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On the Nonlinear Continuum Theory of Dislocations: A Gauge Field Theoretical Approach

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Abstract A nonlinear continuum theory of material bodies with continuously distributed dislocations is presented, based on a gauge theoretical approach. Firstly, we derive the canonical conservation laws that correspond to the group of translations and rotations in the material space using Noether's theorem. These equations give us the canonical Eshelby stress tensor as well as the total canonical angular momentum tensor. The canonical Eshelby stress tensor is neither symmetric nor gauge-invariant. Based on the Belinfante-Rosenfeld procedure, we obtain the gauge-invariant Eshelby stress tensor which can be symmetric relative to the reference configuration only for isotropic materials. The gauge-invariant angular momentum tensor is obtained as well. The decomposition of the gauge-invariant Eshelby stress tensor in an elastic and in a dislocation part gives rise to the derivation of the famous Peach-Koehler force.

Keywords Dislocations · Eshelby stress tensor · Peach-Koehler force · Gauge theory

Mathematics Subject Classification (2000) 49S05 · 74A99 · 74B99

1 Introduction

The purpose of this paper is to investigate the nature of the Peach-Koehler force [20] within the framework of nonlinear continuum theory of dislocations. It is well-known that there are different opinions in the scientific literature about the nature of the Peach-Koehler force. Clearly, it is not our aim to repeat once again these viewpoints. One can find a nice exposition concerning this matter in the paper of Steinmann [24].

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Dislocation theory has the status of a physical field theory. The hyperelastostatic case of materials with continuous distributions of dislocations which are otherwise homogeneous is examined. It has been shown [4, 9, 11, 13] that the dislocation field theory can be understood as the gauge theory of the three-dimensional translation group. One advantage when developing gauge theories of dislocations by means of the field theoretical principle of minimal coupling is that the nature and the choice of the physical state quantities are straightforward and unambiguous. This makes it possible to use the standard techniques of calculus of variation in order to derive the Euler-Lagrange equations as well as the Noether theorem for developing conservation and balance laws. For these reasons, the gauge field theory of dislocations is the appropriate language to clarify the nature of the force acting on a continuum distribution of dislocations. It should be emphasized that the dislocation field theory possesses apart from the incompatible elastic distortion tensor an additional physical state quantity, the dislocation density tensor, which leads to a specific response with the dimension of a moment stress. The pseudomoment stress is the specific response to a distribution of dislocations and it is missing in conventional nonlinear theories of dislocations. Unlike elasticity theory, the dislocation field theory is a theory with pseudomoment stresses in addition to force stresses.

Recently, Lazar and Anastassiadis [12] have proved within the linear version of the translation gauge theory of moving dislocations that the Peach-Koehler force is a material force and it is mathematically analogous to the Lorentz force in the Maxwell field theory of electromagnetic fields. Immediately, the question comes up to study the nature of the Peach-Koehler force in a nonlinear gauge theoretical approach of dislocations.

The paper is organized as follows: in Sect. 2 we give the basic framework concerning the gauge theory of dislocations and we derive the Euler-Lagrange equations for the considered problem. Section 3 starts with the derivation of the general formula of the conserved currents using Noether's theorem. The application to the group of translations in the material space will provide the conservation of the canonical Eshelby stress tensor while the group of rotations will give the total angular-momentum tensor which is conserved in the case that the isotropy condition is fulfilled. Section 3 continues with the construction of the gauge-invariant Eshelby stress tensor using the Belinfante-Rosenfeld procedure. It is proved that the gauge-invariant Eshelby stress tensor is symmetric only for materials which are isotropic relative to the reference configuration. The gauge-invariant angular momentum tensor is obtained as well. Section 4 deals with the corresponding balance laws. Namely, the source term in the translational balance law corresponding to the elastic subsystem is the Peach-Koehler force. It is shown that it is a self-equilibrating force since the source term in the translational balance law for the dislocation subsystem is the negative Peach-Koehler force. The rotational balance laws for the two subsystems are also derived. Section 5 is devoted to a discussion and comparison of our results with results found in the recent literature. The paper ends with the conclusions in the last section.

2 The Basic Framework

We start with a brief presentation of the elastostatics of a continuous distribution of dislocations under the scope of gauge theory. We consider an elastic body whose motion is described by the nonlinear motion deformation map

$$\boldsymbol{\chi} : \mathcal{B}_R \longrightarrow \mathcal{B}_t \quad \text{with } \boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}), \tag{1}$$

where $\mathcal{B}_R$ is the reference configuration and $\mathcal{B}_t$ the current configuration with X^I, $I = 1, 2, 3$ and x^a, $a = 1, 2, 3$ to be the material and the spatial coordinates, respectively. In classical elasticity both indices are holonomic.

In general, *the Lagrangian density* for a hyperelastic body is given by

$$\mathcal{L} = \mathcal{L}(X^I, \chi^a(X^I), \chi^a_{,J}(X^I)), \tag{2}$$

where $\chi^a_{,J}(X^I) = \partial\chi^a(X^I)/\partial X^J$. In classical elasticity, we require that the Lagrangian remains invariant under the homogeneous action of the group of rigid-body translations and rigid-body rotations $\mathcal{G}_0 = \mathrm{SO}(3) \ltimes \mathrm{T}(3)$ in order to take the field equations and the balance law of moment of momentum.

When dislocations are present, the mapping function does not provide anymore a complete characterization of the state of the body and there exists no global natural configuration that is a stress-free configuration of the whole body. Thus, the introduction of an intermediate configuration $\mathcal{B}_{\mathrm{int}}$ is necessary. $\mathcal{B}_{\mathrm{int}}$ has no global coordinate system in other words this configuration is incompatible or anholonomic. We mention here that our reference configuration is a holonomic one. The two configurations are connected by a two-point tensor, *the incompatible elastic distortion tensor* $B^a_{.I}$, as follows

$$\boldsymbol{B}^a = B^a_{.I}(\boldsymbol{X})\,\mathrm{d}X^I, \tag{3}$$

where $\boldsymbol{B}^a$ is a local anholonomic coframe in $T^*\mathcal{B}_{\mathrm{int}}$, which is the cotangent space of $\mathcal{B}_{\mathrm{int}}$, and $\mathrm{d}X^I$ is a holonomic coframe in $T^*\mathcal{B}_R$, which is the cotangent space of $\mathcal{B}_R$. Because the elastic distortion $B^a_{.I}(\boldsymbol{X})$ relates the reference state to the intermediate state, we work in the Lagrangian picture. More generally, $\boldsymbol{B}$ is a multilinear mapping

$$\boldsymbol{B} = B^a_{.I}(\boldsymbol{X})e_a \otimes \mathrm{d}X^I : T^*\mathcal{B}_{\mathrm{int}} \times T\mathcal{B}_R \longrightarrow \mathbb{R}, \tag{4}$$

