ \qquad + (1-2\nu)(n_i r_{.j} - n_j r_{.i})], \end{cases}$$

are Kelvin's fundamental solutions [2, 25], $\nu = \lambda/[2(\lambda + \mu)]$ is the Poisson ratio , $r_{.i}$ is the derivative with respect to x_i, and $r = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$ is the distance between x and y. (2) are obvious singular integral equations. In particular, the second term of the equations represents a Cauchy singularity and the third term is a logarithmic singularity.

A similar setup has been proposed for Laplace [26] equations and for Helmholtz [14] equations in order to obtain high accuracy order $O(h^3)$. The difficulty in solving the integral equations (2) is in dealing with the Cauchy singularity and the logarithmic singularity simultaneously. The elastic equations have been wildly applied in many physical circumstances, such as for cantilever beams, plates with edge notch, edge crack, and so on [3].

After solving (2) for $\bar{u}$ on Γ, the displacement vector and stress tensor in Ω can be calculated [4, 19]

$$\begin{cases} \bar{u}_i(y) = \int_\Gamma h^*_{ij}(y, x)p_j(x)ds_x \\ \qquad - \int_\Gamma k^*_{ij}(y, x)\bar{u}_j(x)ds_x, \quad \forall y \in \Omega, \\ \sigma_{ij}(y) = \int_\Gamma h^*_{ijl}(y, x)p_l(x)ds_x \\ \qquad - \int_\Gamma k^*_{ijl}(y, x)\bar{u}_l(x)ds_x, \quad \forall y \in \Omega, \end{cases} \tag{3}$$

where

$$\begin{cases} k^*_{ijl} = \frac{(1-2\nu)(r_{.j}\delta_{li}+r_{.i}\delta_{lj}-r_{.l}\delta_{ij})+2r_{.i}r_{.j}r_{.l}}{4\pi(1-\nu)r}, \\ h^*_{ijl} = \frac{\mu}{2\pi(1-\nu)r^2}\{2\frac{\partial r}{\partial n}[(1-2\nu)r_{.l}\delta_{ij}+\nu(r_{.j}\delta_{il} \\ \qquad +r_{.i}\delta_{jl})-4r_{.i}r_{.j}r_{.l}]+2\nu(n_i r_{.j}r_{.l}+n_j r_{.i}r_{.l}) \\ \qquad +(1-2\nu)(2n_l r_{.j}r_{.i}+n_j\delta_{il}+n_i\delta_{jl}) \\ \qquad -(1-4\nu)n_l\delta_{ij}\}. \end{cases}$$

Some numerical methods, such as Galerkin methods, collocation methods, Least-squares methods and boundary element methods, often have been applied to solve the differential equations system. The convergent rates of these methods are usually $O(h)$ and $O(h^2)$. Hu and Shi [15] established rectangular nonconforming mixed finite element methods for linear elasticity. Talbot and Crampton [24] approached 2D vibrational problems by a pseudo-spectral method and they transformed the governing partial differential equations into a matrix eigenvalue problem, which is solved by a collocation method. Cai et al. [5] introduced a least-square method for obtaining the solution of linear elastic problems. Chen and Hong [6, 10] reviewed the dual boundary element methods especially for hypersingular integrals, and dealt with dual boundary integral equations in elasticity characterized by geometry degeneracy. Kuo, Chen and Huang [16] used dual boundary element methods to solve true and spurious eigensolutions of a circular cavity problem. Helsing [9] solved mixed boundary conditions elliptic problems by integral equation methods. Li and Nie [17] considered boundary integral methods for solving stressed axisymmetric rod problems. Sidi [22] showed the priority of the quadrature formulas [21] for weakly singular integral equations.

Extrapolation algorithms (EAs) based on asymptotic expansion about errors are effective parallel algorithms, which possesses high accuracy degree, good stability and almost optimal computational complexity. Cheng, Huang and Zeng [7, 8] used extrapolation algorithms to obtain high accuracy order for the Steklov eigenvalue for the Laplace equation. Xu and Zhao [26] established an extrapolation method for solving BIEs related to the Laplace equation of the third kind boundary condition. Huang and Lü established extrapolation algorithms for solving the Steklov eigenvalue problem [12], the Helmholtz equation [14] and the Laplace equation [19].

We firstly use the Sidi quadrature rules [13, 20, 21] to approximate the logarithmic and Cauchy singular operators in (2). Secondly, by Anselone's collective compact and asymptotically compact theory [1], we prove the convergence rate with $O(h^3)$. Finally, based on the asymptotic expansion of errors with odd powers, we establish EAs. After h^3-extrapolation, we obtain the convergence rate with $O(h^5)$. So we not only greatly improve the accuracy of the approximation, but also derive a posteriori error estimate for constructing self-adaptive algorithms. Numerical examples support our algorithms and show that the MQMs are fit for practice.

