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The Effects of Compressibility on Inhomogeneous Deformations for a Class of Almost Incompressible Isotropic Nonlinearly Elastic Materials

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Abstract Experimental data for simple tension suggest that there is a power–law kinematic relationship between the stretches for large classes of slightly compressible (or almost incompressible) non-linearly elastic materials that are homogeneous and isotropic. Here we confine attention to a particular constitutive model for such materials that is of generalized Varga type. The corresponding *incompressible* model has been shown to be particularly tractable analytically. We examine the response of the slightly compressible material to some *nonhomogeneous* deformations and compare the results with those for the corresponding incompressible model. Thus the effects of slight compressibility for some basic nonhomogeneous deformations are explicitly assessed. The results are fundamental to the analytical modeling of almost incompressible hyperelastic materials and are of importance in the context of finite element methods where slight compressibility is usually introduced to avoid element locking due to the incompressibility constraint. It is also shown that even for slightly compressible materials, the volume change can be significant in certain situations.

Keywords Non-linearly elastic materials · Almost incompressible · Power–law kinematic relation · Nonhomogeneous deformations · Generalized Varga model

Mathematics Subject Classifications (2000) 74B20 · 74G55

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1 Introduction

The homogeneous, isotropic, *incompressible* elastic material characterized by the strain-energy density function

$$W = c_1(i_1 - 3) + c_2(i_2 - 3), \tag{1.1}$$

where $c_1 \neq 0$, c_2 are constants ($c_1 + c_2 = 2\mu$, where μ is the infinitesimal shear modulus) has received particular attention in the literature on nonlinear elasticity. Here i_1, i_2 are two of the principal invariants of the stretch tensors, defined in terms of the principal stretches as follows:

$$i_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad i_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1. \tag{1.2}$$

The special case of (1.1) with c_2=0 was introduced by Varga [24] for natural rubber vulcanizates and later by Dickie and Smith [6] for styrene-butadiene rubber. The general form (1.1), called the modified Varga model, was considered by Hill and Arrigo [8] and shown there to have a somewhat better fit with experimental data than the special Varga model. Some remarkable mathematical properties of the model (1.1) have been established in a series of papers by Hill and Arrigo [8, 9] and Arrigo and Hill [1]. See Hill [10] for a review of this work. Since all materials are to some extent compressible, our objective here is to consider a *compressible* version of (1.1) that accounts for *slight compressibility* and to assess the effects of slight compressibility in some *nonhomogeneous* deformations.

The effects of slight compressibility are accounted for by using a model for *almost incompressible materials* suggested recently by Horgan and Murphy [15]. They considered a class of almost incompressible materials characterized by a power–law kinematic relationship between the stretches in simple tension. Specifically, if

$$\lambda_2 = \lambda_1^{-\frac{1}{2}+\varepsilon}, \tag{1.3}$$

where λ_1, λ_2 are the principal stretches parallel and perpendicular to the direction of applied force respectively and ε is such that

$$0 < \varepsilon << 1, \tag{1.4}$$

then the material is considered to be *almost incompressible*. The experimental evidence for assuming (1.3), (1.4), based on the response for some *homogeneous* deformations, is reviewed in [15]. We note that the study of first-order effects due to compressibility has a long history. Early work is due to Oldroyd [22] and Spencer [23]. Extensive work on almost incompressible models was also carried out by Ogden [17–21]. Our particular emphasis is on models that reflect the *power–law* kinematic relation (1.3), (1.4). As mentioned earlier, our objective is to investigate how results for some basic *nonhomogeneous* deformations for the almost incompressible model compare with those for the incompressible model (1.1). Apart from the fundamental analytical importance of this issue, the results are of significance in the context of finite element methods for nonlinear elasticity problems. It is well known that the constraint of incompressibility leads to computational difficulties due to element locking and slight compressibility is usually introduced to circumvent this problem. Our work provides an analytical basis that serves to justify this procedure for some basic nonhomogeneous deformations. However, it is also shown that the volume change can be significant for slightly compressible materials in certain situations.

The standard *compressible* version of (1.1) is written as

$$W = c_1(i_1 - 3) + c_2(i_2 - 3) + h(i_3), h(i_3) \neq 0, \tag{1.5}$$

where $c_1 \neq 0$, c_2 are constants, h is an arbitrary function and $i_3 = \lambda_1 \lambda_2 \lambda_3$. In order for the strain-energy and the stress to vanish in the undeformed state, it is required that

$$h(1) = 0, h'(1) = -(c_1 + 2c_2). \tag{1.6}$$

This *generalized compressible Varga model* has been the subject of many investigations as it has been possible to obtain explicit exact solutions to a variety of boundary-value problems for this particular compressible material model. See, e.g., Aron [2], Carroll [3], Carroll and Murphy [4], Carroll *et al* [5], Haughton [7], Horgan [11–14] and references cited therein. It is worth noting that *cavitation phenomena* for compressible materials can be investigated in a particularly simple analytic way for this model [13].