where e_a is an anholonomic frame in the tangent space $T\mathcal{B}_{\mathrm{int}}$ of $\mathcal{B}_{\mathrm{int}}$ and $T\mathcal{B}_R$ is the tangent space of $\mathcal{B}_R$. Thus, $\boldsymbol{B}$ is a vector-valued 1-form. Because the elastic distortion tensor is incompatible or anholonomic, it violates the compatibility condition and, therefore, the so-called *object of anholonomity* is non-zero. In such a situation, the object of anholonomity, which actually measures how much a given coframe fails to be holonomic [8], plays the role of *the dislocation density tensor* or *torsion tensor*

$$A^a_{.IJ} = B^a_{.J,I} - B^a_{.I,J}, \tag{5}$$

which is a two-point tensor of rank three. From the differential-geometric point of view, it is a vector-valued 2-form

$$\boldsymbol{A} = \frac{1}{2}A^a_{.IJ}(\boldsymbol{X})e_a \otimes \mathrm{d}X^I \wedge \mathrm{d}X^J : T^*\mathcal{B}_{\mathrm{int}} \times T\mathcal{B}_R \times T\mathcal{B}_R \longrightarrow \mathbb{R}, \tag{6}$$

where $\wedge$ denotes the so-called wedge product (skew-symmetric tensor product). Equation (5) satisfies the *translational Bianchi identity* [11]

$$\epsilon^{IJK}A^a_{.IJ,K} = 0, \tag{7}$$

where ϵ^{IJK} is the Levi-Civita tensor. The condition (7) ensures that the dislocation density tensor has the form (5). One can say, it is the compatibility condition for $\boldsymbol{A}$. Physically, it means that dislocations do not end inside the body. In addition, (5) can be written as a two-point tensor of rank two as follows

$$A^{aK} = \frac{1}{2}\epsilon^{IJK} A^a_{.IJ} = \epsilon^{KJI} B^a_{.I,J}, \quad \text{or} \quad \boldsymbol{A} = \text{Curl}\boldsymbol{B}. \tag{8}$$

The inverse relation of (8) is given by

$$A^a_{.IJ} = \epsilon_{IJK} A^{aK}. \tag{9}$$

The dislocation density tensor can describe single dislocations and distributions of dislocations. The *Burgers vector* with respect to an anholonomic frame is defined by integration around an arbitrary, closed curve $\mathcal{C}_R$ (Burgers circuit) in $\mathcal{B}_R$ encircling dislocations

$$b^a = \oint_{\mathcal{C}_R} B^a_{.I}(\boldsymbol{X})\,\mathrm{d}X^I = \frac{1}{2}\int_{\mathcal{A}_R} A^a_{.IJ}(\boldsymbol{X})\,\mathrm{d}X^I \wedge \mathrm{d}X^J, \tag{10}$$

where $\mathcal{A}_R$ is any smooth surface with boundary $\mathcal{C}_R = \partial\mathcal{A}_R$.

Dislocations usually are viewed as relative translations between neighboring particles, so they can arise from the breaking of the homogeneity of the action of the translation[1] group T(3). The passage from the global-homogeneous action of the group $\mathcal{G}_0$ to the local-inhomogeneous action is achieved with the *gauge group* $\mathcal{G}$ that is obtained by allowing the transformations of the group $\mathcal{G}_0$ to be space-dependent. Due to the local action of the group $\mathcal{G}$, the invariance of the Lagrangian under the action of the group $\mathcal{G}_0$ does not get transferred into the invariance of the Lagrangian under the action of $\mathcal{G}$. In other words, the derivatives that enter in the Lagrangian are altered by a change of a coframe and we need the Lagrangian to remain unchanged under changes of coframe. It is clear that the invariance of the Lagrangian has to be restored otherwise the balance laws of linear momentum and moment of momentum are lost since Noether's theorem can be applied only when the Lagrange function is invariant under the action of a group. The restoration of the invariance of the Lagrangian is achieved with *the minimal replacement construct of gauge theory of the translation group*. Thus, the requirement that $\mathcal{L}$ should not be influenced by choices of coframes leads to the minimal replacement construct [3, 4] that is generally expressed as follows

$$\begin{aligned}&\mathcal{L}(X^I, \chi^a(X^I), \chi^a_{,J}(X^I))\\ &\quad\longrightarrow \mathcal{L}(X^I, \chi^a(X^I), \nabla_J\chi^a(X^I)) + \mathcal{L}_G(X^I, \phi^a_{.I}(X^I), \phi^a_{.I,J}(X^I)),\end{aligned} \tag{11}$$

where $\nabla_J\chi^a$ is *the gauge-covariant derivative* of the translation group and $\phi^a_{.I} = \phi^a_{.I}(\boldsymbol{X})$ is the *gauge potential* of the translation group that arises through the breaking of the homogeneity of the action of the group T(3). It is noted that $\mathcal{L}_G$ depends only on the fields $\phi^a_{.I}$ and their first derivatives. One can see more about the gauge theory and the general formula of the minimal replacement construct for elasticity in [4, 9, 11].

In what follows, we focus on a material with a continuous distribution of dislocations which is otherwise homogeneous in the static case. First of all, we require that the Lagrangian should remain invariant under the transformation

$$\tilde{\chi}^a = \chi^a + f^a, \tag{12}$$

[1] In order for the reader to have a more comprehensive view, we mention here that disclinations are considered as relative rotations between systems of neighboring particles, so that they can be thought to arise from the breaking of the homogeneity of the action of the rotation group SO(3).

where $f^a = f^a(\boldsymbol{X})$ is an arbitrary vector function. Second, the minimal replacement construct has the precise form

$$\chi^a_{,I} \longmapsto B^a_{.I} = \nabla_I \chi^a = \chi^a_{,I} + \phi^a_{.I}. \tag{13}$$

$\phi^a_{.I}$ has to transform as follows

$$\tilde{\phi}^a_{.I} = \phi^a_{.I} - f^a_{,I}. \tag{14}$$

Due to the appearance of the gauge potential we can define the *translational field strength tensor* $\boldsymbol{A}$ as follows

$$A^a_{.IJ} = \phi^a_{.J,I} - \phi^a_{.I,J}, \tag{15}$$

which can be identified with the dislocation density tensor (5) since the gradient of $\boldsymbol{\chi}$ does not give a contribution to the dislocation density tensor. $\boldsymbol{B}$ and $\boldsymbol{A}$ are physical state quantities and, of course, are gauge-invariant by construction under the transformations (12) and (14).

The presence of dislocations brings a contribution to the stored energy via eigenenergy (or energy of the dislocation core region), and this leads immediately to the presence of pseudomoment stresses. Therefore, *the total stored energy density* per unit volume in the reference configuration can be given as[2]

$$W = W(\boldsymbol{B}(\boldsymbol{X}), \boldsymbol{A}(\boldsymbol{X})) \tag{16}$$

and it is accompanied by the constitutive equations

$$T_a^{.I} = \frac{\partial W}{\partial B^a_{.I}}, \qquad H_a^{.IJ} = 2\frac{\partial W}{\partial A^a_{.IJ}} \tag{17}$$

for *the first Piola-Kirchhoff stress tensor* $T_a^{.I}$ and *the pseudomoment stress tensor* $H_a^{.IJ}$. The last tensor is skewsymmetric with respect to the indices I and J.

