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On Formulating and Assessing Continuum Theories of Electromagnetic Fields in Elastic Materials

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Abstract My aim is to explore some ideas about the foundations of electromagnetic theory for elastic materials and to suggest some ways of assessing theories of this kind. I will describe some old ideas that seem to have been forgotten, about forces exerted by matter and fields on each other, and a similar idea about energies. Among other things, I will trace Toupin's thinking about elastic dielectrics, showing how he moved toward using these ideas, although he did not explicitly recognize them. Further, I will explain how his dynamical theory can be interpreted to be consistent with them, although this is not obvious from what he wrote.

Key words elastic dielectrics · foundations of electromagnetic theory

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1 Introduction

There is a growing interest in nonlinear continuum theories covering deformation and electromagnetic effects, with theories of this kind proliferating. Different theories have been proposed that are intended to cover about the same range of phenomena. Such theories tend to be quite complicated, making it hard to produce many exact predictions. I think it important to try to develop ways to find weak spots in existing theories and to sort out better bases for formulating theories, to guard against such flaws. So, I will discuss some ideas for doing this.

For one thing, I have found discussions of stress in various expositions of electromagnetic theory somewhat confusing and admit that my thoughts about this have not been very clear. For example, Truesdell and Toupin [1, Sections 542–544] present what are essentially Cauchy's ideas about stresses in matter. Then, their Equation (284.7) describes something called an electromagnetic stress tensor that does not vanish outside matter. Other expositions

Dedicated to the memory of Ronald Rivlin.

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also do this, for example those by Kovetz [2] and Wang [3]. I do not blame such expositors for this, believing that they describe the state of the art. I will cover some ideas that might be useful for improving this.

Various writers, for example Brown [4, pp. 51 ff.] express concern that electromagnetic interactions involve long range forces and that it is hard to cleanly separate these from the contact forces described by stress tensors. He argues that this leads to different, incompatible descriptions of stress that are, nevertheless, physically equivalent. I will comment on this. Actually, I find it harder to relate ideas about energetics used by some different writers. I thought it likely that some older workers might have some interesting thoughts about these matters, and found some in an old textbook by Page [5]. They seem to have been overlooked by many later writers. From the discussion of Whittaker [6, Volume 1, pp. 274–275], I infer that ideas much like Page's occur in much older literature, being stimulated by thinking about radiation pressure. I will discuss these and other thoughts that seem to me to be helpful for better understanding these matters. It is not my intention to give a comprehensive review of the subject or to present results that are really new, although some might well be unfamiliar to you. Toupin is well known for his theories of elastic dielectrics and I will describe how his thinking has moved toward using basic ideas to be presented.

2 Basic Equations

As is common in such discussions, I will use formal reasoning, not mentioning continuity assumptions etc. unless I want to draw attention to particular kinds of exceptions. We need to consider basic equations of electromagnetic theory and, for this, I will use the conventions of Truesdell and Toupin [1, Chapter F] and Kovetz [2], among others. In the classical form that I assume you are familiar with, they have Maxwell's equations as

$$\nabla\cdot B = 0, \tag{2.1}$$

$$\frac{\partial B}{\partial t} + \nabla \times E = 0, \tag{2.2}$$

$$\nabla\cdot D = Q, \tag{2.3}$$

$$\nabla \times H - \frac{\partial D}{\partial t} = J, \tag{2.4}$$

where the charge density Q and current density J include both free and bound charges and currents. To this, add the aether relations

$$B = \mu_0 H,\ D = \varepsilon_0 E,\ c^2 = \frac{1}{\varepsilon_0 \mu_0}, \tag{2.5}$$

where ε_0 and μ_0 are the vacuum constants, c being the speed of light in vacuum. In special relativity theory, these apply in any inertial frame with the fourth coordinate taken as t In non-relativistic theory, the idea is that they apply in aether frames that qualify as inertial frames in both the special relativistic and non-relativistic senses, (2.5) not being invariant under Galilean transformations, although (2.1–2.4) are invariant under time dependent orthogonal transformations and more general transformations to be mentioned later. Here, I will only consider non-relativistic theories. It was Lorentz who originated the idea that (2.5) should hold in matter as well as vacuum, part of the reason that this is called the Maxwell–Lorentz theory, as this is now interpreted, which also includes some ideas about constitutive equations.

