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Mathematical Justification of the Obstacle Problem in the Case of a Shallow Shell

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Abstract This paper deals with the asymptotic formulation and justification of a mechanical model for a shallow shell in frictionless unilateral contact with an obstacle. The first three parts of the paper concern the formulation of the equilibrium problem. Special attention is paid to the contact conditions, which are usual within two or three dimensional elasticity, but which are not so usual in shell theories. Lastly the limit problem is formulated in the main part of the paper and a convergence result is presented. Two points are worth stressing here. First, we point out that unlike classical bilateral shell models justifications, the functional framework of the present analysis involves cones. Secondly, while the cones result from a positivity condition on the boundary as long as the thickness parameter is finite, leading to a Signorini problem in the Sobolev space H^1, the cone results from a positivity condition in the domain, giving rise to a so-called obstacle problem in the Sobolev space H^2 at the limit.

Keywords Asymptotic justification · Obstacle problem · Shallow shell · Signorini problem · Unilateral contact

Mathematics Subject Classifications (2000) 74B99 · 74K25 · 74M15

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1 Introduction

The aim of this study was to provide an asymptotic model for an elastic shallow shell in unilateral contact with an obstacle and to prove the validity of this model.

Bilateral models for structures have been justified by formal asymptotic methods or by variational analyses, as in the case of plates and shells with usual bilateral boundary conditions. Many studies have been published in this field. To quote only one very close to the topic addressed in the present paper, we cite [8] in the case of shallow shells.

Unilateral contact conditions introduce a strong nonlinearity. The corresponding equilibrium problems arising in the case of two or three-dimensional bodies (the so-called *Signorini problem*) have been widely studied, and existence and uniqueness results have been obtained in the context of linear elasticity (e.g. [14] or [13]). The latter results will be used as a basic tool in the present work.

Models for structures with unilateral contact conditions (the so-called *obstacle problems*) have been used and analysed mathematically in some cases, even within nonlinear strains, but the equations of these models have not been mathematically justified. We recall that the main difference between a *Signorini problem* and an *obstacle problem* is that in the former case the unilateral constraints hold on the boundary, whereas in the second case they hold inside the domain. A physical model for the former situation is given by any three-dimensional body resting on a support. The simplest physical model of the second kind is an inflated membrane placed close to a wall. Many studies have been carried out on the obstacle problems associated with the harmonic operator, especially in the case of membranes [18] where the solution evolves as the loading parameter increases [12]. This path following approach was used in [11] in the case of a linear elastic beam located above an obstacle. Models for von Kármán plates with unilateral constraints were also studied (e.g. [10]) based on [4] in the bilateral case, but to our knowledge, unilateral conditions have never been used so far in shell models.

We will first give a brief description of the model problem we are dealing with. We focus here on the case of a shallow shell. In the example studied, we consider a case where the shell is in unilateral contact with the plane of the reference open set, which is not restrictive for the present purpose. The aim is to obtain an asymptotic limit from the three-dimensional elasticity where the equilibrium problem is known to have a single solution, as in the bilateral case.

The first three parts of the paper deal with the formulation of the problem. As usual in shell theory, the geometrical data involve a fairly smooth open set of the plane, denoted by ω, and a map depending on a thickness parameter ε, denoted by θ^{ε}, the domain of which is the reference set ω and the values are in $\mathbb{R}^3$ (or in $\mathbb{R}$ in the present case of shallow shells). This makes it possible to obtain the reference configuration of the shell, denoted by $\overline{\hat{\Omega}^{\varepsilon}}$, in which we write the equilibrium problem. Since θ^{ε} is assumed to be sufficiently regular, we can transpose the problem onto a cylindrical domain Ω^{ε} having middle cross-section ω and height 2ε. Using appropriate scalings, we transform the domain into a new cylinder Ω independent of ε. Accordingly, the elements of the function spaces defined in Ω, Ω^{ε} and $\hat{\Omega}^{\varepsilon}$ will be referred to as v, v^{ε} and $\hat{v}^{\varepsilon}$ respectively. Special attention is paid to formulating the unilateral contact conditions, here without friction, at the boundary of each of these three-dimensional domains. The last part contains the main result obtained, namely that as the thickness parameter ε tends to zero, the scaled displacement converges strongly in a convex cone which is a subset of $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ intersected by a positivity condition. In addition, we establish that the limit is a Kirchhoff–Love displacement field.

These steps are not classical for two main reasons. First, the functional framework is a cone, and not a vector space. Secondly, as long as the thickness is finite the problem is a Signorini problem, while the limit is an obstacle problem. The latter point is, in a way, the counterpart of the usual bilateral case where all the domains are three-dimensional as long as the thickness is finite while the limit is two-dimensional.

2 The Reference Configuration $\hat{\Omega}^\varepsilon$ of the Shell

The first step consists in building the reference configuration of the shell. We recall that a general framework to carry out the construction of a shell model involves a three-dimensional Euclidian space $\mathbb{E}^3$, a two-dimensional vector space, identified with $\mathbb{R}^2$ and containing a domain ω, and an injective immersion $\varphi \in C^3(\overline{\omega}; \mathbb{E}^3)$. According to basic concepts of differential geometry, the image $\varphi(\omega)$ is a surface in $\mathbb{E}^3$. In the particular case of the so-called shallow shell theory, this framework is restricted to maps φ of the form:

$$(x_1, x_2) \in \omega, \quad \varphi(x_1, x_2) = (x_1, x_2, \theta(x_1, x_2)), \quad \theta \in C^3(\overline{\omega}; \mathbb{R}).$$

From now onwards the map θ is assumed to depend on a small parameter $\varepsilon > 0$ in a way which will be specified later and will be denoted by θ^ε. Accordingly, the image $\varphi^\varepsilon(\omega)$, referred to as $\hat{\omega}^\varepsilon$, is the following surface:

$$(x_1, x_2) \in \omega, \quad \hat{\omega}^\varepsilon = \{x^\varepsilon \mid x^\varepsilon = (x_1, x_2, \theta^\varepsilon(x_1, x_2))\}.$$

In the present analysis θ^ε is assumed to be positive in $\overline{\omega}$. It will be clear in what follows that this assumption is necessary but not restrictive since, if it were not satisfied, it could always be recovered by a translation.

The reference domain is defined in the following way: let $\gamma_0 = \partial\omega$ be the boundary of ω. Given $\varepsilon > 0$, the reference domain $\Omega^\varepsilon \subset \mathbb{R}^3$ is a three-dimensional domain, the boundary of which is $\partial\Omega^\varepsilon \equiv \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon \cup \Gamma_0^\varepsilon$, with

$$\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[, \quad \Gamma_+^\varepsilon = \omega \times \{\varepsilon\}, \quad \Gamma_-^\varepsilon = \omega \times \{-\varepsilon\}, \quad \Gamma_0^\varepsilon = \gamma_0 \times [-\varepsilon, \varepsilon]. \tag{1}$$

Any point in Ω^ε will be denoted by $x^\varepsilon=(x_1,x_2,x_3^\varepsilon)$ where $(x_1,x_2)\in\omega$ and $x_3^\varepsilon \in]-\varepsilon, \varepsilon[$. We then introduce $\hat{\boldsymbol{d}}^\varepsilon$ as the unit vector normal to $\hat{\omega}^\varepsilon$.

$$\hat{\boldsymbol{d}}^\varepsilon := \frac{1}{\sqrt{\alpha^\varepsilon}}(-\partial_1\theta^\varepsilon, -\partial_2\theta^\varepsilon, 1) \text{ with } \alpha^\varepsilon := |\partial_1\theta^\varepsilon|^2 + |\partial_2\theta^\varepsilon|^2 + 1, \tag{2}$$

where ∂_α stands for a derivative with respect to x_α.[1]

[1] The Latin indexes and exponents used here have values in the set $\{1, 2, 3\}$, while Greek indexes and exponents, except ε, have values in the set $\{1, 2\}$. The Einstein convention will be applied to repeated indexes and exponents, and bold symbols are used for vector fields and space vectors.