Furthermore, according to the minimal replacement construct (see (11)) the total stored energy density may be decomposed into an elastic part $W(\boldsymbol{B})$, depending only on $\boldsymbol{B}$, and a dislocation part $W(\boldsymbol{A})$ which depends on $\boldsymbol{A}$, that is

$$W = W(\boldsymbol{B}(\boldsymbol{X}), \boldsymbol{A}(\boldsymbol{X})) = W(\boldsymbol{B}) + W(\boldsymbol{A}). \tag{18}$$

The energy functional for a homogeneous body for the considered problem is

$$\mathcal{I} = \int_\Omega \mathcal{L}(\boldsymbol{\chi}^{(1)}, \boldsymbol{\phi}^{(1)})\, \mathrm{d}\boldsymbol{X}, \tag{19}$$

where the integrand $\mathcal{L}(\boldsymbol{\chi}^{(1)}, \boldsymbol{\phi}^{(1)})$ is a smooth function of $\boldsymbol{\chi}, \boldsymbol{\phi}$ and the derivatives of $\boldsymbol{\chi}, \boldsymbol{\phi}$ up to first order, $\boldsymbol{\chi}^{(1)}, \boldsymbol{\phi}^{(1)}$, and Ω is an open, connected subset of $\mathcal{B}_R$ with smooth boundary $\partial\Omega$ and $\mathrm{d}\boldsymbol{X} = \mathrm{d}X_1\,\mathrm{d}X_2\,\mathrm{d}X_3$. The *Euler-Lagrange equations* are

$$E_a(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial \chi^a} - \mathrm{D}_I \frac{\partial \mathcal{L}}{\partial \chi^a_{,I}} = 0, \tag{20}$$

[2]Such a total stored energy, depending on the physical state quantities $\boldsymbol{B}$ and $\boldsymbol{A}$ was also proposed and used by Kröner [10].

$$E_a^{\cdot J}(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial \phi^a_{.J}} - \mathrm{D}_I \frac{\partial \mathcal{L}}{\partial \phi^a_{.J,I}} = 0, \tag{21}$$

where D_I is the total derivative with respect to X^I. The above equations, using the constitutive relations (17) and having in mind the consideration $\mathcal{L} = -W$, give respectively

$$\mathrm{D}_I T_a^{\cdot I} = 0, \tag{22}$$

$$\mathrm{D}_I H_a^{\cdot JI} + T_a^{\cdot J} = 0. \tag{23}$$

Equation (22) represents *the force equilibrium equation* and (23) *the stress balance of dislocations*. Thus, the stress tensor $T_a^{\cdot J}$ is the source for the pseudomoment stress tensor. Due to the fact that the pseudomoment stress tensor is skewsymmetric in the last two indices, (22) can also follow from (23) as an integrability condition.

3 Conservation Laws

For the derivation of the conservation laws we use the Noether theorem [18], according to which "invariance" of a variational principle under a group of transformations implies conservation laws for the solutions of the associated Euler-Lagrange equations. Of course, not every symmetry of a system of Euler-Lagrange equations gives rise to a conservation law. One needs the group of transformations to satisfy an additional "variational" property which leaves the energy functional "invariant", the so-called *infinitesimal criterion of invariance* or *infinitesimal variational symmetry condition*. Here, we shall not dwell in greater detail on this theorem, rather we just give the absolutely necessary formulas for studying our problem. One can read about the theorem of Noether in the books of Olver [19] and Bluman and Kumei [2]. The reader can also find a more extensive application of Noether's theorem to generalized elasticity as a third order variational problem in Agiasofitou and Lazar [1].

We consider a 3-parameter group of transformations G acting on both independent $\boldsymbol{X}$ and dependent $\boldsymbol{\chi}, \boldsymbol{\phi}$ variables

$$(\boldsymbol{X}^*, \boldsymbol{\chi}^*, \boldsymbol{\phi}^*) = (\boldsymbol{\Xi}_{\boldsymbol{\varepsilon}}(\boldsymbol{X}, \boldsymbol{\chi}, \boldsymbol{\phi}), \boldsymbol{\psi}_{\boldsymbol{\varepsilon}}(\boldsymbol{X}, \boldsymbol{\chi}, \boldsymbol{\phi}), \boldsymbol{\Psi}_{\boldsymbol{\varepsilon}}(\boldsymbol{X}, \boldsymbol{\chi}, \boldsymbol{\phi})), \tag{24}$$

where $\boldsymbol{\varepsilon} = (\varepsilon^1, \varepsilon^2, \varepsilon^3)$. *The infinitesimal generator* $\boldsymbol{v}_R$ that corresponds to the parameter ε^R, $R = 1, 2, 3$ is given by the following relation

$$\boldsymbol{v}_R = \xi_R^A \frac{\partial}{\partial X^A} + \psi_R^a \frac{\partial}{\partial \chi^a} + (\Psi^a_{.I})_R \frac{\partial}{\partial \phi^a_{.I}}, \tag{25}$$

where

$$\begin{aligned}
\xi_R^A(\boldsymbol{X}, \boldsymbol{\chi}, \boldsymbol{\phi}) &= \left.\frac{\partial \Xi_{\boldsymbol{\varepsilon}}^A(\boldsymbol{X}, \boldsymbol{\chi}, \boldsymbol{\phi})}{\partial \varepsilon^R}\right|_{\boldsymbol{\varepsilon}=0}, \\
\psi_R^a(\boldsymbol{X}, \boldsymbol{\chi}, \boldsymbol{\phi}) &= \left.\frac{\partial \psi_{\boldsymbol{\varepsilon}}^a(\boldsymbol{X}, \boldsymbol{\chi}, \boldsymbol{\phi})}{\partial \varepsilon^R}\right|_{\boldsymbol{\varepsilon}=0}, \\
(\Psi^a_{.I})_R(\boldsymbol{X}, \boldsymbol{\chi}, \boldsymbol{\phi}) &= \left.\frac{(\partial \Psi^a_{.I})_{\boldsymbol{\varepsilon}}(\boldsymbol{X}, \boldsymbol{\chi}, \boldsymbol{\phi})}{\partial \varepsilon^R}\right|_{\boldsymbol{\varepsilon}=0},
\end{aligned} \tag{26}$$

and $\Xi_{\boldsymbol{\varepsilon}}^A$, $\psi_{\boldsymbol{\varepsilon}}^a$ and $(\Psi^a_{.I})_{\boldsymbol{\varepsilon}}$ are the components of $\boldsymbol{\Xi}_{\boldsymbol{\varepsilon}}$, $\boldsymbol{\psi}_{\boldsymbol{\varepsilon}}$ and $\boldsymbol{\Psi}_{\boldsymbol{\varepsilon}}$, respectively.