As is well known and is not hard to verify, other equations are implied by these, one set being

$$\frac{\partial p}{\partial t} - \nabla \cdot t_F = f_F, \tag{2.6}$$

with

$$p = D \times B,\ t_F = t_F^T = H \otimes B + E \otimes D - \frac{1}{2}(H \cdot B + D \cdot E)1,\ f_F = -QE - J \times B. \tag{2.7}$$

Here, p is the density of electromagnetic momentum, t_F is an electromagnetic stress tensor, and f_F is an electromagnetic body force. Another such equation is

$$\frac{\partial e_F}{\partial t} - \nabla \cdot \mathbf{e}_F = -J \cdot E, \tag{2.8}$$

with

$$e_F = \frac{1}{2}(D \cdot E + H \cdot B), \mathbf{e}_F = -E \times H. \tag{2.9}$$

Here, e_F is electromagnetic energy density, $\mathbf{e}_F$ a flux of electromagnetic energy. These do use (2.5), whereas I shall give reasons not to, in considering constitutive equations. I will employ direct notations for the most part, using Cartesian tensor components in a few places. Actually, Page used a slightly different form, replacing D, H, Q and J by the parts describing free charges and currents. For the simple theory he uses, this amounts to replacing ε_0 by a material constant in matter. Integral versions of these for material regions are given by Truesdell and Toupin [1, Section 284], for example. These are that, for a material region Ω.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} p\mathrm{d}v = \int_{\partial\Omega} (t_F + p \otimes v)\mathrm{d}s + \int_{\Omega} f_F \mathrm{d}v \tag{2.10}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} e_F \mathrm{d}v = \int_{\partial\Omega} (\boldsymbol{e}_F + \boldsymbol{e}_F v)\cdot\mathrm{d}s - \int_{\Omega} J \cdot E\mathrm{d}v; \tag{2.11}$$

where v denotes the velocity field and ds the outward directed vector element of area. In continuum mechanics, we are accustomed to thinking of stress as playing two roles, representing force and being involved in describing power and the latter does not fit comfortably with (2.10) and (2.11). Generally, those dealing with complete theories make some use of both interpretations, although different workers have different ideas about how best to do this.

Suppose that we are trying to use these ideas to formulate constitutive equations or assess versions proposed by others. There is the old problem that we want these to be invariant under Galilean transformations and, sometimes, the group of rigid motions or to use transformations to material coordinates, transformations under which (2.5) is not invariant. Here, we ignore (2.5). After we settle on the constitutive equations, we enforce (2.5). Essentially, this is how the Maxwell–Lorentz theory is often interpreted in getting various theories that are considered to be successful. Of course, when (2.5) does not apply, neither (2.6) nor (2.8) holds. However, with conventional transformation rules, (2.1–2.4) are invariant if properly interpreted and f_F transforms as a vector, making it reasonable to use this in considering constitutive equations. On the other hand, $J\cdot E$ does not transform as a scalar under the aforementioned transformations, so our considerations do not suggest a similar way of dealing with it. Thus, this is a difficulty that I will address. My limited experience is that different workers like different variations on (2.8) and this can lead to inequivalent theories. For example, in matter, Hutter and van de Ven [7, p. 51] recommend replacing e_F by something of a similar form that is invariant under the group of rigid motions. They [7, pp. 55, 61, 72] present three possibilities. As motivation for this, we have their statement on p. 51

> "Basically, balance laws of mass and momentum are derived from a global energy balance by postulating certain invariance properties."

At least in their formulation, it is important for this that the field energy have the indicated invariance. My view is that e_F describes the field energy in or outside matter. Thus, in this alone, we have a basic disagreement about energetics. They argue that various other writers have proposed theories equivalent to those they discuss. On the other hand, it is easy to find theories not using their ideas about energetics, for example Kovetz [2, Section 54]. This is not to say that his ideas about energetics agree with mine. Later, I will mention a disagreement. I will also discuss how Toupin's ideas on elastic dielectrics fit comfortably with mine. I do not think it fruitful to continue the discussion of such disagreements. Hutter and van de Ven [7] discuss transforming to material coordinates in some detail. Equations (2.1–2.4) take the same form in these.

When (2.5) does not apply, one can use (2.1–2.4) to deduce a replacement for (2.6) in the form

$$\frac{\partial p}{\partial t} - \nabla\cdot\widetilde{t}_F = f_F + g_F, \tag{2.12}$$

where $\widetilde{t}_F$ is the asymmetric tensor obtained by copying the prescription for t_F in (2.7) and

$$2g_F = (\nabla H)^T B - (\nabla B)^T H + (\nabla D)^T E - (\nabla E)^T D. \tag{2.13}$$

Similarly, (2.8) gets replaced by

$$E\cdot\frac{\partial D}{\partial t} + H\cdot\frac{\partial B}{\partial t} - \nabla\cdot e_F = -J\cdot E. \tag{2.14}$$

This equation has some interesting features, but I will not pursue this here. In the remainder of this section, I assume that the fields are smooth enough to qualify as classical solutions of all of these equations. Then, (2.6) and (2.8) might be viewed as identities and many writers so regard them. Granted this, you could replace f_M by the left side of (2.6) whenever you like, assuming (2.5) is enforced. Later, I will reconsider this. Here, I will note that in, say, linear elasticity theory, I could take the divergence of the stress tensor as a body force and use the usual constitutive equations to calculate a constitutive equation for it. I suspect that you would not be comfortable in accepting that this is a completely equivalent formulation. If this is correct, ask your self why this is. In these equations, there is no need to distinguish between free and bound charges, to make any particular assumptions about constitutive equations for matter or about how the fields are produced.

