J Elast (2008) 93: 39–61
DOI 10.1007/s10659-008-9164-2

On Extension and Torsion of Strain-Stiffening Rubber-Like Elastic Circular Cylinders

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Received: 19 March 2008 / Published online: 26 April 2008

Abstract This paper is concerned with investigation of the effects of strain-stiffening on the response of solid circular cylinders in the combined deformation of torsion superimposed on axial extension. The cylinders are composed of incompressible isotropic nonlinearly elastic materials. Our primary focus is on materials that undergo *severe* strain-stiffening in the stress-stretch response. In particular, we consider two particular phenomenological constitutive models for such materials that reflect limiting chain extensibility at the molecular level. The axial stretch γ and twist that can be sustained in cylinders composed of such materials are shown to be constrained in a coupled fashion. It is shown that, in the absence of an additional axial force, a *transition value* $\gamma = \gamma_t$ of the axial stretch exists such that for $\gamma < \gamma_t$, the stretched cylinder tends to *elongate* on twisting whereas for $\gamma > \gamma_t$, the stretched cylinder tends to *shorten* on twisting. These results are in sharp contrast with those for classical models such as the Mooney-Rivlin (and neo-Hookean) models that predict that the stretched circular cylinder always tends to *further elongate* on twisting. We also obtain results for materials modeled by the well-known exponential strain-energy widely used in biomechanics applications. This model reflects a strain-stiffening that is less abrupt than that for the limiting chain extensibility models. Surprisingly, it turns out that the results in this case are somewhat more complicated. For a *fixed* stiffening parameter, provided that the stretch is sufficiently small, the stretched bar *always tends to elongate* on twisting in the absence of an additional axial force. However, *for sufficiently large stretch*, the cylinder tends to *shorten* on undergoing sufficiently *small twist* but then tends to *elongate* on further twisting. These results are of interest in view of the widespread use of exponential models in the context of the mechanics of soft biological tissues. The special case of pure torsion is also briefly considered. In this case, the resultant axial force required to maintain pure torsion is *compressive* for all the models discussed here. In the absence of such a force, the bar would elongate on twisting reflecting the celebrated Poynting effect.

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Keywords Strain-stiffening isotropic nonlinearly elastic materials · Incompressible rubber-like materials and soft biological tissues · Extension and torsion of solid circular cylinders · Pure torsion

Mathematics Subject Classification (2000) 74B20 · 74G55

1 Introduction

Our concern in this paper is with investigating the effects of strain-stiffening on the response of solid circular cylinders in the combined deformation of torsion superimposed on axial extension. The bars are composed of isotropic hyperelastic incompressible materials that undergo severe *strain-stiffening* in their stress-stretch response at large stretch. The constitutive models that we employ reflect *limiting chain extensibility* at the molecular level and thus are appropriate for modeling non-crystallizing elastomers and soft biological tissues.

In the next Section, we discuss some preliminaries from the theory of nonlinear hyperelasticity for isotropic incompressible solids. In particular, we describe some phenomenological constitutive models that exhibit strain-stiffening at large strains. The first class of models reflects limiting chain extensibility at the molecular level and gives rise to *severe* strain-stiffening in the stress-stretch response. The second class exhibits a less abrupt strain-stiffening e.g., the exponential models widely used in biomechanics. In Sect. 3, we summarize results for the problem of torsion superimposed on axial extension of a solid circular cylinder composed of an incompressible isotropic hyperelastic material. This problem has been extensively investigated in the literature on nonlinear elasticity following the pioneering theoretical investigations of Rivlin (see, e.g., [34, 35]) and experimental work of Rivlin and Saunders [36] for the case of small twist. We summarize the results of Rivlin [35] for a general strain-energy density. These results are then specialized for two particular strain-energy densities that have been proposed to model limiting chain extensibility both of which involve a single limiting chain extensibility parameter. The axial stretch γ and twist that can be sustained in cylinders composed of such materials are shown to be constrained in a coupled fashion. It is shown that, in the absence of an additional axial force, a *transition value* $\gamma = \gamma_t$ of the axial stretch exists such that for $\gamma < \gamma_t$, the stretched cylinder tends to *elongate* on twisting whereas for $\gamma > \gamma_t$, the stretched cylinder tends to *shorten* on twisting. These results are in sharp contrast with those for classical models such as the Mooney-Rivlin (and neo-Hookean) models that predict that the stretched circular cylinder always tends to *further elongate* on twisting. The foregoing results have been substantiated experimentally in a recent paper by Gent and Hua [12] where a non-monotone relation between axial force and applied extension has been observed giving rise to a transition stretch of the type described above. (We note that focus of [12] was on an instability that can arise for large twist but we shall not address such instability issues here.) Our results are also reminiscent of those of Shield [37] for *small* twist superimposed on extension of bars of *general cross-section*. For example, for the case of sufficiently slender elliptical cross-sections, it is shown in [37] that a transition stretch exists for two particular strain-energies. Here we have confined our attention to *circular* cylinders but do not assume small twist. Thus we have established results analogous to those of Shield [37] in the wider context of large twist where here the strain-stiffening material property plays a key role.

In the remaining part of Sect. 3, we obtain results for the well-known exponential strain-energy due to Fung [9] and Demiray [7]. This model (see (2.15)) reflects a strain-stiffening that is less abrupt than that for the limiting chain extensibility models. Surprisingly, it turns out that the results in this case are somewhat more complicated. For a *fixed* stiffening parameter, provided that the stretch is sufficiently small, the stretched bar *always tends to elongate* on twisting in the absence of an additional axial force. However, *for sufficiently large stretch*, the cylinder tends to *shorten* on undergoing sufficiently *small twist* but then tends to *elongate* on further twisting. These results are of interest in view of the widespread use of exponential models in the context of the mechanics of soft biological tissues (see, e.g., [14, 25, 26, 38]). In particular, the extension and torsion of cylinders has been proposed in [25] and [38] as a useful loading protocol to determine normal and shear properties of papillary muscles in the heart.

The final section of this paper provides a brief discussion of the special case of *pure torsion*. It is shown that the shearing stress (and resultant applied twisting moment) for both of the limiting chain extensibility models are *identical* provided that the limiting chain parameters are related to reflect equal ultimate torsional extensibility. Results for the exponential model are shown to be similar to those for the limiting chain models for small twist but diverge with increasing twist as might be anticipated. The resultant axial force necessary to maintain pure torsion is *compressive* for all of the models discussed here. In the absence of such a force, the bar tends to elongate on twisting reflecting the celebrated Poynting effect [32].

2 Preliminaries

The mechanical properties of elastomeric materials are described in continuum mechanics in terms of a strain-energy density function W. On denoting the left Cauchy-Green tensor by $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$, where $\boldsymbol{F}$ is the gradient of the deformation and $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches, then, for an isotropic material, W is a function of the strain invariants

$$I_1 = \operatorname{tr}\boldsymbol{B} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad I_2 = \frac{1}{2}[(\operatorname{tr}\boldsymbol{B})^2 - \operatorname{tr}(\boldsymbol{B}^2)] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2,$$
$$I_3 = \det\boldsymbol{B} = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{2.1}$$

Rubber-like materials are often assumed to be *incompressible* provided that the hydrostatic stress does not become too large and so the admissible deformations must be isochoric, i.e., $\det\boldsymbol{F} = 1$ so that $I_3 = 1$. The response of an incompressible isotropic elastic material can be determined by applying the standard constitutive law (see, e.g., Atkin and Fox [2], Ogden [30], Beatty [3], Holzapfel [13])

$$\boldsymbol{T} = -p\boldsymbol{1} + 2\frac{\partial W}{\partial I_1}\boldsymbol{B} - 2\frac{\partial W}{\partial I_2}\boldsymbol{B}^{-1}, \tag{2.2}$$

where p is a hydrostatic pressure term associated with the incompressibility constraint and $\boldsymbol{T}$ denotes the Cauchy stress.

The classical strain-energy density for *incompressible* rubber is the Mooney-Rivlin strain-energy

$$W^{MR} = \frac{1}{2}\mu[\alpha(I_1 - 3) + (1-\alpha)(I_2 - 3)], \tag{2.3}$$

where $\mu > 0$ is the constant shear modulus for infinitesimal deformations and $0 < \alpha \leq 1$ is a dimensionless constant. When $\alpha = 1$ in (2.3), one obtains the neo-Hookean strain-energy

$$W^{nH} = \frac{\mu}{2}(I_1 - 3), \tag{2.4}$$

which corresponds to a Gaussian statistical mechanics model. The theoretical predictions based on (2.3) do not adequately describe experimental data for rubber especially at high values of strain. For modeling of soft biological tissue, where *rapid* strain stiffening occurs even at moderate stretches (see, e.g., [14, 20, 25, 26]), classical models are completely inappropriate. To model such *stiffening,* a number of alternative models have been proposed. In the molecular theory of elasticity (see, e.g., [8] for a review) these models are usually called non-Gaussian, because they introduce a distribution function for the end-to-end distance of the polymeric chain composing the rubber-like material which is not Gaussian. The non-Gaussian models of concern here reflect limiting chain extensibility at the molecular level.