Since the functional (19) contains derivatives up to the first order, we need *the first prolongation* of $\boldsymbol{v}_R$,

$$\mathrm{pr}^{(1)}\boldsymbol{v}_R = \mathrm{pr}^{(1)}\boldsymbol{v}_{QR} + \xi^A_R \mathrm{D}_A, \tag{27}$$

where

$$\mathrm{pr}^{(1)}\boldsymbol{v}_{QR} = Q^a_R \frac{\partial}{\partial \chi^a} + (Q^a_{.I})_R \frac{\partial}{\partial \phi^a_{.I}} + \frac{\mathrm{D}Q^a_R}{\mathrm{D}X^A}\frac{\partial}{\partial \chi^a_{,A}} + \frac{\mathrm{D}(Q^a_{.I})_R}{\mathrm{D}X^A}\frac{\partial}{\partial \phi^a_{.I,A}} \tag{28}$$

with

$$\begin{aligned} Q^a_R(\boldsymbol{X}, \boldsymbol{\chi}^{(1)}, \boldsymbol{\phi}^{(1)}) &= \psi^a_R - \xi^A_R \chi^a_{,A}, \\ (Q^a_{.I})_R(\boldsymbol{X}, \boldsymbol{\chi}^{(1)}, \boldsymbol{\phi}^{(1)}) &= (\Psi^a_{.I})_R - \xi^A_R \phi^a_{.I,A} \end{aligned} \tag{29}$$

to be *the characteristics* of the vector field $\boldsymbol{v}_R$.

The well-known *infinitesimal variational symmetry condition* [19] for a first order variational problem states that the group of transformations G is a *variational symmetry* of the functional (19) if and only if

$$\mathrm{pr}^{(1)}\boldsymbol{v}_R(\mathcal{L}) + \mathcal{L}\mathrm{D}_A \xi^A_R = 0 \tag{30}$$

for all $(\boldsymbol{X}, \boldsymbol{\chi}^{(1)}, \boldsymbol{\phi}^{(1)})$ and every infinitesimal generator of the group G. If G is a variational symmetry group of the functional (19), then G is a symmetry group of the Euler-Lagrange equations. The inverse of this statement is not true.

Eventually, *the Noether theorem* gives the conservation laws in characteristic form for the Euler-Lagrange equations

$$\frac{\mathrm{D}\mathcal{J}^{.B}_R}{\mathrm{D}X^B} = -Q^a_R E_a(\mathcal{L}) - (Q^a_{.I})_R E^{.I}_a(\mathcal{L}). \tag{31}$$

If the Euler-Lagrange equations are satisfied, then we have *the conservation laws*

$$\frac{\mathrm{D}\mathcal{J}^{.B}_R}{\mathrm{D}X^B} = 0 \tag{32}$$

with the conserved quantities or *currents* to be given by the specific formula

$$\mathcal{J}^{.B}_R = Q^a_R \frac{\partial \mathcal{L}}{\partial \chi^a_{,B}} + (Q^a_{.I})_R \frac{\partial \mathcal{L}}{\partial \phi^a_{.I,B}} + \mathcal{L}\xi^B_R. \tag{33}$$

3.1 Translation–Canonical Eshelby Stress Tensor

The group of translations in the material space is given by

$$\begin{aligned} (X^A)^* &= X^A + \varepsilon^R \delta_R^{\ A}, \\ (\chi^a)^* &= \chi^a, \\ (\phi^a_{.I})^* &= \phi^a_{.I}, \end{aligned} \tag{34}$$

where $\delta_R^{\ A}$ is the Kronecker delta. The components of the infinitesimal generators, using the relations (26), are

$$\xi^A_R = \delta_R^{\ A}, \qquad \psi^a_R = 0, \qquad (\Psi^a_{.I})_R = 0.$$

Hence, the characteristics (29) give

$$Q^a_R = -\chi^a_{,R}, \qquad (Q^a_{.I})_R = -\phi^a_{.I,R}. \tag{35}$$

By means of the above equations (35) and the relation

$$\frac{\partial W}{\partial \phi^a_{.I,B}} = H_a^{.BI}, \tag{36}$$

the current (33) takes the precise form

$$\mathcal{J}_R^{.B} = -W\delta_R^{\ B} + \chi^a_{,R} T_a^{.B} + \phi^a_{.I,R} H_a^{.BI} \tag{37}$$

in order to constitute *the translational conservation law*

$$\frac{\mathrm{D}P_R^{.B}}{\mathrm{D}X^B} = 0 \tag{38}$$

with

$$P_R^{.B} := -\mathcal{J}_R^{.B} = W\delta_R^{\ B} - \chi^a_{,R} T_a^{.B} + \phi^a_{.I,R} H_a^{.IB} \tag{39}$$

the canonical Eshelby stress tensor for the dislocation gauge theory. It is worth pointing out that $\boldsymbol{P}$ is neither symmetric nor gauge-invariant.

3.2 Rotation—Canonical Angular Momentum Tensor

The group of rotations in the material space is given by

$$\begin{aligned} (X^A)^* &= X^A + \epsilon^A_{\ RB}\varepsilon^R X^B, \\ (\chi^a)^* &= \chi^a, \\ (\phi^a_{.I})^* &= \phi^a_{.I} + \epsilon_{IR}^{\ \ J}\varepsilon^R \phi^a_{.J}. \end{aligned} \tag{40}$$

One can see that $\boldsymbol{\phi}$ transforms under the rotation group in the material space like a vector field in the material space. The components of the infinitesimal generators for the above group are

$$\xi^A_R = \epsilon^A_{\ RB} X^B, \qquad \psi^a_R = 0, \qquad (\Psi^a_{.I})_R = \epsilon_{IR}^{\ \ J}\phi^a_{.J} \tag{41}$$

and consequently the characteristics (29) take the following form

$$Q^a_R = -\epsilon^A_{\ RB} X^B \chi^a_{,A}, \qquad (Q^a_{.I})_R = \epsilon_{IR}^{\ \ J}\phi^a_{.J} - \epsilon^A_{\ RB} X^B \phi^a_{.I,A}. \tag{42}$$

The current (33) due to (41) and (42) gives

$$\mathcal{J}_R^{.B} = -\epsilon^I_{\ RJ} X^J P_I^{.B} + \epsilon_{IR}^{\ \ J}\phi^a_{.J} H_a^{.IB} \tag{43}$$

and essentially represents *the total canonical angular momentum tensor*

$$\mathcal{M}_R^{.B} := -\mathcal{J}_R^{.B} = \epsilon_{RJ}^{\ \ \ I}(X^J P_I^{.B} + \phi^a_{.I} H_a^{.JB}). \tag{44}$$

The first part in (44) is *the orbital angular momentum tensor* and the second one is the *intrinsic or spin angular momentum tensor* which can be rewritten in the following form

$$M^{I.B}_{.J.} = -\phi^{aI} H_{aJ}^{..B} + \phi^{a}_{.J} H_a^{.IB}. \tag{45}$$

If we calculate the divergence of the total angular momentum tensor and also making use of (38) and the Euler-Lagrange equation (23), we derive *the rotational balance law*

$$\frac{\mathrm{D}\mathcal{M}_R^{.B}}{\mathrm{D}X^B} = -\epsilon_{RJ}{}^{I}(B^a_{.I} T_a^{.J} + A^a_{.BI} H_a^{.BJ}). \tag{46}$$