This paper is organized as follows: in Sect. 2 we construct the MQMs to deal with the logarithmic and Cauchy singularity and give the proof about convergence; in Sect. 3 we obtain an asymptotic expansion of the errors to construct EAs; in Sect. 4 numerical examples show the significance of the algorithms.

2 Mechanical Quadrature Methods

We define boundary integral operators on Γ as follows:

$$\begin{cases} (K_{ij}w)(y) = \int_\Gamma k^*_{ij}(y,x)w(x)ds_x, & y \in \Gamma, i,j = 1,2, \\ (H_{ij}w)(y) = \int_\Gamma h^*_{ij}(y,x)w(x)ds_x, & y \in \Gamma, i,j = 1,2. \end{cases}$$

Also, we convert (2) into the following operator equations:

$$W\begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}, \tag{4}$$

where

$$W = \begin{bmatrix} \frac{1}{2}I_0 + K_{11} + c_{11}H_{11} & K_{12} + c_{11}H_{12} \\ K_{21} + c_{22}H_{21} & \frac{1}{2}I_0 + K_{22} + c_{22}H_{22} \end{bmatrix},$$

$g_i = H_{ij}\bar{g}_j, \quad i,j = 1,2$, and I_0 is an identity operator.

Assume that Γ is a smooth closed curve described by a regular parameter mapping $x(s) = (x_1(s), x_2(s)) : [0, 2\pi] \to \Gamma$, satisfying $|x'(s)|^2 = |x_1'(s)|^2 + |x_2'(s)|^2 > 0$. Let $C^{2m}[0, 2\pi]$ denote the set of $2m$ times differentiable periodic functions with the periodic 2π and $x_i(s) \in C^{2m}[0, 2\pi]$, $i = 1, 2$. Define the following integral operators on $C^{2m}[0, 2\pi]$:

$$(A_0\omega)(t) = \int_0^{2\pi} a_0(t,\tau)\omega(\tau)|x'(\tau)|d\tau,$$

with $a_0(t,\tau) = \bar{c}_0 \ln|2e^{-1/2}\sin\frac{t-\tau}{2}|$, $\bar{c}_0 = -(3-4\nu)/[8\pi\mu(1-\nu)]$ and

$$(B_0\omega)(t) = \int_0^{2\pi} b_0(t,\tau)\omega(\tau)|x'(\tau)|d\tau,$$

with $b_0(t,\tau) = \bar{c}_0[\ln|x(t)-x(\tau)| - \ln|2e^{-1/2}\sin\frac{t-\tau}{2}|]$, and

$$(B_{ij}\omega)(t) = \int_0^{2\pi} b_{ij}(t,\tau)\omega(\tau)|x'(\tau)|d\tau,$$

with $b_{ij}(t,\tau) = c_1 r_{.i} r_{.j}$, $c_1 = 1/[8\pi\mu(1-\nu)]$, and

$$(C_0\omega)(t) = \int_0^{2\pi} c_0(t,\tau)\omega(\tau)|x'(\tau)|d\tau,$$

with $c_0(t,\tau) = c_2(n_1 r_{.2} - n_2 r_{.1})/r$, $c_2 = -(1-2\nu)/[4\pi(1-\nu)]$, and

$$(M_{ii}\omega)(t) = \int_0^{2\pi} m_{ii}(t,\tau)\omega(\tau)|x'(\tau)|d\tau, \quad i = 1,2,$$

with $m_{ii}(t,\tau) = c_3\frac{\partial r}{\partial n}[(1-2\upsilon)+2r_{.i}r_{.i}]/r$, $c_3 = -1/[4\pi(1-\nu)]$, and

$$(M_{ij}\omega)(t) = \int_0^{2\pi} m_{ij}(t,\tau)\omega(\tau)|x'(\tau)|d\tau, \quad i,j = 1,2, i \neq j,$$

with $m_{ij}(t,\tau) = c_3 \frac{\partial r}{\partial n}(2r_{.i}r_{.j})/r$.

Then (4) is equivalent to

$$\left(\frac{1}{2}I + C + A + B + M\right)u = f, \tag{5}$$

where $u(t) = \bar{u}(x(t))$, $f(t) = (f_1(t), f_2(t))^T = g(x(t))$ and

$$I = \begin{bmatrix} I_0 & 0 \\ 0 & I_0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & C_0 \\ -C_0 & 0 \end{bmatrix}, \qquad A = c\begin{bmatrix} A_0 & 0 \\ 0 & A_0 \end{bmatrix},$$

$$B = c\begin{bmatrix} B_0 + B_{11} & B_{12} \\ B_{21} & B_0 + B_{22} \end{bmatrix}, \qquad M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

As $t \to s$, depending on the properties of the kernel $a_0(t,\tau)$ and using a Taylor expansion, we know that A_0 is a logarithmic singular operator. Because

$$\frac{n_i r_{.j} - n_j r_{.i}}{r} = (-1)^i \frac{1 + O(t-s)}{(t-s) + O(t-s)}, \quad i \neq j,$$

we see that C_0 is a Cauchy singularity operator. Moreover, B_0, B_{ij}, M_{ij} are smooth operators.