On using the method in Murphy [16], it can be shown that a general strain-energy density of the form $W = W(i_1, i_2, i_3)$ gives rise to the power–law kinematic relation (1.3) in simple tension if

$$W = A\left(i_1 + \frac{4\varepsilon}{1-2\varepsilon} i_3^{2\varepsilon - 1/4\varepsilon} - \frac{3-2\varepsilon}{1-2\varepsilon}, i_2 + \frac{2\varepsilon}{1-2\varepsilon} i_3^{2\varepsilon-1/2\varepsilon} - \frac{4\varepsilon}{1+2\varepsilon} i_3^{2\varepsilon+1/4\varepsilon} - \frac{3-2\varepsilon}{1-4\varepsilon^2}\right), \tag{1.7}$$

where A (. , .) is an arbitrary smooth function of its indicated arguments. On linearization, we obtain

$$\begin{aligned} W = {} & d_1\left(i_1 + \frac{4\varepsilon}{1-2\varepsilon} i_3^{2\varepsilon-1/4\varepsilon} - \frac{3-2\varepsilon}{1-2\varepsilon}\right) \\ & + d_2\left(i_2 + \frac{2\varepsilon}{1-2\varepsilon} i_3^{2\varepsilon-1/2\varepsilon} - \frac{4\varepsilon}{1+2\varepsilon} i_3^{2\varepsilon+1/4\varepsilon} - \frac{3-2\varepsilon}{1-4\varepsilon^2}\right), \end{aligned} \tag{1.8}$$

where d_1 and d_2 are arbitrary constants. To ensure compatibility with the linear theory, it is required that

$$2\mu = d_1 + d_2, \quad 3\kappa = \frac{3-2\varepsilon}{4\varepsilon}(d_1 + d_2), \tag{1.9}$$

where μ, κ are the infinitesimal shear and bulk moduli respectively. It therefore follows that $d_1 + d_2 > 0$ and that

$$\frac{\mu}{\kappa} = \frac{6\varepsilon}{3-2\varepsilon} \tag{1.10}$$

which conforms with the intuitive expectation that the ratio μ/κ is 'small' for almost incompressible materials. A comparison of (1.8) with (1.5) shows that (1.8) is a generalized Varga material if we identify d_1, d_2 with c_1, c_2 and take

$$\begin{aligned} h(i_3) = {} & c_1\left(\frac{4\varepsilon}{1-2\varepsilon} i_3^{(2\varepsilon-1)/4\varepsilon} - \frac{4\varepsilon}{1-2\varepsilon}\right) \\ & + c_2\left(\frac{2\varepsilon}{1-2\varepsilon} i_3^{(2\varepsilon-1)/2\varepsilon} - \frac{4\varepsilon}{1+2\varepsilon} i_3^{(2\varepsilon+1)/4\varepsilon} + \frac{2\varepsilon(1-6\varepsilon)}{1-4\varepsilon^2}\right). \end{aligned} \tag{1.11}$$

Thus the particular compressible Varga model (1.5) with $h(i_3)$ given by (1.11) gives rise to the power–law kinematic relation (1.3) in simple tension. Henceforth, for simplicity, we shall refer to this as the almost incompressible Varga model. From (1.11) we obtain

$$h'(i_3) = -c_1 i_3^{-(2\varepsilon+1)/4\varepsilon} - c_2 i_3^{-1/2\varepsilon} - c_2 i_3^{(1-2\varepsilon)/4\varepsilon}, \tag{1.12}$$

$$h''(i_3) = c_1\left(\frac{2\varepsilon+1}{4\varepsilon}\right) i_3^{-(1+6\varepsilon)/4\varepsilon} + \frac{c_2}{2\varepsilon}\left(i_3^{-(1+2\varepsilon)/2\varepsilon} - \frac{1-2\varepsilon}{2} i_3^{(1-6\varepsilon)/4\varepsilon}\right). \tag{1.13}$$

The nonhomogeneous deformations that we will consider in the sequel are special cases of the well-known families of nonhomogeneous controllable deformations that are known to hold for all *incompressible* isotropic materials. Two approaches will be adopted: when the deformation fields are independent of the material, as in the case for both spherical and cylindrical inflation (Sections 2 and 3), then representative boundary value problems will be solved to examine the effects of slight compressibility on the behavior of the incompressible material (1.1). When the deformation fields depend on the small parameter ε (Sections 4 and 5), it will be shown that the controllable deformations for incompressible materials are recovered on letting ε→0.

Since all elastomers are to some extent compressible, the incompressibility constraint is an idealization, albeit one that usually works exceedingly well in practice. The results described here provide a quantitative assessment of the accuracy of this idealization for some important inhomogeneous deformations.

2 Spherical Inflation

For spherically symmetric deformations of an elastic sphere, the principal stretches are given by

$$\lambda_1 = \frac{dr}{dR}, \lambda_2 = \lambda_3 = \frac{r(R)}{R}, \tag{2.1}$$

where radial spherical polar coordinates of a typical point in the reference and current configurations are denoted by R and r respectively. It was shown by Carroll [3] and independently by Horgan [11] that spherically symmetric deformations for the complete class of compressible Varga materials (1.5) are described by a radial deformation $r(R)$ of the form

$$r^3 = \alpha R^3 + \beta, \tag{2.2}$$

where α, β are integration constants. This result was also established by Haughton [7] for the special Varga model where $c_2=0$ in (1.5). For a review of explicit solutions of this type for a variety of deformations and valid for wide classes of compressible materials, we refer to the book chapter of Horgan [14]. The solution (2.2) will now be used to investigate the effect of slight compressibility for the problem of internal pressurization of a hollow sphere. See also [7] for a discussion of some aspects of this issue for the special Varga material (1.5) with $c_2=0$.