Therefore, if the *isotropy condition* relative to the reference configuration is fulfilled that is

$$\epsilon_{RJ}{}^{I}(B^a_{.I} T_a^{.J} + A^a_{.BI} H_a^{.BJ}) = 0, \tag{47}$$

then the following *rotational conservation law* holds

$$\frac{\mathrm{D}\mathcal{M}_R^{.B}}{\mathrm{D}X^B} = 0. \tag{48}$$

From the above equation and using the translational conservation law (38), we have

$$\epsilon_{RJ}{}^{I}[P_I^{.J} + (\phi^a_{.I} H_a^{.JB})_{,B}] = 0. \tag{49}$$

Multiplying the above equation with the Levi-Civita tensor, we get the following equation

$$P^I_{.J} - P_J^{.I} = (\phi^a_{.J} H_a^{.IB} - \phi^{aI} H_{aJ}^{..B})_{,B} \tag{50}$$

which alternatively can be written in the form

$$P^I_{.J} + (\phi^{aI} H_{aJ}^{..B})_{,B} = P_J^{.I} + (\phi^a_{.J} H_a^{.IB})_{,B}. \tag{51}$$

If we define

$$(P^g)_J^{.I} = P_J^{.I} + (\phi^a_{.J} H_a^{.IB})_{,B}, \tag{52}$$

then (51) shows that the tensor $\boldsymbol{P}^g$ is symmetric

$$(P^g)_J^{.I} = (P^g)^I_{.J}. \tag{53}$$

In fact, $\boldsymbol{P}^g$ is *the symmetrized Eshelby stress tensor*. If we proceed a little more with the calculations in (52) taking into account (39), (23), (15) and (13), it can be seen that $\boldsymbol{P}^g$ is actually given in terms of the physical state quantities $\boldsymbol{B}$ and $\boldsymbol{A}$ as following

$$(P^g)_J^{.I} = W\delta_J{}^I - B^a_{.J} T_a^{.I} - A^a_{.BJ} H_a^{.BI} \tag{54}$$

and, of course, it is gauge-invariant. The result of this simple procedure for the derivation of the symmetrized Eshelby stress tensor is in agreement with the result that one can obtain by using the symmetrization procedure of Belinfante and Rosenfeld as we will see in the next subsection. The reason that we can easily obtain the $\boldsymbol{P}^g$ is that the spin part of the angular momentum tensor has a simple structure.

It should be emphasized that the above procedure for the derivation of the symmetric tensor $\boldsymbol{P}^g$ is based on the isotropy condition. The natural question that arises at this point is what happens if the material is anisotropic. Is $\boldsymbol{P}^g$ also symmetric for an anisotropic material? We will also deal with this question in the next subsection.

3.3 The Belinfante-Rosenfeld Procedure—Gauge-Invariant Eshelby Stress Tensor

We construct here the gauge-invariant Eshelby stress tensor $\boldsymbol{P}^g$ using the so-called Belinfante-Rosenfeld procedure (see, e.g., [5, 17, 23]). We have already seen that the total canonical angular momentum tensor (44) contains a spin part and that the canonical Eshelby stress tensor $\boldsymbol{P}$ is neither symmetric nor gauge-invariant.

The construction of $\boldsymbol{P}^g$ begins with the spin angular momentum tensor (45). From the definition (45) we see that the spin part is antisymmetric in its first two indices. Thus, by construction, the tensor

$$S_J^{\cdot IB} = \frac{1}{2}(M^{I.B}_{.J.} + M_J^{\cdot BI} + M^{IB}_{\cdot\cdot\, J}) \tag{55}$$

is antisymmetric in its last two indices. Using the Belinfante-Rosenfeld procedure, the gauge-invariant Eshelby stress tensor is defined by

$$(P^g)_J^{\cdot I} = P_J^{\cdot I} + S_J^{\cdot IB}{}_{,B}. \tag{56}$$

Since $S_J^{\cdot IB}$ is antisymmetric in the indices I and B, the conservation of $\boldsymbol{P}^g$ is equivalent to the conservation of $\boldsymbol{P}$:

$$(P^g)_J^{\cdot I}{}_{,I} = P_J^{\cdot I}{}_{,I} + S_J^{\cdot IB}{}_{,BI} = P_J^{\cdot I}{}_{,I} = 0. \tag{57}$$

On the other hand, from the analytical expression of the spin part of the angular momentum tensor (45) we obtain that the tensor $S_J^{\cdot IB}$ (55) is given as follows

$$S_J^{\cdot IB} = \phi^a_{.J} H_a^{\cdot IB} \tag{58}$$

and

$$S_J^{\cdot IB}{}_{,B} = \phi^a_{.J,B} H_a^{\cdot IB} + \phi^a_{.J} H_a^{\cdot IB}{}_{,B}. \tag{59}$$

If we substitute (59) into (56) and also use (39), (23), (15) and (13), we obtain for the Eshelby stress tensor $\boldsymbol{P}^g$:

$$(P^g)_J^{\cdot I} = W\delta_J^{\cdot I} - B^a_{.J} T_a^{\cdot I} - A^a_{.BJ} H_a^{\cdot BI} \tag{60}$$

or

$$\boldsymbol{P}^g = W\mathbf{1}_R - \boldsymbol{B}^T \cdot \boldsymbol{T} - \boldsymbol{A}^T : \boldsymbol{H}, \tag{61}$$

which is gauge-invariant. The formula (61) gives *the gauge-invariant Eshelby stress tensor* and it is worthwhile to note that it is expressed in terms of the physical state quantities.

In addition, the antisymmetric part of (60) is given by

$$\epsilon_{RI}{}^{J} (P^g)_J^{\cdot I} = -\epsilon_{RI}{}^{J} (B^a_{.J} T_a^{\cdot I} + A^a_{.BJ} H_a^{\cdot BI}), \tag{62}$$

which is nothing but the isotropy condition (47). Thus, in the anisotropic case the Belinfante-Rosenfeld procedure gives a gauge-invariant Eshelby stress tensor which is generally non-symmetric. If the isotropy condition (47) is fulfilled, then $\boldsymbol{P}^g$ is symmetric. Therefore, only for materials, which are isotropic relative to the reference configuration, the gauge-invariant Eshelby stress tensor $\boldsymbol{P}^g$ of the dislocation gauge theory is symmetric, $\boldsymbol{P}^g = (\boldsymbol{P}^g)^T$, where $(\boldsymbol{P}^g)^T$ denotes the transpose of $\boldsymbol{P}^g$. Also, we would like to mention that from the field theoretical point of view the gauge-invariant Eshelby stress tensor is the true (static) energy-momentum tensor in contrast to the canonical one.