2.1 Nyström's Approximation

Let $h = \pi/n$, $(n \in N)$ be the mesh width and $t_j = \tau_j = jh$, $(j = 0, 1, \ldots, 2n-1)$ be the nodes. Since B_0, B_{ij}, M_{ij} are smooth integral operators with the period 2π, we obtain high accuracy Nyström's approximations by the trapezoidal rule [2, 21, 23]. For example, the Nyström approximation operator B_0^h of B_0 is defined as:

$$(B_0^h \omega)(t) = h\sum_{j=0}^{2n-1} b_0(t,\tau_j)\omega(\tau_j), \tag{6}$$

and the error is

$$(B_0\omega)(t) - (B_0^h\omega)(t) = O(h^{2m}). \tag{7}$$

The Nyström approximation B_{ij}^h of B_{ij} and M_{ij}^h of M_{ij} can be defined similarly.

The continuous approximation kernel $a_n(t,\tau)$ of the logarithmic singular operator A_0 is defined as:

$$a_n(t,\tau) = \begin{cases} a_0(t,\tau), & \text{for } |t-\tau| \geq h, \\ \bar{c}_0 h \ln|e^{-1/2}\frac{h}{2\pi}|, & \text{for } |t-\tau| < h. \end{cases}$$

By Sidi's quadrature rules [21, 23], its Nyström approximation operator can be defined as:

$$(A_0^h\omega)(t) = h\sum_{j=0}^{2n-1} a_n(t,\tau_j)\omega(\tau_j)|x'(\tau_j)|, \tag{8}$$

which has the following error estimate

$$(A_0\omega)(t) - (A_0^h\omega)(t) = 2\sum_{\mu=1}^{m-1}\frac{\varsigma'(-2\mu)}{(2\mu)!}\omega^{(2\mu)}(t)h^{2\mu+1} + O(h^{2m}), \tag{9}$$

where $\varsigma'(t)$ is the derivative of the Riemann zeta function.

Because C_0 is a Cauchy singular operator, its Nyström's approximation operator C_0^h can be defined [20] as:

$$(C_0^h\omega)(t_i) = 2c_2 a_1(t_i, t_i)h \sum_{j=0}^{2n-1} \cot((t_j - t_i)/2)\omega(t_j)|x'(t_j)|\varepsilon_{ij}, \tag{10}$$

where $a_1(t, s) = 2[(t - s) + O(t - s)]^{-1}\tan((t - s)/2)$ when $s \to t$, and

$$\varepsilon_{ij} = \begin{cases} 1, & \text{if } |i - j| \text{ is odd number,} \\ 0, & \text{if } |i - j| \text{ is even number.} \end{cases}$$

The error estimate of the operator is [20]

$$(C_0\omega)(t_i) - (C_0^h\omega)(t_i) = O(h^{2m}). \tag{11}$$

Thus we obtain the numerical approximation of (5):

$$\left(\frac{1}{2}I + C^h + A^h + B^h + M^h\right)u_h = f_h, \tag{12}$$

where A^h, B^h, C^h and M^h are discrete matrices of order $4n$ corresponding to the operators A, B, C and M, respectively.

2.2 The Asymptotically Compact Convergence

Since the approximate operators are no longer in the field of projection theory, the existence and convergence about the numerical approximations have to be studied by collectively compact convergent [1] and asymptotically compact convergent theory. Define $D_1^h = \operatorname{diag}(a_1(t_0, t_0), \ldots, a_1(t_{2n-1}, t_{2n-1}))$, and define the circulant matrix C_1^h as:

$$C_1^h = 2h \text{ circulant}\left(0, -\cot\frac{\pi}{2n}, 0, \ldots, 0, -\cot\frac{(2n-1)\pi}{2n}\right).$$

Then let

$$D^h = \operatorname{diag}(D_1^h, D_1^h), \qquad C_2^h = \begin{pmatrix} 0 & C_1^h \\ -C_1^h & 0 \end{pmatrix},$$

and so

$$C^h = c_2 D^h C_2^h = \begin{pmatrix} 0 & c_2 D_1^h C_1^h \\ -c_2 D_1^h C_1^h & 0 \end{pmatrix}.$$

Lemma 1 [14] *The eigenvalues of the matrix C_1^h consist of*

$$\rho_k = \begin{cases} 0, & \text{if } k = 0, n; \\ 2\pi i, & \text{if } 1 \le k \le n - 1; \\ -2\pi i, & \text{if } n + 1 \le k \le 2n - 1; \end{cases}$$

where $i = \sqrt{-1}$.

Corollary 1 *The eigenvalues of C_2^h consist of* 0 *and* $\pm 2\pi$.