The reference configuration $\hat{\Omega}^\varepsilon$ of the shell is the following three-dimensional domain obtained from $\hat{\omega}^\varepsilon$:

$$\hat{\Omega}^\varepsilon = (x_1, x_2, \theta^\varepsilon(x_1, x_2)) + x_3^\varepsilon \hat{\boldsymbol{d}}^\varepsilon(x_1, x_2), \quad \forall x^\varepsilon \in \Omega^\varepsilon. \tag{3}$$

Introducing an application Θ^ε which maps the cylinder Ω^ε onto the shell $\hat{\Omega}^\varepsilon$, we shall write: $\hat{\Omega}^\varepsilon = \Theta^\varepsilon(\Omega^\varepsilon)$. In the same way as for Ω^ε, the boundary of $\hat{\Omega}^\varepsilon$ is $\partial\hat{\Omega}^\varepsilon = \hat{\Gamma}_-^\varepsilon \cup \hat{\Gamma}_+^\varepsilon \cup \hat{\Gamma}^\varepsilon$, where in particular

$$\hat{\Gamma}_-^\varepsilon = (x_1, x_2, \theta^\varepsilon(x_1, x_2)) - \varepsilon \hat{\boldsymbol{d}}^\varepsilon(x_1, x_2), \quad \forall (x_1, x_2) \in \omega.$$

Of course, these definitions make sense only if Θ^ε is globally injective, but it is precisely known that Θ^ε is a C^1-diffeomorphism and that $\det\{\nabla^\varepsilon \Theta^\varepsilon(x^\varepsilon)\} > 0, \forall x^\varepsilon \in \overline{\Omega}^\varepsilon$, if the immersion φ is itself globally injective on $\overline{\omega}$ and ε is small enough (a proof based on inverse function theorem is for instance given in [7]).

We notice that we have in particular $(\Theta^\varepsilon)^{-1}(\hat{\Gamma}_-^\varepsilon) = \Gamma_-^\varepsilon$ and that the image $\boldsymbol{w}^\varepsilon$ defined in Ω^ε of a function $\hat{\boldsymbol{w}}^\varepsilon$ defined in $\hat{\Omega}^\varepsilon$ can be written as follows:

$$\boldsymbol{w}^\varepsilon = \hat{\boldsymbol{w}}^\varepsilon \circ \Theta^\varepsilon.$$

Remark 1 In the present analysis, θ^ε is assumed to be a $\boldsymbol{C}^3(\overline{\omega})$ map, which is the classical assumption made in asymptotic analyses to give a L^2-regularity to the strain tensor of the limit model. However, we bear in mind the fact that shell models have been built with a much weaker smoothness in order to take into account discontinuities of curvature as in [3] or folds as in [2], but no asymptotic justification of these models has yet been obtained and these nonsmooth shells are beyond the scope of the present study.

3 Equilibrium Problem in the Reference Configuration $\hat{\boldsymbol{\Omega}}^\varepsilon$

We now give the equations and conditions involved in the equilibrium problem.

1. We will focus here on linear elasticity. The three-dimensional elasticity tensor will be denoted by $\boldsymbol{c} = (c_{ijkl})$. For the sake of simplicity, the present study deals with isotropic and homogeneous materials which means that we will use the explicit expression $c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ whenever necessary, where λ and μ are the Lamé coefficients.

2. The loading consists of body forces $\hat{\boldsymbol{f}}^\varepsilon \in \boldsymbol{L}^2(\hat{\Omega}^\varepsilon)$ and surface forces $\hat{\boldsymbol{g}}^\varepsilon \in \boldsymbol{L}^2(\hat{\Gamma}_+^\varepsilon)$ exerted on the upper part $\hat{\Gamma}_+^\varepsilon$ of the boundary.

3. The lateral part $\hat{\Gamma}_0^\varepsilon$ of the boundary is assumed to be clamped, that is $\hat{\boldsymbol{u}}^\varepsilon = \boldsymbol{0}$ on $\hat{\Gamma}_0^\varepsilon$ (clamping only on a nonzero measure subset of $\hat{\Gamma}_0^\varepsilon$ would only add slight complications). The lower part $\hat{\Gamma}_-^\varepsilon$ involves unilateral contact conditions that are dealt with below.

3.1 The Contact Condition

As indicated in the introduction, we restrict the problem here to cases where the shell is in unilateral contact with the plane of the reference set ω. Let us write the

Cartesian components of a vector $O\hat{x}_-^\varepsilon$ as follows, where $\hat{x}_-^\varepsilon$ is a point of the part $\hat{\Gamma}_-^\varepsilon$ of the boundary:

$$O\hat{x}_-^\varepsilon = \begin{pmatrix} x_1^\varepsilon + \varepsilon \dfrac{\partial_1 \theta^\varepsilon}{\sqrt{\alpha^\varepsilon}} \\ x_2^\varepsilon + \varepsilon \dfrac{\partial_2 \theta^\varepsilon}{\sqrt{\alpha^\varepsilon}} \\ x_3^\varepsilon - \varepsilon \dfrac{1}{\sqrt{\alpha^\varepsilon}} \end{pmatrix} = \begin{pmatrix} x_1 + \varepsilon \dfrac{\partial_1 \theta^\varepsilon}{\sqrt{\alpha^\varepsilon}} \\ x_2 + \varepsilon \dfrac{\partial_2 \theta^\varepsilon}{\sqrt{\alpha^\varepsilon}} \\ \theta^\varepsilon - \varepsilon \dfrac{1}{\sqrt{\alpha^\varepsilon}}. \end{pmatrix}. \tag{4}$$

We define the following maps:

$$\hat{\theta}^\varepsilon = \theta^\varepsilon \circ (\Theta^\varepsilon)^{-1}, \;\; \hat{\alpha}^\varepsilon = \alpha^\varepsilon \circ (\Theta^\varepsilon)^{-1}.$$

Then the unilateral contact conditions first mean that the displacement on $\hat{\Gamma}_-^\varepsilon$ must satisfy a nonpenetrability condition $\left(O\hat{x}_-^\varepsilon + \hat{\boldsymbol{u}}(\hat{x}_-^\varepsilon)\right) \cdot \boldsymbol{e}_3 \geq -\varepsilon$ which reads:

$$\forall \hat{x}_-^\varepsilon \in \hat{\Gamma}_-^\varepsilon, \; \left(\hat{\theta}^\varepsilon - \frac{\varepsilon}{\sqrt{\hat{\alpha}^\varepsilon}}\right)(\hat{x}_-^\varepsilon) + \hat{u}_3^\varepsilon(\hat{x}_-^\varepsilon) \geq -\varepsilon. \tag{5}$$

Remark 2 For any given ε we then have a pair which consists of a shell of thickness 2ε and a plane horizontal obstacle at the level $-\varepsilon$.

The so-called Signorini conditions, which give the full description of the unilaterality, are classically obtained by adding the following constraints to the nonpenetrability condition, which mean that:

- No tensile forces but only compressive forces are exerted on the boundary by the obstacle;
- Either a point on the boundary $\hat{\Gamma}_-^\varepsilon$ is not in contact with the obstacle, so that the nonpenetrability inequality (5) is strictly satisfied, and the reaction of the obstacle at this point therefore vanishes, or the point on $\hat{\Gamma}_-^\varepsilon$ is in contact, so that condition (5) is an equality, and the reaction of the obstacle can therefore differ from zero.