3.4 Gauge-Invariant Angular Momentum Tensor

Using the gauge-invariant Eshelby stress tensor (60), we may define *the gauge-invariant angular momentum tensor* as

$$(\mathcal{M}^g)_R^{\cdot B} := \epsilon_{RJ}{}^I X^J (P^g)_I^{\cdot B} \tag{63}$$

or

$$\boldsymbol{M}^g = \boldsymbol{X} \times \boldsymbol{P}^g \tag{64}$$

with

$$\frac{\mathrm{D}(\mathcal{M}^g)_R^{\cdot B}}{\mathrm{D}X^B} = -\epsilon_{RJ}{}^I (B^a_{.I} T_a^{\cdot J} + A^a_{.BI} H_a^{\cdot BJ}). \tag{65}$$

Equation (65) gives rise to an isotropy condition, which is actually the same as (47). Thus, the total canonical and the gauge-invariant angular momentum tensors lead to the same isotropy condition.

4 Balance Laws

In this section we calculate the translational and rotational balance laws of the elastic and dislocation subsystems. In this way, we determine also the configurational forces and the moments produced by dislocations.

4.1 Translational Balance Laws—Peach-Koehler Force

Due to the decomposition of the total stored energy density (18), the gauge-invariant Eshelby stress tensor (60) can be written as the sum of two parts, one involving only the elastic distortion tensor $\boldsymbol{B}$ and the other involving only the dislocation density tensor $\boldsymbol{A}$ according to

$$(P^g)_J^{\cdot I} = (P^g_{\mathrm{el}})_J^{\cdot I} + (P^g_{\mathrm{di}})_J^{\cdot I} \tag{66}$$

with *the elastic part of the gauge-invariant Eshelby stress tensor*

$$(P^g_{\mathrm{el}})_J^{\cdot I} = W(\boldsymbol{B})\delta_J{}^I - B^a_{.J} T_a^{\cdot I} \tag{67}$$

and *the dislocation part of the gauge-invariant Eshelby stress tensor*

$$(P^g_{\mathrm{di}})_J^{\cdot I} = W(\boldsymbol{A})\delta_J{}^I - A^a_{.BJ} H_a^{\cdot BI}. \tag{68}$$

A straightforward computation gives *the translational balance law for the elastic subsystem*

$$\frac{\mathrm{D}(P^g_{\mathrm{el}})_J^{\cdot I}}{\mathrm{D}X^I} = -T_a^{\cdot I} A^a_{.IJ} = \epsilon_{JIK} T_a^{\cdot I} A^{aK} \tag{69}$$

or

$$\mathrm{Div}\, \boldsymbol{P}^g_{\mathrm{el}} = \boldsymbol{F}^{\mathrm{PK}}, \tag{70}$$

where the right hand side of (69) is the famous *Peach-Koehler force density* as given by Rogula [21] and Maugin [17]

$$\boldsymbol{F}^{\mathrm{PK}} = \boldsymbol{T} \times \boldsymbol{A}. \tag{71}$$

An analogous calculation gives *the translational balance law for the dislocation subsystem*

$$\frac{\mathrm{D}(P^g_{\mathrm{di}})^{\cdot I}_{J}}{\mathrm{D}X^I} = T^{\cdot I}_{a} A^a_{.IJ} = -\epsilon_{JIK} T^{\cdot I}_{a} A^{aK} \tag{72}$$

or

$$\mathrm{Div}\, \boldsymbol{P}^g_{\mathrm{di}} = -\boldsymbol{F}^{\mathrm{PK}}. \tag{73}$$

Because the sum of (69) and (72) is zero, the Peach-Koehler force is self-equilibrating. This fact is known from field theories where if the total system is conserved, then the interaction forces between the subsystems are self-equilibrating (see, e.g., [3]). The gauge-invariant Eshelby stress tensors of the elastic and the dislocation subsystems are not conserved separately, while the gauge-invariant Eshelby stress tensor of the total system is conserved. There is an exchange between the elastic and dislocation subsystems. The interaction force between these two subsystems is the Peach-Koehler force.

Finally, it is worth pointing out that it is the divergence of the gauge-invariant Eshelby stress tensor of the elastic or the dislocation subsystem that gives the Peach-Koehler force and not the divergence of the canonical Eshelby stress tensor.

4.2 Rotational Balance Laws—Configurational Moments

The decomposition of the gauge-invariant Eshelby stress tensor (66) induces a similar decomposition of the gauge-invariant angular momentum tensor (63) according to

$$(\mathcal{M}^g)^{\cdot B}_{R} = (\mathcal{M}^g_{\mathrm{el}})^{\cdot B}_{R} + (\mathcal{M}^g_{\mathrm{di}})^{\cdot B}_{R} \tag{74}$$

with

$$(\mathcal{M}^g_{\mathrm{el}})^{\cdot B}_{R} = \epsilon_{RJ}{}^{I} X^J (P^g_{\mathrm{el}})^{\cdot B}_{I} \tag{75}$$

and

$$(\mathcal{M}^g_{\mathrm{di}})^{\cdot B}_{R} = \epsilon_{RJ}{}^{I} X^J (P^g_{\mathrm{di}})^{\cdot B}_{I}. \tag{76}$$

The rotational balance law of the elastic subsystem is calculated as

$$\frac{\mathrm{D}(\mathcal{M}^g_{\mathrm{el}})^{\cdot B}_{R}}{\mathrm{D}X^B} = -\epsilon_{RJ}{}^{I} (X^J T^{\cdot B}_{a} A^a_{.BI} + B^a_{.I} T^{\cdot J}_{a}). \tag{77}$$

The first term on the right hand side of (77) is the moment produced by the Peach-Koehler force (71) and the second term is the elastic part of the isotropy condition (47).

In addition, *the rotational balance law of the dislocation subsystem* is given as

$$\frac{\mathrm{D}(\mathcal{M}^g_{\mathrm{di}})^{\cdot B}_{R}}{\mathrm{D}X^B} = \epsilon_{RJ}{}^{I} (X^J T^{\cdot B}_{a} A^a_{.BI} - A^a_{.BI} H^{\cdot BJ}_{a}). \tag{78}$$

Analogously, the first term on the right hand side of (78) is the moment produced by the negative Peach-Koehler force (71) and the second term is the dislocation part of the isotropy condition (47). The sum of (77) and (78) gives the balance law (65) of the total system.

5 Discussion and Comparison with Other Results

Recently, a number of papers have been devoted to nonlinear continuum theories of dislocations of which we mention [14, 15, 22, 24]. There are different proposals in these mentioned works which we would like to discuss briefly and relate them with the present gauge approach as far as this is possible.

Steinmann [24] has used a stored energy density per unit volume of the reference configuration

$$W = W(\boldsymbol{B}), \tag{79}$$

which is in agreement with the energy of the elastic subsystem considered in this paper. He used the definition of the dislocation density tensor as incompatibility condition for an otherwise homogeneous material with incompatible elastic deformations. In the conventional way of continuum mechanics, he derived the Peach-Kohler force which is in agreement with our result. In this picture, the Peach-Koehler force is caused by incompatibilities. More precisely, the so-called *spatial dislocation Peach-Koehler force* in Steinmann's notation coincides with ours.