Corollary 2 $(1/2)I + C_2^h$ *is invertible, and* $((1/2)I + C_2^h)^{-1}$ *is uniformly bounded.*

Lemma 2 [11] *Let Y, Z be regular matrices of order m, and $X = Y + Z$. Then*

$$|\lambda(X) - \lambda_j(Z)| \leq \max_{1\leq j\leq m} |\lambda_j(Y)|, \quad 1 \leq j \leq m,$$

where $\lambda(X)$, $\lambda(Z)$ and $\lambda(Y)$ are the eigenvalues of matrices X, Z and Y, respectively. Especially, if a complex number β cannot satisfy

$$|\beta - \lambda_j(Z)| \leq \max_{1\leq j\leq m} |\lambda_j(Y)|, \quad 1 \leq j \leq m, \tag{13}$$

then β is not the eigenvalue of matrix X.

Corollary 3 $(1/2)I + C^h$ *is invertible and* $((1/2)I + C^h)^{-1}$ *is uniformly bounded.*

Proof First, we have the property

$$\frac{1}{2}I + C^h = \frac{1}{2}(I + 2c_2 D^h C_2^h) = \frac{1}{2}D^h((D^h)^{-1} + 2c_2 C_2^h).$$

Next, we discuss the eigenvalues of $(D^h)^{-1} + 2c_2 C_2^h$. Since

$$\frac{1}{a_1(t,t)} = 1 > \frac{1-2\nu}{1-\nu} \geq 2c_2 \max_{1\leq j\leq 4n} |\lambda_j(C_2^h)|,$$

then for any real number $\alpha \in (0, \nu/(1-\nu))$, we have

$$\left|\frac{1}{a_1(t,t)} - \alpha\right| \geq 1 - \alpha > \frac{1-2\nu}{1-\nu} \geq 2c_2 \max_{1\leq j\leq 4n} |\lambda_j(C_2^h)|.$$

From Lemma 2 we obtain $\rho((D^h)^{-1} + 2c_2 C_2^h) > \nu/(1-\nu)$. This means that $\|((D^h)^{-1} + 2c_2 C_2^h)^{-1}\| \leq (1-\nu)/\nu$. Also since D^h is invertible and uniformly bounded, $(1/2)I + C^h$ is invertible and uniformly bounded. □

Then (5) and (12) can be rewritten as follows: find $u \in V^{(0)}$ which satisfies

$$(I + L)u = \bar{f}, \tag{14}$$

and find u_h which satisfies

$$(I + L^h)u_h = \bar{f}_h, \tag{15}$$

where $\bar{f} = (\frac{1}{2}I + C)^{-1} f$, $\bar{f}_h = (\frac{1}{2}I + C^h)^{-1} f_h$, $L = (\frac{1}{2}I + C)^{-1}(A + B + M)$, $L^h = (\frac{1}{2}I + C^h)^{-1}(A^h + B^h + M^h)$, and the space $V^{(m)} = C^{(m)}[0, 2\pi] \times C^{(m)}[0, 2\pi]$, $m = 0, 1, 2, \ldots$.

Theorem 1 *The approximate operator sequence $\{L^h\}$ is an asymptotically compact sequence and convergent to L in $V^{(0)}$, i.e.,*

$$L^h \stackrel{a.c}{\to} L, \tag{16}$$

where $\stackrel{a.c}{\to}$ denotes asymptotically compact convergence.

Proof Since the kernels of B_0 B_{ij} and M_{ij} $(i, j = 1, 2)$ are continuous functions, we have the collectively compact convergence [19, 21, 23]

$$B_0^h \stackrel{c.c}{\to} B_0, \quad B_{ij}^h \stackrel{c.c}{\to} B_{ij}, \quad \text{and} \quad M_{ij}^h \stackrel{c.c}{\to} M_{ij} \text{ in } C[0, 2\pi], \quad \text{as } n \to \infty.$$

Also since $a_n(t, \tau)$ is a continuous approximate of $a(t, \tau)$, the approximate operator $\{A_0^h\}$ is an asymptotically compact sequence, convergent to A_0 (see [26]), i.e., $A_0^h \stackrel{a.c}{\to} A_0$ in $C[0, 2\pi]$, as $n \to \infty$. Then in $V^{(0)}$ we have

$$A^h \stackrel{a.c}{\to} A,$$

and

$$B^h + M^h \stackrel{c.c}{\to} B + M.$$

This implies that for any bounded sequence $\{y_m \in V^{(0)}\}$ there exists a convergent subsequence in $\{ (A^h + B^h + M^h)y_m\}$. Without loss of generality, we assume $(A^h + B^h + M^h)y_m \to z$, as $m \to \infty$. From the properties of asymptotically compact convergence and quadrature rules [19, 21, 23], we have

$$\begin{aligned} \left\| L^h y_m - \left(\frac{1}{2}I + C\right)^{-1} z \right\| &\le \left\| \left(\frac{1}{2}I + C^h\right)^{-1} \right\| \|(A^h + B^h + M^h)y_m - z\| \\ &\quad + \left\| \left(\frac{1}{2}I + C^h\right)^{-1} (C - C^h) \left(\frac{1}{2}I + C\right)^{-1} z \right\| \to 0, \end{aligned}$$

as $m \to \infty$ and $h \to 0$,

where $\|\cdot\|$ is the norm of $\$(V^{(0)}, V^{(0)})$. This shows that $\{L^h : V^{(0)} \to V^{(0)}\}$ is an asymptotically compact operator sequence.