In his discussion of radiation pressure, Page [5, Sections 157–158] presents an idea that seems to me reasonable, given (2.6), which is that

(a) Electromagnetic fields can sustain forces.

For this, f_F is thought of as a force exerted by matter on fields, granted that Q=0 and J= 0 where there is no matter. Similarly, t_F can be thought of as describing a stress acting on fields. Another idea that Page presents is that

(b) The concept of action and reaction applies, in the sense that the fields exert a body force

$$f_m = -f_F = QE + J \times B \tag{2.15}$$

on matter. Properly interpreted, this also seems reasonable to me. From the discussion by Whittaker [6, Volume 1, pp. 274–275], I infer that ideas much like this occur in much older works. He does not mention anyone using (2.15), but I have not searched for this. Later, I will discuss a theory that uses this in formulating constitutive equations and also uses (2.6) after they are formulated. I will discuss how one can also use (2.8) for it. Of course, for matter, other kinds of forces need to be accounted for. However, this one does not depend on what kind of material is considered and other relevant forces are not likely to have this property, in my experience. I will consider generalizing this to apply the analogous ideas to (2.8), as formalized by

(c) $-\mathbf{J}\cdot\mathbf{E}$ represents a supply of energy by matter to fields, $\mathbf{J}\cdot\mathbf{E}$ a corresponding supply by fields to matter.

I do not know whether this idea has been used explicitly before. One argument for this is that, in special relativity theory, (2.6) and (2.8) fit together as a four-vector equation, and it seems odd to treat the parts differently. I find these ideas appealing because of their conceptual simplicity, but many theories are not really consistent with them.

Since we see bodies change shape, etc. when fields are applied, it seems natural to think that fields exert forces on matter. By the same token, we know that matter can alter fields, so is it not natural to think that matter exerts forces on fields? If we accept the idea that t_F represents some kind of force, its form suggests that fields can also exert forces on fields. While I have not found these ideas discussed in recent writings, I do think that they are worth considering, so I will explore them a bit.

Now, I consider another line of thought to be important. Electromagnetic theory is a field theory. One idea used in field theories is that of moving toward the goal of replacing action at a distance by appropriate field actions of a local nature, although classical electromagnetic theory does not quite achieve this goal. Consider the very simple static theory of dielectrics, employing a constitutive equation equivalent to

$$P = \alpha E, \ \alpha = \text{const.} \tag{2.16}$$

where P is the polarization density. Obviously, given a value of E at a point you get a value of P there, so this is a local relation. Of course, one needs to use additional theory to determine the values for a particular situation and, typically, this involves considering long range interactions, but this is not relevant to the constitutive equation. A similar situation occurs with constitutive equations in many theories dealing with electromagnetic interactions with matter so they are, in this sense, local theories. Similarly, the quantities occurring in (2.6) and (2.8) are prescribed by local equations.

With these ideas, we are getting closer to fitting t_F to Cauchy's idea of stress. However, his view associates stress with material surfaces and these are not defined outside matter, making it hard to know whether there is a good analog of Cauchy's stress. In non-relativistic fluid mechanics, workers sometimes use control surfaces that are at rest in some inertial frame that could be an aether frame. This involves rearranging Cauchy's equations of motion to be of the form which is more like (2.6),

$$\frac{\partial \rho v}{\partial t} - \nabla \cdot t_R = f, \tag{2.17}$$

where ρ is the mass density, v the velocity, f is a body force and t_R is related to Cauchy's stress t by

$$t_R = t - \rho v \otimes v. \tag{2.18}$$

The difference $\rho v \otimes v$ is often called the Reynolds stress, making it reasonable to call t_R a stress tensor. In a similar way, conventional energy equations can be arranged to look more like (2.8). In analyzing shock waves, we consider stresses associated with surfaces moving in a rather arbitrary way. So, even in old continuum theories, we use "stress" to denote things that are conceptually different. Such thoughts suggest to me that we should find a better way of thinking about such matters. In the special theory of relativity, workers often introduce stress-energy-momentum tensors, but I have not seen an analog of Cauchy's argument for stress or more sophisticated versions of it for existence of these. Obviously, it is important that the entities involved be considered as additive set functions, but what else do we want to say about them, granted that we accept the space–time structure used in special relativity theory? I think that filling this gap would be a step in the right direction. Or one might do this for the parts used in non-relativistic theories, avoiding reliance on material regions. I note that, in the kinetic theory of gases, a transport of linear momentum is, like the Reynolds stress, interpreted as a stress. In the rest of this paper, I will emphasize the lines of thought labeled as (a), (b) and (c), although I will discuss other things that seem useful for assessing theories.