(a) *Internal pressurization of a hollow, incompressible sphere*

For the internal pressurization of an incompressible hollow elastic sphere, where the radial Cauchy stress is specified to be $-P$ at the inner surface, the equations of equilibrium can be integrated to give

$$P = \int_{r_1}^{r_2} \frac{2}{r}\left(\lambda_2 \frac{\partial W}{\partial \lambda_2} - \lambda_1 \frac{\partial W}{\partial \lambda_1}\right) dr, \tag{2.3}$$

where the inner and outer spherical surfaces are denoted by the subscripts 1 and 2 respectively. In (2.3), the derivatives of W are evaluated at the values of the principal stretches given by (2.1). Equation (2.3) provides a relationship between the applied pressure and the inner radius of the deformed sphere since the incompressibility condition gives

$$r_2^3 = R_2^3 + r_1^3 - R_1^3. \tag{2.4}$$

On letting $u \equiv r/R$ (u is the circumferential stretch) equation (2.3) can be rewritten as

$$P = \int_{u_1}^{u_2} \frac{2}{u(1-u^3)}\left(\lambda_2 \frac{\partial W}{\partial \lambda_2} - \lambda_1 \frac{\partial W}{\partial \lambda_1}\right)\bigg|_{\lambda_1 = u^{-2}, \lambda_2 = \lambda_3 = u} du, \tag{2.5}$$

where

$$u_1 \equiv \frac{r_1}{R_1}, \quad u_2 \equiv \frac{r_2}{R_2} = \left(1 - \gamma + \gamma u_1^3\right)^{1/3}, \quad \gamma \equiv R_1^3/R_2^3 < 1. \tag{2.6}$$

For the incompressible Varga model (1.1), the pressure versus deformed inner radius relation (2.5) has the particularly simple form

$$\widehat{P} = \frac{P}{c_1} = \left(\frac{1}{u_2} + c\right)^2 - \left(\frac{1}{u_1} + c\right)^2, \quad c \equiv \frac{c_2}{c_1}. \tag{2.7}$$

(b) *Internal pressurization of an almost incompressible sphere*

Consider now the internal pressurization of a sphere of a *compressible* material of the form (1.5), (1.11) so that the corresponding deformation field is given by (2.2). On using (2.1) and (2.2) and the definition of i_3, we obtain

$$i_3 = \alpha. \tag{2.8}$$

The principal components of the Cauchy stress $\boldsymbol{T}$ for compressible materials are given by

$$T_i = \frac{\lambda_i}{\lambda_1 \lambda_2 \lambda_3} \frac{\partial W}{\partial \lambda_i} \text{ (no sum on } i). \tag{2.9}$$

Therefore we find that the radial and hoop stresses are given by

$$T_{rr} = c_1 R^2/r^2(R) + 2c_2 R/r(R) + h'(\alpha), \tag{2.10}$$

$$T_{\theta\theta} = T_{\phi\phi} = c_1 r(R)/\alpha R + c_2\left[R/r(R) + r^2(R)/\alpha R^2\right] + h'(\alpha), \tag{2.11}$$

where $r(R)$ is given in (2.2). On using the same notation $u = r(R)/R$ introduced for the incompressible case and on employing the boundary condition of zero applied traction on the outer surface we obtain

$$h'(\alpha) + \frac{c_1}{u_2^2} + \frac{2c_2}{u_2} = 0, \tag{2.12}$$

where from (1.12) we have

$$h'(\alpha) = -c_1\alpha^{-\frac{(1+2\varepsilon)}{4\varepsilon}} - c_2\left(\alpha^{\frac{1-2\varepsilon}{4\varepsilon}} + \alpha^{-\frac{1}{2\varepsilon}}\right). \tag{2.13}$$

If the radial Cauchy stress (2.10) is prescribed to be $-P$ on the inner surface, we obtain

$$\widehat{P} = \frac{P}{c_1} = \left(\frac{1}{u_2} + c\right)^2 - \left(\frac{1}{u_1} + c\right)^2, \quad c \equiv \frac{c_2}{c_1}, \tag{2.14}$$

which is *identical* in form to (2.7) but now the u_1 versus u_2 relationship is given by (2.2) so that

$$u_2 = \left[\alpha(1-\gamma) + \gamma u_1^3\right]^{1/3}, \gamma \equiv R_1^3/R_2^3 < 1. \tag{2.15}$$

For a fixed ε, on substitution from (2.15) into (2.12) and making use of (2.13) one obtains α as an implicit function of u_1. However the resulting equation is highly non-linear and treating α as a function of u_1 does not appear to be particularly instructive. An easier, more intuitive approach is adopted instead based on the result (2.8). Because almost incompressible materials are being considered, we use (2.8) to write

$$\alpha = 1 + \varepsilon\delta \quad \text{for} \quad \varepsilon << 1, \tag{2.16}$$

where δ is a number of order 1. Substitution of (2.16) into (2.15) then yields

$$\begin{aligned} u_2 &= \left(1 - \gamma + \gamma u_1^3\right)^{1/3}\left(1 + \varepsilon\delta\frac{1-\gamma}{1-\gamma+\gamma u_1^3}\right)^{1/3} \\ &= u_2^{incomp}\left(1 + \varepsilon\delta\frac{1-\gamma}{\left(u_2^{incomp}\right)^3}\right)^{1/3}, \end{aligned} \tag{2.17}$$

where (2.6) has been used to obtain the last expression in (2.17). To first order in ε, we deduce from (2.17) that

$$u_2 = u_2^{incomp} + \varepsilon\delta\frac{1-\gamma}{3\left(u_2^{incomp}\right)^3}. \tag{2.18}$$

Since the ε term is never likely to be large, we see from (2.18) that the ratio of the outer deformed radius to the outer undeformed radius is virtually the same for the incompressible and almost incompressible models, as one might expect intuitively. By virtue of (2.14) and (2.7), the pressure versus deformed inner radius relation for both models is also virtually identical.