These constraints read:

$$\begin{aligned} -\left(\hat{\boldsymbol{\sigma}}^\varepsilon(\hat{x}_-^\varepsilon).\hat{\boldsymbol{d}}^\varepsilon\right).\boldsymbol{e}_3 \; &\equiv \hat{\sigma}_3^\varepsilon \;\geq\; 0, \;\; \forall \hat{x}^\varepsilon \in \hat{\Gamma}_-^\varepsilon, \\ \hat{\sigma}_3^\varepsilon \left(\hat{\theta}^\varepsilon - \frac{\varepsilon}{\sqrt{\hat{\alpha}^\varepsilon}} + \hat{u}_3^\varepsilon + \varepsilon\right) \; &= \; 0. \end{aligned} \tag{6}$$

The equilibrium problem we are dealing with finally reads

$$\left\{ \begin{array}{l} div \hat{\boldsymbol{\sigma}}^\varepsilon + \hat{\boldsymbol{f}}^\varepsilon = \boldsymbol{0} \\ \hat{\boldsymbol{\sigma}}^\varepsilon = \boldsymbol{c}\, \hat{\boldsymbol{e}}^\varepsilon \qquad\qquad \text{in } \hat{\Omega}^\varepsilon, \\ \hat{\boldsymbol{e}}^\varepsilon = \frac{1}{2}(\nabla \hat{\boldsymbol{u}}^\varepsilon + \nabla (\hat{\boldsymbol{u}}^\varepsilon)^T) \\ \\ \hat{\boldsymbol{u}}^\varepsilon = \boldsymbol{0} \quad \text{on } \hat{\Gamma}_0^\varepsilon, \\ \hat{\boldsymbol{\sigma}}^\varepsilon . \hat{\boldsymbol{d}}^\varepsilon = \hat{\boldsymbol{g}}^\varepsilon \quad \text{on } \hat{\Gamma}_+^\varepsilon, \\ \hat{\theta}^\varepsilon - \dfrac{\varepsilon}{\sqrt{\hat{\alpha}^\varepsilon}} + \hat{u}_3^\varepsilon + \varepsilon \geq 0, \; \hat{\sigma}_3^\varepsilon \geq 0, \; \hat{\sigma}_3^\varepsilon \left(\hat{\theta}^\varepsilon - \dfrac{\varepsilon}{\sqrt{\hat{\alpha}^\varepsilon}} + \hat{u}_3^\varepsilon + \varepsilon\right) = 0 \;\text{ on } \hat{\Gamma}_-^\varepsilon. \end{array} \right. \tag{7}$$

3.2 The Variational Inequality in $\hat{\Omega}^\varepsilon$

The natural functional framework for the equilibrium problem (7) is the convex cone $\hat{\boldsymbol{K}}(\hat{\Omega}^\varepsilon)$ defined as

$$\hat{\boldsymbol{K}}^\varepsilon\left(\hat{\Omega}^\varepsilon\right) = \left\{\hat{\boldsymbol{v}}^\varepsilon \in \boldsymbol{H}^1\left(\hat{\Omega}^\varepsilon\right),\ \hat{\boldsymbol{v}}^\varepsilon = \boldsymbol{0} \text{ on } \hat{\Gamma}_0^\varepsilon,\ \hat{v}_3^\varepsilon \geq -\hat{\theta}^\varepsilon - \varepsilon + \frac{\varepsilon}{\sqrt{\hat{\alpha}^\varepsilon}} \text{ on } \hat{\Gamma}_-^\varepsilon\right\}. \tag{8}$$

From problem (7) a variational inequality is then classically obtained in $\hat{\boldsymbol{K}}^\varepsilon(\hat{\Omega}^\varepsilon)$ (e.g., [13]), which reads

$$\begin{cases} \text{Find } \hat{\boldsymbol{u}}^\varepsilon \in \hat{\boldsymbol{K}}^\varepsilon(\hat{\Omega}^\varepsilon) \text{ such that} \\ \displaystyle\int_{\hat{\Omega}^\varepsilon} c_{ijkl}\hat{e}_{kl}^\varepsilon(\hat{\boldsymbol{u}}^\varepsilon)\hat{e}_{ij}^\varepsilon(\hat{\boldsymbol{v}}^\varepsilon - \hat{\boldsymbol{u}}^\varepsilon)d\hat{x}^\varepsilon \geq \int_{\hat{\Omega}^\varepsilon} \hat{\boldsymbol{f}}^\varepsilon \cdot (\hat{\boldsymbol{v}}^\varepsilon - \hat{\boldsymbol{u}}^\varepsilon)d\hat{x}^\varepsilon \\ \qquad\qquad\qquad\qquad\qquad + \displaystyle\int_{\hat{\Gamma}_+^\varepsilon} \hat{\boldsymbol{g}}^\varepsilon \cdot (\hat{\boldsymbol{v}}^\varepsilon - \hat{\boldsymbol{u}}^\varepsilon)d\hat{a}^\varepsilon \\ \forall \hat{\boldsymbol{v}}^\varepsilon \in \hat{\boldsymbol{K}}^\varepsilon(\hat{\Omega}^\varepsilon). \end{cases} \tag{9}$$

Based on classical arguments, problems (7) and (9) can be taken to be equivalent [see eg. [15] for details of the proof of the equivalence and for the corresponding exact meaning of (7) as the strong problem]. Moreover, problem (9) has a single solution for any finite ε.

4 Scaling and Equilibrium Equations in the Fixed Domain Ω

4.1 The Problem in Ω^ε

Using the one to one map $(\Theta^\varepsilon)^{-1}$, the reference configuration of the shell $\hat{\Omega}^\varepsilon$ is mapped onto the cylindrical domain Ω^ε. Because of the same diffeomorphism $(\Theta^\varepsilon)^{-1}$, the cone $\hat{\boldsymbol{K}}^\varepsilon(\hat{\Omega}^\varepsilon)$ is mapped onto another cone $\boldsymbol{K}^\varepsilon(\Omega^\varepsilon)$ defined as follows (see [1]):

$$\boldsymbol{K}^\varepsilon(\Omega^\varepsilon) = \left\{\boldsymbol{v}^\varepsilon \in \boldsymbol{H}^1(\Omega^\varepsilon), \boldsymbol{v}^\varepsilon = \boldsymbol{0} \text{ on } \Gamma_0^\varepsilon,\ v_3^\varepsilon \geq -\theta^\varepsilon + \frac{\varepsilon}{\sqrt{\alpha^\varepsilon}} - \varepsilon \text{ on } \Gamma_-^\varepsilon\right\}. \tag{10}$$

In order to rewrite problem (9) over the domain Ω^ε and, later, in a fixed domain Ω, we recall that the transformation of the volume element yields

$$d\hat{x}^\varepsilon = \delta^\varepsilon dx^\varepsilon, \text{ where } \delta^\varepsilon = \delta^\varepsilon(x^\varepsilon) = \det\{\nabla^\varepsilon\Theta^\varepsilon(x^\varepsilon)\},\ x^\varepsilon \in \Omega^\varepsilon,$$

where the superscript ε in ∇^ε means derivatives with respect to x^ε. Likewise, a surface element of Γ_+^ε is mapped onto a surface element of $\hat{\Gamma}_+^\varepsilon$ by:

$$d\hat{a}^\varepsilon = \delta^\varepsilon \left\{b_{3j}^\varepsilon b_{3j}^\varepsilon\right\}^{1/2} da^\varepsilon,$$

where the matrix $b_{ij}^\varepsilon(x^\varepsilon)$ is given by

$$b_{ij}^\varepsilon(x^\varepsilon) := \left(\{\nabla^\varepsilon\Theta^\varepsilon(x^\varepsilon)\}^{-1}\right)_{ij}.$$

Because of the chain rule $\hat{\partial}_j^\varepsilon \hat{v}_i^\varepsilon(\hat{x}^\varepsilon) = b_{kj}^\varepsilon(x^\varepsilon)\partial_k^\varepsilon v_i^\varepsilon$ we obtain the expression for the linearized strain tensor

$$\hat{e}_{ij}^\varepsilon(\hat{\boldsymbol{v}}^\varepsilon) = e_{ij}^\varepsilon(\boldsymbol{v}^\varepsilon) = \frac{1}{2}\left(b_{ki}^\varepsilon \partial_k^\varepsilon v_j^\varepsilon + b_{lj}^\varepsilon \partial_l^\varepsilon v_i^\varepsilon\right),$$

which finally changes problem (9) into the following variational problem:

$$\begin{cases} \text{Find } \boldsymbol{u}^\varepsilon \in \boldsymbol{K}^\varepsilon(\Omega^\varepsilon) \text{ such that:} \\ \displaystyle\int_{\Omega^\varepsilon} c_{ijkl}\, e_{kl}^\varepsilon(\boldsymbol{u}^\varepsilon) e_{ij}^\varepsilon(\boldsymbol{v}^\varepsilon - \boldsymbol{u}^\varepsilon)\delta^\varepsilon dx^\varepsilon \;\geq\; \int_{\Omega^\varepsilon} \boldsymbol{f}^\varepsilon \cdot (\boldsymbol{v}^\varepsilon - \boldsymbol{u}^\varepsilon)\delta^\varepsilon dx^\varepsilon \\ \qquad\qquad\qquad\qquad\qquad\qquad + \displaystyle\int_{\Gamma_+^\varepsilon} \boldsymbol{g}^\varepsilon \cdot (\boldsymbol{v}^\varepsilon - \boldsymbol{u}^\varepsilon)\delta^\varepsilon \{b_{3j}^\varepsilon b_{3j}^\varepsilon\}^{1/2} da^\varepsilon \\ \forall \boldsymbol{v}^\varepsilon \in \boldsymbol{K}^\varepsilon(\Omega^\varepsilon). \end{cases} \tag{11}$$

Here, the applied volume and surface forces $\boldsymbol{f}^\varepsilon$ and $\boldsymbol{g}^\varepsilon$ are defined by $\boldsymbol{f}^\varepsilon = \hat{\boldsymbol{f}}^\varepsilon \circ \Theta^\varepsilon$ and $\boldsymbol{g}^\varepsilon = \hat{\boldsymbol{g}}^\varepsilon \circ \Theta^\varepsilon$.