We shall not attempt here to describe the most general framework within which limiting chain extensibility models can be developed but rather describe some *specific* phenomenological models that have been shown to be particularly useful in modeling strain-stiffening phenomena. For a review of such models, see [17, 22, 23]. One class of *isotropic* incompressible models with a maximum achievable length of the polymeric molecular chains composing the material is described by strain-energies of the form $W(I_1, I_2, I^*)$ where I^* is a limiting chain extensibility parameter. The function W is such that the stress components are unbounded as $f(I_1, I_2) \to I^*$ for some specified function f and so one must impose the *constraint*

$$f(I_1, I_2) < I^*, \tag{2.5}$$

on admissible deformations. One model of this type is a *three-parameter* model due to Gent [10, 11], who proposed the strain-energy density

$$W = \frac{\mu}{2}\left[-\alpha J_m \ln\left(1 - \frac{I_1 - 3}{J_m}\right) + (1-\alpha)(I_2 - 3)\right], \tag{2.6}$$

where μ is the shear modulus for infinitesimal deformations, α $(0 < \alpha \leq 1)$ is a dimensionless constant and J_m is the limiting value for $I_1 - 3$, taking into account limiting polymeric chain extensibility so that $I^* = 3 + J_m$ for this model. On taking the limit as $J_m \to \infty$ in (2.6) we recover the Mooney-Rivlin model (2.3). Other related three-parameter models with more elaborate dependence on I_2 are discussed in [31, 33]. Thus, on using a dependence on I_2 suggested by Gent and Thomas, the model

$$W = -C_1 J_m \ln\left(1 - \frac{I_1 - 3}{J_m}\right) + C_2 \ln(I_2/3), \quad I_1 < J_m + 3, \tag{2.7}$$

has been proposed by Gent [11] and by Pucci and Saccomandi [33], where the constants C_1 and C_2 are related to the infinitesimal shear modulus by

$$C_1 + C_2/3 = \mu/2. \tag{2.8}$$

It has been shown [31] that the model (2.7) can be used to accurately fit the classical data of Treloar for vulcanized rubber over the full range of deformations for uniaxial, equibiaxial and general biaxial extension.

For the case when $\alpha = 1$ in (2.6), first proposed by Gent [10], we obtain the *two-parameter* generalized neo-Hookean model (i.e., $W = W(I_1)$)

$$W^G = -\frac{\mu}{2} J_m \ln\left(1 - \frac{I_1 - 3}{J_m}\right), \quad I_1 < J_m + 3, \tag{2.9}$$

(henceforth called *the basic Gent model*), and one recovers the neo-Hookean model on taking the limit as $J_m \to \infty$ in (2.9). The stress response for (2.9) in simple extension is described in [10] and elastic instabilities of inflated rubber shells have been discussed in [11]. For rubber, typical values for the dimensionless parameter J_m for simple extension range from 30–100 whereas for biological tissue, much smaller values of J_m are appropriate. For example, for human arterial wall tissue, values on the order of 0.4 to 2.3 have been suggested [20]. The basic Gent model (2.9) gives theoretical predictions similar to the Arruda and Boyce eight chain model based on inverse Langevin function compact support statistics (see, e.g., [1, 4, 5]). A molecular basis for the basic Gent model was given in [19], where it is shown that the infinitesimal shear modulus is $\mu = nkT$ where k is the Boltzmann constant, T is the absolute temperature, and n is the chain density, while $J_m = 3(N-1)$, where N is the number of links in a single chain. Thus the basic Gent phenomenological model predicts similar behavior to the molecular models, has a clear microscopic interpretation for the constitutive coefficients and is tractable analytically. On using (2.2), we find that the Cauchy stress associated with (2.9) is given by

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu \frac{J_m}{J_m - (I_1 - 3)} \boldsymbol{B}, \tag{2.10}$$

so that the stress has a singularity as $I_1 \to J_m + 3$, reflecting the rapid strain stiffening observed in experiments.

An alternative *two-parameter* limiting chain extensibility model with $W(I_1, I_2, J)$ has been proposed in [21] (see also [15]) for which

$$W = W^{HS} = -\frac{\mu}{2}\frac{(J-1)^2}{J} \ln\left(\frac{1}{(J-1)^3}\left(J^3 - J^2 I_1 + J I_2 - 1\right)\right), \tag{2.11}$$

where the limiting chain extensibility parameter J is such that $J > 1$. On using the principal stretches of the deformation we can write W^{HS} as

$$W^{HS} = -\frac{\mu}{2}\frac{(J-1)^2}{J} \ln\left(\frac{(1-\frac{\lambda_1^2}{J})(1-\frac{\lambda_2^2}{J})(1-\frac{\lambda_3^2}{J})}{(1-\frac{1}{J})^3}\right), \quad \lambda_1\lambda_2\lambda_3 = 1. \tag{2.12}$$

In (2.11), (2.12), μ is the shear modulus for infinitesimal deformations. Note that these definitions differ from those in [21–24] by a multiplicative factor of $(J-1)^2/J^2$. In order for the logarithm in (2.11) or (2.12) to be well defined, we impose the constraint

$$J I_1 - I_2 < \frac{J^3 - 1}{J}, \tag{2.13}$$

or alternatively

$$\max\left(\lambda_1^2, \lambda_2^2, \lambda_3^2\right) < J. \tag{2.14}$$

Thus, from (2.14), the limiting chain extensibility parameter J can be interpreted as the square of the maximum stretch allowed by the finite extensibility of the chains. Again, in the limit as $J \to \infty$ in (2.11) or (2.12), we recover the neo-Hookean model (2.4). It is important to point out the difference between the *constraint* (2.14) and the *constraint* $I_1 < J_m + 3$ arising in connection with the Gent model. As already pointed out in [21–24], the limiting chain condition expressed in terms of the principal invariant is less physically accessible than (2.14). Furthermore, the absence of the dependence on the *second* invariant in the basic Gent model entails some physical limitations. Thus the model (2.11) has advantages over the basic Gent model. Note that (2.12) belongs to the class of models for which $W(\lambda_1, \lambda_2, \lambda_3, J)$, with $\lambda_1\lambda_2\lambda_3 = 1$ because of incompressibility. For such models, the limiting chain extensibility constraint is given in terms of the principal stretches directly and this has some advantages from a physical point of view. This alternative approach to constitutive model development reflecting limiting chain extensibility has received some attention in the literature [18, 21, 29]. The response of the W^{HS} model in *homogeneous* deformations such as simple extension, simple shear and equibiaxial extension was examined in Horgan and Schwartz [24]. See also Kanner and Horgan [27]. It was shown that the response of the Gent and W^{HS} models was virtually identical for these deformations except at small values of J_m and J. The inhomogeneous deformation of pure bending was investigated in [28].

While our discussion will focus on limiting chain extensibility models such as the above that exhibit *severe* strain-stiffening, we note that there are numerous strain-hardening constitutive models that have been successfully employed to investigate the effects of a *less abrupt* strain-stiffening. A generalized neo-Hookean model of this type widely used in the biomechanics literature is the two-parameter *exponential* strain-energy density first proposed by Fung [9] and then extended to three dimensions in Demiray [7], namely

$$W^F = \frac{\mu}{2b}\{\exp[b(I_1 - 3)] - 1\}, \tag{2.15}$$

where the dimensionless constant $b > 0$. The parameter b provides a measure for the rate of strain-stiffening. On taking the limit as $b \to 0$ in (2.15) one recovers the neo-Hookean model (2.4). We shall henceforth refer to (2.15) as the Fung (or exponential) model.

We observe that, while the exponential model reflects strain-stiffening, it does *not* exhibit the *rapid strain-stiffening* characteristic of the limiting chain extensibility models. This is an important difference between these models. Chagnon et al. [6] suggest that both types of models are essentially equivalent but as was discussed in Horgan and Saccomandi [23] there are several significant differences in their predictions. For example, as described in detail in [17], the shear stresses at the tip of a Mode III crack are *singular* for the model (2.15) whereas these crack tip stresses are *bounded* for the Gent model (2.9). It was shown by Horgan and Schwartz [24] that similar results hold for fracture toughness in the tearing test for rubber. See the review article by Horgan and Saccomandi [23] for further discussion.

3 Extension and Torsion of a Solid Circular Cylinder

Consider a long, solid, circular cylinder of radius A composed of an incompressible isotropic hyperelastic material subjected to a stretch in the axial direction and then to a twist at its ends. On using cylindrical coordinates (R, Θ, Z) in the undeformed configuration and (r, θ, z) in the current configuration, we may thus write

$$r = \gamma^{-1/2} R, \qquad \theta = \Theta + \tau\gamma Z, \qquad z = \gamma Z, \tag{3.1}$$

where τ denotes the twist per unit length of the stretched rod and γ denotes the axial stretch. The coefficient $\gamma^{-1/2}$ appears in $(3.1)_1$ in order to maintain incompressibility, so that for extension $(\gamma > 1)$, the cylinder necessarily contracts laterally. The deformed radius a is given by $a = \gamma^{-1/2} A$. Corresponding to (3.1), one has the left Cauchy-Green tensor $\boldsymbol{B}$ and its inverse

$$\boldsymbol{B}=\begin{bmatrix}\gamma^{-1} & 0 & 0\\ 0 & \gamma^{-1}+\gamma\tau^2R^2 & \gamma^{3/2}\tau R\\ 0 & \gamma^{3/2}\tau R & \gamma^2\end{bmatrix},\qquad \boldsymbol{B}^{-1}=\begin{bmatrix}\gamma & 0 & 0\\ 0 & \gamma & -\gamma^{1/2}\tau R\\ 0 & -\gamma^{1/2}\tau R & \gamma^{-2}+\tau^2R^2\end{bmatrix}, \tag{3.2}$$

and so the principal invariants are

$$I_1=\gamma^2+2\gamma^{-1}+\gamma\tau^2R^2,\qquad I_2=2\gamma+\gamma^{-2}+\tau^2R^2,\qquad I_3=1. \tag{3.3}$$

For convenience of the reader, we briefly summarize the results of Rivlin [34, 35] for general W and specialized for the classical Mooney-Rivlin (and neo-Hookean) models. Our main focus is then on the strain-stiffening models described earlier. The non-zero stresses are given by Rivlin [35] as

$$T_{rr}=-2\gamma\tau^2\int_R^A sW_1(s)ds, \tag{3.4}$$

$$T_{\theta\theta}=-2\gamma\tau^2\int_R^A sW_1(s)ds+2\gamma\tau^2R^2W_1, \tag{3.5}$$

$$T_{zz}=-2\gamma\tau^2\int_R^A sW_1(s)ds+2(\gamma^2-\gamma^{-1})W_1+2(\gamma-\gamma^{-2}-\tau^2R^2)W_2, \tag{3.6}$$

$$T_{z\theta}=2\gamma^{3/2}\tau RW_1+2\gamma^{1/2}\tau RW_2, \tag{3.7}$$

where the subscript on W denotes differentiation with respect to the corresponding invariant and these derivatives are evaluated at (3.3).