Moreover, we will show that $\{L^h\}$ is pointwise convergent to L, as $n \to \infty$. In fact, since $A^h + B^h + M^h \stackrel{a.c}{\to} A + B + M$ for $\forall y \in V^{(0)}$, we obtain

$$\|(A^h + B^h + M^h)y - (A + B + M)y\| \to 0, \quad \text{as } h \to 0.$$

From Corollary 3 and quadrature rules [19, 21, 23], we derive

$$\begin{aligned} \|L^h y - Ly\| &\le \left\| \left(\frac{1}{2}I + C^h\right)^{-1} \right\| \cdot \|(A^h + B^h + M^h)y - (A + B + M)y\| \\ &\quad + \left\| \left(\frac{1}{2}I + C^h\right)^{-1} (C^h - C) \left(\frac{1}{2}I + C\right)^{-1} (A + B + M)y \right\| \to 0, \end{aligned}$$

as $h \to 0$.

Since the $\{L^h\}$ is an asymptotically compact sequence and pointwise convergent to L, the proof of Theorem 1 is completed. □

Corollary 4 [14] *Under the assumption of Theorem 1, we have*

$$\|(L^h - L)L\| \to 0 \quad \textit{and} \quad \|(L^h - L)L^h\| \to 0, \quad \textit{as } h \to 0.$$

Corollary 5 [14] *Assume that h is sufficiently small, then there exists a unique solution u_h in (29). Under the norm of $V^{(2m)}[0, 2\pi]$, we have the following error bound:*

$$\|u_h - u\| \leq \|(I+L)^{-1}\| \frac{\|(L^h - L)\bar{f}\| + \|(L^h - L)L^h u\|}{1 - \|(I+L^h)^{-1}(L^h - L)L^h\|}.$$

3 Asymptotic Expansions of Errors and Extrapolation Algorithms

3.1 Asymptotic Expansions

Theorem 2 *Suppose $u(s) \in V^{(2m)}$, then we have the following asymptotic expansion*

$$(L^h - L)u(s) = \sum_{j=1}^{m-1} \psi_j(s) h^{2j+1} + O(h^{2m}), \tag{17}$$

where $\psi_j(s) \in V^{(2m-2j)}$, $j = 1, \ldots, m-1$, are functions independent of h.

Proof By Sidi's quadrature rules [21, 23], there exist $\omega^{(2\mu)} \in V^{(2m-2\mu)}$, $\mu = 1, \ldots, m-1$ satisfying the error expansion for the logarithmic singular operator A:

$$(A\omega)(t) - (A^h\omega)(t) = 2\sum_{\mu=1}^{m-1} \frac{\varsigma'(-2\mu)}{(2\mu)!} \omega^{(2\mu)}(t) h^{2\mu+1} + O(h^{2m}). \tag{18}$$

Since B and M are smooth operators, following (7) and (18), we derive

$$(A + B + M)u(t) - (A^h + B^h + M^h)u(t) = \sum_{j=1}^{m-1} \varphi_j(t) h^{2j+1} + O(h^{2m}), \tag{19}$$

where $\varphi_j(t) = \frac{\varsigma'(-2j)}{(2j)!} u^{(2j)}(t) \in V^{(2m-2j)}$, $j = 1, \ldots, m-1$, are functions independent of h.

We also have an error estimate according to the trapezoidal rule for Cauchy singular operator

$$\max_{0 \leq s \leq 2\pi} |(C - C^h)\phi(s)| = \|(C - C^h)\phi\| = O(h^{2m}), \quad \forall \phi \in V^{(2m)}, \tag{20}$$

and the identity

$$\begin{aligned} L^h u - Lu = {} & \left(\frac{1}{2}I + C\right)^{-1} \cdot \left((A^h + B^h + M^h)u - (A + B + M)u\right) \\ & + \left(\frac{1}{2}I + C^h\right)^{-1} (C - C^h) \left(\frac{1}{2}I + C\right)^{-1} \\ & \times \left((A^h + B^h + M^h)u - (A + B + M)u\right) \\ & + \left(\frac{1}{2}I + C^h\right)^{-1} (C - C^h) \left(\frac{1}{2}I + C\right)^{-1} \cdot (A + B + M)u. \end{aligned}$$

Substituting (19), (20) into the above equation, and letting $\psi_j(s) = (\frac{1}{2}I + C)^{-1}\varphi_j(s)$, we complete the proof of Theorem 2. □

Theorem 3 *Suppose the hypothesis of Theorem 2 holds and $x(t), g(t) \in V^{2m}[0, 2\pi]$, Then there exists functions $\bar{\omega}_l \in V^{2m-2l}[0, 2\pi], l = 1, \ldots, m$ independent of h, such that*

$$(u - u_h)|_{t=t_j} = \sum_{l=1}^{m-1} h^{2l+1}\bar{\omega}_l|_{t=t_j} + O(h^{2l}). \tag{21}$$