The foregoing considerations can be made more explicit for the special Varga model where c_2=0. In this case, (2.12), (2.13) and (2.15) yield

$$\gamma u_1^3 = \alpha(\gamma - 1) + \alpha^{\frac{3(1+2\varepsilon)}{8\varepsilon}}, \tag{2.19}$$

which gives u_1=$u_1(\alpha)$. By virtue of $(1.9)_1$ with d_2≡c_2=0, we have c_1=2μ. When c_2=0, we see from (2.14) that c=0 and on using $(2.15)_1$, we find that the applied pressure versus inner deformed radius relation (2.14) simplifies to

$$\widehat{P} = \frac{P}{2\mu} = \frac{1}{\left(\alpha(1-\gamma) + \gamma u_1^3\right)^{2/3}} - \frac{1}{u_1^2}, \tag{2.20}$$

where $\alpha(u_1)$ is given implicitly by (2.19). The corresponding relation for the *incompressible* case with c_2=0 (i.e., (2.7) with c=0) can be obtained formally from (2.20) on setting α=1.

In Fig. 1, we plot the relation (2.20) for $\gamma=1/2$ (recall from its definition in (2.15) that that $0<\gamma<1$) and for $\varepsilon=0.05$ and $\varepsilon=0.25$ and compare with the corresponding results for the incompressible case. It is seen from Fig. 1 that, while all the inflation curves are qualitatively similar, increasing compressibility is manifested in a lowering of the maximum pressure. More importantly, we see that the pressure–stretch relation for the incompressible material is obtained uniformly in the limit as $\varepsilon\to 0$ from the corresponding curves in the almost incompressible case. This therefore supports the case for the use of general strain-energy functions of the form (1.7) in the modeling of almost incompressible materials. As described in [15], the motivation for adopting (1.7) was the matching of theoretical predictions with experimental data for some basic *homogeneous* deformations.

A quantity of particular interest in the modeling of almost incompressible materials is the volume change. By virtue of (2.8), we see that the volume change for spherical inflation is given implicitly as a function of the inner hoop stretch by (2.19) for the case when $c_2=0$. To investigate this volume change, we first rewrite (2.19) as

$$\alpha = \left[\gamma u_1^3 + \alpha(1-\gamma)\right]^{8\varepsilon/3(1+2\varepsilon)}. \tag{2.21}$$

On letting $\varepsilon\to 0$ we obtain $\alpha=1$, i.e., the material preserves volume in spherical inflation. To obtain the first order term for the volume change, let

$$\alpha = 1 + \varepsilon\alpha_1, \tag{2.22}$$

and substitution into (2.21) yields

$$\alpha_1 = \frac{8}{3}\ln\left[1 + \gamma\left(u_1^3 - 1\right)\right]. \tag{2.23}$$

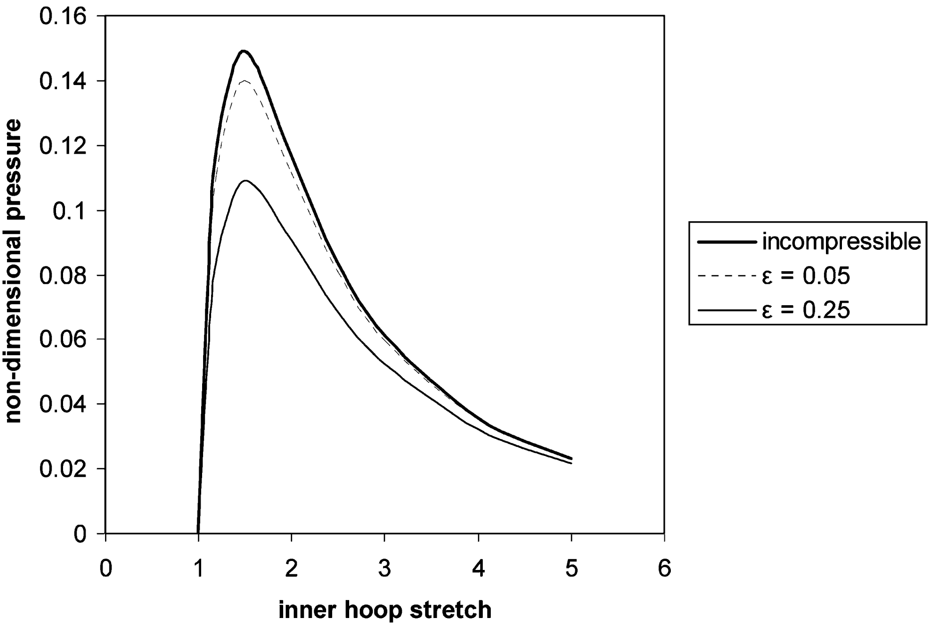


Fig. 1 Pressure versus inner circumferential stretch for a hollow sphere with $\gamma \equiv R_1^3/R_2^3 = 1/2$