3 Jump Discontinuities

In continuum mechanics, it is rather common to analyze jump conditions associated with stress waves, shock waves, twinning of crystals, etc. and the need for this remains when we add electromagnetic effects. So, it is pertinent to consider how well a theory can do in handling these. Commonly, electromagnetic fields undergo other kinds of jump discontinuities at boundaries of bodies and sometimes in the interior. For example, in dielectrics, Q is commonly taken to be the polarization charge $-\nabla \cdot P$. By common consent, at a surface of jump discontinuity, the normal component of the jump in $-P$ is interpreted as a surface charge, implying that $\nabla \cdot P$ should be viewed as a delta function. In (2.7), this is multiplied by E which is likely to undergo a jump discontinuity. So, this body force is not likely to be bounded or even well defined at such places. On the other hand, P and t_F will just undergo jump discontinuities and we are used to dealing with situations of this kind in continuum theory. Also, we have a contribution J_F to the current, given by

$$J_F = \frac{\partial P}{\partial t} + \nabla \times (P \times v), \tag{3.1}$$

what Truesdell and Toupin [1, p. 686] call "the current of polarization." Similarly, this can make the right side of (2.8) unbounded or ill-defined at surfaces of jump discontinuity. In magnetized materials, the amperian current can cause similar problems. For such reasons, I think it unwise to accept the idea that (2.6) and (2.8) are just identities and we have some reason to prefer the left sides. However, in considering constitutive equations, we generally consider the fields to be smooth, use different kinds of reasoning and disregard (2.5), so it can then be better not to do so. In an example, Truesdell and Toupin [1, Section 286] do use the left side of (2.6), but do not mention how this affects analysis of jump discontinuities or the idea that (2.6) and (2.8) are identities, which they accepted.

I suggest that, when you find a writer claiming that two theories are equivalent, pay attention to whether this accounts for relevant jump discontinuities. Some writers overlook this. Bear in mind that, in considering boundary conditions for vacuum-matter interfaces, this is somewhat like considering internal stress discontinuities in mechanics since those field stresses are generally not zero in vacuum. Physically, when one asserts that two theories are equivalent, one is implying that it is impossible to design an experiment to decide between them. So, we should expect a convincing argument that this is the case. Even given this, there is a possibility that someone might find a good theoretical reason to favor one over the other. So, I think it healthy to be somewhat skeptical about such claims.

4 Some Particular Theories

To illustrate some of the ideas that have been presented, I find it instructive to examine how experts have wrestled with these issues in a way that provides some test of ideas discussed in Section 2. I will pick just one possibility. Toupin is well known for his theories of (non-magnetizable) elastic dielectrics and his thinking has moved toward using the general ideas discussed in Section 2. He did not explicitly endorse (a), (b) and (c) but, in effect, he came to use (a) and (b). At first, he [8] proposed a static theory, using a principle of virtual work, introducing as a field body force f and body couple 1 given by

$$f = \left(\nabla\overline{E}\right)P, \ \left(f_i = \overline{E}_{i,j}P_j\right), \ 1 = P \times \overline{E}, \tag{4.1}$$

where $\overline{E}$ is a given smooth vacuum field in the union of regions the dielectric is allowed to occupy, so $\nabla\overline{E} = \left(\nabla\overline{E}\right)^T$. This is rather conventional. In this work, material coordinates are used quite a bit, although spatial coordinates are also used. Obviously, f and 1 remain bounded where P undergoes a jump discontinuity, merely suffering jump discontinuities. He also introduces as a field stress in matter one attributed to Maxwell, the asymmetric tensor.

$$\widehat{t} = \varepsilon_0\widehat{E} \otimes \widehat{E} + \widehat{E} \otimes P - \frac{1}{2}\varepsilon_0\widehat{E}\cdot\widehat{E}1 \tag{4.2}$$

where the hats denote self fields for the dielectric. In this first effort, he uses a Lagrangian density of the form

$$\rho\Sigma - \widehat{E}\cdot P - \frac{1}{2}\varepsilon_0\left|\widehat{E}\right|^2 1, \widehat{E} = -\nabla\varphi, \tag{4.3}$$

with a constitutive equation of the form

$$\Sigma = \overline{\Sigma}(F, P), \tag{4.4}$$

where F is the deformation gradient used in nonlinear elasticity theory, this function being invariant under the group of rotations. Actually, there is a difference in sign between the second term in (4.3) and the correspondent in Toupin's Equation (10.10), which I attribute to his typo. My version but not his is consistent with his (10.22), that I consider to be correct.

Brown [4, Section 2] criticized (4.2), mentioning that, even in Maxwell's time, experts questioned this. Also, he [4, p. 78] notes that (4.3) is not well-suited for treating stability of equilibrium because extremals that it generates do not correspond to minima or maxima, being more like those associated with saddle points. Brown also complained about typos in Toupin's paper: I just mentioned one that is relevant here.