Proof Because $((1/2)I + C^h)^{-1}$ is uniformly bounded, and $A^h \stackrel{a.c}{\to} A$, $B^h \stackrel{c.c}{\to} B$ in $V^{(0)}$, then there exists the asymptotic expansion

$$(\bar{f} - \bar{f}_h)|_{t=t_j} = h^3\omega_1|_{t=t_j} + h^5\omega_2|_{t=t_j} + \ldots + O(h^{2m}), \tag{22}$$

where $\omega_l \in V^{2m-2l}[0, 2\pi], l = 1, \ldots, m-1$ and $\bar{f} = (\frac{1}{2}I + C)^{-1}(A + B)\bar{g}(x(t))$.

Because u and u_h satisfy (14) and (15), respectively, we obtain

$$\begin{aligned}(I + L^h)(u_h - u)|_{t=t_j} &= \Big[(I + L^h)u_h - (I + L)u + (I + L)u - (I + L^h)u\Big]\Big|_{t=t_j} \\ &= (\bar{f}_h - \bar{f})|_{t=t_j} + (L - L^h)u|_{t=t_j} \\ &= h^3\phi_1|_{t=t_j} + h^5\phi_2|_{t=t_j} + \ldots + O(h^{2m}),\end{aligned} \tag{23}$$

where $\phi_l \in V^{2m-2l}[0, 2\pi]$.

Define an auxiliary equation

$$(I + L)\bar{\omega}_l = \phi_l, \quad l = 1, \ldots, m-1, \tag{24}$$

and its approximate equation

$$(I + L^h)\bar{\omega}_l^h = \phi_l^h, \quad l = 1, \ldots, m-1. \tag{25}$$

Substituting (25) into (23) we find

$$(I + L^h)\left(u_h - u - \sum_{l=1}^{m-1} h^{2l+1}\bar{\omega}_l^h\right)\Bigg|_{t=t_j} = O(h^{2m}). \tag{26}$$

Noticing $\bar{\omega}_l^h \in V^{2m-2l}[0, 2\pi]$, we obtain

$$(\bar{\omega}_l - \bar{\omega}_l^h)(t_i) = O(h^{2m-2l}). \tag{27}$$

Then, substituting $\bar{\omega}_l^h$ with $\bar{\omega}_l$ and following Theorem 1, we have

$$\left[u_h - u - \sum_{l=1}^{m-1} h^{2l+1}\bar{\omega}_l\right]\Bigg|_{t=t_j} = O(h^{2m}), \tag{28}$$

and we complete the proof. □

3.2 Extrapolation Algorithms

The asymptotic expansion (21) implies that the extrapolation algorithms can be applied to the solution of (2) to improve the approximate order. Moreover, the high accuracy order $O(h^5)$ can be obtained by computing some coarse grids on Γ in parallel. The EAs are described as follows:

Taking h and $h/2$, and solving (12) in parallel, we obtain that $u_h(t_i), u_{h/2}(t_i)$ are the solutions on Γ.

Compute the solution at coarse grid points, so the EAs [14, 18] are given by

$$u_h^*(t_i) = \frac{1}{7}(8u_{h/2}(t_i) - u_h(t_i)), \tag{29}$$

and the error is $|u_h^*(t_i) - u(t_i)| = O(h^5)$;

Following (21), a higher accuracy order also can be achieved by EAs:

$$\bar{u}_h^*(t_i) = \frac{1}{31}(32u_{h/2}^*(t_i) - u_h^*(t_i)), \tag{30}$$

and the error is $|\bar{u}_h^*(t_i) - u(t_i)| = O(h^7)$.

Moreover, using $|u^*(t_i) - u(t_i)| = O(h^5)$, we obtain the a posteriori error estimate

$$\begin{aligned}
&|u(t_i) - u_{h/2}(t_i)| \\
&\quad\le |u(t_i) - \frac{1}{7}(8u_{h/2}(t_i) - u_h(t_i))| \\
&\quad+ \frac{1}{7}|u_{h/2}(t_i) - u_h(t_i)| \\
&\quad\le \frac{1}{7}|u_{h/2}(t_i) - u_h(t_i)| + O(h^5).
\end{aligned}$$

Note that this can be used to construct self-adaptive algorithms.

4 Numerical Examples

We first introduce some notation for $i = 1, 2$: $e_i^h(P) = |u_{ih}(P) - u_i(P)|$ is the error of the displacement; $r_i^h(P) = e_i^h(P)/e_i^{h/2}(P)$ is the error ratio; $\bar{e}_i^h(P) = |u_{ih}^*(P) - u_i^*(P)|$ is the error after one-step EAs; and $p_i^h(P) = \frac{1}{7}|u_{ih/2}(P) - u_{ih}(P)|$ is the a posteriori error estimate.