By virtue of its definition in (2.15), the limit $\gamma \to 0$ corresponds to two physical situations i.e. the case of a thick walled sphere or a spherical cavity in an infinite medium. In either case, (2.23) yields $\alpha_1=0$ so that the volume change will be small. However for thin walled spherical shells (corresponding to $\gamma \to 1$), we see from (2.23) that α_1 is large for $u_1 \gg 1$. Thus, for a given slightly compressible special Varga material, the volume change can be significant *for sufficiently severe deformations of thin shells.* Volume changes for an intermediate value of γ are illustrated in Fig. 2, where the volume changes for $\varepsilon=0.05$ and $\varepsilon=0.25$ are plotted for the case $\gamma=1/2$. As might be expected, at fixed values of the inner hoop stretch, the volume change decreases as ε decreases. For both values of ε we see that the volume change increases with increasing stretch and is significant even along the strain-hardening sections of the corresponding pressure–stretch relation which, as seen from Fig. 1, corresponds approximately to stretches in the range [1, 1.5].

3 Cylindrical Inflation Accompanied by an Axial Stretch

For the general Varga material (1.5), Carroll [3] has shown that the equations of equilibrium for inflation, accompanied by a constant axial stretch λ, of hollow circular cylinders, have the solution

$$r^2 = \frac{\alpha}{\lambda} R^2 + \beta, \theta = \Theta, z = \lambda Z, \tag{3.1}$$

where the radial cylindrical polar coordinates of a typical point in the reference and current configurations are given by R and r respectively and α, β are constants of integration. On noting that $i_3=\alpha$, the corresponding principal radial and axial Cauchy stresses are given by

$$T_{rr} = \frac{R}{\lambda r} c_1 + \left(\frac{R}{r} + \frac{1}{\lambda}\right) c_2 + h'(\alpha), \quad T_{zz} = \frac{\lambda}{\alpha} c_1 + \left(\frac{R}{r} + \frac{\lambda r}{\alpha R}\right) c_2 + h'(\alpha). \tag{3.2}$$

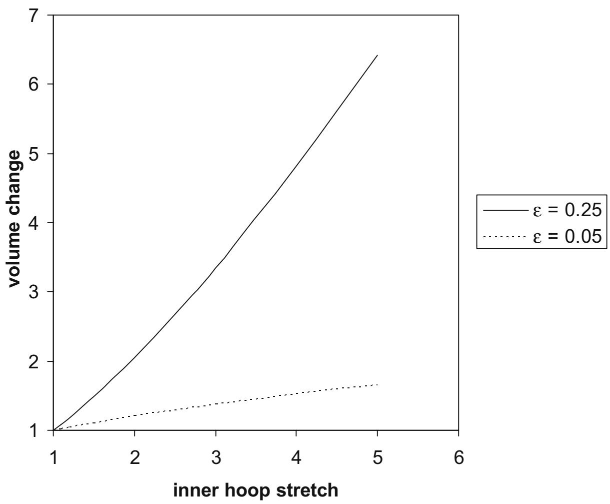


Fig. 2 Volume changes for $\varepsilon=$ 0.05 and 0.25 for the same hollow sphere as in Fig. 1

Only plane stress problems will be considered here and it was observed by Carroll *et. al.* [5] that such problems can be solved *exactly* for the Varga material (1.5) in the case when c_2=0. This will be assumed here and so the plane stress condition T_{zz}=0 then yields

$$h'(\alpha) = -\frac{\lambda}{\alpha}c_1. \tag{3.3}$$

For the particular almost incompressible model given by (1.5), (1.11) with c_2=0, the condition (3.3) reduces to

$$\lambda = \alpha^{\frac{2\varepsilon-1}{4\varepsilon}}. \tag{3.4}$$

It follows immediately from (3.1) and (3.4), on using similar notation as before, i.e., $u \equiv r/R$, that

$$u_2^2 = \alpha^{\frac{2\varepsilon+1}{4\varepsilon}}(1-\gamma_c) + \gamma_c u_1^2, \quad \gamma_c \equiv R_1^2/R_2^2 < 1 \tag{3.5}$$

Only internal pressurization of tubes will be considered here. Thus, on using (3.2) with c_2=0, the boundary condition of zero traction on the outer surface yields

$$\gamma_c u_1^2 = \alpha^{\frac{2\varepsilon+1}{4\varepsilon}}(\gamma_c - 1) + \alpha^{\frac{1}{\varepsilon}}, \tag{3.6}$$

which determines $\alpha = i_3$ implicitly as a function of u_1. Equation (3.6) is the counterpart of (2.19) obtained for spherical inflation. If the radial Cauchy stress is prescribed to be $-P$ on the inner boundary, we find that

$$\widehat{P} = \frac{P}{2\mu} = \alpha^{\frac{1-2\varepsilon}{4\varepsilon}}\left(\frac{1}{u_2} - \frac{1}{u_1}\right). \tag{3.7}$$

It follows from (3.5), (3.6) that both $\widehat{P}$, α are implicitly functions of u_1 and so by virtue of (3.4) we see that λ is an implicit function of u_1. Examples will be given shortly and the results compared with the corresponding incompressible model.