On the other hand, Sfyris et al. [22] have used the following stored energy function per unit volume of the reference configuration

$$\hat{W} = \hat{W}(\boldsymbol{F}, \boldsymbol{X}) = J_K^{-1} W(\boldsymbol{F}\boldsymbol{K}(\boldsymbol{X})), \tag{80}$$

where $\boldsymbol{F}$ denotes the deformation gradient which is considered compatible, the local map $\boldsymbol{K}$ represents the local structural rearrangement and $J_K = \det\boldsymbol{K}$. According to them, the presence of the continuous distributions of dislocations within the body is realized through the inhomogeneity of $\hat{W}$. The energy (80) describes compatible deformation of an inhomogeneous body. Their Eshelby stress tensor, which is derived from the energy (80), has the following form

$$\boldsymbol{b} = \hat{W}\mathbf{1}_R - \boldsymbol{F}^T \cdot \hat{\boldsymbol{T}}, \tag{81}$$

where $\hat{\boldsymbol{T}} = \partial\hat{W}/\partial\boldsymbol{F}$ and its divergence gives the material momentum equation of a "dislocated" medium as follows

$$\mathrm{Div}\boldsymbol{b} = -\boldsymbol{M} : \boldsymbol{\Gamma}, \tag{82}$$

where $\boldsymbol{M} = \boldsymbol{F}^T \cdot \hat{\boldsymbol{T}}$ is *the Mandel stress tensor* and $\boldsymbol{\Gamma} := -\boldsymbol{K}^{-1}\mathrm{Grad}\boldsymbol{K}$ the (geometrical) *connection* based on $\boldsymbol{K}$. The connection $\boldsymbol{\Gamma}$ is not necessarily symmetric, and it is well-known from differential geometry that $\boldsymbol{\Gamma}$ may contain a skew-symmetric part called *Cartan's torsion tensor* (see, e.g., [16]).

To sum up, their Eshelby stress tensor is the usual Eshelby stress tensor of inhomogeneous compatible elasticity which is different from the Eshelby stress tensor of incompatible elasticity used by Steinmann [24] as well as in the present paper concerning the elastic subsystem (see formula (67)). For that reason, it is not surprising that the divergence of different Eshelby stress tensors results in different configurational forces. Also, the first Piola-Kirchhoff stress tensor $\hat{\boldsymbol{T}}$, used by Sfyris et al. [22], is based on $\boldsymbol{F}$ and consequently it is not the "eigenstress". In their approach [22] even if they use the notion of incompatibility, they essentially consider that the driving force on a dislocation is due to inhomogeneities of the body and not due to incompatibilities. However, from the definition of the dislocation density tensor (see, e.g., (5)), a dislocation is an incompatibility and not an inhomogeneity. Thus, this interpretation of a dislocation is misleading and inconvenient.

In addition, Sfyris et al. [22] have used Epstein and Maugin's approach of material uniformity and inhomogeneity [6, 7, 17] which is based on the following stored energy density per unit volume of the reference configuration

$$\hat{W}(\boldsymbol{F},\boldsymbol{X}) = J_K^{-1} W(\boldsymbol{F}\boldsymbol{K}(\boldsymbol{X})) = \bar{W}(\boldsymbol{F},\boldsymbol{K}) \tag{83}$$

with the volume change $J_K = \det\boldsymbol{K}$. Sfyris et al. [22] have used the same theoretical arguments as Epstein and Maugin [7], however they claim that they have obtained a different result. In fact, their result is a special case of the result of Epstein and Maugin [7]. The result derived by Epstein and Maugin [7] is valid in the general case where the local structural rearrangement represented by the local map $\boldsymbol{K}$ involves also a change of the volume. In this case the divergence of the Eshelby stress tensor is given by

$$\mathrm{Div}\boldsymbol{b} = \boldsymbol{b} : \boldsymbol{\Gamma}. \tag{84}$$

If the local rearrangement is isochoric ($J_K = 1$), there remains only the negative Mandel stress on the right hand side of (84) and one can recover the result of Sfyris et al. [22], that is (82). This case is realized in finite strain plasticity with the multiplicative decomposition of $\boldsymbol{F}$ as: $\boldsymbol{F} = \boldsymbol{F}^{\mathrm{e}} \cdot \boldsymbol{F}^{\mathrm{p}}$ ($\boldsymbol{F}^{\mathrm{e}}$ and $\boldsymbol{F}^{\mathrm{p}}$ denote the elastic and plastic deformations). Here we can identify $\boldsymbol{K}^{-1} \equiv \boldsymbol{F}^{\mathrm{p}}$ and also $\boldsymbol{F}\boldsymbol{K} \equiv \boldsymbol{F}^{\mathrm{e}}$ and, thus, the plastic deformation involves no volume change. From the mathematical point of view their difference is based on the treatment of $\boldsymbol{F}$ and $\boldsymbol{K}$. Epstein and Maugin [7] have considered $\boldsymbol{F}$ and $\boldsymbol{K}$ as independent variables while Sfyris et al. [22] as dependent ones.

Moreover, Sfyris et al. [22] claimed that a third configuration which is actually the intermediate configuration is missing in Steinmann's approach [24]. This cannot be correct because Steinmann [24] used an incompatible elastic distortion, which is a two-point tensor, connecting the reference configuration with the intermediate one.

In the framework of elastoplasticity, Le and Stumpf [14] postulated the following stored energy for a body with continuously distributed dislocations

$$\hat{W} = \hat{W}(\boldsymbol{F}^{\mathrm{p}}, \boldsymbol{C}, \boldsymbol{T}^{\mathrm{p}}), \tag{85}$$

where $\boldsymbol{C}$ is the right Cauchy-Green tensor pertaining to $\boldsymbol{F}$ and $\boldsymbol{T}^{\mathrm{p}}$ is the torsion tensor defined in terms of $\boldsymbol{F}^{\mathrm{p}}$. However, $\boldsymbol{C}$ and $\boldsymbol{F}^{\mathrm{p}}$ are not physical state quantities. Le and Stumpf [14, 15] derived a configurational stress tensor which they call "*the* (*internal*) *driving stress tensor*"

$$\boldsymbol{J} = (\boldsymbol{F}^{\mathrm{p}})^{-1}\big[\hat{W}\boldsymbol{1}_R - \boldsymbol{C}^T \cdot \boldsymbol{S} - (\boldsymbol{T}^{\mathrm{p}})^T : \boldsymbol{S}^{\mathrm{d}}\big], \tag{86}$$

where $\boldsymbol{S} = 2\partial\hat{W}/\partial\boldsymbol{C}$ corresponds to the second Piola-Kirchhoff stress tensor and $\boldsymbol{S}^{\mathrm{d}} = 2\partial\hat{W}/\partial\boldsymbol{T}^{\mathrm{p}}$ to a microstress tensor containing microcouples. The configurational stress tensor (86) is the pull-back of the tensor in the square brackets to the so-called reference crystal. In order to compare the results, we rewrite the gauge-invariant Eshelby stress tensor (61) as

$$\boldsymbol{P}^g = W\boldsymbol{1}_R - \boldsymbol{C}^{\mathrm{e}} \cdot \tilde{\boldsymbol{T}} - \boldsymbol{A}^T : \boldsymbol{H}, \tag{87}$$

where $\boldsymbol{C}^{\mathrm{e}}$ is the right Cauchy-Green tensor pertaining to $\boldsymbol{B}$ and $\tilde{\boldsymbol{T}}$ is the second Piola-Kirchhoff stress tensor. One can see that the gauge-invariant Eshelby stress tensor (87) is similar to the tensor in the square brackets in (86). If a dislocation density tensor gives a contribution to the stored energy density (like in (16) and (85)), then a response quantity with respect to dislocations (like $\boldsymbol{H}$ or $\boldsymbol{S}^{\mathrm{d}}$) appears in the physical arena of dislocation theory. One consequence is that in such a dislocation theory the new response quantity, which is called in our framework the pseudomoment stress tensor, appears in the Euler-Lagrange equations. Another consequence is that the Eshelby stress tensor of the total system contains a part given in terms of the dislocation density tensor (like in (87) and (86)).