4.2 Scaling

We now change the domain Ω^ε having thickness 2ε into a fixed domain Ω independent of ε via the simple geometrical transformation defined as

$$\begin{cases} \Omega^\varepsilon \ni \boldsymbol{x}^\varepsilon = (x_1, x_2, x_3^\varepsilon) \longrightarrow \boldsymbol{x} = (x_1, x_2, x_3) \in \Omega, \\ x_3 = \dfrac{1}{\varepsilon} x_3^\varepsilon. \end{cases} \tag{12}$$

The domain Ω is then a cylinder having cross-section ω and height 2. With obvious notations, its boundary is $\partial\Omega = \Gamma_- \cup \Gamma_+ \cup \Gamma_0$. In addition to the definition of this fixed domain, we introduce the scaled displacement $\boldsymbol{u}(\varepsilon)$ and scaled test functions $\boldsymbol{v}$ defined as

$$\begin{cases} u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x),\; u_3^\varepsilon(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x), \\ v_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 v_\alpha(x), \qquad v_3^\varepsilon(x^\varepsilon) = \varepsilon v_3(x). \end{cases} \tag{13}$$

Along with this scaling procedure, we take $\boldsymbol{e}(\varepsilon) = (e_{ij}(\varepsilon))$ to denote the scaled linearized strain tensor, the components of which are found to be (see [8])

$$\begin{cases} e_{\alpha\beta}(\varepsilon)(v(x)) = \varepsilon^2\{e_{\alpha\beta}^\theta(v(x)) + \varepsilon^2 e_{\alpha\beta}^\sharp(\varepsilon, \theta; \boldsymbol{v}(x))\}, \\ e_{\alpha 3}(\varepsilon)(v(x)) = \varepsilon\{e_{\alpha 3}^\theta(v(x)) + \varepsilon^2 e_{\alpha 3}^\sharp(\varepsilon, \theta; \boldsymbol{v}(x))\}, \\ e_{3\alpha}(\varepsilon)(v(x)) = \varepsilon\{e_{3\alpha}^\theta(v(x)) + \varepsilon^2 e_{3\alpha}^\sharp(\varepsilon, \theta; \boldsymbol{v}(x))\}, \\ e_{33}(\varepsilon)(v(x)) = e_{33}^\theta(v(x)) + \varepsilon^2\left(\partial_\alpha\theta\partial_\alpha v_3 + b_{33}^\sharp(\varepsilon, \theta)\partial_3 v_3\right) + \varepsilon^4 e_{33}^\sharp(\varepsilon, \theta; \boldsymbol{v}(x)), \end{cases} \tag{14}$$

where the explicit values of the terms e^θ will be given further on and where we shall see that the $e^\sharp$ are bounded with respect to the norm of $\boldsymbol{v}$ in $\boldsymbol{H}^1(\Omega)$.

4.3 Assumptions about the Data

In order to obtain a nontrivial limit problem in asymptotic analyses, it is essential to scale the data accordingly with the scalings of the unknowns. We assume that there exist functions $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{g} \in \boldsymbol{L}^2(\Gamma_+)$ and $\theta \in C^3(\overline{\omega})$ not depending on ε such that

$$\begin{cases} f^\varepsilon_\alpha(x^\varepsilon) = f_\alpha(\varepsilon)(x) = \varepsilon^2 f_\alpha(x), \;\; f^\varepsilon_3(x^\varepsilon) = f_3(\varepsilon)(x) = \varepsilon^3 f_3(x), \\ g^\varepsilon_\alpha(x^\varepsilon) = g_\alpha(\varepsilon)(x) = \varepsilon^3 g_\alpha(x), \;\; g^\varepsilon_3(x^\varepsilon) = g_3(\varepsilon)(x) = \varepsilon^4 g_3(x), \\ \theta^\varepsilon(x_1, x_2) = \varepsilon\theta(x_1, x_2). \end{cases} \tag{15}$$

The last assumption about $\theta^\varepsilon(x_1, x_2)$ results from an analysis of the exact mathematical definition of the shallowness in shell theory (see [8] where it is also shown that this assumption is the only one that gives a shell model at the limit).

Remark 3 Basically the last assumption of formula (15) means that the deepness of the shell is of the same order as its thickness. But this does not mean at all that the limit problem will be the same as the one of a plate. In particular, we shall see that the obstacle problem will involve the function $\theta(x_1, x_2)$, which is any nonvanishing $C^3(\overline{\omega})$ function.

We add the following notations

$$\begin{cases} b_{ij}(\varepsilon)(x) = b^\varepsilon_{ij}(x^\varepsilon), \\ \delta(\varepsilon)(x) \quad = \delta^\varepsilon(x^\varepsilon). \end{cases} \tag{16}$$

4.4 The Contact Condition (continued)

As we have seen in Section 2, the lower part Γ^ε_- of the boundary is the image of $\hat{\Gamma}^\varepsilon_-$ by $(\Theta^\varepsilon)^{-1}$. Based on (1), Γ^ε_- is the set

$$\Gamma^\varepsilon_- = \{(x_1, x_2, -\varepsilon), (x_1, x_2) \in \omega\}$$

which is parallel to the plane of the reference set. After the scaling process, the nonpenetrability condition, which holds now on Γ_-, reads

$$v_3(x_1, x_2, -1) \geq -\theta(x_1, x_2) + \frac{1}{\sqrt{\alpha^\varepsilon}} - 1, \;\; \forall (x_1, x_2) \in \omega. \tag{17}$$

4.5 The Equilibrium Problem in the Fixed Domain Ω

After the scaling processes (12) and (13) and given the nonpenetrability condition (17), the cone $\boldsymbol{K}^\varepsilon(\Omega^\varepsilon)$ becomes $\boldsymbol{K}(\varepsilon)(\Omega)$, which is defined as follows:

$$\boldsymbol{K}(\varepsilon)(\Omega) = \left\{ \boldsymbol{v}^\varepsilon \in \boldsymbol{H}^1(\Omega), \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0, \;\; v_3(x_1, x_2, -1) \geq -\theta + \frac{1}{\sqrt{\alpha^\varepsilon}} - 1 \right\}.$$

The equilibrium problem now consists in looking for a solution to a problem set over Ω which reads

$$\begin{cases} \text{Find } \boldsymbol{u}(\varepsilon) \in \boldsymbol{K}(\varepsilon)(\Omega) \\ \varepsilon \displaystyle\int_{\Omega} c_{ijkl} e_{kl}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{ij}(\varepsilon)(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\delta(\varepsilon) dx \geq \varepsilon \int_{\Omega} \boldsymbol{f}(\varepsilon) \cdot (\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\delta(\varepsilon) dx \\ + \displaystyle\int_{\Gamma_+} \boldsymbol{g}(\varepsilon) \cdot (\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\delta(\varepsilon) \left\{ b_{3j}(\varepsilon) b_{3j}(\varepsilon) \right\}^{1/2} da \\ \forall \boldsymbol{v} \in \boldsymbol{K}(\varepsilon)(\Omega), \end{cases} \tag{18}$$

where da is the area element of the upper boundary Γ_+, i.e., $da^{\varepsilon} = da$.