The resultant applied moment and axial force necessary to maintain the deformation are given by

$$M\equiv\int_0^{2\pi}\int_0^a T_{z\theta}r^2drd\theta=4\pi\tau\int_0^A R^3(W_1+\gamma^{-1}W_2)dR, \tag{3.8}$$

and

$$N\equiv\int_0^{2\pi}\int_0^a T_{zz}rdrd\theta=4\pi(\gamma-\gamma^{-2})\int_0^A R(W_1+\gamma^{-1}W_2)dR-2\pi\tau^2\times\int_0^A R^3(W_1+2\gamma^{-1}W_2)dR, \tag{3.9}$$

respectively.

For generalized neo-Hookean materials where $W=W(I_1)$, (3.6)–(3.9) simplify so that

$$T_{rr}=-2\gamma\tau^2\int_R^A sW_1(s)ds,\qquad T_{\theta\theta}=-2\gamma\tau^2\int_R^A sW_1(s)ds+2\gamma\tau^2R^2W_1, \tag{3.10}$$

$$T_{zz}=-2\gamma\tau^2\int_R^A sW_1(s)ds+2(\gamma^2-\gamma^{-1})W_1,\qquad T_{z\theta}=2\gamma^{3/2}\tau RW_1, \tag{3.11}$$

$$M=4\pi\tau\int_0^A R^3W_1dR,\qquad N=4\pi(\gamma-\gamma^{-2})\int_0^A RW_1dR-2\pi\tau^2\int_0^A R^3W_1dR. \tag{3.12}$$

For the Mooney-Rivlin model (2.3) (and its specialization to the neo-Hookean case), we have the well-known results (see, e.g., Rivlin [35])

$$T_{rr}^{MR} = -\frac{\mu\alpha\gamma\tau^2}{2}(A^2 - R^2), \qquad T_{\theta\theta}^{MR} = -\frac{\mu\alpha\gamma\tau^2}{2}(A^2 - 3R^2), \tag{3.13}$$

$$T_{zz}^{MR} = \mu\left[-\frac{\alpha\gamma\tau^2}{2}(A^2 - R^2) + \alpha(\gamma^2 - \gamma^{-1}) + (1-\alpha)(\gamma - \gamma^{-2} - \tau^2 R^2)\right], \tag{3.14}$$

$$T_{z\theta}^{MR} = \mu\gamma^{1/2}\tau R[\alpha(\gamma - 1) + 1], \qquad M^{MR} = \frac{\mu\pi\tau A^4}{2}[\alpha + (1-\alpha)\gamma^{-1}], \tag{3.15}$$

$$N^{MR} = \mu\pi A^2\left\{(\gamma - \gamma^{-2})[\alpha + (1-\alpha)\gamma^{-1}] - \frac{\tau^2 A^2}{4}[\alpha + 2(1-\alpha)\gamma^{-1}]\right\}, \tag{3.16}$$

and

$$T_{rr}^{nH} = -\frac{\mu\gamma\tau^2}{2}(A^2 - R^2), \qquad T_{\theta\theta}^{nH} = -\frac{\mu\gamma\tau^2}{2}(A^2 - 3R^2), \tag{3.17}$$

$$T_{zz}^{nH} = \mu\left[-\frac{\gamma\tau^2}{2}(A^2 - R^2) + \gamma^2 - \gamma^{-1}\right], \qquad T_{z\theta}^{nH} = \mu\gamma^{3/2}\tau R, \tag{3.18}$$

$$M^{nH} = \frac{\mu\pi\tau A^4}{2}, \qquad N^{nH} = \mu\pi A^2\left(\gamma - \gamma^{-2} - \frac{\tau^2 A^2}{4}\right), \tag{3.19}$$

respectively. For the classical models, the axial force N depends on τ in a simple quadratic fashion and decreases as τ increases *regardless of the amount of stretch* γ.

For the Gent model (2.9), the limiting chain extensibility constraint $I_1 - 3 < J_m$ (see (2.9)) can be written, on using (3.3), as

$$\gamma^2 + 2\gamma^{-1} + \gamma\tau^2 R^2 < J_m + 3, \tag{3.20}$$

so that the twist τ and axial stretch γ are constrained in a *coupled* fashion. To ensure that the *local* constraint (3.20) holds for all R in $[0, A]$, we henceforth assume that (3.20) holds at $R = A$ and so impose the stronger *global* constraint

$$\tau^2 A^2 < \frac{1}{\gamma}(J_m - \gamma^2 - 2\gamma^{-1} + 3). \tag{3.21}$$

For a given extensibility parameter J_m and a given stretch γ, (3.21) constrains the total angle of twist τA that the strain-stiffening cylinder can undergo. Two special cases of (3.21) for which the twist τ and axial stretch γ are *uncoupled* are worth noting:

(1) When $\tau = 0$ we have *pure extension* of the cylinder and (3.21) reduces to

$$\gamma^2 + 2\gamma^{-1} < J_m + 3, \tag{3.22}$$

which is the usual constraint for simple extension for the Gent model (see, e.g., [10, 23]).

(2) When $\gamma = 1$, we have *pure torsion*, which we will consider in detail in Sect. 4. In this case, the total angle of twist τA is constrained by

$$\tau^2 A^2 < J_m. \tag{3.23}$$

For the Gent model, where $W = W(I_1)$, the stresses (3.10)–(3.11) are

$$T_{rr}^G = -\frac{\mu}{2} J_m \ln\left(\frac{J_m - \gamma^2 - 2\gamma^{-1} + 3 - \gamma\tau^2 R^2}{J_m - \gamma^2 - 2\gamma^{-1} + 3 - \gamma\tau^2 A^2}\right), \tag{3.24}$$

$$T_{\theta\theta}^G = \frac{\mu}{2} J_m \left[\frac{2\gamma\tau^2 R^2}{J_m - \gamma^2 - 2\gamma^{-1} + 3 - \gamma\tau^2 R^2} - \ln\left(\frac{J_m - \gamma^2 - 2\gamma^{-1} + 3 - \gamma\tau^2 R^2}{J_m - \gamma^2 - 2\gamma^{-1} + 3 - \gamma\tau^2 A^2}\right)\right], \tag{3.25}$$

$$T_{zz}^G = \frac{\mu}{2} J_m \left[\frac{2(\gamma^2 - \gamma^{-1})}{J_m - \gamma^2 - 2\gamma^{-1} + 3 - \gamma\tau^2 R^2} - \ln\left(\frac{J_m - \gamma^2 - 2\gamma^{-1} + 3 - \gamma\tau^2 R^2}{J_m - \gamma^2 - 2\gamma^{-1} + 3 - \gamma\tau^2 A^2}\right)\right], \tag{3.26}$$

$$T_{z\theta}^G = \mu \frac{J_m \gamma^{3/2} \tau R}{J_m - \gamma^2 - 2\gamma^{-1} + 3 - \gamma\tau^2 R^2}, \tag{3.27}$$

and the relations (3.12) now read

$$M^G = -\frac{\mu\pi J_m}{\gamma\tau^3}\left\{\tau^2 A^2 + \frac{1}{\gamma}(J_m - 2\gamma^{-1} - \gamma^2 + 3)\ln\left[\frac{J_m - 2\gamma^{-1} - \gamma^2 + 3 - \gamma\tau^2 A^2}{J_m - 2\gamma^{-1} - \gamma^2 + 3}\right]\right\}, \tag{3.28}$$

$$N^G = \frac{\mu\pi J_m}{2\gamma\tau^2}\left\{\tau^2 A^2 + \frac{1}{\gamma}(J_m - 3\gamma^2 + 3)\ln\left[\frac{J_m - 2\gamma^{-1} - \gamma^2 + 3 - \gamma\tau^2 A^2}{J_m - 2\gamma^{-1} - \gamma^2 + 3}\right]\right\}. \tag{3.29}$$

For the HS model, the constraint (2.13) or (2.14) will be assumed to hold at $R = A$ thus ensuring that it holds for all R in $[0, A]$. This constraint, again involving a coupling of τ and γ, can be written as

$$\tau^2 A^2 < \frac{(J - \gamma^2)(J - \gamma^{-1})}{J\gamma}, \tag{3.30}$$

where the limiting chain parameter J can be shown to necessarily satisfy

$$J > \gamma^2 \quad \text{and} \quad J > \gamma^{-1}. \tag{3.31}$$

The constraint (3.30) is the analog of (3.21). Again two special cases of (3.30) are of interest:

(1) When $\tau = 0$ (i.e., *pure extension*), the constraint (3.30) is equivalent to (3.31) which we write, for $\gamma > 1$, as

$$\gamma < J^{1/2}, \tag{3.32}$$

and so the stretch in pure extension cannot exceed $J^{1/2}$ (cf. [15, 21–24]).