Toupin deduces that the matter stress is described by

$$\widehat{t}_{Mij} = \rho F^i_\alpha \frac{\partial \widehat{\Sigma}}{\partial F^j_\alpha}, \tag{4.5}$$

where the Greek indices refer to material coordinates, an equation for the total electric field equivalent to

$$E = \rho \frac{\partial \widehat{\Sigma}}{\partial P} \tag{4.6}$$

and the equations

$$\widehat{t}_M = \widehat{t}^T_M = P \otimes E = E \otimes P, \nabla \cdot t + \left(\nabla \overline{E}\right) P = 0, \left(t_{ij \cdot j} + \overline{E}_{i,j} P_j = 0\right), t = \widehat{t} + \widehat{t}_M. \tag{4.7}$$

On his own, Toupin [9] did eliminate the assumption that (4.2) holds, deriving formulae for stress tensors in a rather general way, using a variational method. So I regard this assumption as obsolete. I think that this negates the first objection by Brown mentioned above. This includes a proposal for elastic dielectrics based on a choice of a Lagrangian which, again, is not well suited for using minimum energy criteria for stability. This seems reasonable for getting equilibrium equations etc. and is preferable to assuming (4.2). In this, he uses spatial coordinates as independent variables and does not split the electric field into internal and external parts. For the dielectric, this gives a symmetric total stress tensor. The field stress is given in part by (2.6), but the term $E \otimes P$ is added to this. He uses an equivalent of $\widehat{t}_M$ as the matter stress. Clearly, the difference is something that could be regarded as a contribution to field stress but, in the theory to be considered next, Toupin found a natural way to regard it as part of the matter stress. This merely exemplifies the old idea that one can get equivalent theories with different descriptions of field stress, when the difference is accounted for in describing contributions for matter. This does make it a little tricky to decide when or whether to use (2.7) as a basic description of field stress.

A third version, this time covering dynamical theory of dielectrics, was presented by Truesdell and Toupin [1, Section 312]. This does not introduce a field stress, using the body force f_M given by (2.15) with

$$Q = -\nabla \cdot P, J = J_F, \tag{4.8}$$

using (3.1). For reasons given above, I think it better to replace this body force by the left side of (2.6). However, in a later presentation of this, Toupin [10] does replace the body force by this left side, when he considers jump conditions, as I would do. There is a difficulty which might seem to be hard to overcome. As a volume source of energy, he does not use $J \cdot E$, postulating instead, for a reduced energy equation,

$$(J - Qv) \cdot (E + v \times B) = J \cdot E - (QE + J \times B) \cdot v, \tag{4.9}$$

which does fit neatly with his ideas about constitutive equations. I know of no complaints about the latter. Shortly, I will reconcile this with (c) in Section 2. The left side of (4.9) does transform as a scalar under the group of rigid motions and obviously reduces to $J{\cdot}E$ when v =0 So, this is how he dealt with the difficulty in item (c) described in Section 2. However, this is not likely to be well defined at jump discontinuities in the fields, so I will come back to this. This is not very important for what he does. For smooth fields, his energy equation reduces to an identity. At least for interior surfaces of jump discontinuity, one should use the idea that energy can be dissipated on these to get admissibility conditions for these. So, this is a rough spot in his theory. In various places, Hutter and van de Ven [7] discuss different forms of entropy inequalities somewhat like the Clausius–Duhem inequality that are likely be useful for fixing this and including temperature effects, although this involves deciding which version to use.

His treatment of forces is consistent with the ideas of action and reaction discussed in Section 2. Here, he replaces (4.4) by the function

$$\Sigma = \widetilde{\Sigma}(F, \pi) \text{ with } P = \left(\det\ F^{-1}\right)F\pi, \tag{4.10}$$

π being a representation of P in material coordinates, with the analogous assumption for other constitutive equations. He deduces that the matter stress $\widetilde{t}_M$ is obtained by replacing $\widehat{\Sigma}$ by $\widetilde{\Sigma}$ in (4.4), from which it follows that

$$\widetilde{t}_M = \widetilde{t}^{\,T}_M. \tag{4.11}$$

With this, it follows that his total stress tensors are symmetric when the left side of (2.6) is used. One could use the traditional argument to get this symmetry, but reasoning based on energetics seems to me to be satisfactory, for this kind of theory. He does not say much about the relation between this theory and his equilibrium theories, which use very different ideas. Whether the Lagrangians he uses in these is related in a natural way to the energies considered in this version is a question left to the reader. He uses spatial coordinates for some analyses, material ordinates for others, depending on which is simpler to use. He uses material coordinates in considering constitutive equations and in his nice treatment of linearized theories, which does make the analyses neater. However, in considering jump conditions, he uses spatial coordinates, which is simpler. For what he does, it is not useful to decompose the fields into external and interior parts.

There are some advantages to using material coordinates, but there are also some disadvantages, related to the fact that (2.5) does not hold in these. Ponder how you would do the equivalent of Toupin's treatment of jump conditions using material coordinates and you might agree. Hutter and van de Ven [7, Section 1.5] do include some jump conditions obtained using material coordinates, based on (2.1–2.4). Earlier, Toupin [8] tried doing something like this, abandoning it in later work. So, I think that it is good to understand how to do the transformations relating spatial and material coordinates to best fit the kind of analysis being considered, as Toupin [10] did.