First, however, the volume change, as determined by (3.6), will be considered. On writing α in the form given by (2.22), (3.6) becomes

$$\gamma_c u_1^2 = \left((1+\varepsilon\alpha_1)^{\frac{1}{\varepsilon}}\right)^{\frac{2\varepsilon+1}{4}}(\gamma_c - 1) + (1+\varepsilon\alpha_1)^{\frac{1}{\varepsilon}}. \tag{3.8}$$

To first order in ε, we obtain

$$\gamma_c u_1^2 = (\gamma_c - 1)e^{\alpha_1/4} + e^{\alpha_1}, \tag{3.9}$$

which, for a given γ_c, can be viewed as a quartic equation for $e^{\alpha_1/4}$. For the special case of an infinitely thick cylinder, on letting $\gamma_c \to 0$ in (3.9), we find that

$$0 = e^{\alpha_1} - e^{\alpha_1/4}, \tag{3.10}$$

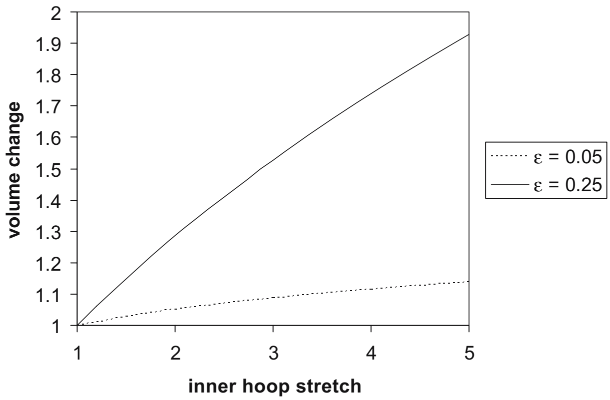
which yields α_1=0 and thus for thick cylinders the volume change will be infinitesimal. On the other hand, for a thin tube, on setting γ_c=1 in (3.9) we obtain

$$\alpha_1 = 2\ln u_1. \tag{3.11}$$

Thus we conclude that if u_1 is large enough, then α_1 is also large and therefore the corresponding volume change is appreciable. This implies that *for thin cylindrical shells under sufficiently large deformations*, significant volume changes occur for the slightly compressible materials considered here.

In general, for given values of the inner hoop stretch and ε, (3.6) can be solved numerically to obtain the corresponding volume change. For γ_c=1/2, these changes are plotted in Fig. 3 for ε=0.05 and ε=0.25. The same qualitative features are obtained as for

Fig. 3 Volume changes for ε= 0.05 and 0.25 for a hollow cylinder with $\gamma \equiv R_1^2/R_2^2 = 1/2$



spherical inflation but, for a given inner circumferential stretch, the volume change is smaller in cylindrical inflation, as might be anticipated.

Internal pressurization of tubes composed of the *incompressible* Varga material (1.1) with c_2=0, i.e.,

$$W = 2\mu(i_1 - 3), \tag{3.12}$$

is now considered. The plane stress condition is again imposed and yields the following implicit relation for λ in terms of u_1:

$$\gamma_c u_1^2 \lambda^4 + (1 - \gamma_c)\lambda^3 - 1 = 0, \tag{3.13}$$

on using the same notation as for the almost incompressible case. The relation between the internal pressure and the inner hoop stretch is easily shown to be

$$\widehat{P} = \frac{P}{2\mu} = \frac{1}{\lambda}\left(\frac{1}{u_2} - \frac{1}{u_1}\right), \tag{3.14}$$

where

$$\lambda u_2^2 = 1 - \gamma_c + \lambda \gamma_c u_1^2. \tag{3.15}$$

A plot of the axial stretch versus u_1 is given in Fig. 4 for the incompressible case and for the almost incompressible case with ε=0.05 and ε=0.25 for γ_c=1/2, where we recall from (3.5) that γ_c is the squared ratio of the inner to the outer undeformed radius. Again as ε→0, there is uniform convergence to the incompressible result. The same feature can be seen in Fig. 5 where plots of the internal pressure versus u_1 for the incompressible case and for the almost incompressible material with ε=0.05 and ε=0.25 are given for the same tube, i.e., for γ_c=1/2.

For the deformations considered so far, the deformation fields have been independent of the form of the arbitrary function $h(i_3)$ for the general Varga material (1.5) and, in particular, have been independent of the material constants c_1, c_2, ε in the proposed model (1.5), (1.11) for an almost incompressible material. In the next two sections, deformations will be considered where this is not the case.

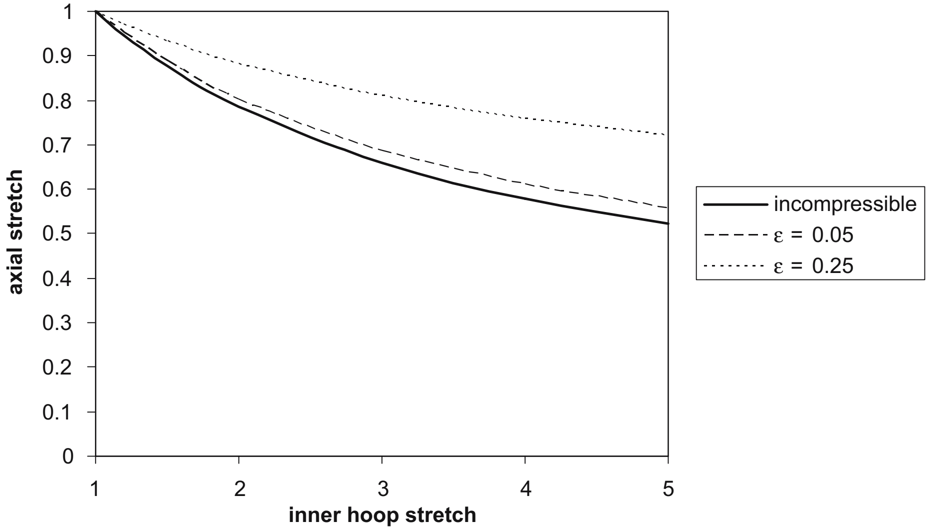


Fig. 4 Axial stretch versus inner hoop stretch for the same tube as in Fig. 3

4 Straightening of a Cylindrical Sector

The straightening of a cylindrical sector, with initial radii R_1, R_2, into a rectangular block, accompanied by an axial stretch, is described by