(2) When $\gamma = 1$ (i.e., *pure torsion*) we find from (3.30) that the total angle of twist τA is constrained as

$$\tau^2 A^2 < \frac{(J - 1)^2}{J}. \tag{3.33}$$

From (3.4)–(3.7) we obtain

$$T_{rr}^{HS} = -\frac{\mu}{2}\frac{(J - 1)^2}{J - \gamma^{-1}} \ln\left[\frac{(J - \gamma^2)(J - \gamma^{-1}) - J\gamma\tau^2 R^2}{(J - \gamma^2)(J - \gamma^{-1}) - J\gamma\tau^2 A^2}\right], \tag{3.34}$$

$$T_{\theta\theta}^{HS} = \frac{\mu}{2}\frac{(J-1)^2}{J-\gamma^{-1}}\left\{\frac{2J\gamma\tau^2R^2}{(J-\gamma^2)(J-\gamma^{-1})-J\gamma\tau^2R^2} - \ln\left[\frac{(J-\gamma^2)(J-\gamma^{-1})-J\gamma\tau^2R^2}{(J-\gamma^2)(J-\gamma^{-1})-J\gamma\tau^2A^2}\right]\right\}, \tag{3.35}$$

$$T_{zz}^{HS} = \frac{\mu}{2}\frac{(J-1)^2}{J-\gamma^{-1}}\left\{2\frac{J(\gamma^2-\gamma^{-1})-(\gamma-\gamma^{-2}-\tau^2R^2)}{(J-\gamma^2)(J-\gamma^{-1})-J\gamma\tau^2R^2} - \ln\left[\frac{(J-\gamma^2)(J-\gamma^{-1})-J\gamma\tau^2R^2}{(J-\gamma^2)(J-\gamma^{-1})-J\gamma\tau^2A^2}\right]\right\}, \tag{3.36}$$

$$T_{z\theta}^{HS} = \mu\frac{(J-1)^2\gamma^{3/2}\tau R}{(J-\gamma^2)(J-\gamma^{-1})-J\gamma\tau^2R^2}, \tag{3.37}$$

and (3.8)–(3.9) now yield

$$M^{HS} = -\frac{\mu\pi(J-1)^2}{J^2\gamma^2\tau^3}\left\{J\gamma\tau^2A^2 + (J-\gamma^2)(J-\gamma^{-1}) \times \ln\left[\frac{(J-\gamma^2)(J-\gamma^{-1})-J\gamma\tau^2A^2}{(J-\gamma^2)(J-\gamma^{-1})}\right]\right\}, \tag{3.38}$$

$$N^{HS} = \frac{\mu\pi(J-1)^2}{2J^2\gamma\tau^2}\left\{\frac{J(J-2\gamma^{-1})}{(J-\gamma^{-1})}\tau^2A^2 + (J^2\gamma^{-1}-3J\gamma+2) \times \ln\left[\frac{(J-\gamma^2)(J-\gamma^{-1})-J\gamma\tau^2A^2}{(J-\gamma^2)(J-\gamma^{-1})}\right]\right\}. \tag{3.39}$$

In the limit as $J_m \to \infty$ or as $J \to \infty$ in the foregoing it can be verified that one recovers the results (3.17)–(3.19) for the *neo-Hookean* material.

For the Fung exponential model (2.15) we find from (3.10)–(3.12) that

$$T_{rr}^F = \frac{\mu}{2b}e^{b(\gamma^2+2\gamma^{-1}-3)}(e^{b\gamma\tau^2R^2} - e^{b\gamma\tau^2A^2}), \tag{3.40}$$

$$T_{\theta\theta}^F = \frac{\mu}{2b}e^{b(\gamma^2+2\gamma^{-1}-3)}[(1+2b\gamma\tau^2R^2)e^{b\gamma\tau^2R^2} - e^{b\gamma\tau^2A^2}], \tag{3.41}$$

$$T_{zz}^F = \frac{\mu}{2b}e^{b(\gamma^2+2\gamma^{-1}-3)}[(1+2b\gamma^2-2b\gamma^{-1})e^{b\gamma\tau^2R^2} - e^{b\gamma\tau^2A^2}], \tag{3.42}$$

$$T_{z\theta}^F = \mu\gamma^{3/2}\tau Re^{b(\gamma^2+2\gamma^{-1}-3+\gamma\tau^2R^2)}, \tag{3.43}$$

$$M^F = \frac{\mu\pi}{b^2\gamma^2\tau^3}e^{b(\gamma^2+2\gamma^{-1}-3)}[(b\gamma\tau^2A^2-1)e^{b\gamma\tau^2A^2}+1], \tag{3.44}$$

$$N^F = \frac{\mu\pi}{2b^2\gamma^2\tau^2}e^{b(\gamma^2+2\gamma^{-1}-3)} \times [(2b\gamma^2-2b\gamma^{-1}+1-b\gamma\tau^2A^2)e^{b\gamma\tau^2A^2} - 2b\gamma^2+2b\gamma^{-1}-1]. \tag{3.45}$$

3.1 Discussion

In order to investigate how the resultant moment M and axial force N depend on the twist τ and the stretch γ, we define $N_0(\gamma_0) \equiv N|_{\gamma=\gamma_0, \tau=0}$ so that $N_0(\gamma_0)$ represents the axial force necessary to achieve an axial stretch $\gamma = \gamma_0$ in the absence of a torsional moment, i.e., when $\tau = 0$. Thus N_0 is the axial force necessary to produce a stretch γ_0 in uniaxial extension. We also define the quantity N_T such that

$$N(\gamma_0, \tau) = N_0(\gamma_0) + N_T(\gamma_0, \tau), \tag{3.46}$$

and so N_T represents the *additional axial force* that is necessary to *maintain* a constant stretch of $\gamma = \gamma_0$ upon application of the twist τ. We see from (3.9) that

$$N_0(\gamma_0) = N|_{\substack{\gamma=\gamma_0 \\ \tau=0}} = 2\pi A^2(\gamma_0 - \gamma_0^{-2})(W_1 + \gamma_0^{-1} W_2), \tag{3.47}$$

where W_1 and W_2 are evaluated at $I_1 = \gamma_0^2 + 2\gamma_0^{-1}$ and $I_2 = 2\gamma_0 + \gamma_0^{-2}$. On using (3.9) and (3.47), (3.46) yields an expression for N_T. *In the absence of the additional force* N_T (i.e., when the axial force N is *constant*), the stretch γ will, in general, differ from γ_0 upon torsion. Whether a *stretched* cylinder tends to *further lengthen* or to *shorten upon increasing torsion* depends on the sign of $\partial N/\partial\tau (= \partial N_T/\partial\tau)$. Since N_0 does not depend on τ and $N_T|_{\tau=0} = 0$, we see that if N is a monotonically increasing function of τ then N_T will be positive, while if N is a monotonically decreasing function of τ then N_T will be negative.

For the Mooney-Rivlin material (and its specialization to the neo-Hookean case), we have

$$\frac{\partial N^{MR}}{\partial\tau} = -\frac{\mu\pi\tau A^4}{2}[\alpha + 2(1-\alpha)\gamma^{-1}], \tag{3.48}$$

and

$$\frac{\partial N^{nH}}{\partial\tau} = -\frac{\mu\pi\tau A^4}{2}, \tag{3.49}$$

respectively. Clearly, for both classical models $\partial N/\partial\tau < 0$ regardless of the amount of stretch γ or the amount of torsion τ, and so N_T is always compressive. Under a *constant* axial force (i.e., in the absence of the *additional* force N_T), the stretched cylinder thus always has a tendency to *further elongate upon twisting,* i.e., $\gamma > \gamma_0$ for $\tau > 0$.