I will now present my way of rationalizing that energy supply and better dealing with those jump conditions. It involves accepting that there are two kinds of body forces that I [11, 12] called external and internal. The idea is that, an external body force f_E can be prescribed in different ways for the same material, whereas an internal body force f_I is intimately related to constitutive equations, as is E in (4.6). The total body force is $f = f_E + f_I$. In the contribution of this to power in the energy equation, use $f_E{\cdot}v$ not $f{\cdot}v$. In Toupin's theory, start with equations of motion of the form

$$\rho\frac{\mathrm{d}v}{\mathrm{d}t} = \nabla{\cdot}t_M + f_E + f_I, \tag{4.12}$$

where d/dt denotes the material derivative and t_M is assumed to be a function of F and his π. As suggested in Section 2, take

$$f_I = f_M = QE + J \times B. \tag{4.13}$$

To get to his energy equation, I start with a form that is consistent with the ideas mentioned in Section 2,

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\left(\widetilde{\Sigma} + \frac{1}{2}|v|^2\right) = J\cdot E + \nabla\cdot\left(t_M^T v\right) + f_E\cdot v, \tag{4.14}$$

with Toupin's assumptions about the general form of constitutive equations. Toupin does not present this equation. Hutter and van de Ven [7. p.71] cite writers that have used Toupin's description of the sources. While I have not used entropy inequalities, I will give my interpretation of implications of some that have been proposed. I note that, if we accepted Kovetz' [2, Section 55] recommendation, we should add the divergence of an electromagnetic energy flux, which is of the form $-(E + v \times B) \times (H - v \times D)$, also implying that something is wrong with Toupin's description of energetics. Kovetz clearly states that this is an hypothesis. On the other hand, by what I consider to be better reasoning, Hutter and van de Ven [7, Equation (2.133)] suggest using instead $-(E + v \times B) \times M$ for theories rather similar to those considered here, M denoting the magnetization density. Clearly, this vanishes for the non-magnetizable materials considered here, as is tacitly assumed in (4.14).

In many theories, f_M is given by a constitutive equation, as is the case in Toupin's theory. However, he describes an old equation proposed by Voigt for time dependent fields in matter at rest, one that is perhaps best regarded as an equation of motion for P. Here, I refer to the special case involving just one molecular species. In it, f_M is one term. Granted this interpretation, this is not really a constitutive equation for f_M. As is now commonly done in considering constitutive equations, I assume f_E can be assigned at will, so solve (4.12) for this, substitute this in (4.14) and use (4.9), to get the reduced energy equation

$$\rho \frac{\mathrm{d}\widetilde{\Sigma}}{\mathrm{d}t} = (J - Qv)\cdot(E + v \times B) + t_{Mij} v_{i,j}. \tag{4.15}$$

To get his prescriptions for Q and J given by (4.8), use (2.3) and (2.4) in the form

$$D = D' - P, H = H' + P \times v \tag{4.16}$$

with

$$\nabla\cdot D' = 0, \nabla \times H' - \frac{\partial D'}{\partial t} = 0. \tag{4.17}$$

This gives the equivalent of (4.15) used by Toupin, who requires that it be satisfied identically, to get his constitutive equations. His treatment of this does not require that (2.1), (2.2) or (2.5) be satisfied. After getting the constitutive equations, enforce these, as he does in his treatment of jump conditions. This involves replacing f_m by the left side of (2.6) in (4.12). Similarly, one can replace $J\cdot E$ by the left side of (2.8) in (4.14), which reintroduces the field momentum and energy given in (2.9). The fact that these are not even invariant under Galilean transformations makes nontrivial differences between this theory and various others occurring in the literature. In any event, this evades the difficulty that these sources are not likely to be well defined at jump discontinuities. So, this describes how the Maxwell–Lorentz theory can be interpreted for this theory, in a manner that is consistent with the ideas discussed in Section 2. My view is that one should take the reformulation without the volume sources as more basic. So, why not start with this? To get (2.6) and (2.8), we needed to use (2.5) and, in treating constitutive equations, transformations are used under which (2.5) is not invariant, for example transformations to material coordinates. So one has to use some judgment as to how best to deal with this. My discussion should make it easier to compare Toupin's theory with others that are somewhat similar.