5 Convergence

The aim of this section is to establish that when ε tends to zero, the sequence $\{\boldsymbol{u}(\varepsilon)\}$ converges to a limit $\boldsymbol{u}$ which solves a two-dimensional obstacle problem. An important preliminary point here is the following lemma, which is a version of Korn's inequality for shallow shells.

Lemma 5.1 *Let $\theta \in C^3(\overline{\omega})$ be a given function, and let the functions $e^{\theta}_{ij}(\boldsymbol{v})$ be defined as, for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$*

$$\begin{cases} e^{\theta}_{\alpha\beta}(\boldsymbol{v}) = e_{\alpha\beta}(\boldsymbol{v}) - \dfrac{1}{2}(\partial_{\beta}\theta\partial_3 v_{\alpha} + \partial_{\alpha}\theta\partial_3 v_{\beta}) \\ e^{\theta}_{\alpha 3}(\boldsymbol{v}) = e^{\theta}_{3\alpha}(\boldsymbol{v}) = e_{\alpha 3}(\boldsymbol{v}) - \dfrac{1}{2}\partial_{\alpha}\theta\partial_3 v_3 \\ e^{\theta}_{33}(\boldsymbol{v}) = e_{33}(\boldsymbol{v}). \end{cases}$$

Then, for any ε, the mapping

$$\boldsymbol{v} \longrightarrow \left\{ \sum_{i,j} \left| e^{\theta}_{ij}(\boldsymbol{v}) \right|^2_{0,\Omega} \right\}^{1/2}$$

is a norm over the cone $\boldsymbol{K}(\varepsilon)(\Omega)$, which is equivalent to the norm induced by $\|.\|_{1,\Omega}$.

A similar result has been given in [8] for the bilateral case. In the present case, the proof follows from the fact that the cone $\boldsymbol{K}(\varepsilon)(\Omega)$ is a closed subset of the vector space $\{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0\}$.

The convergence theorem can now be stated.

Theorem 5.2 *Assume*

$$\boldsymbol{f} \in \boldsymbol{L}^2(\Omega),\ \boldsymbol{g} \in \boldsymbol{L}^2(\Gamma_+),\ \theta \in C^3(\overline{\omega}).$$

Then

1. *As ε tends to 0, the family $\{\boldsymbol{u}(\varepsilon)\}_{\varepsilon>0}$ converges strongly in the cone*

$$\boldsymbol{K}(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0,\ v_3(x_1, x_2, -1) \geq -\theta \text{ on } \Gamma_-\right\}.$$

2. *Let*

$$\begin{aligned} \boldsymbol{V}_H(\omega) &= \left\{\boldsymbol{\eta}_H = (\eta_\alpha) \in \boldsymbol{H}^1(\omega),\ \boldsymbol{\eta}_H = 0 \text{ on } \gamma_0\right\}, \\ K_3(\omega) &= \left\{\eta_3 \in H^2(\omega), \eta_3 = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0, \eta_3 \geq -\theta \text{ in } \omega\right\}. \end{aligned}$$

Then the limit of $\boldsymbol{u}(\varepsilon)$ as ε tends to 0 is a Kirchhoff–Love displacement field, namely

$$u_\alpha = \zeta_\alpha - x_3 \partial_\alpha \zeta_3,\ u_3 = \zeta_3,$$

with $(\boldsymbol{\zeta}_\alpha) \in \boldsymbol{V}_H(\omega)$ and $\zeta_3 \in K_3(\omega)$.

3. *The function $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_\alpha, \zeta_3)$ solves the following problem:*

$$\begin{aligned} &\text{Find } \zeta_3 \in K_3(\omega) \\ &-\int_\omega m_{\alpha\beta}(\zeta_3)\partial_{\alpha\beta}(\eta_3 - \zeta_3)d\omega + \int_\omega n^\theta_{\alpha\beta}(\zeta_3)\partial_\alpha\theta\partial_\beta(\eta_3 - \zeta_3)d\omega \\ &\geq \int_\omega p_3(\eta_3 - \zeta_3)d\omega + \int_\omega s_\alpha \partial_\alpha(\eta_3 - \zeta_3)d\omega, \quad \forall \eta_3 \in K_3(\omega), \\ &\text{and find } \zeta_\alpha \in \boldsymbol{V}_{\mathrm{H}}(\omega) \\ &\int_\omega n^\theta_{\alpha\beta}(\boldsymbol{\zeta})\partial_\beta \eta_\alpha d\omega = \int_\omega p_\alpha \eta_\alpha d\omega, \qquad \forall \boldsymbol{\eta}_H = (\eta_\alpha) \in \boldsymbol{V}_H(\omega). \end{aligned} \tag{19}$$

Using the explicit values of the elasticity coefficients in the homogeneous isotropic case, quantities $m_{\alpha\beta}$, $n^\theta_{\alpha\beta}$, $e^\theta_{\alpha\beta}(\boldsymbol{\zeta})$, p_i and s_α read:

$$\begin{cases} m_{\alpha\beta} := \dfrac{4\lambda\mu}{3(\lambda + 2\mu)}\Delta\zeta_3\delta_{\alpha\beta} + \dfrac{4}{3}\mu\partial_{\alpha\beta}\zeta_3, \\ n^\theta_{\alpha\beta} := \dfrac{4\lambda\mu}{\lambda + 2\mu} e^\theta_{\rho\rho}(\boldsymbol{\zeta})\delta_{\alpha\beta} + 4\mu e^\theta_{\alpha\beta}(\boldsymbol{\zeta}), \\ e^\theta_{\alpha\beta}(\boldsymbol{\zeta}) := \dfrac{1}{2}(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha) + \dfrac{1}{2}(\partial_\alpha\theta\partial_\beta\zeta_3 + \partial_\beta\theta\partial_\alpha\zeta_3), \\ p_i := \displaystyle\int_{-1}^1 f_i dy_3 + g_i^+, \\ s_\alpha := \displaystyle\int_{-1}^1 y_3 f_\alpha dy_3 + g_\alpha^+. \end{cases} \tag{20}$$

Proof The proof is broken into five parts. In the first one, we introduce a new scaled strain tensor. Recalling some boundedness results we establish that the sequence $\{\boldsymbol{u}(\varepsilon)\}$ converges weakly to a limit $\boldsymbol{u}$ which is a Kirchhoff–Love field. The second one introduces a new variational problem built from problem (18) thanks to the use of the new scaled strain tensor. The third part deals with technical results about the components of this strain tensor. In the next part we show that in fact the convergence of the family $\{\boldsymbol{u}(\varepsilon)\}$ towards the Kirchhoff–Love field $\boldsymbol{u}$ is strong. The last part completes the proof by deducing the variational problem of which the strong limit $\boldsymbol{u}$ is the single solution. □

- *Step I* We first give without proof the following results, the detailed proof of which can be found in [8].

Lemma 5.3 *The norms $|\boldsymbol{u}(\varepsilon)|_{1,\Omega}$ are bounded uniformly in ε.*

Let us now introduce the following symmetric tensor $\boldsymbol{R}^\theta(\varepsilon) = \left(R^\theta_{ij}(\varepsilon)\right) \in \boldsymbol{L}^2(\Omega)$ given by:

$$\begin{cases} R^\theta_{\alpha\beta}(\varepsilon) = e^\theta_{\alpha\beta}(\boldsymbol{u}(\varepsilon)), \\ R^\theta_{\alpha 3}(\varepsilon) = \dfrac{1}{\varepsilon} e^\theta_{\alpha 3}(\boldsymbol{u}(\varepsilon)), \\ R^\theta_{33}(\varepsilon) = \dfrac{1}{\varepsilon^2} e^\theta_{33}(\boldsymbol{u}(\varepsilon)) + \partial_\alpha\theta\,\partial_\alpha u_3(\varepsilon), \end{cases} \tag{21}$$

which is a new scaled strain tensor.