For the Gent model, however, we find that

$$\begin{aligned}\frac{\partial N^G}{\partial\tau} &= -\frac{\mu\pi J_m}{\gamma^2\tau^3}(J_m - 3\gamma^2 + 3)\\ &\quad\times\left[\frac{\gamma\tau^2A^2}{J_m - \gamma^2 - 2\gamma^{-1} + 3 - \gamma\tau^2A^2} - \ln\left(\frac{J_m - \gamma^2 - 2\gamma^{-1} + 3}{J_m - \gamma^2 - 2\gamma^{-1} + 3 - \gamma\tau^2A^2}\right)\right],\end{aligned} \tag{3.50}$$

which may be re-written as

$$\frac{\partial N^G}{\partial\tau} = -\frac{\mu\pi J_m}{\gamma^2\tau^3}(J_m - 3\gamma^2 + 3)\left[\frac{\Delta}{1-\Delta} - \ln\left(\frac{1}{1-\Delta}\right)\right], \tag{3.51}$$

where we have defined $\Delta \equiv \gamma\tau^2A^2/(J_m - \gamma^2 - 2\gamma^{-1} + 3)$. Note that the constraint (3.21) guarantees that $0 \le \Delta < 1$. On using power series expansions, we may rewrite (3.51) as

$$\frac{\partial N^G}{\partial\tau} = -\frac{\mu\pi J_m}{\gamma^2\tau^3}(J_m - 3\gamma^2 + 3)\left(\frac{\Delta^2}{2} + \frac{2\Delta^3}{3} + \frac{3\Delta^4}{4} + \cdots\right). \tag{3.52}$$

We see from (3.52) that

$$\partial N^G/\partial\tau < 0 \quad \text{if and only if} \quad J_m - 3\gamma^2 + 3 > 0, \tag{3.53}$$

so that

$$N_T^G < 0 \quad \text{if and only if} \quad \gamma < \sqrt{\frac{J_m+3}{3}} \equiv \gamma_t^G. \tag{3.54}$$

Since $J_m > 0$, we have $\gamma_t^G > 1$. Similarly for the HS model, we find that

$$\begin{aligned}\frac{\partial N^{HS}}{\partial\tau} &= -\frac{\mu\pi(J-1)^2}{J^2\gamma\tau^3}(J^2\gamma^{-1} - 3J\gamma + 2)\\ &\quad\times\left[\frac{J\gamma\tau^2A^2}{(J-\gamma^2)(J-\gamma^{-1}) - J\gamma\tau^2A^2} - \ln\left(\frac{(J-\gamma^2)(J-\gamma^{-1})}{(J-\gamma^2)(J-\gamma^{-1}) - J\gamma\tau^2A^2}\right)\right].\end{aligned} \tag{3.55}$$

On using (3.30) and (3.31), we find that

$$N_T^{HS} < 0 \quad \text{if and only if} \quad \gamma < \frac{1+\sqrt{1+3J^3}}{3J} \equiv \gamma_t^{HS}. \tag{3.56}$$

To ensure that the quantity γ_t^{HS} defined in (3.56) is such that $\gamma_t^{HS} > 1$, we require that

$$J > 2, \tag{3.57}$$

which is assumed henceforth.

We have just shown that for the Gent and HS models, in the absence of the additional axial force N_T, the stretched cylinder has a tendency to further extend with increased twisting for $\gamma < \gamma_t$, but for $\gamma > \gamma_t$ the cylinder tends to shorten upon twisting. For $\gamma = \gamma_t$, $N_T = 0$ and $N = N_0$ regardless of the amount of torsion τ. We may think of γ_t as a transition stretch that exists for such strain-stiffening stretched cylinders. The results (3.54) and (3.56) are depicted graphically in Fig. 1, where the transition value γ_t is shown as a function of the parameter J_m for the Gent model and as a function of the parameter J for HS model.

The foregoing results have been substantiated experimentally for vulcanized rubber in a recent paper by Gent and Hua [12] where a non-monotone relation between axial force and applied extension has been observed giving rise to a transition stretch of the type predicted above (see Fig. 2 of [12]). Our results are also analogous to those obtained by Shield [37] in the context of *small* twist superimposed on extension of bars of *general cross-section*. As remarked on p. 76 of [37]: "Whether an extended cylinder will tend to elongate or shorten when given a small twist with the axial force held constant depends on the material, the geometry of the cross-section of the cylinder and the amount of initial extension." Here we have restricted our attention to a *circular* cross-section but our results do not assume small twist. We have thus demonstrated the validity of Shield's assertion in the wider context of large twist where here the strain-stiffening material property plays a key role. Figure 1 of the present paper is analogous to Fig. 4 of [37] where a transition stretch is found for sufficiently slender elliptical cross-sections for two particular strain-energy densities.

For the exponential model (2.15) we find from (3.45) that

$$\frac{\partial N^F}{\partial\tau} = \frac{\mu\pi}{b^2\gamma^2\tau^3}\beta e^{b(\gamma^2+2\gamma^{-1}-3)}\left[1 - e^x\left(\frac{x^2}{\beta} - x + 1\right)\right], \tag{3.58}$$

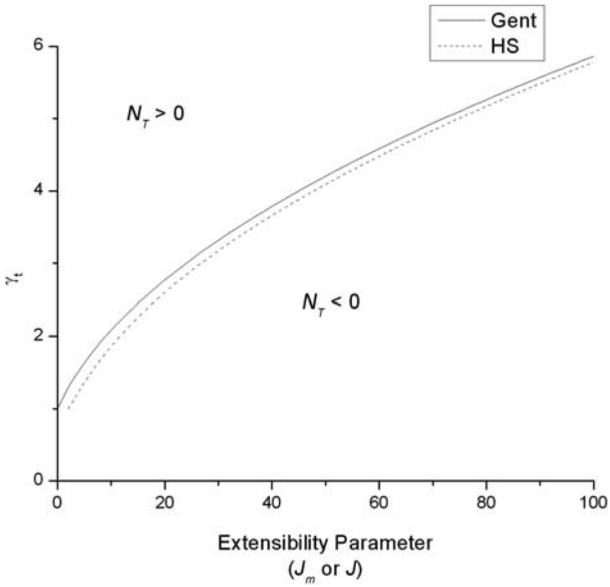


Fig. 1 The transition value γ_t is shown as a function of the parameter J_m for the Gent model and as a function of the parameter J for HS model. For a stretch less than γ_t the stretched cylinder tends to further extend upon torsion while for a stretch greater than γ_t, the cylinder tends to shorten upon torsion

where we have defined the quantities

$$x \equiv b\gamma\tau^2 A^2 \geq 0 \quad \text{and} \quad \beta \equiv 2b(\gamma^2 - \gamma^{-1}) + 1 \geq 1. \tag{3.59}$$

We see from (3.58) that $\partial N^F/\partial\tau < 0$ if and only if

$$e^x\left(\frac{x^2}{\beta} - x + 1\right) - 1 > 0. \tag{3.60}$$

In Fig. 2, we plot the left hand side of (3.60) for various values of β. It may be shown (see Fig. 2) that whenever the stretch is such that $\beta \leq 2$, i.e.,

$$\gamma^2 - \gamma^{-1} \leq \frac{1}{2b}, \tag{3.61}$$

then (3.60) holds *over the entire range* of $\tau^2 A^2$, and so N^F is a monotonically decreasing function of τ in this case. Thus, under the condition (3.61), N_T^F is *always compressive*, and the stretched cylinder would further elongate upon twisting in the absence of such a force. The condition (3.61) may be interpreted in two ways: (1) For a given stiffening parameter $b > 0$, the inequality (3.61) requires the stretch γ (> 1) to be sufficiently small. For large b, (3.61) holds only for γ close to unity. Note that in the limit as $b \to 0$, i.e., as $\beta \to 1$, the inequality (3.61) holds for all γ and we recover the previously discussed result for the neo-Hookean model. (2) Alternatively, for a given stretch $\gamma > 1$, the inequality (3.61) holds if the stiffening parameter b is sufficiently small. However, when (3.61) does not hold (i.e., $\beta > 2$) so that

$$\gamma^2 - \gamma^{-1} > \frac{1}{2b}, \tag{3.62}$$

the situation is more complicated. Again, the condition (3.62) can be interpreted in two ways analogous to those described above in connection with (3.61). In this case (see Fig. 2), it

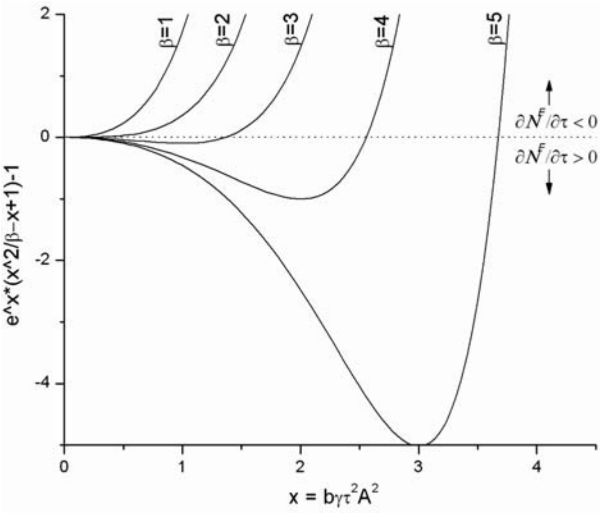


Fig. 2 The left hand side of (3.60) plotted versus $x = b\gamma\tau^2 A^2$ for various values of β defined in $(3.59)_2$. Note that for $\beta \leq 2$, the inequality (3.60) holds *for all* x, while for $\beta > 2$, (3.60) holds only for sufficiently large x

may be shown that N^F is no longer a monotonic function of τ. One finds that $\partial N^F/\partial\tau > 0$ for small twist, while $\partial N^F/\partial\tau < 0$ for sufficiently large twist. This transition from N^F increasing to N^F decreasing occurs for the value of τ such that

$$e^x\left(\frac{x^2}{\beta} - x + 1\right) - 1 = 0, \tag{3.63}$$

where x and β are defined in (3.59). The exponential model thus predicts that, for a given stiffening parameter b, in the absence of the additional force N_T a *sufficiently stretched cylinder* will *shorten* on undergoing sufficiently small twist, but then tends to *elongate* upon further twisting.

These somewhat complicated predictions for the Fung exponential model are rather surprising and illustrate again the differences that can occur in the behavior of both classes of strain-stiffening models (cf. the remarks made at the end of Sect. 2).

