J Elast (2011) 103: 289–294
DOI 10.1007/s10659-010-9276-3

RESEARCH NOTE

Further Developments of Physically Based Invariants for Nonlinear Elastic Orthotropic Solids

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Received: 8 June 2010 / Published online: 12 October 2010

Abstract Recently, Rubin and Jabareen (J. Elast. 90:1–18, 2008) introduced six physically based invariants for nonlinear elastic orthotropic solids which are measures of distortions that cause deviatoric Cauchy stress. Three of these invariants include three dependent functions that characterize the distortion in a hydrostatic state of stress. In particular, these invariants can be used without the need to place additional restrictions on the strain energy function to model the distortion in a hydrostatic state of stress. The objective of this research note is to modify the definitions of the remaining three invariants. These new invariants have clear physical interpretations that can be measured in experiments.

Keywords Elasticity · Invariants · Nonlinear · Orthotropic

Mathematics Subject Classification (2000) 74B20 · 15A72 · 34B15

1 Introduction

Criscione and his coworkers [2–4] have been developing invariants for nonlinear elastic isotropic and transversely isotropic solids which are physically based and yield nearly orthogonal response. Recently, Rubin and Jabareen [7] introduced physically based invariants for nonlinear elastic orthotropic materials. In that formulation use was made of the work of Flory [6] to separate the kinematics of dilatation and distortion and the strain energy function was taken to depend on seven invariants: the dilatation J and six measures of distortion β_i $(i = 1, 2, \dots, 6)$. The invariants β_i are measures of distortion which cause deviatoric stress. Specifically, $\{\beta_1, \beta_2, \beta_3\}$ depend on three dependent functions $\eta_i(J)$ $(i = 1, 2, 3)$ which

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model distortion in a hydrostatic state of stress and the remaining invariants $\{\beta_4, \beta_5, \beta_6\}$ were similar to those previously used in the literature [1, 5]. Most importantly, the invariants β_i $(i = 1, 2, 3)$ and the functions η_i $(i = 1, 2, 3)$ can be used to model the distortion in a hydrostatic state of stress independently of the form of the strain energy function. The objective of this research note is to introduce modified expressions for the invariants $\{\beta_4, \beta_5, \beta_6\}$ which have clear physical interpretations.

2 Background and New Invariants

Using standard notation, let $\mathbf{X}$ denote the location of a material point in an unstressed referenced configuration and let $\mathbf{x}$ denote the location of the same material point in the present configuration. Then, the deformation gradient $\mathbf{F}$, dilatation J, and right Cauchy-Green deformation tensor $\mathbf{C}$ are defined by

$$\mathbf{F} = \partial \mathbf{x}/\partial \mathbf{X}, \qquad J = \det(\mathbf{F}), \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F}. \tag{2.1}$$

Moreover, using the work of Flory [6] it is possible to define the unimodular tensor $\mathbf{C}'$ as a pure measure of distortion by

$$\mathbf{C}' = J^{-2/3}\mathbf{C}, \qquad \det(\mathbf{C}') = 1. \tag{2.2}$$

Next, let $\mathbf{a}_i$ be an orthonormal triad of vectors which characterizes the principal directions of anisotropy of an orthotropic solid in its reference configuration and define the structural tensors $\mathbf{N}_i$ by

$$\mathbf{N}_i = \mathbf{a}_i \otimes \mathbf{a}_i \quad (i = 1, 2, 3,\ \text{no sum on } i), \tag{2.3}$$

where $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product of the two vectors $\mathbf{a}$ and $\mathbf{b}$. For the class of materials considered in Rubin and Jabareen [7] the orthotropic solid will be in a hydrostatic state of stress if and only if the distortional deformation tensor $\mathbf{C}'$ has the form

$$\mathbf{C}' = \eta_1^2\mathbf{N}_1 + \eta_2^2\mathbf{N}_2 + \eta_3^2\mathbf{N}_3, \tag{2.4}$$

where η_i are positive constitutive functions of the dilatation J satisfying the restrictions

$$\eta_i(J) > 0, \qquad \eta_1\eta_2\eta_3 = 1, \qquad \eta_i(1) = 1. \tag{2.5}$$

Then, the three physically based invariants $\{\beta_1, \beta_2, \beta_3\}$ defined in Rubin and Jabareen [7] are given by

$$\beta_i = \left(\frac{1}{\eta_i^2}\mathbf{C}' + \eta_i^2\mathbf{C}'^{-1}\right) \cdot \mathbf{N}_i, \quad \beta_i \geq 2\ (i = 1, 2, 3), \tag{2.6}$$

where $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}(\mathbf{A}\mathbf{B}^T)$ is the inner product between two second order tensors $\{\mathbf{A}, \mathbf{B}\}$.