6 Conclusion

In this paper, we have investigated a dislocation field theory based on the stored energy density per unit volume of the reference state:

$$W = W(\boldsymbol{B}(\boldsymbol{X}), \boldsymbol{A}(\boldsymbol{X})) = W(\boldsymbol{B}) + W(\boldsymbol{A}).$$

The total stored energy density can be decomposed into an elastic part, depending only on $\boldsymbol{B}$, and a dislocation part, which depends on $\boldsymbol{A}$. This decomposition is based on the minimal replacement construct. First we have examined the total system. Using W, we derived the variational formulation for the dislocation field theory. In the field theory of dislocations we have two kinds of stresses: the first Piola-Kirchhoff stress, which plays the role of the "eigenstress" of dislocations, and the pseudomoment stress, which contains moment stresses. Thus, a dislocation field theory is a theory with force and pseudomoment stresses. We have shown that the Euler-Lagrange equations provide the balance of stresses of dislocations in addition to the balance of forces. In the former one, pseudomoment stresses enter in the field equations, a fact which is usually ignored in the literature. From the mathematical point of view, the consideration of the total system comes out to be very important since the corresponding Euler-Lagrange equations give a closed system of partial differential equations for the canonical variables $\boldsymbol{\chi}$ and $\boldsymbol{\phi}$. Usually, in incompatible elasticity where just the balance of forces appeared, one is forced to give an a priori value to the dislocation density tensor or to the plastic deformation tensor in order to solve the equations of equilibrium. Here, this is not necessary because we have the equations (23) arising from the consideration of $\boldsymbol{\phi}$ as additional field variable. This is one of the advantages of the gauge theoretical approach in describing models of dislocations.

Moreover, Noether's theorem has been used for the derivation of the conservation laws. We have calculated the canonical Eshelby stress and angular momentum tensors. Using the Belinfante-Rosenfeld procedure, we have found the gauge-invariant Eshelby stress and angular momentum tensors. If the isotropy condition relative to the reference configuration is fulfilled, then the gauge-invariant Eshelby stress tensor is symmetric. We have decomposed the gauge-invariant Eshelby stress tensor into elastic and dislocation parts in order to find the expression for the Peach-Koehler force defined in nonlinear dislocation field theory. The gauge-invariant Eshelby stress tensor of the incompatible elastic or dislocation subsystem gives the Peach-Koehler force as a source term in the corresponding balance law. It is clear that in this framework the Peach-Koehler force is a material force caused by dislocations as incompatibilities.

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References

1. Agiasofitou, E.K., Lazar, M.: Conservation and balance laws in linear elasticity of grade three. J. Elast. **94**, 69–85 (2009)
2. Bluman, G.W., Kumei, S.: Symmetries and Differential Equations. Springer, Berlin (1989)
3. Edelen, D.G.B.: Applied Exterior Calculus. Wiley, New York (1985)
4. Edelen, D.G.B., Lagoudas, D.C.: Gauge theory and defects in solids. In: Sih, G.C. (ed.) Mechanics and Physics of Discrete System, vol. 1. North-Holland, Amsterdam (1988)
5. Eshelby, J.D.: The elastic energy-momentum tensor. J. Elast. **5**, 321–335 (1975)
6. Epstein, M., Elżanowski, M.: Material Inhomogeneities and Their Evolution. Springer, Berlin (2007)
7. Epstein, M., Maugin, G.A.: The energy-momentum tensor and material uniformity in finite elasticity. Acta Mech. **83**, 127–133 (1990)
8. Hehl, F.W., McCrea, J.D., Mielke, E.W., Ne'eman, Y.: Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance. Phys. Rep. **258**, 1–171 (1995)
9. Kadić, A., Edelen, D.G.B.: A Yang-Mills type minimal coupling theory for materials with dislocations and disclinations. Int. J. Eng. Sci. **20**, 433–438 (1982)
10. Kröner, E.: Dislocations in crystals and in continua: a confrontation. Int. J. Eng. Sci. **33**, 2127–2135 (1995)
11. Lazar, M.: Dislocation theory as a 3-dimensional translation gauge theory. Ann. Phys. (Leipz.) **9**, 461–473 (2000)
12. Lazar, M., Anastassiadis, C.: The gauge theory of dislocations: conservation and balance laws. Philos. Mag. **88**, 1673–1699 (2008)
13. Lazar, M., Anastassiadis, C.: The gauge theory of dislocations: static solutions of screw and edge dislocations. Philos. Mag. **89**, 199–231 (2009)
14. Le, K.C., Stumpf, H.: Nonlinear continuum theory of dislocations. Int. J. Eng. Sci. **34**, 339–358 (1996)
15. Le, K.C., Stumpf, H.: A model of elastoplastic bodies with continuously distributed dislocations. Int. J. Plast. **12**, 611–627 (1996)
16. Lovelock, D., Rund, H.: Tensors, Differential Forms and Variational Principles. Wiley, New York (1975)
17. Maugin, G.A.: Material Inhomogeneities in Elasticity. Chapman & Hall, London (1993)
18. Noether, E.: Invariante variationspropleme. Goettinger Nachr. Math.-Phys. Kl. **2**, 235–256 (1918); [English translation: Invariant variation problems. Transp. Theory Stat. Phys. **1**, 186–207 (1971)]
19. Olver, P.J.: Applications of Lie Groups to Differential Equations. Springer, New York (1986)
20. Peach, M.O., Koehler, J.S.: Forces extended on dislocations and the stress fields produced by them. Phys. Rev. **80**, 436–439 (1950)
21. Rogula, D.: Forces in material space. Arch. Mech. **29**, 705–713 (1977)
22. Sfyris, D., Charalambakis, N., Kalpakides, V.K.: Variational arguments and Noether's theorem on the nonlinear continuum theory of dislocations. Int. J. Eng. Sci. **44**, 501–512 (2006)
23. Soper, D.E.: Classical Field Theory. Wiley, New York (1976)
24. Steinmann, P.: On spatial and material settings of hyperelastostatic crystal defects. J. Mech. Phys. Solids **50**, 1743–1766 (2002)