In Figs. 3–8 we plot the nondimensional applied moment M and axial force N versus τA for varying amounts of constant stretch $\gamma = \gamma_0$. This is done for the Gent model with $J_m =$ 2.289, 30, and 97.2 (for which to $\gamma_t = 1.3$, 3.3, and 5.8) and for the exponential model with $b = 0.55$, 0.035, and 0.01. These values of J_m are chosen in accordance with the discussion of simple extension in Horgan and Schwartz [24] where it was pointed out that $J_m = 2.289$ is representative of human arterial wall tissue of low extensibility (see [20]), $J_m = 97.2$ is representative of vulcanized rubber [10], and $J_m = 30$ is an intermediate value for less extensible rubber. The values of the parameter b in the exponential model were chosen to qualitatively match the moment curves for the Gent model at $\gamma_0 = 1$, i.e., for *pure torsion*. To simplify Figs. 3–8, we have omitted analogous plots for the HS model as they are very similar to those for the Gent model. Note that as γ_0 increases, the vertical asymptotes for the Gent model occur at decreasing values of τA as expected from the constraint (3.21), which limits the allowable twist τ and axial stretch γ in a coupled fashion. In Figs. 4, 6, and 8 we see that N is a decreasing function of τ when $\gamma_0 < \gamma_t$ and increasing when $\gamma_0 > \gamma_t$, where γ_t is the transition stretch defined in (3.54). We also see a loss of monotonicity in the N versus τ curve for the exponential model when $\gamma_0^2 - \gamma_0^{-1} > \frac{1}{2b}$ (i.e., when $\gamma = 1.5$, 1.75, and 2 in Fig. 4 and when $\gamma = 4$ in Fig. 6), as expected from the discussion following (3.62). Each curve in Figs. 4, 6, and 8 intercepts the vertical axis at $N = N_0(\gamma_0)$.

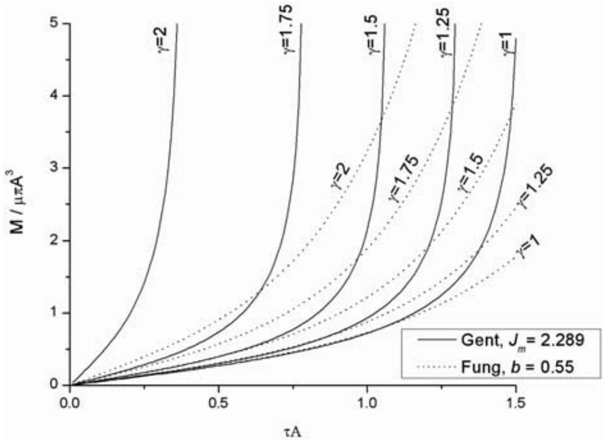


Fig. 3 Nondimensional resultant applied moment M versus τA for varying amounts of constant stretch $\gamma = \gamma_0$. *Solid curves* correspond to the Gent model with $J_m = 2.289$. *Dotted curves* correspond to the Fung model with $b = 0.55$

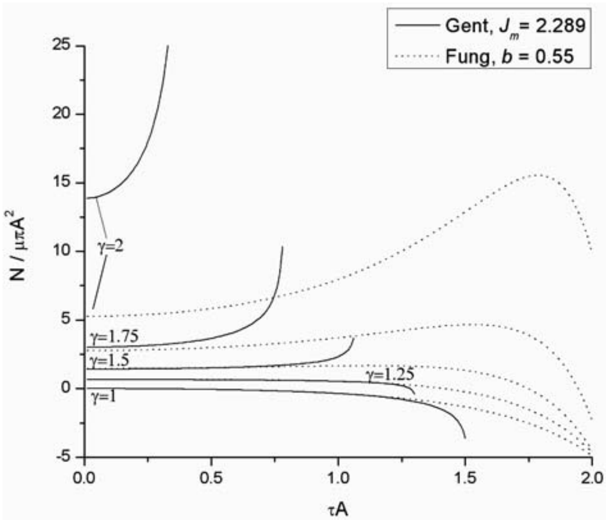


Fig. 4 Nondimensional resultant axial force N versus τA for varying amounts of constant stretch $\gamma = \gamma_0$. *Solid curves* correspond to the Gent model with $J_m = 2.289$ (thus $\gamma_t \approx 1.3$). *Dotted curves* correspond to the Fung model with $b = 0.55$

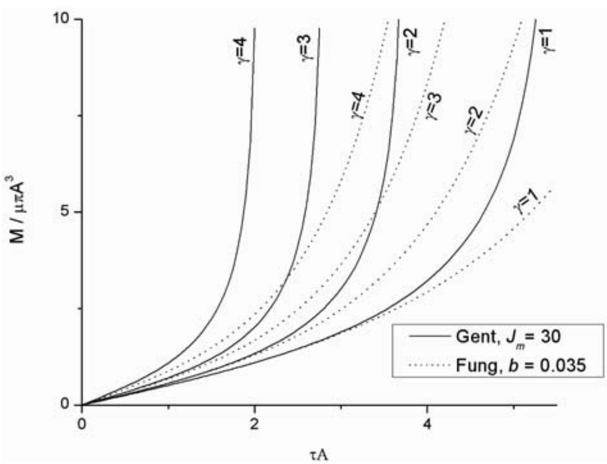


Fig. 5 Nondimensional resultant applied moment M versus τA for varying amounts of constant stretch $\gamma = \gamma_0$. *Solid curves* correspond to the Gent model with $J_m = 30$. *Dotted curves* correspond to the Fung model with $b = 0.035$

Fig. 6 Nondimensional resultant axial force N versus τA for varying amounts of constant stretch $\gamma = \gamma_0$. *Solid curves* correspond to the Gent model with $J_m = 30$ (thus $\gamma_t \approx 3.3$). *Dotted curves* correspond to the Fung model with $b = 0.035$

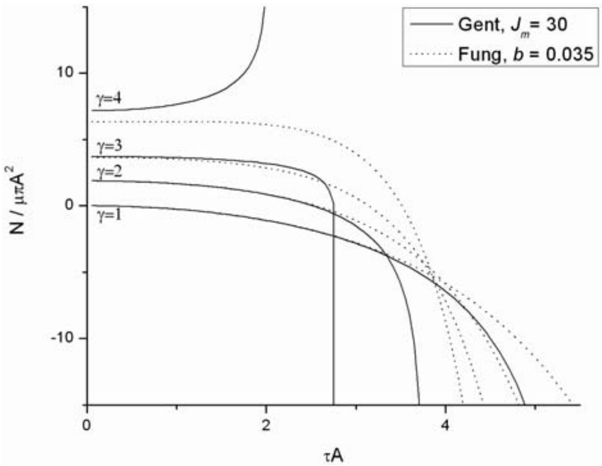


Fig. 7 Nondimensional resultant applied moment M versus τA for varying amounts of constant stretch $\gamma = \gamma_0$. *Solid curves* correspond to the Gent model with $J_m = 97.2$. *Dotted curves* correspond to the Fung model with $b = 0.01$

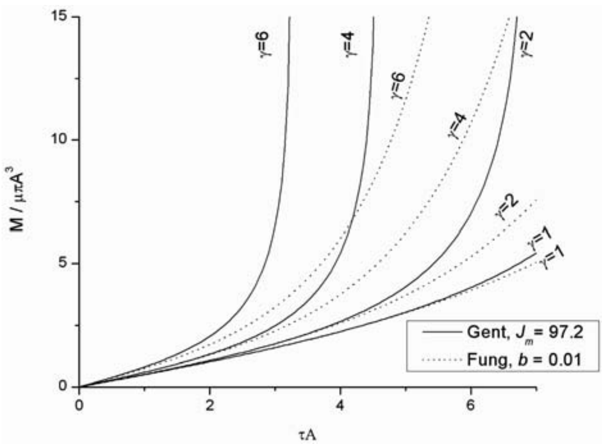
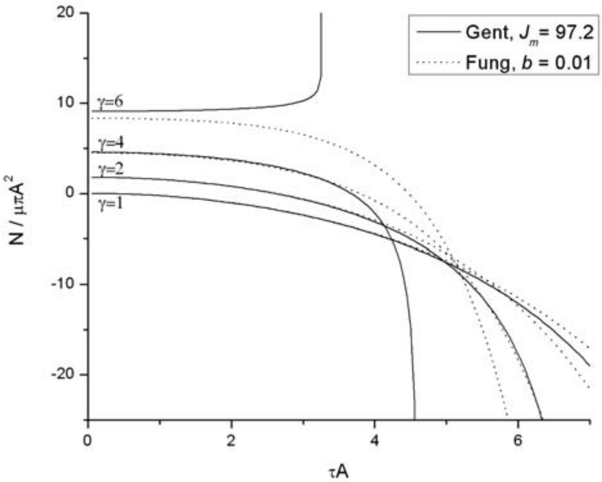


Fig. 8 Nondimensional resultant axial force N versus τA for varying amounts of constant stretch $\gamma = \gamma_0$. *Solid curves* correspond to the Gent model with $J_m = 97.2$ (thus $\gamma_t \approx 5.8$). *Dotted curves* correspond to the Fung model with $b = 0.01$