Here, the additional invariants $\{\beta_4, \beta_5, \beta_6\}$ are defined by the modified expressions

$$\begin{aligned}
\beta_4 &= 1 - \frac{(\mathbf{C}'^{-1} \cdot \mathbf{N}_3)}{(\mathbf{C}' \cdot \mathbf{N}_1)(\mathbf{C}' \cdot \mathbf{N}_2)} = \frac{(\mathbf{N}_1\mathbf{C}\mathbf{N}_2 + \mathbf{N}_2\mathbf{C}\mathbf{N}_1) \cdot \mathbf{C}}{2(\mathbf{C} \cdot \mathbf{N}_1)(\mathbf{C} \cdot \mathbf{N}_2)}, \\
\beta_5 &= 1 - \frac{(\mathbf{C}'^{-1} \cdot \mathbf{N}_2)}{(\mathbf{C}' \cdot \mathbf{N}_1)(\mathbf{C}' \cdot \mathbf{N}_3)} = \frac{(\mathbf{N}_1\mathbf{C}\mathbf{N}_3 + \mathbf{N}_3\mathbf{C}\mathbf{N}_1) \cdot \mathbf{C}}{2(\mathbf{C} \cdot \mathbf{N}_1)(\mathbf{C} \cdot \mathbf{N}_3)}, \\
\beta_6 &= 1 - \frac{(\mathbf{C}'^{-1} \cdot \mathbf{N}_1)}{(\mathbf{C}' \cdot \mathbf{N}_2)(\mathbf{C}' \cdot \mathbf{N}_3)} = \frac{(\mathbf{N}_2\mathbf{C}\mathbf{N}_3 + \mathbf{N}_3\mathbf{C}\mathbf{N}_2) \cdot \mathbf{C}}{2(\mathbf{C} \cdot \mathbf{N}_2)(\mathbf{C} \cdot \mathbf{N}_3)},
\end{aligned} \tag{2.7}$$

Letting $C_{ij} = \mathbf{C} \cdot (\mathbf{a}_i \otimes \mathbf{a}_j)$ be the components of $\mathbf{C}$ relative to the principal directions of anisotropy $\mathbf{a}_i$, it can be shown that

$$\beta_4 = \frac{C_{12}^2}{C_{11}C_{22}}, \qquad \beta_5 = \frac{C_{13}^2}{C_{11}C_{33}}, \qquad \beta_6 = \frac{C_{23}^2}{C_{22}C_{33}}, \qquad 0 \le \beta_{i+3} < 1 \ (i = 1, 2, 3). \tag{2.8}$$

Consequently, these new invariants $\{\beta_4, \beta_5, \beta_6\}$ can be interpreted physically as the squares of the cosines of the angles between two material fibers which in the reference configuration were orthogonal and parallel to the principal directions of orthotropy.

3 Constitutive Equations

For a general nonlinear elastic solid with strain energy function $\Sigma(\mathbf{C})$ per unit mass, the Cauchy stress $\mathbf{T}$ is determined by the hyperelastic form

$$\mathbf{T} = 2\rho \mathbf{F} \frac{\partial \Sigma}{\partial \mathbf{C}} \mathbf{F}^T = -p\mathbf{I} + \mathbf{T}', \quad \mathbf{T}' \cdot \mathbf{I} = 0, \tag{3.1}$$

where ρ is the current mass density, p is the pressure and $\mathbf{T}'$ is the deviatoric part of $\mathbf{T}$. Here, the strain energy function for an orthotropic elastic solid is taken in the form

$$\Sigma = \hat{\Sigma}(J, \beta_i). \tag{3.2}$$

In order to evaluate the expression (3.1) it is convenient to determine the derivatives of the invariants with respect to $\mathbf{C}$. From Rubin and Jabareen [7] it is recalled that J and the auxiliary functions $n_i(J)$ satisfy the equations

$$\begin{aligned}
&\frac{\partial J}{\partial \mathbf{C}} = \frac{1}{2} J \mathbf{C}^{-1}, \qquad n_i = \frac{3J}{\eta_i}\frac{d\eta_i}{dJ}, \qquad \frac{\partial \eta_i}{\partial \mathbf{C}} = \frac{1}{6} n_i \eta_i J^{-2/3} \mathbf{C}'^{-1} \\
&\quad (\text{no sum on } i;\ i = 1, 2, 3), \\
&n_1 + n_2 + n_3 = 0.
\end{aligned} \tag{3.3}$$