Example 1 Consider a circular isotropic elastic body Ω with radius $a = 1$ in plane strain deformation. Its boundary Γ is described as $x = \cos(t), y = \sin(t), t \in [0, 2\pi]$. Let $\lambda = \mu = 2.5, \nu = 1/4$, the coefficient matrix $c = \mathrm{diag}(1, 1)$, and $g_1 = \cos(t), g_2 = 2\sin(t), t \in [0, 2\pi]$.

We calculate the boundary numerical solutions $u_h = (u_{1h}, u_{2h})^T$ on Γ following (12). Table 1 lists the approximate values of $u_{1h}(P)$ at points $P_1 = (\cos 0, \sin 0)$ and $P_2 = (\cos\frac{\pi}{4}, \sin\frac{\pi}{4})$. Table 2 lists the approximate values of $u_{2h}(P)$ at points $P_2 = (\cos\frac{\pi}{4}, \sin\frac{\pi}{4})$ and $P_3 = (\cos\frac{\pi}{2}, \sin\frac{\pi}{2})$.

From Tables 1–2, we can numerically see:

$$\log_2 r_i^h(P) \approx 3,$$

and

$$\log_2 \bar{r}_i^h(P) \approx 5,$$

Table 1 The errors, errors ratio of $u_{1h}(P)$ at points $P = P_1, P_2$

n	4	8	16	32	64	128
$e_1^h(P_1)$	1.033E-3	1.278E-4	1.594E-5	1.991E-6	2.488E-7	3.110E-8
$r_1^h(P_1)$		8.081	8.020	8.005	8.001	8.000
$\bar{e}_1^h(P_1)$		1.48E-06	4.59E-08	1.43E-09	4.48E-11	1.40E-12
$p_1^h(P_1)$		1.293E-4	1.598E-5	1.992E-6	2.488E-7	3.110E-8
$e_1^h(P_2)$	7.303E-4	9.037E-5	1.127E-5	1.408E-6	1.759E-7	2.199E-8
$r_1^h(P_2)$		8.081	8.020	8.005	8.001	8.000
$\bar{e}_1^h(P_2)$		1.04E-06	3.24E-08	1.01E-09	3.17E-11	9.94E-13
$p_1^h(P_2)$		9.141E-5	1.130E-5	1.409E-6	1.760E-7	2.199E-8

Table 2 The errors, errors ratio of $u_{2h}(P)$ at points $P = P_2, P_3$

n	4	8	16	32	64	128
$e_2^h(P_2)$	1.476E-3	1.829E-4	2.281E-5	2.849E-6	3.561E-7	4.451E-8
$r_2^h(P_2)$		8.070	8.019	8.005	8.001	8.000
$\bar{e}_2^h(P_2)$		1.83E-06	6.13E-08	1.98E-09	6.31E-11	1.99E-12
$p_2^h(P_2)$		1.847E-4	2.287E-5	2.851E-6	3.561E-7	4.451E-8
$e_2^h(P_3)$	2.087E-3	2.586E-4	3.225E-5	4.029E-6	5.036E-7	6.294E-8
$r_2^h(P_3)$		8.070	8.019	8.005	8.001	8.000
$\bar{e}_2^h(P_3)$		2.59E-06	8.67E-08	2.81E-09	8.93E-11	2.81E-12
$p_2^h(P_3)$		2.612E-4	3.234E-5	4.032E-6	5.036E-7	6.294E-8

Table 3 The errors, errors ratio of $(u_{1h}, u_{2h})^T$ at points $P = P_0$

n	4	8	16	32	64	128
$e_1^h(P_0)$	9.132E-5	4.774E-6	5.953E-7	7.437E-8	9.295E-9	1.162E-9
$r_1^h(P_0)$		19.13	8.019	8.005	8.001	8.000
$\bar{e}_1^h(P_0)$		7.59E-06	1.60E-09	5.15E-11	1.64E-12	5.20E-14
$p_1^h(P_0)$		1.236E-5	5.969E-7	7.442E-8	9.296E-9	1.162E-9
$e_2^h(P_0)$	1.074E-4	1.592E-6	1.984E-7	2.479E-8	3.098E-9	3.872E-10
$r_2^h(P_0)$		67.50	8.022	8.005	8.001	8.000
$\bar{e}_2^h(P_0)$		1.35E-05	6.18E-10	1.91E-11	5.77E-13	1.73E-14
$p_2^h(P_0)$		1.512E-5	1.990E-7	2.481E-8	3.098E-9	3.872E-10

which shows that the convergent rate of the approximation solution is $O(h^3)$, and is $O(h^5)$ after EAs.

Substituting the displacement $u_h = (u_{1h}, u_{2h})^T$ on Γ into the boundary condition of (1), we'll obtain the normal derivative $p_h = (p_{1h}, p_{2h})^T$ on Γ. So following (3), we can obtain the displacement $u_h = (u_{1h}, u_{2h})^T$ in Ω. Table 3 shows approximate values of the displacement $(u_{1h}(P_0), u_{2h}(P_0))^T$ at a inner point $P_0 = (\sqrt{2}/8)(\cos\frac{\pi}{3}, \sin\frac{\pi}{3})$ in Ω.