Lemma 5.4 *The tensor $\boldsymbol{R}^\theta(\varepsilon) = (R^\theta_{ij}(\varepsilon)) \in \boldsymbol{L}^2(\Omega)$ is bounded uniformly in ε in the space $\boldsymbol{L}^2(\Omega)$.*

Because of Lemmas 5.3 and 5.4 there exist two subsequences, still indexed by ε, a limit displacement field $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and a limit scaled strain tensor $\boldsymbol{R}^\theta \in \boldsymbol{L}^2(\Omega)$ such that we have the weak convergence

$$\boldsymbol{u}(\varepsilon) \rightharpoonup \boldsymbol{u} \text{ in } \boldsymbol{H}^1(\Omega), \quad \text{and } \boldsymbol{R}^\theta(\varepsilon) \rightharpoonup \boldsymbol{R}^\theta \text{ in } \boldsymbol{L}^2(\Omega).$$

Moreover, from the definition of $\boldsymbol{R}^\theta(\varepsilon)$, we easily show that the weak limit $\boldsymbol{u}$ satisfies $e^\theta_{i3}(\boldsymbol{u}) = \boldsymbol{0}$, which in turn implies that $\boldsymbol{u}$ is a Kirchhoff–Love field, i.e., that there exists $\zeta_\alpha \in H^1(\omega)$ and $\zeta_3 \in H^2(\omega)$ such that:

$$u_\alpha = \zeta_\alpha - x_3\partial_\alpha\zeta_3, \quad u_3 = \zeta_3.$$

- *Step II* We first observe that for all $\boldsymbol{v} \in \boldsymbol{K}(\varepsilon)(\Omega)$ we have $v_3(x_1, x_2, -1) \geq -\theta + \frac{1}{\sqrt{\alpha^\varepsilon}} - 1$. Since α^ε, given by (2), is such that $\lim_{\varepsilon\longrightarrow 0} \alpha^\varepsilon = 1$ and $\alpha^\varepsilon > 1$, we can introduce the following subset of $\boldsymbol{K}(\varepsilon)(\Omega)$ defined by

$$\boldsymbol{K}(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0,\ v_3(x_1, x_2, -1) \geq -\theta \text{ on } \Gamma_-\right\}.$$

Let us insert the scaled strain tensor $\boldsymbol{R}^\theta(\varepsilon)$ into problem (18). After tedious but straightforward computations (see [8] for a complete derivation), the variational inequality (18) reads:

$$\int_\Omega \left\{\lambda R^\theta_{ii}(\varepsilon)\delta_{\alpha\beta}+2\mu R^\theta_{\alpha\beta}(\varepsilon)\right\}$$

$$\times\left\{\partial_\alpha(v_\beta-u_\beta(\varepsilon))-\frac{1}{2}\Big[\partial_\beta\theta\partial_3(v_\alpha-u_\alpha(\varepsilon))+\partial_\alpha\theta\partial_3(v_\beta-u_\beta(\varepsilon))\Big]\right\}dx$$

$$+\int_\Omega\left\{\lambda R^\theta_{\alpha\alpha}(\varepsilon)+(\lambda+2\mu)R^\theta_{33}(\varepsilon)\right\}\partial_\beta\theta\partial_\beta(v_3-u_3(\varepsilon))dx$$

$$+\frac{1}{\varepsilon}\int_\Omega 2\mu R^\theta_{\alpha 3}(\varepsilon)[\partial_3(v_\alpha-u_\alpha(\varepsilon))+\partial_\alpha(v_3-u_3(\varepsilon))-\partial_\alpha\theta\partial_3(v_3-u_3(\varepsilon))]dx$$

$$+\frac{1}{\varepsilon^2}\int_\Omega\left\{\lambda R^\theta_{\beta\beta}(\varepsilon)+(\lambda+2\mu)R^\theta_{33}(\varepsilon)\right\}\partial_3(v_3-u_3(\varepsilon))dx$$

$$+B^\sharp(\varepsilon,\theta,\boldsymbol{u}(\varepsilon),\boldsymbol{R}^\theta(\varepsilon);\boldsymbol{v}-\boldsymbol{u}(\varepsilon))\ \geq L(\boldsymbol{v}-\boldsymbol{u}(\varepsilon))+\varepsilon^2L^\sharp(\varepsilon,\theta;\boldsymbol{v}-\boldsymbol{u}(\varepsilon))$$

$$\forall\boldsymbol{v}\in\boldsymbol{K}(\varepsilon)(\Omega),\tag{22}$$

where the linear form $L(.)$ is defined as

$$L(\boldsymbol{v})=\int_\Omega f_iv_idx+\int_{\Gamma_+}g_iv_ida$$

and where the quantities $B^\sharp(\varepsilon,\theta,\boldsymbol{u}(\varepsilon))$ and $L^\sharp(\varepsilon,\theta;\boldsymbol{v})$ stand for remainders which are explicitly given and estimated in Appendix. Remainder $B^\sharp(\varepsilon,\theta,\boldsymbol{u}(\varepsilon))$ is uniformly bounded, i.e., there exists a positive constant $C(\theta)$ independent of ε such that, for all $\boldsymbol{u}\in\boldsymbol{K}(\Omega)$, $\boldsymbol{v}\in\boldsymbol{K}(\Omega)$, $\boldsymbol{R}^\theta\in\boldsymbol{L}^2(\Omega)$ we have

$$\sup_{0<\varepsilon\leq\varepsilon_0}B^\sharp(\varepsilon,\theta,\boldsymbol{u}(\varepsilon),\boldsymbol{R}^\theta(\varepsilon),\boldsymbol{v})\leq C(\theta)\left(|\boldsymbol{R}^\theta|_{0,\Omega}+|\boldsymbol{u}|_{1,\Omega}\right)|\boldsymbol{v}|_{1,\Omega}.\tag{23}$$

- *Step III* By taking appropriate test functions $\boldsymbol{v}$ in (22) we characterize the limit $\boldsymbol{R}^\theta$ of the sequence $\{\boldsymbol{R}^\theta(\varepsilon)\}$. More specifically, since $R^\theta_{\alpha\beta}(\varepsilon)=e^\theta_{\alpha\beta}(\boldsymbol{u}(\varepsilon))$ and $\boldsymbol{u}(\varepsilon)\rightharpoonup\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$, we first have

$$R^\theta_{\alpha\beta}=e^\theta_{\alpha\beta}(\boldsymbol{u}).$$

The components $R^\theta_{3\alpha}$ and $R^\theta_{\alpha 3}$ are obtained by multiplying (22) by ε, using estimate (23) and choosing $v_3=u_3(\varepsilon)$ in (22). This gives

$$\lim_{\varepsilon\longrightarrow 0}\int_\Omega 2\mu R^\theta_{\alpha 3}(\varepsilon)\partial_3(v_\alpha-u_\alpha(\varepsilon))dx\geq 0,\ \forall\, v_\alpha\in H^1(\Omega),\, v_\alpha=0\text{ on }\Gamma_0.$$

Using the fact that the components v_α belong to a vector space this inequality can be rewritten as:

$$\lim_{\varepsilon\longrightarrow 0}\int_\Omega 2\mu R^\theta_{\alpha 3}(\varepsilon)\partial_3v_\alpha dx\geq 0,\ \forall\, v_\alpha\in H^1(\Omega),\, v_\alpha=0\text{ on }\Gamma_0,$$

so that, at the limit:

$$\int_\Omega 2\mu R^\theta_{\alpha 3}\partial_3v_\alpha dx=0,\ \forall\, v_\alpha\in H^1(\Omega),\, v_\alpha=0\text{ on }\Gamma_0,$$

and we get

$$R^\theta_{\alpha 3} = R^\theta_{3\alpha} = 0 \tag{24}$$

because of a classical lemma of the calculus of variations (given in [6], pages 19–20).
On the other hand, let us choose $v_\alpha = u_\alpha(\varepsilon)$, multiply (22) by ε^2 and pass to the limit. This gives:

$$\begin{cases} \lim\limits_{\varepsilon \longrightarrow 0} \displaystyle\int_\Omega \left\{\lambda R^\theta_{\alpha\alpha}(\varepsilon) + (\lambda + 2\mu) R^\theta_{33}(\varepsilon)\right\} \partial_3 (v_3 - u_3(\varepsilon)) dx \geq 0 \\ \forall v_3 \in K \overset{def}{\equiv} \left\{v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma_0,\ v \geq -\theta \text{ on } \Gamma_-\right\}. \end{cases} \tag{25}$$

From (25), the calculation of R^θ_{33} results from the following lemma:

Lemma 5.5 *Let K be the convex cone defined in* (25) *and let* $u \in L^2(\Omega)$ *and* $w \in K$ *such that*

$$\int_\Omega u \partial_3 (v - w) dx \geq 0 \ \ \forall v \in K. \tag{26}$$

Then, $u = 0$.