Also, the derivatives of β_i are given by

$$\frac{\partial \beta_i}{\partial \mathbf{C}} = \mathbf{B}_i \quad (i = 1, 2, \ldots, 6), \tag{3.4}$$

where the symmetric tensors $\mathbf{B}_i$ now take the forms

$$\begin{aligned}
\mathbf{B}_i &= J^{-2/3}\Bigg[\bigg\{\frac{1}{\eta_i^2}\mathbf{N}_i - \eta_i^2\mathbf{C}'^{-1}\mathbf{N}_i\mathbf{C}'^{-1}\bigg\} \\
&\quad - \frac{1}{3}(1+n_i)\bigg\{\frac{1}{\eta_i^2}(\mathbf{C}'\cdot\mathbf{N}_i) - \eta_i^2(\mathbf{C}'^{-1}\cdot\mathbf{N}_i)\bigg\}\mathbf{C}'^{-1}\Bigg] \\
&\quad (\text{no sum on } i;\ i=1,2,3), \\
\mathbf{B}_4 &= \frac{[(\mathbf{N}_1\mathbf{C}\mathbf{N}_2 + \mathbf{N}_2\mathbf{C}\mathbf{N}_1) - \beta_4(\mathbf{C}\cdot\mathbf{N}_2)\mathbf{N}_1 - \beta_4(\mathbf{C}\cdot\mathbf{N}_1)\mathbf{N}_2]}{(\mathbf{C}\cdot\mathbf{N}_1)(\mathbf{C}\cdot\mathbf{N}_2)}, \\
\mathbf{B}_5 &= \frac{[(\mathbf{N}_1\mathbf{C}\mathbf{N}_3 + \mathbf{N}_3\mathbf{C}\mathbf{N}_1) - \beta_5(\mathbf{C}\cdot\mathbf{N}_3)\mathbf{N}_1 - \beta_5(\mathbf{C}\cdot\mathbf{N}_1)\mathbf{N}_3]}{(\mathbf{C}\cdot\mathbf{N}_1)(\mathbf{C}\cdot\mathbf{N}_3)}, \\
\mathbf{B}_6 &= \frac{[(\mathbf{N}_2\mathbf{C}\mathbf{N}_3 + \mathbf{N}_3\mathbf{C}\mathbf{N}_2) - \beta_6(\mathbf{C}\cdot\mathbf{N}_3)\mathbf{N}_2 - \beta_6(\mathbf{C}\cdot\mathbf{N}_2)\mathbf{N}_3]}{(\mathbf{C}\cdot\mathbf{N}_2)(\mathbf{C}\cdot\mathbf{N}_3)}.
\end{aligned} \tag{3.5}$$

It then follows that the stresses associated with the strain energy function (3.2) become

$$\begin{aligned}
p &= -\rho_0\frac{\partial\hat{\Sigma}}{\partial J} - \frac{2}{3}\rho\sum_{i=1}^{3}\frac{\partial\hat{\Sigma}}{\partial\beta_i}(\mathbf{F}\mathbf{B}_i\mathbf{F}^T\cdot\mathbf{I}), \\
\mathbf{T}' &= 2\rho\sum_{i=1}^{6}\frac{\partial\hat{\Sigma}}{\partial\beta_i}\bigg[\mathbf{F}\mathbf{B}_i\mathbf{F}^T - \frac{1}{3}(\mathbf{F}\mathbf{B}_i\mathbf{F}^T\cdot\mathbf{I})\mathbf{I}\bigg],
\end{aligned} \tag{3.6}$$

where it is noted that $(\mathbf{F}\mathbf{B}_i\mathbf{F}^T\cdot\mathbf{I})$ vanishes for $(i=4,5,6)$. Next, with the help of (2.6), (2.7), (3.5) and (3.6) it follows that if $\mathbf{C}'$ has the form (2.4) then the invariants β_i are constants and the tensors $\mathbf{B}_i$ vanish

$$\beta_i = 2, \quad \beta_{i+3} = 0 \quad \text{for } i=1,2,3, \qquad \mathbf{B}_i = 0 \quad \text{for } i=1,2,\ldots,6. \tag{3.7}$$

Also, the material is in a state of hydrostatic stress with

$$p = -\rho_0\frac{\partial\hat{\Sigma}}{\partial J}, \qquad \mathbf{T}' = 0. \tag{3.8}$$