$$x = \widehat{x}(R), y = B\Theta, z = \lambda Z, \tag{4.1}$$

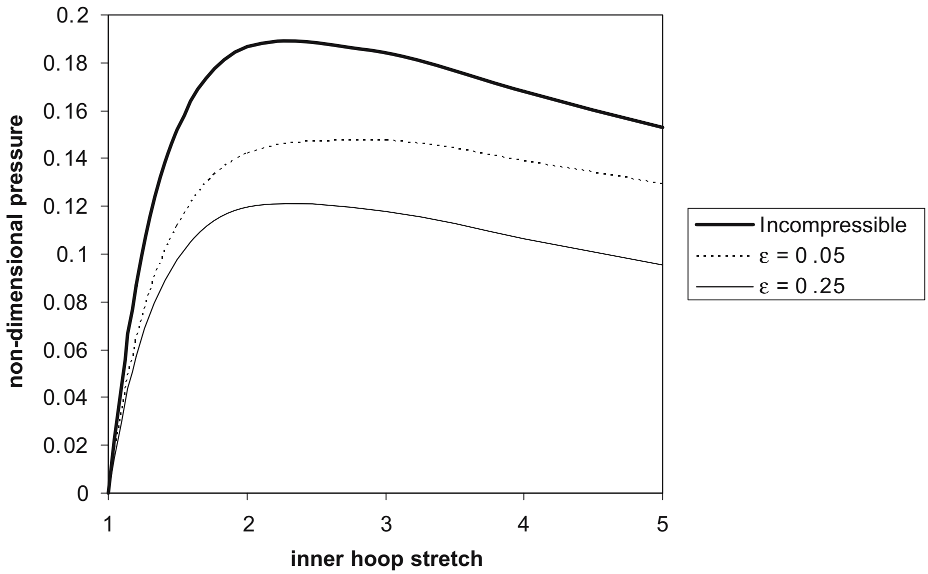


Fig. 5 Pressure versus inner hoop stretch for the same tube as in Figs. 3 and 4

where $dx/dR>0$, $\lambda>0$, $B>0$. Here (R, Θ, Z) are the cylindrical coordinates of a typical particle before deformation and (x, y, z) denote the Cartesian coordinates of the particle after deformation. The principal stretches are

$$\lambda_1 = dx/dR, \lambda_2 = B/R, \lambda_3 = \lambda. \tag{4.2}$$

It is shown in Carroll [3] that the principal Cauchy stresses for the generalized Varga model (1.5) are given by

$$\begin{aligned} T_{xx} &= c_1 R/\lambda B + c_2(R/B + 1/\lambda) + h'(i_3), \\ T_{yy} &= c_1/\lambda(dR/dx) + c_2(dR/dx + 1/\lambda) + h'(i_3), \\ T_{zz} &= c_1 R/B(dR/dx) + c_2(dR/dx + R/B) + h'(i_3), \end{aligned} \tag{4.3}$$

where

$$i_3 = \frac{\lambda B}{R}\frac{dx}{dR}. \tag{4.4}$$

Thus, in the absence of body forces, the equations of equilibrium $div\boldsymbol{T}=\boldsymbol{0}$ in the y and z directions are satisfied identically and that for the x direction is $dT_{xx}/dx=0$ which may be integrated to give

$$\frac{R}{\lambda B}c_1 + \left(\frac{R}{B} + \frac{1}{\lambda}\right)c_2 + h'(i_3) = \widehat{a}, \quad (\widehat{a} \text{ constant}). \tag{4.5}$$

Thus a quadrature solution for x can be obtained, provided that h' is invertible. An examination of (1.12) shows that for the almost incompressible Varga material, h' is not easily invertible. However, one special case immediately suggests itself, namely the special Varga model where $c_2=0$. On setting $c_2=0$ in (4.5) and (1.12) we get

$$\frac{R}{\lambda B} - i_3^{-(2\varepsilon+1)/4\varepsilon} = -a, a \equiv -\frac{\widehat{a}}{2\mu}. \tag{4.6}$$

On substitution of (4.4) in (4.6) the resulting first-order ordinary differential equation can be easily integrated to yield

$$x = \frac{\lambda B(1+2\varepsilon)^2}{2(1-2\varepsilon)}\left(\frac{R}{\lambda B} + a\right)^{\frac{1-2\varepsilon}{1+2\varepsilon}}\left(\frac{R}{\lambda B}\frac{1-2\varepsilon}{1+2\varepsilon} - a\right) + b, \quad (b \text{ constant}). \tag{4.7}$$