I continue the discussion for equilibrium theory of non-magnetizable dielectrics. At least for equilibrium theory, there is a reason to consider the electric field as the sum of an external field $\overline{E}$ and a self field $\widehat{E}$. Very often, workers introduce as $\overline{E}$ a vacuum field, given in some bounded region of space, restricting regions occupied by the body of interest to be sub regions of it. At least tacitly, it is assumed that the source of it is not affected by $\widehat{E}$. Suppose that $\overline{E}$ is generated by some distribution of charge away from the body. Then, $\widehat{E}$ will exert a force on this which is not likely to leave the distribution unchanged. So, one is assuming that such effects are negligible, giving an approximate theory, hopefully good, when it is used. I do not advocate discarding all calculations of this kind. As another point, I agree with the view of Hutter and van de Ven [7] to the effect that any electromagnetic problem should be solved in all of space but, often, workers make compromises with this that can be reasonable. Consider the field energy e_F given by (2.9) with $E = \overline{E} + \widehat{E}$. Being a quadratic, the total is not just the sum of the energies for the individual parts and a similar remark applies to t_F or to likely replacements preferred by some. Obviously, with $\overline{E}$ given only in a bounded region, we cannot calculate the value of the integral of e_F over all of space, for example. I will describe one way this difficulty has been dealt with in an equilibrium theory. Generally, it is easy to see how workers have done such calculations but, often, it is not easy to see how this can be reconciled with comparable theories not splitting the fields into the two parts.

In theories of stability of equilibrium, I think that it is generally accepted that one should account for self fields, which extend throughout space. Commonly, workers do consider minimizing the density of some thermodynamic potential, but this is certainly not sufficient to treat many instabilities. As a step toward bringing equilibrium theory into line with the general ideas of field energy, I [13] decided to try revising Toupin's equilibrium theories to enable use of minimum energy ideas, in cases where only an external field acts on the dielectric, when the source of the external field is not in contact with the latter. I accepted the simplification just discussed. In the analyses, I used spatial coordinates. In particular, my assumptions exclude having conductors contacting the boundary of the body of interest, which is involved in some important applications. I assumed that the fields considered are smooth except for isolated jump discontinuities. For this kind of theory, $B = H = 0$. We want to minimize energy in all of space. For the field energy, we start with what the prescription given by $(2.9)_1$, using (2.5) to put it in the form

$$e_F = \frac{|D|^2}{2\varepsilon_0}, \tag{4.18}$$

which is obviously bounded below. As noted before, this fits Toupin's dynamical theory reasonably well. In some bounded region $\overline{\Omega}$ we are given the external field, a smooth field satisfying

$$\overline{D} = \varepsilon_0 \overline{E}, \nabla \cdot \overline{E} = 0, \nabla \times \overline{E} = 0 \tag{4.19}$$

and we have to deal with the difficulty that this is not known elsewhere in space. The body will occupy a variable region $\widehat{\Omega} \subset \overline{\Omega}$ and involve static self fields satisfying

$$\widehat{D} = D' - P = \varepsilon_0 \widehat{E}, \nabla \cdot D' = 0, \tag{4.20}$$

as well as being consistent with the static version of Maxwell's equations. The field energy (4.18) can then be put in the form

$$\frac{|D' - P|^2}{2\varepsilon_0} + (D' - P)\cdot\overline{E} + \frac{\overline{D}\cdot\overline{E}}{2}, \tag{4.21}$$

at least where $\overline{E}$ is given, with $P=0$ outside $\widehat{\Omega}$

Now, I will sketch the plausibility arguments I used to deal with our difficulty. Consider the last term in (4.2) as the field energy density of the source. Generally, self fields are smooth, static vacuum fields away from the source, approaching zero at infinity fast enough for the total energy to converge. In the source, allow jump discontinuities and $(4.19)_2$ to be violated. With our simplification, one could do a correct calculation of it with the dielectric absent. Integrating the last term over all of space then gives an energy

$$\overline{e} = \int_{E_3} \frac{\overline{D}\cdot\overline{E}}{2}\mathrm{d}v = \text{const.} \tag{4.22}$$

So, we can drop this term without affecting minimizers. Another difficulty occurs with the term

$$D'\cdot\overline{E}, \tag{4.23}$$

now thought of as involving the two self-fields, one being smooth where the other suffers jumps, which should satisfy the usual condition that tangential components of E are continuous. With all this, integrating this over $\widehat{\Omega}_C$, the complement of $\widehat{\Omega}$ introducing a vector potential $\mathbf{A}$ for D' using the divergence theorem and accounting for jump discontinuities gives

$$\int_{\widehat{\Omega}_C} D'\cdot\overline{E}\,\mathrm{d}v = \int_{\widehat{\Omega}_C} \nabla\times A\cdot\overline{E}\,\mathrm{d}v = -\int_{\partial\widehat{\Omega}} A\times\overline{E}\cdot\mathrm{d}s, \tag{4.24}$$

where the latter integrand is evaluated on the side exterior to $\widehat{\Omega}$, ds being taken outward relative to this region. With these evaluations, the total field energy can be calculated using given values of $\overline{E}$. Then, assuming as a jump condition that the normal component of D' is continuous, equivalent to Toupin's [8] Equation 6.7, I used similar arguments to conclude that the field energy density can be simplified to the form

$$\widetilde{e}_F = \frac{|D' - P|^2}{2\varepsilon_0} - P\cdot\overline{E}. \tag{4.25}$$