From Tables 1–3, we can numerically see that although the h^3-Richardson extrapolation algorithms are not very complex, they are effective to obtain high accuracy approximate solutions.

Table 4 The errors, errors ratio of $u_{1h}(P)$ at points $P = P_4, P_5$

n	16	32	64	128	256	512
$e_1^h(P_4)$	1.851E-5	2.267E-6	2.820E-7	3.521E-8	4.401E-9	5.500E-10
$r_1^h(P_4)$		8.166	8.039	8.009	8.000	8.000
$\bar{e}_1^h(P_4)$		5.37E-08	1.57E-09	4.87E-11	1.43E-12	9.89E-14
$p_1^h(P_4)$		2.321E-6	2.836E-7	3.526E-8	4.401E-9	5.500E-10
$e_1^h(P_5)$	4.537E-5	5.649E-6	7.055E-7	8.816E-8	1.102E-8	1.377E-9
$r_1^h(P_5)$		8.031	8.008	8.002	8.000	8.000
$\bar{e}_1^h(P_5)$		2.48E-08	8.18E-10	2.58E-11	8.24E-13	4.16E-14
$p_1^h(P_5)$		5.674E-6	7.063E-7	8.819E-8	1.102E-8	1.377E-9

Table 5 The errors, errors ratio of $u_{2h}(P)$ at points $P = P_6, P_7$

n	16	32	64	128	256	512
$e_2^h(P_6)$	3.466E-5	4.278E-6	5.330E-7	6.657E-8	8.320E-9	1.040E-9
$r_2^h(P_6)$		8.101	8.026	8.008	8.002	8.000
$\bar{e}_2^h(P_6)$		6.20E-08	2.00E-09	6.39E-11	2.08E-12	1.06E-13
$\bar{r}_2^h(P_6)$			31.02	31.29	32.37	18.56
$p_2^h(P_6)$		4.340E-6	5.350E-7	6.664E-8	8.322E-9	1.040E-9
$e_2^h(P_7)$	1.012E-4	1.263E-5	1.578E-6	1.972E-7	2.465E-8	3.081E-9
$r_2^h(P_7)$		8.012	8.003	8.001	8.000	8.000
$\bar{e}_2^h(P_7)$		2.18E-08	7.50E-10	2.40E-11	7.53E-13	2.40E-14
$\bar{r}_2^h(P_7)$			29.00	31.22	30.93	32.40
$p_2^h(P_7)$		1.265E-5	1.579E-6	1.972E-7	2.465E-8	3.081E-9

Example 2 Consider an isotropic elliptical body Ω with $a = 3, b = 2$ in plane strain deformation. The boundary Γ is described as $x = 3\cos(t), y = 2\sin(t), t \in [0, 2\pi]$. We also let $\lambda = \mu = 2.5, \nu = 1/4$, the coefficient matrix $c = \mathrm{diag}(1, 1)$, and $g_1 = \cos(t), g_2 = 2\sin(t), t \in [0, 2\pi]$.

We calculate the boundary numerical solutions $u_h = (u_{1h}, u_{2h})^T$ on Γ following (12). Table 4 lists the approximate values of $u_{1h}(P)$ at points $P_4 = (a\cos 0, b\sin 0)$ and $P_5 = (a\cos\frac{\pi}{4}, b\sin\frac{\pi}{4})$. Table 5 lists the approximate values of $u_{2h}(P)$ at points $P_6 = (a\cos\frac{\pi}{8}, b\sin\frac{\pi}{8})$ and $P_7 = (a\cos\frac{\pi}{2}, b\sin\frac{\pi}{2})$.

From Tables 4–5, we can numerically see that $\log_2(r_i^h(P)) \approx 3$, and $\log_2(\bar{r}_i^h(P)) \approx 5$, which agree with Theorem 3.

5 Conclusion

The following conclusions can be drawn concerning the mechanical quadrature method:

(a) Computing entry of discrete matrices is simple and straightforward, without any singular integrals. The mechanical quadrature method involves a high accuracy algorithm with convergent rate $O(h^3)$. However, the analysis of the mechanical quadrature method is no longer within the framework of projection theory.

(b) The larger the scale of the problem, the more precise are the results that can be obtained according to the numerical results. The extrapolation algorithm is not very complex, but it is very effective.
(c) In this paper we only discuss the mechanical quadrature method and the EAs for problems with smooth boundary conditions. It can be viewed as the first step toward general singular problems such as those for notches, cracks, and so on.

Acknowledgements Project is supported by the National Natural Science Foundation of China (10871034), and is supported by natural science foundation project of CQ (CSTC20-10BB8270), and is partially supported by the Chongqing Jiaotong University Foundation of China (xn(2009)18).

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