The constant b can be set equal to 0 without loss of generality as it corresponds to a rigid body translation. On setting $\varepsilon=0$ in (4.7) we recover, to within an additive constant, the corresponding controllable deformation for incompressible materials, i.e.,

$$x = R^2/2\lambda B. \tag{4.8}$$

On imposing traction-free boundary conditions on the lateral surfaces, i.e., $T_{xx} = 0$ for $x = x(R_1), x(R_2)$, one finds that $a=0$ and (4.7) yields

$$x = \frac{1+2\varepsilon}{2}\lambda B\left(\frac{R}{\lambda B}\right)^{\frac{2}{1+2\varepsilon}}, \tag{4.9}$$

for the deformation of the almost incompressible special Varga material (1.5), (1.11) with $c_2=0$. This solution can be shown to be identical to that obtained by Aron [2]. The non-zero principal stresses are given by

$$T_{yy} = 2\mu\left(\frac{1}{\lambda}\left(\frac{R}{\lambda B}\right)^{\frac{2\varepsilon-1}{2\varepsilon+1}} - \frac{R}{\lambda B}\right), \quad T_{zz} = 2\mu\left(\lambda\left(\frac{R}{\lambda B}\right)^{\frac{4\varepsilon}{2\varepsilon+1}} - \frac{R}{\lambda B}\right). \tag{4.10}$$

On observing that the plane stress boundary condition *cannot* be satisfied exactly here, we will be content to consider plane strain problems only for which $\lambda=1$. On denoting the resultant applied moment by M, we obtain

$$\widehat{M} \equiv \frac{M}{2\mu L} = \frac{1}{2\mu} \int_{x(R_1)}^{x(R_2)} xT_{yy}dx = \frac{(1+2\varepsilon)^2}{2} B^{\frac{2\varepsilon-3}{2\varepsilon+1}} \left(\frac{B^{\frac{2}{1+2\varepsilon}} R^{\frac{3+2\varepsilon}{1+2\varepsilon}}}{3+2\varepsilon} - \frac{R^{\frac{5+2\varepsilon}{1+2\varepsilon}}}{5+2\varepsilon} \right) \Bigg|_{R=R_1}^{R=R_2}, \quad (4.11)$$

where L is the width of the sector in the Z-direction before deformation. As $\varepsilon \to 0$ in (4.11), we obtain the corresponding result for the *incompressible* Varga model (1.1) with $c_2=0$ that is

$$\widehat{M} \equiv \frac{M}{2\mu L} = \frac{1}{2B^3} \left(\frac{B^2 R^3}{3} - \frac{R^5}{5} \right) \Bigg|_{R=R_1}^{R=R_2}. \quad (4.12)$$

The corresponding volume change can be found from (4.6) with $a=0$ and $\lambda=1$, i.e.,

$$i_3 = \left(\frac{B}{R} \right)^{\frac{4\varepsilon}{1+2\varepsilon}}, \quad (4.13)$$

which, to the first order in ε, has the form

$$i_3 = 1 + 4\varepsilon \ln \frac{B}{R}. \quad (4.14)$$

On noting that the ε term is a monotonically decreasing function of R, we see that if the geometrical ratio, B/R_1, of the undeformed sector is sufficiently large, then again the volume change is no longer small for the slightly compressible special Varga material considered here.

5 The Bending of a Rectangular Block

The related problem of the bending of a rectangular block is described by

$$r = \widehat{r}(X), \quad \theta = BY, \quad z = \lambda Z, \quad (5.1)$$

where $dr/dR>0$, $\lambda>0$, $B>0$. Here (X, Y, Z) are rectangular Cartesian coordinates before deformation and (r, θ, z) are cylindrical polar coordinates after deformation. For the general Varga model (1.5), Carroll [3] has shown that the equations of equilibrium reduce to

$$i_3 \frac{dh'(i_3)}{dr} = \lambda B \left(\frac{c_1}{\lambda} + c_2 \right), \quad i_3 = \lambda Br \frac{dr}{dX}. \quad (5.2)$$

This equation can reduced to a quadrature, provided that the function $H(i_3) = \int^{i_3} sh''(s)ds$ is invertible. Consideration of (1.13) shows that this is unlikely for the almost incompressible Varga material (1.5), (1.11) but is easily achieved for the special case when $c_2=0$ in which case $H(i_3) = c_1 \frac{2\varepsilon+1}{4\varepsilon} i_3^{-\frac{2\varepsilon+1}{4\varepsilon}}$. Then (5.2) can be integrated to yield

$$\lambda Br \frac{dr}{dX} = \left(\frac{2\varepsilon-1}{2\varepsilon+1} Br + k \right)^{\frac{4\varepsilon}{2\varepsilon-1}}, \quad (k \text{ constant}) \quad (5.3)$$

which yields the implicit solution

$$\left(\frac{2\varepsilon-1}{2\varepsilon+1} Br + k \right)^{\frac{1+2\varepsilon}{1-2\varepsilon}} (Br+k) = -\frac{2B}{\lambda(1+2\varepsilon)} X + K, \quad (K \text{ constant}). \quad (5.4)$$

Solution of associated boundary value problems is complicated here because the solution (5.4) is an inverse solution, giving the undeformed variable explicitly in terms of the deformed variable. We will be therefore content to note that on setting ε=0 in (5.4), we recover, to within an additive constant, the corresponding controllable solution for incompressible materials, namely,

$$r^2 = 2X/\lambda B. \tag{5.5}$$

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