Note that this looks quite different from the (4.21) I started with. As was noted earlier, Toupin [8] added an energy depending on deformation gradients and polarizations, as he also did in [9]. This is a common practice in theories of elastic dielectrics, and I used an equivalent of the form

$$w = \rho\Sigma = \widehat{w}\left(F^{-1}, P\right). \tag{4.26}$$

Employing rather standard techniques in the calculus of variations, I generated equilibrium equations and jump conditions and, from these, inferred a description of a total stress tensor t' satisfying

$$\nabla\cdot t' + \left(\nabla\overline{E}\right)P = 0, \left(t'_{ij,j} + \overline{E}_{i,j}P_j = 0\right), \tag{4.27}$$

and

$$t' - t'T = P \otimes \overline{E} - \overline{E} \otimes P, \tag{4.28}$$

fitting a common way of accounting for body forces and couples associated with $\overline{E}$, in theories using the kind of simplification I am discussing. Here, t' can be put in the form

$$t' = \widehat{t}_M + \overline{E} \otimes D' - \left| P \cdot \left(\widehat{E} + \overline{E} \right) + \frac{\varepsilon_0 \left| \widehat{E} \right|^2}{2} \right| 1, \tag{4.29}$$

$\widehat{t}_M$ being given in terms of derivatives of $\widehat{w}$ by a formula equivalent to that used in (4.5). Obviously, this is not the same as that obtained with the Maxwell stress given by (4.2) used by Toupin [8], but I noted that, had he used his Lagrangian with ideas similar to mine or his [9], he could have deduced an equivalent of (4.29). So, there is some consistency with his oldest views. There is still a difference, in that my formulation is more suitable for using minimum energy criteria for stability, although he allowed for mechanical loadings and I did not: it is in fact rather tricky to deal with most of these in a realistic way. While I have attempted to relate this simplified theory to the general theory, I have not provided means of assessing the errors made in using this kind of approximation. One should try to determine how consistent stability theory of this kind is with Toupin's dynamical theory or a variation made consistent with thermodynamics, something I have not tried to do. For example, it would be of some help to know if one can construct a Lyapounov functional based on such theory, indicating that some thermodynamic potential is a non-increasing function of time, for the kinds of situations considered in my equilibrium theory. Hopefully, the potential would be consistent with what I used. In any event, these studies give some qualified support for the idea that one can get reasonable theories using the field contributions given in (2.6–2.9). Whether theories covering magnetism can be treated in a somewhat similar way, using the ideas in Section 2, is not clear from what I have written. This can be done, but it requires using rather different lines of thought and some background not needed here, so I will treat this in another paper.

In the equilibrium theories I have discussed, prescriptions for stress tensors are deduced from energetic considerations, related more to virtual work, essentially ignoring (2.7), which takes no account of constitutive equations. So, conceptually, they are different. In my theory, one could add to the stress tensor any smooth symmetric stress tensor with zero divergence without affecting the equilibrium equations or jump conditions. Adding that related by (2.7) to $\overline{E}$ would make mine more similar to that given in (2.7), for whatever that is worth. In any event, this gives two stress tensors that are equivalent for the particular calculations I considered. Of course, some difference with the general theory might be expected, since my theory uses those hopeful approximations.

In this Section, I have only considered a small part of electromagnetic theory and restricted my attention to Toupin's views, except for my little contributions. As an amateur, I picked this kind of theory because I am fairly familiar with it, and picked Toupin because he is an expert who, in essence, used the ideas about force discussed in Section 2, and I fit the ideas about energy to his theory. So, this illustrates how the ideas covered in Section 2 have been made to work, for a special kind of theory, and some ways of assessing theories.

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energy to his theory. So, this illustrates how the ideas covered in Section 2 have been made to work, for a special kind of theory, and some ways of assessing theories.

Toupin is certainly not the only person to have had trouble with these issues. For example, Brown [4, Section 5.4] caused some consternation by deducing a field stress vector for magnetized materials that was cubic in the normal. I do regard Brown as an expert in this area. Using a mechanistic calculation, Schlömerkemper [14] concluded that he was wrong about this and I agree with her. The theories I have discussed do not allow for a dependence on magnetization, excluding dielectrics that are paramagnetic, for example, which do exist. Other writers have proposed such theories. Hutter and van de Ven [7, Chapters 2–4] compare older theories of this kind. Although this reference is rather old, it does include useful thoughts about theories of this general kind, and they are more expert than am I in electromagnetic theory. In a critique of theories of magnetism neglecting electric effects, I [15] cite some newer references. Also, Kovetz [2, Chapter 15] presents a version of theory of this kind that he favors. I have not tried to collect all theories of this kind.

The first dielectric material that was also paramagnetic was made by Wilson and Wilson [16], who used it in experiments showing that, for matter moving with a speed that is very small compared to the speed of light, correcting for a relativistic effect gives a better fit to their data, a result that is accepted by experts. So, this is one case where a special relativistic correction gives something measurable for slowly moving matter that differs from what is predicted by the non-relativistic theories. I [15] discussed this and a related modern experiment, partly because some writers seem not to be aware of this finding. This is all that I want to say about theories of magnetizable dielectrics.

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