Proof The cone K has its vertex at $-\theta$. We first introduce a convex cone K^0 with vertex at the origin defined from K by a translation

$$K^0 = \left\{v^0 = v + \theta, v \in K\right\},$$

so that formula (26) becomes:

$$\int_\Omega u \partial_3 \left(v^0 - w^0\right) dx \geq 0 \ \ \forall v^0 \in K^0, \tag{27}$$

where $w^0 = w + \theta$. Since the cone K^0 has its vertex at the origin, inequality (27) turns out to be

$$\int_\Omega u \partial_3 v^0 dx \geq 0 \ \ \forall v^0 \in K^0. \tag{28}$$

Let us now introduce $\mathcal{D}(\Omega)$ and $\mathcal{D}(\omega)$ as the sets of indefinitely differentiable functions with compact support respectively in Ω and ω, consider two functions $\phi \in \mathcal{D}(\Omega)$ and $\psi \in \mathcal{D}(\omega)$ such that

$$\psi(x_1, x_2) \geq 0,\ \ (x_1, x_2) \in \bar{\omega},$$

and let us build a function z such as

$$z(x_1, x_2, x_3) = \psi(x_1, x_2) + \int_{-1}^{x_3} \phi(x_1, x_2, t) dt, \quad (x_1, x_2, x_3) \in \overline{\Omega}.$$

We then infer from inequality (28)

$$\int_\Omega u \phi dx \geq 0 \ \ \forall \phi \in \mathcal{D}(\Omega)$$

and hence $u = 0$.

We observe that the same translation by θ can be done in (25), so that the limit as $\varepsilon \longrightarrow 0$ is obvious and, since operator $\lambda R^\theta_{33} + (\lambda + 2\mu) R^\theta_{\alpha\alpha}$ has values in $\boldsymbol{L}^2(\Omega)$, we get from lemma (5.5):

$$R^\theta_{33} = \frac{-\lambda}{\lambda + 2\mu} R^\theta_{\alpha\alpha}. \tag{29}$$

- *Step IV* We now establish that the convergence is strong for the whole family $\{\boldsymbol{u}(\varepsilon)\}_\varepsilon$. For the sake of simplicity, let us take :: to denote the operator associated with the bilinear form of the linear elasticity, that is, given two symmetric tensors $\boldsymbol{S} = (S_{ij})$, $\boldsymbol{T} = (T_{ij})$,

$$\int_\Omega \boldsymbol{S} :: \boldsymbol{T} dx \overset{def}{\equiv} \int_\Omega (\lambda S_{pp} T_{qq} + 2\mu S_{ij} T_{ij}) dx.$$

In order to prove the strong convergence we first estimate $|\boldsymbol{R}^\theta(\varepsilon) - \boldsymbol{R}^\theta|^2_{0,\Omega}$. It is immediately seen that we have:

$$\begin{aligned} 2\mu|\boldsymbol{R}^\theta(\varepsilon) - \boldsymbol{R}^\theta|^2_{0,\Omega} &\leq \int_\Omega (\boldsymbol{R}^\theta(\varepsilon) - \boldsymbol{R}^\theta) :: (\boldsymbol{R}^\theta(\varepsilon) - \boldsymbol{R}^\theta) dx \\ &\leq \int_\Omega \boldsymbol{R}^\theta :: \left(\boldsymbol{R}^\theta - 2\boldsymbol{R}^\theta(\varepsilon)\right) dx + \int_\Omega \boldsymbol{R}^\theta(\varepsilon) :: \boldsymbol{R}^\theta(\varepsilon) dx. \end{aligned} \tag{30}$$

From inequality (30) we clearly get:

$$\lim_{\varepsilon \longrightarrow 0} 2\mu|\boldsymbol{R}^\theta(\varepsilon) - \boldsymbol{R}^\theta|^2_{0,\Omega} \leq -\int_\Omega \boldsymbol{R}^\theta :: \boldsymbol{R}^\theta + \lim_{\varepsilon \longrightarrow 0} \int_\Omega \boldsymbol{R}^\theta(\varepsilon) :: \boldsymbol{R}^\theta(\varepsilon) dx. \tag{31}$$

We already know that the limit $\boldsymbol{u}$ is a Kirchhoff–Love field. Choosing $\boldsymbol{v} = \boldsymbol{u}$ as a test function, inequality (22) gives

$$\begin{aligned} &\int_\Omega \boldsymbol{R}^\theta :: \boldsymbol{R}^\theta dx + \varepsilon B^\sharp_3\left(\varepsilon, \theta, \boldsymbol{R}^\theta(\varepsilon), \boldsymbol{u} - \boldsymbol{u}(\varepsilon)\right) - \int_\Omega \boldsymbol{R}^\theta(\varepsilon) :: \boldsymbol{R}^\theta(\varepsilon) dx \\ &\quad \leq L(\boldsymbol{u} - \boldsymbol{u}(\varepsilon)) + \varepsilon^2 L^\sharp(\varepsilon, \theta; \boldsymbol{u} - \boldsymbol{u}(\varepsilon)), \end{aligned} \tag{32}$$

from which we get at the limit:

$$\int_\Omega \boldsymbol{R}^\theta :: \boldsymbol{R}^\theta dx - \lim_{\varepsilon \longrightarrow 0} \int_\Omega \boldsymbol{R}^\theta(\varepsilon) :: \boldsymbol{R}^\theta(\varepsilon) dx \geq 0. \tag{33}$$

Therefore, using the weak convergence in $\boldsymbol{L}^2(\Omega)$ of sequences $\{\boldsymbol{u}(\varepsilon)\}_\varepsilon$ and $\{\boldsymbol{R}^\theta(\varepsilon)\}_\varepsilon$, we obtain from inequality (30) the strong convergence in $\boldsymbol{L}^2(\Omega)$ of the sequence $\{\boldsymbol{R}^\theta(\varepsilon)\}_\varepsilon$ to $\boldsymbol{R}^\theta$. In addition, using the definition of quantities $R^\theta_{ij}(\varepsilon)$ we have:

$$\begin{aligned} |\boldsymbol{e}^\theta(\boldsymbol{u}(\varepsilon)) - \boldsymbol{e}^\theta(\boldsymbol{u})|^2_{0,\Omega} \leq &\sum_{\alpha,\beta} |R^\theta_{\alpha\beta}(\varepsilon) - R^\theta_{\alpha\beta}|^2_{0,\Omega} \\ &+ 2\varepsilon^2 \sum_\alpha |R^\theta_{\alpha 3}(\varepsilon)|^2_{0,\Omega} \\ &+ \varepsilon^4 |R^\theta_{33}(\varepsilon)|^2_{0,\Omega} + \varepsilon^4 |\partial_\alpha \theta \partial_\alpha u_3(\varepsilon)|^2_{0,\Omega}. \end{aligned} \tag{34}$$

Inequality (34) shows that the sequence $\{\boldsymbol{e}^\theta(\boldsymbol{u}(\varepsilon))\}_\varepsilon$ converges strongly in $\boldsymbol{L}^2(\Omega)$ to $\boldsymbol{e}^\theta(\boldsymbol{u})$. The sequence $\{\boldsymbol{u}(\varepsilon)\}_\varepsilon$ then converges strongly in $\boldsymbol{L}^2(\Omega)$ together with $\{\boldsymbol{e}^\theta(\boldsymbol{u}(\varepsilon))\}_\varepsilon$ and we end up with Korn's inequality.