As a special case consider the strain energy function given by

$$2\rho_0\Sigma = \sum_{i=1}^{3}K_i(\beta_i - 2) + \sum_{i=4}^{6}K_i\beta_i + K_7(J-1)^2, \tag{3.9}$$

where ρ_0 is the reference mass density and K_i are constants. It then follows from (3.6) that the stress becomes

$$\mathbf{T} = J^{-1}\Bigg[\sum_{i=1}^{6}K_i\mathbf{F}\mathbf{B}_i\mathbf{F}^T\Bigg] + K_7(J-1)\mathbf{I}. \tag{3.10}$$

4 Linearized Formulation

For the linearized formulation it is convenient to let $\{E_{ij}, T_{ij}, K_{ijkl}\}$ be the components of the Lagrangian strain $\mathbf{E}$, Cauchy stress $\mathbf{T}$ and constant stiffness tensor $\mathbf{K}$, relative to the principal directions of orthotropy $\mathbf{a}_i$. Moreover, the strain energy function is expressed as a quadratic function of the strain $\mathbf{E}$, such that

$$\rho_0\Sigma = \frac{1}{2}\mathbf{K}\cdot(\mathbf{E}\otimes\mathbf{E}) = \frac{1}{2}K_{ijmn}E_{ij}E_{mn}, \qquad T_{ij} = K_{ijkl}E_{kl}, \tag{4.1}$$

where the usual summation convention is employed over repeated indices.

Now, for small deformations quadratic terms in the strain $\mathbf{E}$ are neglected and it can be shown that

$$\begin{gathered} J = 1+\mathbf{E}\cdot\mathbf{I}, \qquad \mathbf{C} = \mathbf{I}+2\mathbf{E}, \qquad \mathbf{C}' = \mathbf{I}+2\mathbf{E}', \qquad \mathbf{C}'^{-1} = \mathbf{I}-2\mathbf{E}', \qquad \mathbf{E}'\cdot\mathbf{I} = 0, \\ \eta_i = 1+\frac{1}{3}n_{i0}(\mathbf{E}\cdot\mathbf{I}), \qquad \frac{1}{\eta_i^2} = 1-\frac{2}{3}n_{i0}(\mathbf{E}\cdot\mathbf{I}), \qquad \eta_i^2 = 1+\frac{2}{3}n_{i0}(\mathbf{E}\cdot\mathbf{I}), \\ n_{10}+n_{20}+n_{30} = 0, \end{gathered} \tag{4.2}$$

where n_{i0} are constants. Also, the small deformation approximations of $\mathbf{B}_i$ become

$$\begin{gathered} \mathbf{B}_i = 2\left[\mathbf{E}'\mathbf{N}_i + \mathbf{N}_i\mathbf{E}' - \frac{2}{3}n_{i0}(\mathbf{E}\cdot\mathbf{I})\mathbf{N}_i - \frac{2}{3}(1+n_{i0})\left\{\mathbf{E}'\cdot\mathbf{N}_i - \frac{1}{3}n_{i0}(\mathbf{E}\cdot\mathbf{I})\right\}\mathbf{I}\right], \\ (\text{no sum } i = 1,2,3), \\ \mathbf{B}_4 = 2(\mathbf{N}_1\mathbf{E}\mathbf{N}_2 + \mathbf{N}_2\mathbf{E}\mathbf{N}_1), \qquad \mathbf{B}_5 = 2(\mathbf{N}_1\mathbf{E}\mathbf{N}_3 + \mathbf{N}_3\mathbf{E}\mathbf{N}_1), \\ \mathbf{B}_6 = 2(\mathbf{N}_2\mathbf{E}\mathbf{N}_3 + \mathbf{N}_3\mathbf{E}\mathbf{N}_2). \end{gathered} \tag{4.3}$$

Then, following the work in Rubin and Jabareen [7], the constants $\{n_{01}, n_{02}, n_{03}, K_i\}$ can be expressed in terms of the components K_{ijkl}. Specifically, these constants are given by the expressions in Rubin and Jabareen [7] with those for $\{K_4, K_5, K_6\}$ replaced by

$$K_4 = K_{1212} - K_1 - K_2, \qquad K_5 = K_{1313} - K_1 - K_3, \qquad K_6 = K_{2323} - K_2 - K_3. \tag{4.4}$$

5 Conclusions

The strain energy of a nonlinear orthotropic material can be expressed in terms of seven physical invariants: the dilatation J and six invariants β_i $(i = 1, 2, \ldots, 6)$ which measure distortions that cause deviatoric stress. Here, new definitions of the invariants $\{\beta_4, \beta_5, \beta_6\}$ have been proposed which admit physical interpretations as the squares of the cosines of the angles between two material fibers which in the reference configuration were orthogonal and parallel to the principal directions of orthotropy. A strain energy function that depends on $\{J, \beta_i\}$ has the advantage that it automatically predicts the distortion in a hydrostatic state of stress which is determined by the measurable dependent functions $\eta_i(J)$ $(i = 1, 2, 3)$.