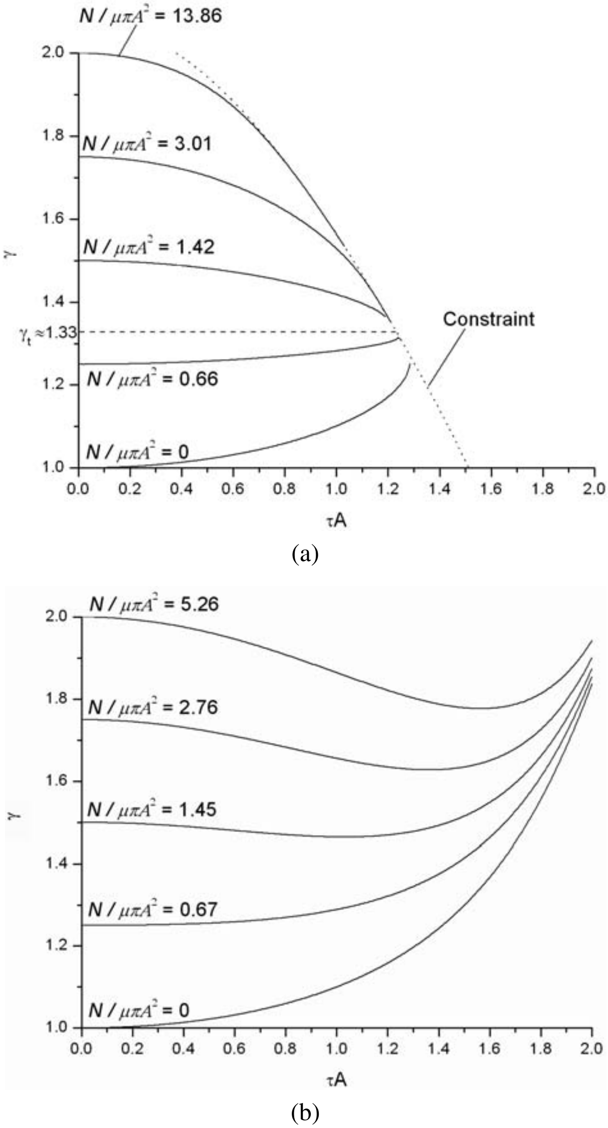


Fig. 9 Stretch γ versus τA for $N = N_0$ for various values of N_0 corresponding to $\gamma_0 = 1$, 1.25, 1.5, 1.75, and 2. (**a**) is for the Gent model with $J_m = 2.289$, which leads to $\gamma_t \approx 1.33$. The *dotted curve* corresponds to the constraint (3.21) so that only deformations on the left of the dotted line are admissible. (**b**) is for the Fung model with $b = 0.55$

Thus far, we have considered γ constant and taken $N = N(\tau)$ in order to maintain such a deformation. Alternatively, instead of keeping the stretch constant, we can consider a constant axial force N and examine how γ changes as a function of τ. If we assume

$$N(\gamma, \tau) = N_0(\gamma_0) \tag{3.64}$$

for all τ, then for a given γ_0, (3.64) gives $\gamma = \gamma(\tau)$ implicitly. We write

$$\gamma(\tau) = \gamma_0 + \gamma_T(\tau), \tag{3.65}$$

where we have introduced the quantity $\gamma_T(\tau)$, which represents the additional stretch induced by torsion in the absence of the additional force N_T. In Fig. 9 we plot γ versus τA

when $N = N_0(\gamma_0)$ for $\gamma_0 = 1$, 1.25, 1.5, 1.75, and 2. Each curve in Fig. 9 intercepts the vertical axis at $\gamma = \gamma_0$. The values of N_0 that lead to these stretches may be calculated from (3.47) and are seen, in general, to depend on the constitutive model being used, with the exception that $\gamma_0 = 1$ implies $N = N_0 = 0$ (in this case, $\gamma_0 = 1$ does *not* imply simple torsion since, in general, $\gamma \neq \gamma_0$ for $\tau > 0$). Results for the Gent model with $J_m = 2.289$ are shown in Fig. 9a and so, from its definition in (3.54), we have $\gamma_t \approx 1.33$. The dashed horizontal straight line in Fig. 9a intercepts the vertical axis at $\gamma_0 = \gamma_t$. This is the transition stretch at which, on application of a constant axial force $N = N_0(\gamma_t) = 0.86\mu\pi A^2$, the stretched cylinder will not change length upon twisting. The dotted curve in Fig. 9a corresponds to the constraint (3.21) so that only deformations to the left of the dotted curve are admissible. We see that for the Gent model the stretch versus total angle of twist curve is monotonic increasing for $\gamma < \gamma_t$ and monotonic decreasing for $\gamma > \gamma_t$ and so the stretched cylinder further extends upon torsion in the former case and shortens in the latter. In Fig. 9b we plot results for the Fung model with $b = 0.55$. In this case there are no constraints on the deformation. For $b = 0.55$, the condition (3.61) implies that $\gamma \leq 1.30$, and so, in the absence of an additional axial force, the stretched cylinder tends to elongate upon torsion for $\gamma \leq 1.30$. When $\gamma > 1.30$, however, the stretch versus total angle of twist curves are non-monotonic and show that the stretched bar shortens for sufficiently small twist and then elongates at larger twist.

4 Simple Torsion

If we set $\gamma = 1$ in the preceding, then one recovers the classic problem of *simple or pure torsion*, where the cylinder is *not allowed to stretch axially or contract radially*. For completeness, we briefly summarize the results for this case. From (3.3), we have

$$I_1 = 3 + \tau^2 r^2, \qquad I_2 = 3 + \tau^2 r^2, \qquad I_3 = 1. \tag{4.1}$$

Since $r = R$ for simple torsion, all quantities in this subsection can be written in terms of r or R without alteration. For simplicity we will use the lower case letter. We denote both the undeformed and the current radius of the cylinder as a. On setting $\gamma = 1$ in (3.4)–(3.7) we recover the results of Rivlin [34, 35] that the non-zero Cauchy stresses are given by

$$T_{rr} = -2\tau^2 \int_r^a sW_1(s)ds, \tag{4.2}$$

$$T_{\theta\theta} = -2\tau^2 \int_r^a sW_1(s)ds + 2\tau^2 r^2 W_1, \tag{4.3}$$

$$T_{zz} = -2\tau^2 \int_r^a sW_1(s)ds - 2\tau^2 r^2 W_2, \tag{4.4}$$

$$T_{z\theta} = 2\tau r(W_1 + W_2), \tag{4.5}$$

where the subscript on W denotes differentiation with respect to the corresponding invariant and these derivatives are evaluated at (4.1). On setting $\gamma = 1$ in (3.8) and (3.9), we find that the resultant applied moment is given by

$$M = \int_0^{2\pi} \int_0^a T_{z\theta} r^2 dr d\theta = 4\pi\tau \int_0^a r^3(W_1 + W_2)dr, \tag{4.6}$$

while the resultant axial force is

$$N = \int_0^{2\pi}\int_0^a T_{zz} r\,dr\,d\theta = -2\pi\tau^2\int_0^a r^3(W_1 + 2W_2)dr. \tag{4.7}$$

For $W = W(I_1)$, (4.4)–(4.7) simplify so that for generalized neo-Hookean materials we have (see e.g., [16, 39])

$$T_{rr} = T_{zz} = -2\tau^2\int_r^a sW_1(s)ds, \qquad T_{\theta\theta} = -2\tau^2\int_r^a sW_1(s)ds + 2\tau^2 r^2 W_1, \tag{4.8}$$

$$T_{z\theta} = 2\tau r W_1, \qquad M = 4\pi\tau\int_0^a r^3 W_1 dr, \qquad N = -2\pi\tau^2\int_0^a r^3 W_1 dr. \tag{4.9}$$

For a Mooney-Rivlin material (and its specialization to the neo-Hookean case), on setting $\gamma = 1$ in (3.13)–(3.16) we recover the well-known results of Rivlin [34], i.e.,

$$T_{rr}^{MR} = -\frac{\mu\alpha\tau^2}{2}(a^2 - r^2), \qquad T_{\theta\theta}^{MR} = -\frac{\mu\alpha\tau^2}{2}(a^2 - 3r^2), \tag{4.10}$$

$$T_{zz}^{MR} = -\mu\left[\frac{\alpha\tau^2}{2}(a^2 - r^2) + (1-\alpha)\tau^2 r^2\right], \qquad T_{z\theta}^{MR} = \mu\tau r, \tag{4.11}$$

$$M^{MR} = \frac{\mu\pi\tau a^4}{2}, \qquad N^{MR} = -\frac{\mu\pi\tau^2 a^4}{4}(2-\alpha), \tag{4.12}$$

and

$$T_{rr}^{nH} = T_{zz}^{nH} = -\frac{\mu\tau^2}{2}(a^2 - r^2), \qquad T_{\theta\theta}^{nH} = -\frac{\mu\tau^2}{2}(a^2 - 3r^2), \tag{4.13}$$

$$T_{z\theta}^{nH} = \mu\tau r, \qquad M^{nH} = \frac{\mu\pi\tau a^4}{2}, \qquad N^{nH} = -\frac{\mu\pi\tau^2 a^4}{4}, \tag{4.14}$$

respectively.

For the Gent model we have the constraint (3.23), i.e.,

$$\tau^2 a^2 < J_m. \tag{4.15}$$

On setting $\gamma = 1$ in (3.24)–(3.29), we find that

$$T_{rr}^{G} = T_{zz}^{G} = -\frac{\mu}{2}J_m \ln\left(\frac{J_m - \tau^2 r^2}{J_m - \tau^2 a^2}\right), \tag{4.16}$$

$$T_{\theta\theta}^{G} = \frac{\mu}{2}J_m\left[\frac{2\tau^2 r^2}{J_m - \tau^2 r^2} - \ln\left(\frac{J_m - \tau^2 r^2}{J_m - \tau^2 a^2}\right)\right], \tag{4.17}$$

$$T_{z\theta}^{G} = \mu\frac{J_m \tau r}{J_m - \tau^2 r^2}, \tag{4.18}$$

$$M^{G} = -\frac{\mu\pi J_m}{\tau^3}\left[\tau^2 a^2 + J_m \ln\left(\frac{J_m - \tau^2 a^2}{J_m}\right)\right], \tag{4.19}$$

$$N^{G} = \frac{\mu\pi J_m}{2\tau^2}\left[\tau^2 a^2 + J_m \ln\left(\frac{J_m - \tau^2 a^2}{J_m}\right)\right]. \tag{4.20}$$

These results were also obtained directly from (4.8), (4.9) in [16]. From (4.19), (4.20) we see that M^G and $N^G \to \infty$ as $\tau a \to \sqrt{J_m}$.