- *STEP V* Let us come back to variational inequality (22) and make use of (24) and (29). As the convergence of $\{\boldsymbol{u}(\varepsilon)\}_\varepsilon$ towards $\boldsymbol{u}$ is strong, we can pass to the limit as $\varepsilon \longrightarrow 0$ in (22) which immediately gives:

$$\int_\Omega \left\{\lambda R^\theta_{ii}\delta_{\alpha\beta} + 2\mu R^\theta_{\alpha\beta}\right\} \left\{\partial_\alpha(v_\beta - u_\beta) + \frac{1}{2}\left[\partial_\beta\theta\partial_3(v_\alpha - u_\alpha) + \partial_\alpha\theta\partial_3(v_\beta - u_\beta)\right]\right\} dx$$
$$\geq \quad L(\boldsymbol{v} - \boldsymbol{u}). \qquad (35)$$

Using the fact that the limit $\boldsymbol{u}$ is a Kirchhoff–Love field, and introducing the quantities defined in (20) into (35), we easily establish that the components (ζ_α, ζ_3) of $\boldsymbol{u}$ solve the variational problem (19). □

Remark 4 It is interesting to observe that Problem (19) consists of both an equality in a vector space for the ζ_α components, that is for the membrane part of the solution, and of an inequality in a cone for the ζ_3 component, that is for the bending part of the solution, which in turn means that the obstacle condition only deals with the bending part.

6 Conclusion

The present study on the case of a shallow shell in unilateral contact with an obstacle gives a two-dimensional limit problem from three-dimensional elasticity, with the specificity that unilaterality holds on the boundary as long as the thickness parameter is finite, while it holds in the domain at the limit. The justification of the model is obtained with a complete convergence result.

Shell models involving unilateral contact, in either the domain or part of the domain are already available in industrial softwares and are being widely used in engineering applications. These applications often involve shells which are nonshallow, nonsmooth, and geometrically nonlinear. It would probably be a long-term project to prove the validity of shell models in all these cases. However, short term studies can yield useful insights into different points which can be listed.

Removing the shallowness assumption leads to geometrical problems with respect to the nonpenetrability condition. We are at present working on these lines.

Some specific difficulties arise when dealing with contact conditions in the case of nonsmooth shells, with folds or conical points for instance, but these difficulties probably occur within large strain nonlinearities and are not very different from those encountered with general shells in the linear case. Nonlinear strains is certainly the most important point as far as engineering applications are concerned, especially in the case of buckling or path following problems. As far as we can see, although proving a justification of a nonlinear unilateral model could be useful, this could only be a formal proof for the moment.

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Appendix

The quantities $B^\sharp$ and $L^\sharp$ which appear in (22) correspond to the remainders resulting from the expansion of $\nabla^\varepsilon\Theta^\varepsilon$ with respect to successive powers of ε. Since $\delta^\varepsilon = \det(\nabla^\varepsilon\Theta^\varepsilon)$ and $b^\varepsilon_{ij} = (\nabla^\varepsilon\Theta^\varepsilon)^{-1}_{ij}$, we can introduce the notations $\delta^\sharp, b^\sharp_{33}$ in order to write:

$$\begin{cases} \delta^\varepsilon(x^\varepsilon) = \delta(\varepsilon)(x) \qquad = 1 + \varepsilon^2\delta^\sharp(\varepsilon,\theta), \\ b^\varepsilon_{33}(x^\varepsilon) = b_{33}(\varepsilon)(x) = 1 + \varepsilon^2 b^\sharp_{33}(\varepsilon,\theta), \end{cases}$$

and there exists $\varepsilon_0 > 0$ and a positive constant C independent of ε such that:

$$\sup_{0\le\varepsilon\le\varepsilon_0} \max_{x\in\overline{\Omega}} \left(\max_{ij} |b^\sharp_{ij}(\varepsilon,\theta)(x)| + |\delta^\sharp(\varepsilon,\theta)(x)| \right) \le C(\theta).$$

The functions indexed by $\sharp$ depend on (ε,θ); from now on, we will skip this dependence for the sake of simplification. Using the same kind of expansions, we introduce the notations $e^\sharp_{\alpha\beta}, e^\sharp_{\alpha 3}, e^\flat_{33}$ and write:

$$\begin{cases} e^\varepsilon_{\alpha\beta}(\hat{\boldsymbol{v}}^\varepsilon)(\hat{x}^\varepsilon) = \varepsilon^2 e^\theta_{\alpha\beta}(\boldsymbol{v})(x) + \varepsilon^4 e^\sharp_{\alpha\beta}(\boldsymbol{v})(x), \\ e^\varepsilon_{\alpha 3}(\hat{\boldsymbol{v}}^\varepsilon)(\hat{x}^\varepsilon) = \varepsilon e^\theta_{\alpha 3}(\boldsymbol{v})(x) + \varepsilon^3 e^\sharp_{\alpha 3}(\boldsymbol{v})(x), \\ e^\varepsilon_{33}(\hat{\boldsymbol{v}}^\varepsilon)(\hat{x}^\varepsilon) = e^\theta_{33}(\boldsymbol{v})(x) + \varepsilon^2 \left(\partial_\alpha\theta\,\partial_\alpha v_3 + b^\sharp_{33}\partial_3 v_3\right)(x) + \varepsilon^4 e^\sharp_{33}(\boldsymbol{v})(x), \end{cases}$$

such that:

$$\sup_{0\le\varepsilon\le\varepsilon_0} \left(\max_{ij} |e^\sharp_{ij}(\varepsilon,\theta;\boldsymbol{v})|_{0,\Omega} + |e^\sharp_{33}(\varepsilon,\theta;\boldsymbol{v})(x)|_{0,\Omega} \right) \le C(\theta)||\boldsymbol{v}||_{1,\Omega}.$$

We are now in a position to examine quantity $B^\sharp$. This quantity consists of three terms as the result of a decomposition of the form

$$B^\sharp = B^\sharp_0 + \varepsilon B^\sharp_1 + \varepsilon^2 B^\sharp_2,$$

where the complete expression of $B^\flat_0$ reads:

$$\begin{aligned} B^\sharp_0(\varepsilon,\theta;\boldsymbol{u}(\varepsilon),\boldsymbol{R}^\theta(\varepsilon),\boldsymbol{v}) = &\int_\Omega \left(\lambda R^\theta_{\alpha\alpha}(\varepsilon) b^\sharp_{33}\partial_3 v_3 + \lambda e^\sharp_{\alpha\alpha}(\boldsymbol{u}(\varepsilon))\partial_3 v_3\right) dx \\ &+ \int_\Omega 2\mu e^\sharp_{\alpha 3}(\boldsymbol{u}(\varepsilon))\,(\partial_3 v_\alpha + \partial_\alpha v_3 - \partial_\alpha\theta\,\partial_3 v_3)\,dx \\ &+ \int_\Omega (\lambda + 2\mu) e^\sharp_{33}(\boldsymbol{u}(\varepsilon))\partial_3 v_3 dx \\ &+ \int_\Omega (\lambda + 2\mu) R^\theta_{33}(\varepsilon) b^\sharp_{33}\partial_3 v_3 dx \\ &+ \int_\Omega \left(\lambda R^\theta_{\beta\beta}(\varepsilon) + (\lambda + 2\mu) R^\theta_{33}(\varepsilon)\right)\partial_3 v_3 \delta^\sharp dx. \end{aligned}$$

The other quantities satisfy the following bounds:
for all $\boldsymbol{u}, \boldsymbol{R}^\theta, \boldsymbol{v} \in \boldsymbol{K}(\Omega) \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{K}(\Omega)$,

$$\sup_{0\le\varepsilon\le\varepsilon_0} |B^\sharp_1(\varepsilon,\theta;\boldsymbol{R}^\theta,\boldsymbol{v})| \le C(\theta)|\boldsymbol{R}^\theta|_{0,\Omega}|\boldsymbol{v}|_{1,\Omega}, \quad \sup_{0\le\varepsilon\le\varepsilon_0} |B^\sharp_2(\varepsilon,\theta;\boldsymbol{u},\boldsymbol{v})| \le C(\theta)|\boldsymbol{u}|_{1,\Omega}|\boldsymbol{v}|_{1,\Omega}.$$

Finally, it can easily be shown that

$$\sup_{0\leq\varepsilon\leq\varepsilon_0} |L^\sharp(\varepsilon,\theta;\boldsymbol{v})| \leq C(\theta)|\boldsymbol{v}|_{1,\Omega} \ \ \forall \boldsymbol{v} \in \boldsymbol{K}(\Omega).$$

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