It is also of interest to express more standard invariants

$$\{\mathbf{C}'\cdot\mathbf{N}_1, \mathbf{C}'\cdot\mathbf{N}_2, \mathbf{C}'\cdot\mathbf{N}_3, \mathbf{C}'^{-1}\cdot\mathbf{N}_1, \mathbf{C}'^{-1}\cdot\mathbf{N}_2, \mathbf{C}'^{-1}\cdot\mathbf{N}_3\}, \tag{5.1}$$

in terms of the new invariants β_i. To this end, (2.7) are solved to obtain

$$\begin{aligned} \mathbf{C}'^{-1}\cdot\mathbf{N}_1 &= (1-\beta_6)(\mathbf{C}'\cdot\mathbf{N}_2)(\mathbf{C}'\cdot\mathbf{N}_3), \\ \mathbf{C}'^{-1}\cdot\mathbf{N}_2 &= (1-\beta_5)(\mathbf{C}'\cdot\mathbf{N}_1)(\mathbf{C}'\cdot\mathbf{N}_3), \\ \mathbf{C}'^{-1}\cdot\mathbf{N}_3 &= (1-\beta_4)(\mathbf{C}'\cdot\mathbf{N}_1)(\mathbf{C}'\cdot\mathbf{N}_2). \end{aligned} \tag{5.2}$$

With the help of these results the expression for $\{\beta_2, \beta_3\}$ in (2.6) can be solved for $(\mathbf{C}'^{-1} \cdot \mathbf{N}_2)$ and $(\mathbf{C}'^{-1} \cdot \mathbf{N}_3)$

$$\begin{aligned} \mathbf{C}' \cdot \mathbf{N}_2 &= \frac{\eta_2^2 \beta_2 - \eta_2^4 \eta_3^2 \beta_3 (1-\beta_5)(\mathbf{C}' \cdot \mathbf{N}_1)}{1 - \eta_2^4 \eta_3^4 (1-\beta_4)(1-\beta_5)(\mathbf{C}' \cdot \mathbf{N}_1)^2}, \\ \mathbf{C}' \cdot \mathbf{N}_3 &= \frac{\eta_3^2 \beta_3 - \eta_2^2 \eta_3^4 \beta_2 (1-\beta_4)(\mathbf{C}' \cdot \mathbf{N}_1)}{1 - \eta_2^4 \eta_3^4 (1-\beta_4)(1-\beta_5)(\mathbf{C}' \cdot \mathbf{N}_1)^2}, \end{aligned} \tag{5.3}$$

which yields the invariants (5.1) in terms of β_i and the value of $(\mathbf{C}' \cdot \mathbf{N}_1)$.

In particular, (5.2) and (5.3) together with the representation

$$\mathbf{C}' \cdot \mathbf{N}_1 = \frac{\alpha}{\eta_2^2 \eta_3^2}, \tag{5.4}$$

can be used to rewrite the equation for β_1 in (2.6) as a fifth order polynomial for α

$$a_5 \alpha^5 + a_4 \alpha^4 + a_3 \alpha^3 + a_2 \alpha^2 + a_1 \alpha + a_0 = 0, \tag{5.5}$$

where with the help of (2.5) the coefficients a_i can be expressed as

$$\begin{aligned} &a_0 = -[\beta_1 - \beta_2 \beta_3 (1-\beta_6)], \qquad a_1 = 1 - [\beta_2^2 (1-\beta_4) + \beta_3^2 (1-\beta_5)](1-\beta_6), \\ &a_2 = (1-\beta_4)(1-\beta_5)[2\beta_1 + \beta_2 \beta_3 (1-\beta_6)], \qquad a_3 = -2(1-\beta_4)(1-\beta_5), \\ &a_4 = -\beta_1 (1-\beta_4)^2 (1-\beta_5)^2, \qquad a_5 = (1-\beta_4)^2 (1-\beta_5)^2, \end{aligned} \tag{5.6}$$

with β_i satisfying the restrictions (2.6) and (2.8). The physical value of $(\mathbf{C}' \cdot \mathbf{N}_1)$ is determined by a real positive root of the fifth order equation (5.4). At this point it is not known how to prove whether the physical solution of (5.4) is unique or not. This result is similar to the one discussed in Appendix A in Rubin and Jabareen [7] associated with the previous invariants.

Acknowledgements This research was partially supported by MB Rubin's Gerard Swope Chair in Mechanics and by the fund for the promotion of research at the Technion.

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