For the HS model we have the constraint (3.33), i.e.,

$$\tau^2 a^2 < \frac{(J-1)^2}{J}. \tag{4.21}$$

On setting $\gamma = 1$ in (3.34)–(3.39) we obtain

$$T_{rr}^{HS} = -\frac{\mu}{2}(J-1)\ln\left[\frac{(J-1)^2 - J\tau^2 r^2}{(J-1)^2 - J\tau^2 a^2}\right], \tag{4.22}$$

$$T_{\theta\theta}^{HS} = \frac{\mu}{2}(J-1)\left\{\frac{2J\tau^2 r^2}{[(J-1)^2 - J\tau^2 r^2]} - \ln\left[\frac{(J-1)^2 - J\tau^2 r^2}{(J-1)^2 - J\tau^2 a^2}\right]\right\}, \tag{4.23}$$

$$T_{zz}^{HS} = \frac{\mu}{2}(J-1)\left\{\frac{2\tau^2 r^2}{[(J-1)^2 - J\tau^2 r^2]} - \ln\left[\frac{(J-1)^2 - J\tau^2 r^2}{(J-1)^2 - J\tau^2 a^2}\right]\right\}, \tag{4.24}$$

$$T_{z\theta}^{HS} = \frac{\mu(J-1)^2 \tau r}{(J-1)^2 - J\tau^2 r^2}, \tag{4.25}$$

$$M^{HS} = \frac{-\mu\pi(J-1)^2}{J^2\tau^3}\left\{J\tau^2 a^2 + (J-1)^2 \ln\left[\frac{(J-1)^2 - J\tau^2 a^2}{(J-1)^2}\right]\right\}, \tag{4.26}$$

$$N^{HS} = \frac{\mu\pi(J-1)^2}{2J^2\tau^2}\left\{\frac{J(J-2)}{(J-1)}\tau^2 a^2 + (J-1)(J-2)\ln\left[\frac{(J-1)^2 - J\tau^2 a^2}{(J-1)^2}\right]\right\}. \tag{4.27}$$

We see that M^{HS} and $N^{HS} \to \infty$ as $\tau a \to (J-1)J^{-1/2}$. Note that for the HS model we have $T_{rr} \neq T_{zz}$, which is in contrast with (4.16) for Gent model. This reflects the fact that W^{HS} depends on I_2. As $J_m \to \infty$ in (4.16)–(4.20) or as $J \to \infty$ in (4.22)–(4.27), one recovers the results for pure torsion of a neo-Hookean material.

For the Fung exponential model (2.15), on setting $\gamma = 1$ in (3.40)–(3.45), we find that

$$T_{rr}^F = T_{zz}^F = \frac{\mu}{2b}(e^{b\tau^2 R^2} - e^{b\tau^2 A^2}), \tag{4.28}$$

$$T_{\theta\theta}^F = \frac{\mu}{2b}[(1 + 2b\tau^2 R^2)e^{b\tau^2 R^2} - e^{b\tau^2 A^2}], \tag{4.29}$$

$$T_{z\theta}^F = \mu\tau R e^{b\tau^2 R^2}, \tag{4.30}$$

$$M^F = \frac{\mu\pi}{b^2\tau^3}[(b\tau^2 A^2 - 1)e^{b\tau^2 A^2} + 1], \tag{4.31}$$

$$N^F = \frac{\mu\pi}{2b^2\tau^2}[(1 - b\gamma\tau^2 A^2)e^{b\tau^2 A^2} - 1]. \tag{4.32}$$

On letting $b \to 0$ in (4.28)–(4.32) and using l'Hopital's rule, we recover results for the neo-Hookean model.

4.1 Discussion

Our first objective is to compare results for the Gent and HS models. As noted above, for the Gent model M and $N \to \infty$ as $\tau a \to \sqrt{J_m}$, and similarly, for the HS model, M and $N \to \infty$

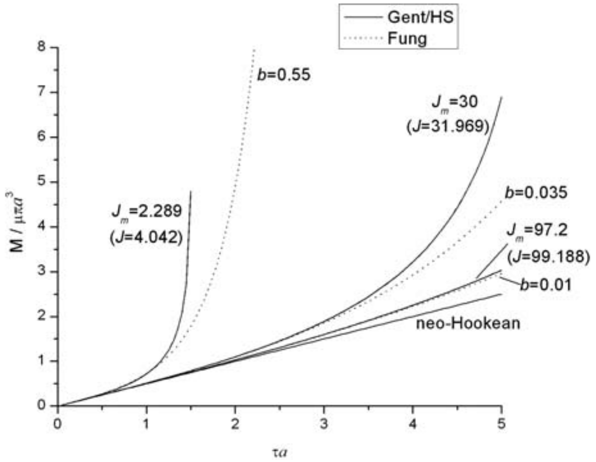


Fig. 10 Resultant applied moment for simple torsion for the Gent model with $J_m = 2.289$, 30, and 97.2, which is also valid for the HS model with $J = 4.042$, 31.969, and 99.188. The vertical asymptotes occur at $\tau a = 1.51$, 5.47, and 9.86 respectively. The *dotted curves* show results for the exponential model with $b = 0.55$, 0.035, and 0.01. The *straight line* corresponds to the neo-Hookean model

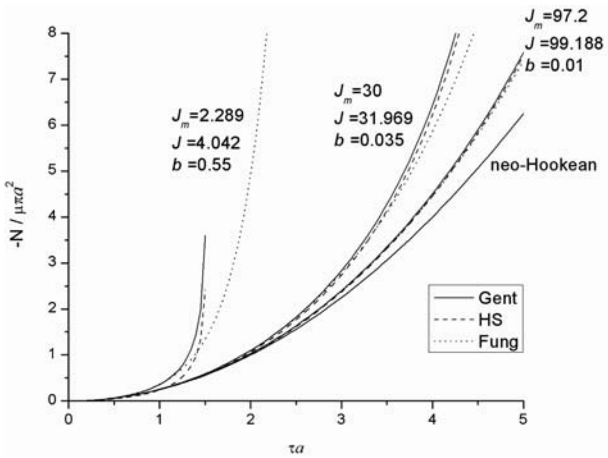


Fig. 11 Resultant compressive axial force for simple torsion for the Gent, HS, and exponential models with J_m, J, and b the same as in Fig. 10. The vertical asymptotes for the first two models occur at the same values as those in Fig. 10. Results for the neo-Hookean model are also shown

as $\tau a \to \frac{J-1}{\sqrt{J}}$. Thus, for materials of *equal ultimate torsional extensibility* as measured by the limiting resultant moment and axial force, the parameter J is related to J_m by

$$J_m = \frac{(J-1)^2}{J} \quad \text{or} \quad J = \frac{J_m + 2 + \sqrt{J_m^2 + 4J_m}}{2}. \tag{4.33}$$

Henceforth, we will assume that (4.33) holds and thus compare results for two different strain-stiffening models with *equal ultimate torsional extensibility*. First, observe that on using the first of (4.33) in (4.18), we find from (4.25) that the shear stress $T_{z\theta}$ is *identical* for both models. Thus, by (4.6), the resultant moment M for both models is also *identical*. This may be verified directly from (4.19) and (4.26). In Fig. 10, we plot M for both models for the three representative values of J_m used in Sect. 3, namely $J_m = 2.289$, 30, and 97.2. The corresponding values of J according to (4.33) are $J = 4.042$, 31.969, and 99.188, respectively. We also plot results for the exponential model for the same values of the parameter b that were used in Sect. 3. As the total angle of twist τa increases, the curves diverge as they must since the exponential model does not reflect limited extensibility and thus does not have the vertical asymptotes characteristic of the Gent and HS models. In Fig. 10, we also show the straight line (see $(4.14)_2$) for the neo-Hookean model.

It may be readily verified that the resultant axial force N required to maintain pure torsion is *compressive* for all the models discussed here. In the absence of such a force, the bar would elongate on twisting reflecting the celebrated Poynting effect [32]. These compressive axial forces are plotted in Fig. 11 for the same parameter values that were used in Fig. 10. As can be seen from Fig. 11, for large values of J_m and J the predictions for both of the severe strain-stiffening models are in very close agreement. In fact, as remarked previously, in the limit as $J_m \to \infty$ and $J \to \infty$ the results for the two models coincide and reduce to those for the neo-Hookean model. However when J_m and J are quite small (very low extensibility) as for soft biological tissues, the Gent model predicts a slightly *more rapid* approach to the vertical asymptote than the HS model. These results are similar to those obtained in [24] for a variety of *homogeneous* deformations. The curves in Fig. 11 for the exponential model show similar trends to those discussed in connection with Fig. 10.

Acknowledgements The research of L.M.K. was supported by Graduate Teaching Assistantships, a Ballard and VEF Fellowship from the University of Virginia, and a Virginia Space Grant Consortium Fellowship.

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